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Extensions of Algorithms for D-Finite Functions DD-Finite and Beyond

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Known Results on D-Finite Functions

Definition. *Let K be a field and $f \in K[[x]]$. We say that f is D-Finite if, for some polynomials $p_0(x), ..., p_d(x)$, it satisfies the linear differential equation*

$$p_0(x)f(x) + p_1(x)f'(x) + ... + p_d(x)f^{(d)}(x) = 0.$$

Closure properties on D-Finite Functions[2, 3]

- *Addition:* if f, g are D-Finite, then $f + g$ is D-Finite.
- *Product:* if f, g are D-Finite, then fg is D-Finite.
- *Derivative:* if f is D-Finite, then $f'(x)$ is D-Finite.
- *Antiderivative:* if $g'(x) = f(x)$ and f is D-Finite, then g is D-Finite.
- *Algebraic:* if $f(x)$ is algebraic over $K(x)$, then f is D-Finite.
- **Algebraic substitution:** let $f(x)$ be D-Finite and $a(x)$ an algebraic function with $a(0) = 0$. Then $f(a(x))$ is D-Finite.

Examples of D-Finite Functions

$$\begin{aligned} e^x &\longrightarrow (e^x)' - e^x = 0 \\ \sin(x) &\longrightarrow \sin''(x) + \sin(x) = 0 \\ J_n(x) &\longrightarrow x^2 J_n''(x) + xJ_n'(x) + (x^2 - n^2)J_n(x) = 0 \end{aligned}$$

Examples of non D-Finite Functions

$e^{e^x - 1}$	*because*	e^x is not algebraic
$\tan(x)$	*because*	$\tan(x)$ is not algebraic
$\Gamma(x+1)$	*because*	has too many singularities

New Theory: *Differentially Definable Functions*

Definition

Let R be a differential subring of $K[[x]]$. We say that $f(x) \in K[[x]]$ is *differentially definable over* R if, for some elements $r_0(x), ..., r_d(x)$ in R, f satisfies the linear differential equation:

$$r_0(x)f(x) + r_1(x)f'(x) + ... + r_d(x)f^{(d)}(x) = 0.$$

- $\mathrm{DD}(R)$: the set of all $f(x)$ differentially definable over R.
- $\mathrm{ord}_R(f)$: the minimal order of equations that f satisfies with coefficients in R.

Characterization Theorem [1] *Let R be a differential subring of $K[[x]]$, F its field of fractions and $f(x) \in K[[x]]$. It is equivalent:*

1. $f(x) \in \mathrm{DD}(R)$.

2. There are $r_0(x), ..., r_d(x) \in R$ and $g \in \mathrm{DD}(R)$ such that $f(x)$ satisfies the **inhomogeneous** *linear differential equation*

$$r_0(x)f(x) + r_1(x)f'(x) + ... + r_d(x)f^{(d)}(x) = g(x).$$

3. The F-vector space generated by $\{f(x), f'(x), f''(x), ...\}$ has **finite** *dimension.*

Theorem (Closure Properties)[1]

Let R be a differential subring of $K[[x]]$ and $F = Q(R)$ its field of fractions. Then

- *Addition:* if $f, g \in \mathrm{DD}(R)$, then $(f + g) \in \mathrm{DD}(R)$.
- *Product:* if $f, g \in \mathrm{DD}(R)$, then $(fg) \in \mathrm{DD}(R)$.
- *Derivative:* if $f \in \mathrm{DD}(R)$, then $f'(x) \in \mathrm{DD}(R)$.
- *Antiderivative:* if $g'(x) = f(x)$ and $f \in \mathrm{DD}(R)$, then $g \in \mathrm{DD}(R)$.
- *Algebraic:* if $f(x)$ is algebraic over F, then $f \in \mathrm{DD}(R)$.
- **Division:** if $f \in R$ and $f(0) \neq 0$, then $1/f \in \mathrm{DD}(R)$.

A Particular Case: *DD-Finite* Functions

Definition

Let K be a field and $f \in K[[x]]$. We say that f is DD-Finite if, for some D-Finite Functions $g_0(x), ..., g_d(x)$, it satisfies the linear differential equation

$$g_0(x)f(x) + g_1(x)f'(x) + ... + g_d(x)f^{(d)}(x) = 0.$$

$$\begin{array}{ccccc} \text{Polynomials} & \longrightarrow & \text{D-Finite} & \longrightarrow & \text{DD-Finite} \\ K[x] & \longrightarrow & \mathrm{DD}(K[x]) & \longrightarrow & \mathrm{DD}(\mathrm{DD}(K[x])) \end{array}$$

Examples of DD-Finite Functions

- **Tangent:** the tangent function $(\tan(x))$ is DD-Finite:

$$\cos^2(x)(\tan(x))'' - 2\tan(x) = 0.$$

- **Rational powers:** let $f(x)$ be D-Finite and $\alpha \in \mathbb{Q}$. Then $g(x) = f^\alpha(x)$ is DD-Finite:

$$f(x)g'(x) - \alpha f'(x)g(x) = 0.$$

- **Mathieu's functions:** let $a, q \in K$. Then any solution $w(x)$ of the following equation is DD-Finite:

$$w''(x) + (a - q\cos(2x))\, w(x) = 0.$$

Unique Representation: *Initial Values*

$$\text{Computer Representation} \longrightarrow \left\{ \begin{array}{c} \text{Differential Equation} \\ + \\ \text{Initial Values} \end{array} \right.$$

$$\begin{array}{rl} f''(x) + f(x) & \longrightarrow 2 \text{ initial values: } f(0), f'(0) \\ xf''(x) + f'(x) + xf(x) & \longrightarrow 1 \text{ initial value: } f(0) \\ xf'(x) - nf(x) & \longrightarrow 1 \text{ initial value: } f^{(n)}(0) \end{array}$$

Problem: Initial Values

- **Problem:** given a linear differential equation with coefficients in $K[[x]]$, determine **how many** and **which** initial values are needed to define a unique solution.
- **Solution:** go to the sequence level:

$$g(x) \in K[[x]] \longrightarrow g(x) = \sum_{n\geq 0} g_n x^n \longrightarrow \mathbf{g} = (g_n)_{n\geq 0}$$

Let $f(x), r_0(x), ..., r_d(x) \in K[[x]]$. Then:

$$r_0(x)f(x) + r_1(x)f'(x) + ... + r_d(x)f^{(d)}(x) = 0, \qquad \forall x$$

$$\Updownarrow$$

$$\sum_{k=0}^{n+d} \underbrace{\left(\sum_{l=\max\{0,k-n\}}^{\min\{d,k\}} k^{\underline{l}} r_{l;n-k+l} \right)}_{m_{n,k}} f_k = 0, \qquad \forall n \in \mathbb{N}$$

Theorem [1]

Let $r_0(x), ..., r_d(x) \in K[[x]]$ such that $r_i(0) \neq 0$ for some i. There are $d_0 \in \{0, ..., d\}$ and $n_0 \in \mathbb{N}$ (both computable) such that the dimension of the solution space for the equation induced by $r_0(x), ..., r_d(x)$ is the dimension of the right nullspace of:

$$M = (m_{n,k})_{0\leq n\leq n_0}^{0\leq k\leq n_0+d_0}.$$

Software

The closure properties can be executed entirely automatically. I have developed and implemented these algorithms in *SAGE*, a free open-source mathematics software system.

References

[1] A. Jiménez-Pastor and V. Pillwein. A computable extension for holonomic functions: DD-Finite functions. *In preparation*, 2017.

[2] M. Kauers and P. Paule. *The Concrete Tetrahedron: Symbolic Sums, Recurrence Equations, Generating Functions, Asymptotic Estimates*. Springer Publishing Company, Incorporated, 1st edition, 2011.

[3] C. Mallinger. Algorithmic Manipulations and Transformations of Univariate Holonomic Functions and Sequences. Master's thesis, RISC, J. Kepler University, August 1996.