Towards a Direct Method for Finding Hypergeometric Solutions of Linear First Order Recurrence Systems

Johannes Middeke   Carsten Schneider
Research Institute for Symbolic Computation
Johannes Kepler University
Altenbergerstraße 69, A-4040 Linz, Austria
jmiddeke@risc.jku.at   cschneid@risc.jku.at

Overview

We consider a difference field extension $(\mathbb{F}, \sigma)$ of a difference field $(\mathbb{K}, \sigma)$ where $\mathbb{F} = \mathbb{K}(x_1, \ldots, x_n)$ and where $\sigma(x_k) = \alpha_k x_k + \beta_k$ with $\alpha_k \in \mathbb{K}^*$, $\beta_k \in \mathbb{K}$ and $(\alpha_k, \beta_k) \neq (1, 0)$ for $k = 1, \ldots, n$.

Definition 1. Let $(\mathbb{G}, \sigma)$ be a difference ring extension of $(\mathbb{F}, \sigma)$. Then $\gamma \in \mathbb{G}^*$ is ***hypergeometric*** over $\mathbb{F}$ if $\sigma(\gamma)/\gamma \in \mathbb{F}$.

We consider a first order ***linear difference system***

$$\sigma(v) = Av \qquad \text{where} \qquad A \in \mathrm{GL}_s(\mathbb{F}). \qquad (\textsc{sys})$$

We are interested in finding hypergeometric solutions of the form $v = \gamma w$ where γ is hypergeometric and $w \in \mathbb{F}^s \setminus \{0\}$.

Other Methods

In the case $n = 1$, there are already methods available for finding hypergeometric solutions: First, one can decouple the system using, for example, [3, 7] and then solve the resulting higher order scalar equations using [6, 4].

Recently, an alternative algorithm was given in [2] where the authors provide a way of deriving scalar equations from a system which is cheaper than the traditional uncoupling method.

In contrast, our approach tries to avoid scalar equations altogether and is suitable for $n \geq 1$.

Initial Results

Let $T \in \mathrm{GL}_s(\mathbb{F})$. If we substitute $v = Tv'$ in equation (SYS), then we obtain

$$\sigma(v') = \sigma(T)^{-1} AT\, v'.$$

That is, we have a new system of the same shape but in new unknowns. We call $T[A] = \sigma(T)^{-1}AT$ the ***gauge transformation*** of A by T.

Definition 2. Assume that for $A \in \mathbb{F}^{s\times s}$, $\lambda \in \mathbb{F}$ and $q \in \mathbb{F}^s \setminus \{0\}$ we have

$$\lambda\sigma(q) = Aq.$$

Then we call λ a ***σ-eigenvalue*** of A and q a ***σ-eigenvector***.

If $q \in \mathbb{F}^s$ is a σ-eigenvector for the σ-eigenvalue λ and if $\alpha \in \mathbb{F}^*$, then

$$\sigma(\alpha)^{-1}\lambda\alpha\, \sigma(\alpha q) = A(\alpha q);$$

that is, αq is also a σ-eigenvector but for the new σ-eigenvalue $\sigma(\alpha)^{-1}\lambda\alpha$. We say that λ and $\sigma(\alpha)^{-1}\lambda\alpha$ are ***gauge equivalent***.

Lemma 3. *σ-Eigenvectors for pairwise not gauge equivalent σ-eigenvalues are linearly independent over* $\mathbb{F}$.

Main Result

In the following we let $\mathbb{A} = \mathbb{K}[x_1, \ldots, x_n]$ denote the polynomials. Note that $(\mathbb{A}, \sigma)$ is a difference subring of $\mathbb{F}$.

Let $a \in \mathbb{F}$ be such that $q = aw \in \mathbb{A}^s$ with $\gcd(q) = 1$ and let $\eta = \gamma/a$. Also η is hypergeometric. Let $\lambda = \sigma(\eta)/\eta$. Then (SYS) becomes

$$Aq = \frac{\sigma(\eta)}{\eta}\sigma(p) = \lambda\sigma(q).$$

That is, q is a σ-eigenvector for λ.

Moreover, we can complete q to a matrix $T = (q, *, \cdots, *) \in \mathbb{A}^{s\times s}$ with $\det T = 1$. We have

$$T[A] = \sigma(T)^{-1}AT = \begin{pmatrix} \lambda & * \\ 0 & * \end{pmatrix}.$$

Multiplying by $d = \mathrm{denom}(A)$ and taking determinants on both sides yields $d\lambda \mid \det dA \in \mathbb{A}$.

Theorem 4. *Let $v = \gamma w$ be any solution of* (SYS) *with hypergeometric γ and $w \in \mathbb{F}^s \setminus \{0\}$. Then we can rewrite v as $v = \eta q$ where*

(a) η is hypergeometric over $\mathbb{F}$ with $\sigma(\eta)/\eta = \lambda \in \mathbb{F}$,
(b) $q \in \mathbb{A}^s$ with $\gcd(q) = 1$,
(c) $d\lambda \mid \det dA$ where $d = \mathrm{denom}(A)$.

Moreover, A has λ as a σ-eigenvalue with σ-eigenvector q; and there exists a gauge transformation $T \in \mathbb{A}^{s\times s}$ such that $\det T = 1$ and

$$T[A] = \begin{pmatrix} \lambda & * \\ 0 & * \end{pmatrix}.$$

Corollary 5. *If A has s linearly independent σ-eigenvectors $q_1, \ldots, q_s$, then with $T = (q_1, \ldots, q_s) \in \mathrm{GL}_s(\mathbb{F})$ we have*

$$T[A] = \mathrm{diag}(\lambda_1, \ldots, \lambda_s)$$

where $\lambda_1, \ldots, \lambda_s$ are the corresponding σ-eigenvalues.

Example

Consider the differential ring $\mathbb{F} = \mathbb{Q}(x, y)$ where the automorphism σ is given by the substitutions $\sigma(x) = x + 1$ and $\sigma(y) = 2y$. (That is, y models 2^n in the sequence world.) Let

$$A = \begin{pmatrix} \frac{2x^2-xy-2y^2+2x+y}{(x+y)^2} & -\frac{(4x^2+3xy+2x+y)y}{(x+y)^2} \\ -\frac{3x+2y}{(x+y)^2} & -\frac{x^2-xy-y^2}{(x+y)^2} \end{pmatrix} \in \mathrm{GL}_2(\mathbb{F}).$$

A hypergeometric solution is given by $\gamma(x, -1)^t$ where γ corresponds to

$$\prod_{k=1}^{n-1} \frac{2k + 2^k}{k + 2^k}$$

in the sequence world. With

$$\lambda = \frac{\sigma(\gamma)}{\gamma} = \frac{2x+y}{x+y} \qquad \text{and} \qquad d = \mathrm{denom}(A) = (x+y)^2$$

we obtain—as the theorem predicts—

$$d\lambda = (x+y)(2x+y) \mid \mathrm{denom}(dA) = -(x+2y+1)(2x+y)(x+y)^2.$$

Algorithmic Concerns

The theorem suggests the following method for finding hypergeometric solution of (SYS):

> For each divisor $d\lambda \mid \det dA$ where $d = \mathrm{denom}(A)$ find all solutions $q \in \mathbb{A}^s$ for $\sigma(q) = \lambda^{-1}Aq$. Each pair (λ, q) yields a hypergeometric solution $v = \eta q$ where $\sigma(\eta)/\eta = \lambda/d$.

Thus, we can reduce the problem of computing (one) hypergeometric solutions to computing (several) polynomial solutions similar to [6]. Computing polynomial solutions can be done with, for example, [1].

Note that we do actually not need to consider all possible divisors since $d\lambda\sigma(q) = dAq$ implies $\deg_{x_j} d\lambda \leq \deg_{x_j} dA$ for $j = 1, \ldots, n$. In the example, we have $\deg_x dA = \deg_y A = 2$ which leaves only 8 (monic) divisors of $\det dA$ instead of 12.

There is still a big problem here: As the example shows, it is not sufficient to look at the monic divisors of $\det dA$. We also need to determine the leading coefficient $c \in \mathbb{K}$.

It is possible to compute c if we are given a bound on $\deg_{x_j} q$ for $j = 1, \ldots, n$. However, at the moment we have no way of algorithmically determining such a bound. Still, we do have a semi-decidable algorithm where we simply keep increasing the bound until a solution is found.

Future Work

After our initial results, Mark van Hoeij and Moulay Barkatou approached us having developed (see [5]) a very similar method. They have been concentrating on the case $n = 1$ and $\sigma(x) = x + 1$ where they have an algorithmic way of determining c.

References

[1] Abramov, S.A., Barkatou, M.: Rational solutions of first order linear difference systems. In: Proceedings of ISSAC'98. Rostock (1998)
[2] Abramov, S.A., Petkovšek, M., Ryabenko, A.A.: Hypergeometric Solutions of First-Order Linear Difference Systems with Rational-Function Coefficients, pp. 1–14. Springer International Publishing (2015).
[3] Barkatou, M.: An algorithm for computing a companion block diagonal form for a system of linear differential equations. Appl. Algebra Engrg. Comm. Comput. 4(3), 185–195 (1993)
[4] van Hoeij, M.: Finite singularities and hypergeometric solutions of linear recurrence equations. J. Pure Appl. Algebra **139**(1-3), 109–131 (1999)
[5] van Hoeij, M., Barkatou M.: Factorization and hypergeometric solutions of linear recurrence systems. Unpublished. https://www.math.fsu.edu/~hoeij/slides/bronstein/slides.pdf
[6] Petkovšek, M.: Hypergeometric solutions of linear recurrences with polynomial coefficients. Journal of Symbolic Computation (14), 243–264 (1992)
[7] Zürcher, B.: Rationale Normalformen von pseudolinearen Abbildungen. Master's thesis, ETH Zürich (1994)