Denominator Bounds for Higher Order Systems of Linear Recurrence Equations

Johannes Middeke
Research Institute for Symbolic Computation
Johannes Kepler University
Altenbergerstraße 69, A-4040 Linz, Austria
jmiddeke@risc.jku.at

Carsten Schneider
Research Institute for Symbolic Computation
Johannes Kepler University
Altenbergerstraße 69, A-4040 Linz, Austria
cschneider@risc.jku.at

Overview

Let $\mathbb{F}$ be a field and $a: \mathbb{F} \to \mathbb{F}$ be an automorphism. We extend $a$ to the rational functions $\mathbb{F}(t)$ by

\[ \sigma(a(t)) = a(t + c) \text{ for some } c \in \mathbb{F}, \]

and to the differential and difference systems of an arbitrary order $\sigma(t)$ term in the equation $\sigma(y)$ and then to apply shifts and linear combinations to the remaining terms. At some point they will be further apart from $\sigma(y)$ then the dispersion of the denominator $d$. When this happens, the denominator can only cancel with the coefficients of the system.

\begin{align*}
\text{Main Result} &
\end{align*}

Assume now that the system (svs) has already been brought into head regular form

\[ A_0y(t) + \ldots + A_ny(t) = b \]

where $\hat{A}_0$ is a regular polynomial matrix, that is, the corresponding subsystem is head regular. Similarly row-/column-reduction with respect to $\sigma^{-1}$ produces a tail regular system. If the original system is already head regular, then the tail regular system will have the same number of nonzero rows and moreover using row operations will be sufficient.

The transformation matrices are unimodular, the solutions of the transformed systems (if any exist) are in one-to-one correspondence to the solutions of the original system.

\begin{align*}
\text{References} &
\end{align*}


\begin{align*}
\text{Previous Work} &
\end{align*}

Most existing algorithms as for instance [3, 1] work by translating the higher order system to a first order system. We only know of one method, [4], dealing directly with higher order systems. Our algorithm is similar to that later work, however, we expand it in several points:

\begin{itemize}
  \item Most importantly, we address the problem for general $\Sigma^n$-extensions instead of concentrating on the case that $\mathbb{F}$ is a constant field and $\sigma$ is the shift operator $t \mapsto t+1$;
  \item In addition our method does not require the system matrices to be square or their rows to be linearly independent.
\end{itemize}

\begin{align*}
\text{Head \& Tail Regularity} &
\end{align*}

We call the system (svs) \textit{head regular} if $m = n$ and det $A_{\sigma} \neq 0$, and we call it \textit{tail regular} if $m = n$ and det $A_{\sigma} = 0$.

We can transform a system to head or tail regular form in the following way: We consider $A = A_0 + \ldots + A_n + \lambda P$ as an operator matrix over the ring $\mathbb{F}[t]/(t^n - 1)$ of Ore Laurent polynomials. Applying row-/column-reduction (see, for example, [5]) with respect to $\sigma$ we obtain unimodular operator matrices $S$ and $T$ such that

\[ SAT = \begin{pmatrix} A_0 & \ldots & A_n & + & \lambda P \\
0 & \ldots & 0 & \lambda P 
\end{pmatrix} \]

where $\hat{A}_0$ is a regular polynomial matrix, that is, the corresponding subsystem is head regular. Similarly row-/column-reduction with respect to $\sigma^{-1}$ produces a tail regular system. If the original system is already head regular, then the tail regular system will have the same number of nonzero rows and moreover using row operations will be sufficient.

Because the transformation matrices are unimodular, the solutions of the transformed systems (if any exist) are in one-to-one correspondence to the solutions of the original system.

\begin{align*}
\text{Spread \& Dispersion} &
\end{align*}

The \textit{spread} of two nonzero polynomials $f$ and $g \in \mathbb{F}[t] \setminus \{0\}$ is defined as

\[ \text{spread}(f, g) = \{ k \geq 0 : \text{gcd}(f, \sigma^k(g)) \notin \mathbb{F} \} \]

The \textit{dispersion} of $f$ and $g$ is

\[ \text{disp}(f, g) = \text{max} \text{ spread}(f, g) \]

with the conventions that max $\emptyset = -\infty$ and max $\emptyset = \infty$ whenever $S$ is infinite.

In the $\Sigma$ case, the spread is always finite. In the $\Pi$ case, there is a problem when $t | f$ and $f | g$. However, that is the only problematic case (see, for example, [7]). Thus, below we will simply assume that in the $\Pi$ case $t$ does not occur as a divisor.

\begin{align*}
\text{Example} &
\end{align*}

We do an example for the case $F$ and $a(t) = 2t$. Consider the system

\[ \begin{pmatrix} -2(4t+1)(2t+1)^2 & 0 \\
2(t+1)(2t+1)(t+2) -9(t+1)(2t+1)(t+2) & 0 
\end{pmatrix} y(t) + 3(2t+1)(t+2)(4t+1) + 3(2t+1)(t+2)(4t+1) y(t) = 0. \]

This system is already both head and tail regular. We obtain

\[ m = 18(4t+1)(2t+1)^2(t+1)(t+2) \]

and

\[ p = 3(4t+1)(2t+1)^2(t+1). \]

The dispersion is $2$. This yields a denominator bound of

\[ (2t+1)(t+1)^2(t+2)^2 \]

which matches the solutions

\[ \frac{1}{(t+1)^2} \quad \text{and} \quad \frac{1}{(t+2)^2} \].