PARTICLEFACE 2018, Valencia, February 26, 2018

Computer Algebra Algorithms for the Evaluation of Feynman Integrals

Carsten Schneider

Research Institute for Symbolic Computation (RISC)
Johannes Kepler University Linz

JOHANNES KEPLER UNIVERSITY LINZ

EUROPEAN COOPERATION IN SCIENCE AND TECHNOLOGY

COST is supported by the EU Framework Programme Horizon 2020

Der Wissenschaftsfonds.

The general tactic

Feynman integrals

The general tactic

Feynman integrals

$\downarrow$ non-trivial transformations (DESY)

multiple sums

The general tactic

Feynman integrals

$\downarrow$ non-trivial transformations (DESY)

multiple sums

$\downarrow$ symbolic summation

compact expression in terms of special functions

$$F(\varepsilon, N) = \int \frac{d^{4+\varepsilon}k_1}{(2\pi)^{4+\varepsilon}} \frac{d^{4+\varepsilon}k_2}{(2\pi)^{4+\varepsilon}} \frac{d^{4+\varepsilon}k_3}{(2\pi)^{4+\varepsilon}} \frac{(\Delta.k_3)^N}{k_2^4((k_1-k_3)^2-m^2)(k_1-k_2)^2((k_3-p)^2-m^2)}$$

$$||?$$

$$F_{-3}(N)\varepsilon^{-3} + F_{-2}(N)\varepsilon^{-2} + F_{-1}(N)\varepsilon^{-1} + F_0(N)\varepsilon^0 + \dots$$

$$F(\varepsilon,N)=\int\frac{d^{4+\varepsilon}k_1}{(2\pi)^{4+\varepsilon}}\frac{d^{4+\varepsilon}k_2}{(2\pi)^{4+\varepsilon}}\frac{d^{4+\varepsilon}k_3}{(2\pi)^{4+\varepsilon}}\frac{(\Delta.k_3)^N}{k_2^4((k_1-k_3)^2-m^2)(k_1-k_2)^2((k_3-p)^2-m^2)}$$

$$||$$

$$\sum_{k=1}^{N}(-1)^k e^{-\frac{3\varepsilon\gamma}{2}}\Gamma\Big(-1-\frac{3\varepsilon}{2}\Big)\times$$
$$\times B\Big(2+k,\frac{\varepsilon}{2}\Big)B(-\varepsilon+k,-\varepsilon)B\Big(1-\frac{\varepsilon}{2}+k,1+\frac{\varepsilon}{2}\Big)\binom{N}{k}$$

where

$$B(a,b)=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

$$F(\varepsilon, N) = \int \frac{d^{4+\varepsilon}k_1}{(2\pi)^{4+\varepsilon}} \frac{d^{4+\varepsilon}k_2}{(2\pi)^{4+\varepsilon}} \frac{d^{4+\varepsilon}k_3}{(2\pi)^{4+\varepsilon}} \frac{(\Delta.k_3)^N}{k_2^4((k_1-k_3)^2-m^2)(k_1-k_2)^2((k_3-p)^2-m^2)}$$

$$||$$

$$\underbrace{\sum_{k=1}^{N}(-1)^k e^{-\frac{3\varepsilon\gamma}{2}}\Gamma\Big(-1-\frac{3\varepsilon}{2}\Big)\times \times B\Big(2+k,\frac{\varepsilon}{2}\Big)B(-\varepsilon+k,-\varepsilon)B\Big(1-\frac{\varepsilon}{2}+k,1+\frac{\varepsilon}{2}\Big)\binom{N}{k}}_{= f_{-3}(N,k)\varepsilon^{-3}+f_{-2}(N,k)\varepsilon^{-2}+f_{-1}(N,k)\varepsilon^{-1}+\dots}$$

G.E. Andrews, R. Askey, R. Roy: Special Functions

$$F(\varepsilon, N) = \int \frac{d^{4+\varepsilon}k_1}{(2\pi)^{4+\varepsilon}} \frac{d^{4+\varepsilon}k_2}{(2\pi)^{4+\varepsilon}} \frac{d^{4+\varepsilon}k_3}{(2\pi)^{4+\varepsilon}} \frac{(\Delta.k_3)^N}{k_2^4((k_1-k_3)^2-m^2)(k_1-k_2)^2((k_3-p)^2-m^2)}$$

$$||$$

$$\underbrace{\sum_{k=1}^{N}(-1)^k e^{-\frac{3\varepsilon\gamma}{2}}\Gamma\Big(-1-\frac{3\varepsilon}{2}\Big)\times \times B\Big(2+k,\frac{\varepsilon}{2}\Big)B(-\varepsilon+k,-\varepsilon)B\Big(1-\frac{\varepsilon}{2}+k,1+\frac{\varepsilon}{2}\Big)\binom{N}{k}}_{= f_{-3}(N,k)\varepsilon^{-3}+f_{-2}(N,k)\varepsilon^{-2}+f_{-1}(N,k)\varepsilon^{-1}+\dots}$$

$$||$$

$$\Big(\sum_{k=1}^{N} f_{-3}(N,k)\Big)\varepsilon^{-3} + \Big(\sum_{k=1}^{N} f_{-2}(N,k)\Big)\varepsilon^{-2} + \Big(\sum_{k=1}^{N} f_{-1}(N,k)\Big)\varepsilon^{-1} + \dots$$

$$F(\varepsilon, N) = \int \frac{d^{4+\varepsilon}k_1}{(2\pi)^{4+\varepsilon}} \frac{d^{4+\varepsilon}k_2}{(2\pi)^{4+\varepsilon}} \frac{d^{4+\varepsilon}k_3}{(2\pi)^{4+\varepsilon}} \frac{(\Delta.k_3)^N}{k_2^4((k_1-k_3)^2-m^2)(k_1-k_2)^2((k_3-p)^2-m^2)}$$

$$||$$

$$\underbrace{\sum_{k=1}^{N}(-1)^k e^{-\frac{3\varepsilon\gamma}{2}}\Gamma\Big(-1-\frac{3\varepsilon}{2}\Big)\times \times B\Big(2+k,\frac{\varepsilon}{2}\Big)B(-\varepsilon+k,-\varepsilon)B\Big(1-\frac{\varepsilon}{2}+k,1+\frac{\varepsilon}{2}\Big)\binom{N}{k}}_{=f_{-3}(N,k)\varepsilon^{-3}+f_{-2}(N,k)\varepsilon^{-2}+f_{-1}(N,k)\varepsilon^{-1}+\dots}$$

$$||$$

$$\Big(\sum_{k=1}^{N} f_{-3}(N,k)\Big)\varepsilon^{-3} + \Big(\sum_{k=1}^{N} f_{-2}(N,k)\Big)\varepsilon^{-2} + \Big(\boxed{\sum_{k=1}^{N} f_{-1}(N,k)}\Big)\varepsilon^{-1} + \dots$$

Simplify

$$F_{-1}(N) = \sum_{k=1}^{N} (-1)^{k+1} \binom{N}{k} \Big(\frac{(2+3k)\big(-2+3k+7k^2+3k^3\big)}{3k^2(1+k)^3} + \frac{2S_2(k)}{1+k} + \frac{\zeta_2}{2(1+k)} \Big)$$

where

$$S_a(N) = \sum_{i=1}^{N} \frac{\mathrm{sign}(a)^i}{i^a} \text{ and } \zeta_a = \sum_{i=1}^{\infty} \frac{1}{i^a}$$

Simplify

$$F_{-1}(N)=\sum_{k=1}^{N}(-1)^{k+1}\binom{N}{k}\Big(\frac{(2+3k)\big(-2+3k+7k^2+3k^3\big)}{3k^2(1+k)^3}+\frac{2S_2(k)}{1+k}+\frac{\zeta_2}{2(1+k)}\Big)$$

$\downarrow$ (summation package Sigma.m)

$$\begin{aligned}&\big(16N^3+144N^2+413N+384\big)(N+1)^2F_{-1}(N)\\&\quad-(N+2)(2N+5)\big(16N^3+112N^2+221N+113\big)F_{-1}(N+1)\\&\quad+(N+3)^2\big(16N^3+96N^2+173N+99\big)F_{-1}(N+2)\\&\quad=\frac{1}{2}\big(4N^2+21N+29\big)\zeta_2+\frac{-64N^5-500N^4-1133N^3+203N^2+3516N+3090}{3(N+2)(N+3)}\end{aligned}$$

Simplify

$$F_{-1}(N)=\sum_{k=1}^{N}(-1)^{k+1}\binom{N}{k}\Big(\frac{(2+3k)\big(-2+3k+7k^2+3k^3\big)}{3k^2(1+k)^3}+\frac{2S_2(k)}{1+k}+\frac{\zeta_2}{2(1+k)}\Big)$$

↓ (summation package `Sigma.m`)

$$\begin{aligned}&\big(16N^3+144N^2+413N+384\big)(N+1)^2F_{-1}(N)\\&\quad-(N+2)(2N+5)\big(16N^3+112N^2+221N+113\big)F_{-1}(N+1)\\&\quad+(N+3)^2\big(16N^3+96N^2+173N+99\big)F_{-1}(N+2)\\&\quad=\frac{1}{2}\big(4N^2+21N+29\big)\zeta_2+\frac{-64N^5-500N^4-1133N^3+203N^2+3516N+3090}{3(N+2)(N+3)}\end{aligned}$$

↓ (summation package `Sigma.m`)

$$\begin{aligned}&\Big\{c_1\frac{1-4N}{N+1}+c_2\frac{-14N-13}{(N+1)^2}\\&+\frac{(4N-1)S_1(N)}{N+1}+\frac{(1-4N)S_1(N)^2}{6(N+1)}+\frac{(14N+13)S_1(N)}{3(N+1)^2}\\&+\frac{175N^2+334N+155}{12(N+1)^3}+\frac{(1-4N)S_2(N)}{6(N+1)}+\frac{\zeta_2}{8(N+1)}\Big|c_1,c_2\in\mathbb{Q}\Big\}\end{aligned}$$

Simplify

$$F_{-1}(N) = \sum_{k=1}^{N} (-1)^{k+1} \binom{N}{k} \Big(\frac{(2+3k)\big(-2+3k+7k^2+3k^3\big)}{3k^2(1+k)^3} + \frac{2S_2(k)}{1+k} + \frac{\zeta_2}{2(1+k)} \Big)$$

$$\bigcap$$

$$\Big\{ c_1 \frac{1-4N}{N+1} + c_2 \frac{-14N-13}{(N+1)^2} + \frac{(4N-1)S_1(N)}{N+1} + \frac{(1-4N)S_1(N)^2}{6(N+1)} + \frac{(14N+13)S_1(N)}{3(N+1)^2} + \frac{175N^2+334N+155}{12(N+1)^3} + \frac{(1-4N)S_2(N)}{6(N+1)} + \frac{\zeta_2}{8(N+1)} \Big| c_1, c_2 \in \mathbb{Q} \Big\}$$

Simplify

$$F_{-1}(N) = \sum_{k=1}^{N} (-1)^{k+1} \binom{N}{k} \left(\frac{(2+3k)\big(-2+3k+7k^2+3k^3\big)}{3k^2(1+k)^3} + \frac{2S_2(k)}{1+k} + \frac{\zeta_2}{2(1+k)} \right)$$

$||$ (recurrence finding and solving)

$$\left(\frac{1}{12} - \frac{1}{8}\zeta_2\right) \frac{1-4N}{N+1} + 1\, \frac{-14N-13}{(N+1)^2} + \frac{(4N-1)S_1(N)}{N+1} + \frac{(1-4N)S_1(N)^2}{6(N+1)} + \frac{(14N+13)S_1(N)}{3(N+1)^2} + \frac{175N^2+334N+155}{12(N+1)^3} + \frac{(1-4N)S_2(N)}{6(N+1)} + \frac{\zeta_2}{8(N+1)}$$

In[1]:= $<<$ **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= $<<$ **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[3]:= $<<$ **EvaluateMultiSums.m**

EvaluateMultiSums by Carsten Schneider © RISC-Linz

In[1]:= $<<$ **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= $<<$ **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[3]:= $<<$ **EvaluateMultiSums.m**

EvaluateMultiSums by Carsten Schneider © RISC-Linz

In[4]:= $\mathbf{mySum} =$

$$\sum_{k=1}^{N}(-1)^k e^{-\frac{3\varepsilon\gamma}{2}}\left(-2-\frac{3\varepsilon}{2}\right)!B\left(2+k,\frac{\varepsilon}{2}\right)B(-\varepsilon+k,-\varepsilon)B\left(1-\frac{\varepsilon}{2}+k,1+\frac{\varepsilon}{2}\right)\binom{N}{k};$$

In[5]:= $\mathbf{EvaluateMultiSum[mySum,\{\},\{N\},\{1\},ExpandIn\to\{\varepsilon,-3,-3\}]}$

In[1]:= $<<$ **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= $<<$ **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[3]:= $<<$ **EvaluateMultiSums.m**

EvaluateMultiSums by Carsten Schneider © RISC-Linz

In[4]:= $\mathbf{mySum} =$

$$\sum_{k=1}^{N}(-1)^k e^{-\frac{3\varepsilon\gamma}{2}}\Big(-2-\frac{3\varepsilon}{2}\Big)!\,B\Big(2+k,\frac{\varepsilon}{2}\Big)B(-\varepsilon+k,-\varepsilon)B\Big(1-\frac{\varepsilon}{2}+k,1+\frac{\varepsilon}{2}\Big)\binom{N}{k};$$

In[5]:= $\mathbf{EvaluateMultiSum[mySum,\{\},\{N\},\{1\},ExpandIn\to\{\varepsilon,-3,-3\}]}$

Out[5]= $\Big\{\frac{59N^2+120N+49}{9(N+1)^2}-\frac{2(N+3)S_1[N]}{3(N+1)}\Big\}$

In[1]:= $<<$ **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= $<<$ **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[3]:= $<<$ **EvaluateMultiSums.m**

EvaluateMultiSums by Carsten Schneider © RISC-Linz

In[4]:= $\mathbf{mySum} =$

$$\sum_{k=1}^{N}(-1)^k e^{-\frac{3\varepsilon\gamma}{2}}\left(-2-\frac{3\varepsilon}{2}\right)!B\left(2+k,\frac{\varepsilon}{2}\right)B(-\varepsilon+k,-\varepsilon)B\left(1-\frac{\varepsilon}{2}+k,1+\frac{\varepsilon}{2}\right)\binom{N}{k};$$

In[5]:= $\mathbf{EvaluateMultiSum[mySum, \{\}, \{N\}, \{1\}, ExpandIn \to \{\varepsilon, -3, -2\}]}$

Out[5]=

$$\left\{\frac{59N^2+120N+49}{9(N+1)^2}-\frac{2(N+3)S_1[N]}{3(N+1)},\right.$$
$$\left.-\frac{2(20N^3+58N^2+57N+22)}{3(N+1)^3}+\frac{2(N+2)(2N-1)S_1[N]}{3(N+1)^2}-\frac{S_1[N]^2}{N+1}-\frac{S_2[N]}{N+1}\right\}$$

In[1]:= $<<$ **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= $<<$ **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[3]:= $<<$ **EvaluateMultiSums.m**

EvaluateMultiSums by Carsten Schneider © RISC-Linz

In[4]:= $\mathbf{mySum} =$

$$\sum_{k=1}^{N}(-1)^{k}e^{-\frac{3\varepsilon\gamma}{2}}\Big(-2-\frac{3\varepsilon}{2}\Big)!B\Big(2+k,\frac{\varepsilon}{2}\Big)B(-\varepsilon+k,-\varepsilon)B\Big(1-\frac{\varepsilon}{2}+k,1+\frac{\varepsilon}{2}\Big)\binom{N}{k};$$

In[5]:= $\mathbf{EvaluateMultiSum[mySum,\{\},\{N\},\{1\},ExpandIn\to\{\varepsilon,-3,-1\}]}$

Out[5]=
$$\Big\{\frac{59N^2+120N+49}{9(N+1)^2}-\frac{2(N+3)S_1[N]}{3(N+1)},$$
$$-\frac{2(20N^3+58N^2+57N+22)}{3(N+1)^3}+\frac{2(N+2)(2N-1)S_1[N]}{3(N+1)^2}-\frac{S_1[N]^2}{N+1}-\frac{S_2[N]}{N+1},$$
$$\Big(\frac{1}{12}-\frac{1}{8}\zeta(2)\Big)\frac{1-4N}{N+1}+\frac{-14N-13}{(N+1)^2}+\frac{(4N-1)S_1(N)}{N+1}+\frac{(1-4N)S_1(N)^2}{6(N+1)}+$$
$$\frac{(14N+13)S_1(N)}{3(N+1)^2}+\frac{175N^2+334N+155}{12(N+1)^3}+\frac{(1-4N)S_2(N)}{6(N+1)}+\frac{\zeta(2)}{8(N+1)}\Big\}$$

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$F(N)=\sum_{k=0}^{N} f(N,k);$$

$f(N,k)$: indefinite nested product-sum in k;
N: extra parameter

FIND a recurrence for $F(N)$

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$F(N) = \sum_{k=0}^{N} f(N,k);$$

$f(N,k)$: indefinite nested product-sum in k;
N: extra parameter

FIND a recurrence for $F(N)$

2. Recurrence solving

GIVEN a recurrence

$a_0(N), \dots, a_d(N), h(N)$:
indefinite nested product-sum expressions.

$$a_0(N)F(N) + \dots + a_d(N)F(N+d) = h(N);$$

FIND all solutions expressible by indefinite nested products/sums
(Abramov/Bronstein/Petkovšek/CS, in preparation)

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$F(N)=\sum_{k=0}^{N} f(N,k);$$

$f(N,k)$: indefinite nested product-sum in k;
N: extra parameter

FIND a recurrence for $F(N)$

2. Recurrence solving

GIVEN a recurrence

$a_0(N),\dots,a_d(N),h(N)$:
indefinite nested product-sum expressions.

$$a_0(N)F(N)+\dots+a_d(N)F(N+d)=h(N);$$

FIND all solutions expressible by indefinite nested products/sums
(Abramov/Bronstein/Petkovšek/CS, in preparation)

3. Find a "closed form"

F(N)=combined solutions in terms of indefinite nested sums.

Sigma.m is based on difference ring/field theory

1. M. Karr. Summation in finite terms. *J. ACM*, 28:305–350, 1981.
2. P. Paule. Greatest factorial factorization and symbolic summation. J. Symbolic Comput. **20**(3), 235–268 (1995)
3. M. Petkovšek, H. S. Wilf, and D. Zeilberger. $A = B$. A. K. Peters, Wellesley, MA, 1996.
4. P. A. Hendriks and M. F. Singer. Solving difference equations in finite terms. *J. Symbolic Comput.*, 27(3):239–259, 1999.
5. M. Bronstein. On solutions of linear ordinary difference equations in their coefficient field. *J. Symbolic Comput.*, 29(6):841–877, 2000.
6. CS. Symbolic summation in difference fields. J. Kepler University, May 2001. PhD Thesis.
7. CS. A collection of denominator bounds to solve parameterized linear difference equations in $\Pi\Sigma$-extensions. *An. Univ. Timişoara Ser. Mat.-Inform.*, 42(2):163–179, 2004.
8. CS. Symbolic summation with single-nested sum extensions. In J. Gutierrez, editor, *Proc. ISSAC'04*, pages 282–289. ACM Press, 2004.
9. CS. Degree bounds to find polynomial solutions of parameterized linear difference equations in $\Pi\Sigma$-fields. *Appl. Algebra Engrg. Comm. Comput.*, 16(1):1–32, 2005.
10. CS. Product representations in $\Pi\Sigma$-fields. *Ann. Comb.*, 9(1):75–99, 2005.
11. CS. Solving parameterized linear difference equations in terms of indefinite nested sums and products. *J. Differ. Equations Appl.*, 11(9):799–821, 2005.
12. CS. Finding telescopers with minimal depth for indefinite nested sums and product expressions. In *Proc. ISSAC'05*, pages 285–292. ACM Press, 2005.
13. CS. Simplifying Sums in $\Pi\Sigma$-Extensions. *J. Algebra Appl.*, 6(3):415–441, 2007.
14. CS. A refined difference field theory for symbolic summation. *J. Symbolic Comput.*, 43(9):611–644, 2008. [arXiv:0808.2543v1].
15. CS. A Symbolic Summation Approach to Find Optimal Nested Sum Representations. In A. Carey, D. Ellwood, S. Paycha, and S. Rosenberg, editors, *Motives, Quantum Field Theory, and Pseudodifferential Operators*, pages 285–308. 2010.
16. CS. Parameterized Telescoping Proves Algebraic Independence of Sums. *Ann. Comb.*, 14(4):533–552, 2010. [arXiv:0808.2596].
17. CS. Structural Theorems for Symbolic Summation. *Appl. Algebra Engrg. Comm. Comput.*, 21(1):1–32, 2010.
18. C. Schneider. Simplifying Multiple Sums in Difference Fields. In: Computer Algebra in Quantum Field Theory: Integration, Summation and Special Functions, J. Blümlein, C. Schneider (ed.), Texts and Monographs in Symbolic Computation, pp. 325-360. Springer, 2013.
19. CS. Fast Algorithms for Refined Parameterized Telescoping in Difference Fields. In *Computer Algebra and Polynomials*, Lecture Notes in Computer Science (LNCS), Springer, 2014. arXiv:1307.7887 [cs.SC].
20. CS. A Difference Ring Theory for Symbolic Summation. J. Symb. Comput. 72, pp. 82-127. 2016.
21. CS. Summation Theory II: Characterizations of $R\Pi\Sigma$-extensions and algorithmic aspects. J. Symb. Comput. 80(3), pp. 616-664. 2017.
22. E.D. Ocansey, CS. Representing (q-)hypergeometric products and mixed versions in difference rings. In: Advances in Computer Algebra,, C. Schneider, E. Zima (ed.), Springer Proceedings in Mathematics & Statistics 226. 2018.
23. S.A. Abramov, M. Bronstein, M. Petkovšek, CS, in preparation

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

Simple sum

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\boxed{\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}}$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\boxed{\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}}$$

$$||$$

$$\boxed{\binom{j+1}{r}\left(\frac{(-1)^r(-j+n-2)!r!}{(n+1)(-j+n+r-1)(-j+n+r)!}+\frac{(-1)^{n+r}(j+1)!(-j+n-2)!(-j+n-1)_r r!}{(n-1)n(n+1)(-j+n+r)!(-j-1)_r(2-n)_j}\right)}$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$\|$$

$$\boxed{\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\binom{j+1}{r}\Big(\frac{(-1)^r(-j+n-2)!r!}{(n+1)(-j+n+r-1)(-j+n+r)!}+\frac{(-1)^{n+r}(j+1)!(-j+n-2)!(-j+n-1)_r r!}{(n-1)n(n+1)(-j+n+r)!(-j-1)_r(2-n)_j}\Big)}$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$\|$$

$$\sum_{j=0}^{n-2}\boxed{\sum_{r=0}^{j+1}\binom{j+1}{r}\Big(\frac{(-1)^r(-j+n-2)!r!}{(n+1)(-j+n+r-1)(-j+n+r)!}+\frac{(-1)^{n+r}(j+1)!(-j+n-2)!(-j+n-1)_r r!}{(n-1)n(n+1)(-j+n+r)!(-j-1)_r(2-n)_j}\Big)}$$

$$\|$$

$$\boxed{\Big(\frac{n^2-n+1}{(n-1)^2n^2(n+1)(2-n)_j}+\frac{\sum_{i=1}^{j}\frac{(2-n)_i}{(-i+n-1)^2(i+1)!}}{(n+1)(2-n)_j}+\frac{(-1)^{j+n}(-j-2)(-j+n-2)!}{(j-n+1)(n+1)^2n!}\Big)(j+1)!-\frac{1}{(n+1)^2(-j+n-1)}}$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\sum_{j=0}^{n-2}\left(\left(\frac{n^2-n+1}{(n-1)^2n^2(n+1)(2-n)_j}+\frac{\sum_{i=1}^{j}\frac{(2-n)_i}{(-i+n-1)^2(i+1)!}}{(n+1)(2-n)_j}+\frac{(-1)^{j+n}(-j-2)(-j+n-2)!}{(j-n+1)(n+1)^2n!}\right)(j+1)!-\frac{1}{(n+1)^2(-j+n-1)}\right)$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

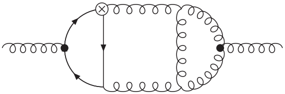
$$||$$

$$\sum_{j=0}^{n-2}\left(\left(\frac{n^2-n+1}{(n-1)^2n^2(n+1)(2-n)_j}+\frac{\sum_{i=1}^{j}\frac{(2-n)_i}{(-i+n-1)^2(i+1)!}}{(n+1)(2-n)_j}+\frac{(-1)^{j+n}(-j-2)(-j+n-2)!}{(j-n+1)(n+1)^2n!}\right)(j+1)!-\frac{1}{(n+1)^2(-j+n-1)}\right)$$

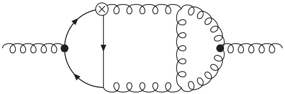
$$||$$

$$\frac{-n^2-n-1}{n^2(n+1)^3}+\frac{(-1)^n\left(n^2+n+1\right)}{n^2(n+1)^3}-\frac{2S_{-2}(n)}{n+1}+\frac{S_1(n)}{(n+1)^2}+\frac{S_2(n)}{-n-1}$$

Note: $S_a(n)=\sum_{i=1}^{N}\frac{\mathsf{sign}(a)^i}{i^{|a|}}$, $a\in\mathbb{Z}\setminus\{0\}$.



$$= F_{-3}(N)\varepsilon^{-3} + F_{-2}(N)\varepsilon^{-2} + F_{-1}(N)\varepsilon^{-1} + \boxed{F_0(N)}$$



$$= F_{-3}(N)\varepsilon^{-3} + F_{-2}(N)\varepsilon^{-2} + F_{-1}(N)\varepsilon^{-1} + \boxed{F_0(N)}$$

Simplify

$$\|$$

$$\sum_{j=0}^{N-3}\sum_{k=0}^{j}\sum_{l=0}^{k}\sum_{q=0}^{-j+N-3}\sum_{s=1}^{-l+N-q-3}\sum_{r=0}^{-l+N-q-s-3}(-1)^{-j+k-l+N-q-3}\times$$

$$\times \frac{\binom{j+1}{k+1}\binom{k}{l}\binom{N-1}{j+2}\binom{-j+N-3}{q}\binom{-l+N-q-3}{s}\binom{-l+N-q-s-3}{r} r!(-l+N-q-r-s-3)!(s-1)!}{(-l+N-q-2)!(-j+N-1)(N-q-r-s-2)(q+s+1)}$$

$$\Big[4S_1(-j+N-1) - 4S_1(-j+N-2) - 2S_1(k)$$

$$-\left(S_1(-l+N-q-2) + S_1(-l+N-q-r-s-3) - 2S_1(r+s)\right)$$

$$+2S_1(s-1) - 2S_1(r+s)\Big] + \textbf{3 further 6–fold sums}$$

$$\boxed{F_0(N)} =$$

$$\frac{7}{12}S_1(N)^4 + \frac{(17N+5)S_1(N)^3}{3N(N+1)} + \Big(\frac{35N^2-2N-5}{2N^2(N+1)^2} + \frac{13S_2(N)}{2} + \frac{5(-1)^N}{2N^2}\Big)S_1(N)^2$$
$$+\Big(-\frac{4(13N+5)}{N^2(N+1)^2} + \Big(\frac{4(-1)^N(2N+1)}{N(N+1)} - \frac{13}{N}\Big)S_2(N) + \Big(\frac{29}{3} - (-1)^N\Big)S_3(N)$$
$$+\big(2+2(-1)^N\big)S_{2,1}(N) - 28S_{-2,1}(N) + \frac{20(-1)^N}{N^2(N+1)}\Big)S_1(N) + \Big(\frac{3}{4} + (-1)^N\Big)S_2(N)^2$$
$$-2(-1)^N S_{-2}(N)^2 + S_{-3}(N)\Big(\frac{2(3N-5)}{N(N+1)} + \big(26+4(-1)^N\big)S_1(N) + \frac{4(-1)^N}{N+1}\Big)$$
$$+\Big(\frac{(-1)^N(5-3N)}{2N^2(N+1)} - \frac{5}{2N^2}\Big)S_2(N) + S_{-2}(N)\Big(10S_1(N)^2 + \Big(\frac{8(-1)^N(2N+1)}{N(N+1)}$$
$$+\frac{4(3N-1)}{N(N+1)}\Big)S_1(N) + \frac{8(-1)^N(3N+1)}{N(N+1)^2} + \big(-22+6(-1)^N\big)S_2(N) - \frac{16}{N(N+1)}\Big)$$
$$+\Big(\frac{(-1)^N(9N+5)}{N(N+1)} - \frac{29}{3N}\Big)S_3(N) + \Big(\frac{19}{2} - 2(-1)^N\Big)S_4(N) + \big(-6+5(-1)^N\big)S_{-4}(N)$$
$$+\Big(-\frac{2(-1)^N(9N+5)}{N(N+1)} - \frac{2}{N}\Big)S_{2,1}(N) + \big(20+2(-1)^N\big)S_{2,-2}(N) + \big(-17+13(-1)^N\big)S_{3,1}(N)$$
$$-\frac{8(-1)^N(2N+1)+4(9N+1)}{N(N+1)}S_{-2,1}(N) - \big(24+4(-1)^N\big)S_{-3,1}(N) + \big(3-5(-1)^N\big)S_{2,1,1}(N)$$
$$+32S_{-2,1,1}(N) + \left(\frac{3}{2}S_1(N)^2 - \frac{3S_1(N)}{N} + \frac{3}{2}(-1)^N S_{-2}(N)\right)\zeta(2)$$

$$\boxed{F_0(N)} =$$

$$\frac{7}{12}S_1(N)^4 + \frac{(17N+5)S_1(N)^3}{3N(N+1)} + \Big(\frac{35N^2-2N-5}{2N^2(N+1)^2} + \frac{13S_2(N)}{2} + \frac{5(-1)^N}{2N^2}\Big)S_1(N)^2$$

$$+ \big(- \ldots \frac{\ldots 1)^N(2N+1)}{\ldots N(N+1)} - \frac{13}{N}\big)S_2(N) + \big(\frac{29}{3} - (-1)^N\big)S_3(N)$$

$$S_1(N) = \sum_{i=1}^{N} \frac{1}{i}$$

$$+ \big(2 + \ldots 28S_{-2,1}(N) + \frac{20(-1)^N}{N^2(N+1)}\big)S_1(N) + \big(\frac{3}{4} + (-1)^N\big)S_2(N)^2$$

$$- 2(-1)^N S_{-2}(N)^2 + S_{-3}(N)\Big(\frac{2(3N-5)}{N(N+1)} + \big(26 + 4(-1)^N\big)S_1(N) + \frac{4(-1)^N}{N+1}\Big)$$

$$+ \Big(\frac{(-1)^N(5-3N)}{2N^2(N+1)} - \frac{5}{2N^2}\Big)S_2(N) + S_{-2}(N)\Big(10S_1(N)^2 + \Big(\frac{8(-1)^N(2N+1)}{N(N+1)}$$

$$+ \frac{4(3N-1)}{N(N+1)}\Big)S_1(N) + \frac{8(-1)^N(3N+1)}{N(N+1)^2} + \big(-22 + 6(-1)^N\big)S_2(N) - \frac{16}{N(N+1)}\Big)$$

$$+ \Big(\frac{(-1)^N(9N+5)}{N(N+1)} - \frac{29}{3N}\Big)S_3(N) + \Big(\frac{19}{2} - 2(-1)^N\Big)S_4(N) + \big(-6 + 5(-1)^N\big)S_{-4}(N)$$

$$+ \Big(-\frac{2(-1)^N(9N+5)}{N(N+1)} - \frac{2}{N}\Big)S_{2,1}(N) + \big(20 + 2(-1)^N\big)S_{2,-2}(N) + \big(-17 + 13(-1)^N\big)S_{3,1}(N)$$

$$- \frac{8(-1)^N(2N+1) + 4(9N+1)}{N(N+1)}S_{-2,1}(N) - \big(24 + 4(-1)^N\big)S_{-3,1}(N) + \big(3 - 5(-1)^N\big)S_{2,1,1}(N)$$

$$+ 32S_{-2,1,1}(N) + \left(\frac{3}{2}S_1(N)^2 - \frac{3S_1(N)}{N} + \frac{3}{2}(-1)^N S_{-2}(N)\right)\zeta(2)$$

$$\boxed{F_0(N)} =$$

$$\frac{7}{12}S_1(N)^4 + \frac{(17N+5)S_1(N)^3}{3N(N+1)} + \Big(\frac{35N^2-2N-5}{2N^2(N+1)^2} + \frac{13S_2(N)}{2} + \frac{5(-1)^N}{2N^2}\Big)S_1(N)^2$$

$$+ \Big(\ldots 1)^N(2N+1)}{N(N+1)} - \frac{13}{N}\Big)S_2(N) + \Big(\frac{29}{3} - (-1)^N\Big)S_3(N)$$

$$+ \big(2 + \ldots 28S_{-2,1}(N) + \frac{20(-1)^N}{N^2(N+1 \ldots} \ldots)^N\big)S_2(N)^2$$

$$- 2(-1)^N S_{-2}(N)^2 + S_{-3}(N)\Big(\frac{2(3N-5)}{N(N+1)} + \big(26 + 4 \ldots)^N}{\ldots +1}\Big)$$

$$+ \Big(\frac{(-1)^N(5-3N)}{2N^2(N+1)} - \frac{5}{2N^2}\Big)S_2(N) + S_{-2}(N)\big(10S_1(N)^2 + \Big(\frac{\ldots (2N+1)}{N(N+1)}$$

$$+ \frac{4(3N-1)}{N(N+1)}\Big)S_1(N) + \frac{8(-1)^N(3N+1)}{N(N+1)^2} + \big(-22 + 6(-1)^N\big)S_2(N) - \frac{16}{N(N+1)}\big)$$

$$+ \Big(\frac{(-1)^N(9N+5)}{N(N+1)} - \frac{29}{3N}\Big)S_3(N) + \Big(\frac{19}{2} - 2(-1)^N\Big)S_4(N) + \big(-6 + 5(-1)^N\big)S_{-4}(N)$$

$$+ \Big(-\frac{2(-1)^N(9N+5)}{N(N+1)} - \frac{2}{N}\Big)S_{2,1}(N) + \big(20 + 2(-1)^N\big)S_{2,-2}(N) + \big(-17 + 13(-1)^N\big)S_{3,1}(N)$$

$$- \frac{8(-1)^N(2N+1) + 4(9N+1)}{N(N+1)}S_{-2,1}(N) - \big(24 + 4(-1)^N\big)S_{-3,1}(N) + \big(3 - 5(-1)^N\big)S_{2,1,1}(N)$$

$$+ 32S_{-2,1,1}(N) + \left(\frac{3}{2}S_1(N)^2 - \frac{3S_1(N)}{N} + \frac{3}{2}(-1)^N S_{-2}(N)\right)\zeta(2)$$

$$S_1(N) = \sum_{i=1}^{N}\frac{1}{i}$$

$$S_2(N) = \sum_{i=1}^{N}\frac{1}{i^2}$$

$$\boxed{F_0(N)} =$$

$$\frac{7}{12}S_1(N)^4 + \frac{(17N+5)S_1(N)^3}{3N(N+1)} + \left(\frac{35N^2-2N-5}{2N^2(N+1)^2} + \frac{13S_2(N)}{2} + \frac{5(-1)^N}{2N^2}\right)S_1(N)^2$$

$$+ \Big(\ldots \frac{\ldots 1)^N(2N+1)}{\ldots N(N+1)} - \frac{13}{N}\Big)S_2(N) + \Big(\frac{29}{3} - (-1)^N\Big)S_3(N)$$

$$+ \big(2 + \ldots 28S_{-2,1}(N) + \frac{20(-1)^N}{N^2(N+1 \ldots} \ldots {}^{N}\big)S_2(N)^2$$

$$- 2(-1)^N S_{-2}(N)^2 + S_{-3}(N)\Big(\frac{2(3N-5)}{N(N+1)} + (26 + 4 \ldots \Big)$$

$$+ \Big(\frac{(-1)^N(5-3N)}{2N^2 \ldots} - \frac{5}{\ldots}\Big)S_2(N) + S_{-2}(N)\Big(10S_1(N)^2 + \Big(\frac{\ldots (2N+1)}{N(N+1)}$$

$$+ \frac{4(3N - \ldots}{N(N + \ldots} \ldots -1)^N\big)S_2(N) - \frac{16}{N(N+1)}\Big)$$

$$+ \Big(\frac{(-1)^{\ldots}}{N(\ldots} \ldots (N) + \big(-6 + 5(-1)^N\big)S_{-4}(N)$$

$$+ \Big(- \frac{2(-\ldots}{\ldots} \ldots \Big)S_{2,-2}(N) + \big(-17 + 13(-1)^N\big)S_{3,1}(N)$$

$$- \frac{8(-1)^N \ldots}{N(N+1)} S_{-2,1}(N) - \big(24 + 4(-1)^N\big)S_{-3,1}(N) + \big(3 - 5(-1)^N\big)S_{2,1,1}(N)$$

$$+ 32S_{-2,1,1}(N) + \left(\frac{3}{2}S_1(N)^2 - \frac{3S_1(N)}{N} + \frac{3}{2}(-1)^N S_{-2}(N)\right)\zeta(2)$$

$$S_1(N) = \sum_{i=1}^{N} \frac{1}{i}$$

$$S_2(N) = \sum_{i=1}^{N} \frac{1}{i^2}$$

$$S_{-2,1,1}(N) = \sum_{i=1}^{N} \frac{(-1)^i \sum_{j=1}^{i} \frac{\sum_{k=1}^{j} \frac{1}{k}}{j}}{i^2}$$

The general tactic

Feynman integrals

↓ non-trivial transformations (DESY)

multiple sums

↓ symbolic summation

compact expression in terms of special functions

Tactic 1: Expand the summand and simplify

Ablinger, Blümlein, Klein, CS, LL2010, arXiv:1006.4797 [math-ph]
Blümlein, Hasselhuhn, CS, RADCOR'10, arXiv:1202.4303 [math-ph]
CS, ACAT 2013, arXiv:1310.0160 [cs.SC]

The general tactic

Feynman integrals

↓ non-trivial transformations (DESY)

multiple sums

↓ symbolic summation

compact expression in terms of special functions

Tactic 2: Expand a recurrence in ε

Blümlein, Klein, CS, Stan, J. Symbol. Comput. 2012; arXiv:1011.2656 [cs.SC]
Ablinger, Blümlein, Round, CS, LL2012, arXiv:1210.1685 [cs.SC]

$$F(N)=\sum_{k=1}^{N}(-1)^k e^{-\frac{3\varepsilon\gamma}{2}}\Gamma\big(-1-\frac{3\varepsilon}{2}\big)B\big(2+k,\frac{\varepsilon}{2}\big)B(-\varepsilon+k,-\varepsilon)B\big(1-\frac{\varepsilon}{2}+k,1+\frac{\varepsilon}{2}\big)\binom{N}{k}$$

↓ (summation package `Sigma.m`)

$$2(N+1)^2F(N)+\big(3\varepsilon^2+3\varepsilon N+9\varepsilon-4N^2-12N-8\big)F(N+1)$$
$$-(2\varepsilon-N-1)(\varepsilon+2N+6)F(N+2)=0\varepsilon^{-3}-\frac{16}{3}\varepsilon^{-2}+\frac{40}{3}\varepsilon^{-1}-\big(2\zeta_2-\frac{68}{3}\big)\varepsilon^0+\dots$$

$$F(N)=\sum_{k=1}^{N}(-1)^k e^{-\frac{3\varepsilon\gamma}{2}}\Gamma\big(-1-\frac{3\varepsilon}{2}\big)B\big(2+k,\frac{\varepsilon}{2}\big)B(-\varepsilon+k,-\varepsilon)B\big(1-\frac{\varepsilon}{2}+k,1+\frac{\varepsilon}{2}\big)\binom{N}{k}$$

$\downarrow$ (summation package `Sigma.m`)

$$2(N+1)^2F(N)+\big(3\varepsilon^2+3\varepsilon N+9\varepsilon-4N^2-12N-8\big)F(N+1)$$
$$-(2\varepsilon-N-1)(\varepsilon+2N+6)F(N+2)=0\varepsilon^{-3}-\frac{16}{3}\varepsilon^{-2}+\frac{40}{3}\varepsilon^{-1}-\big(2\zeta_2-\frac{68}{3}\big)\varepsilon^0+\dots$$

$$F(1)=\frac{2}{3}\varepsilon^{-3}-\frac{11}{6}\varepsilon^{-2}+\big(\frac{\zeta_2}{4}+\frac{79}{24}\big)\varepsilon^{-1}+\dots$$
$$F(2)=\frac{8}{9}\varepsilon^{-3}-\frac{73}{27}\varepsilon^{-2}+\big(\frac{\zeta_2}{3}+\frac{1415}{324}\big)\varepsilon^{-1}+\dots$$

$\downarrow$

$$\boxed{F(N)=F_{-3}(N)\,\varepsilon^{-3}+F_{-2}(N)\,\varepsilon^{-2}+F_{-1}(N)\,\varepsilon^{-1}+\dots}$$

Ansatz (for power series)

$$
\begin{aligned}
&a_0(\varepsilon, N)\Big[F(N)\Big] \\
+&a_1(\varepsilon, N)\Big[F(N+1)\Big] \\
+& \\
&\vdots \\
+&a_d(\varepsilon, N)\Big[F(N+d)\Big]
\end{aligned}
$$

$$= h_0(N) + h_1(N)\varepsilon + h_1(N)\varepsilon^2 + \dots$$

given (in terms of indefinite nested sums and products)

Ansatz (for power series)

$$
\begin{aligned}
&a_0(\varepsilon,N)\Big[F_0(N)+F_1(N)\varepsilon+F_2(N)\varepsilon^2+\dots\Big]\\
+&a_1(\varepsilon,N)\Big[F(N+1)\Big]\\
+&\\
&\vdots\\
+&a_d(\varepsilon,N)\Big[F(N+d)\Big]\\
&\qquad\qquad\qquad\qquad = h_0(N)+h_1(N)\varepsilon+h_1(N)\varepsilon^2+\dots
\end{aligned}
$$

given (in terms of indefinite nested sums and products)

Ansatz (for power series)

$$
\begin{aligned}
&a_0(\varepsilon,N)\Big[F_0(N)+F_1(N)\varepsilon+F_2(N)\varepsilon^2+\dots\Big]\\
+&a_1(\varepsilon,N)\Big[F_0(N+1)+F_1(N+1)\varepsilon+F_2(N+1)\varepsilon^2+\dots\Big]\\
+&\\
&\vdots\\
+&a_d(\varepsilon,N)\Big[F(N+d)\Big]\\
&\qquad\qquad = h_0(N)+h_1(N)\varepsilon+h_1(N)\varepsilon^2+\dots
\end{aligned}
$$

given (in terms of indefinite nested sums and products)

Ansatz (for power series)

$$
\begin{aligned}
&a_0(\varepsilon, N)\Big[F_0(N) + F_1(N)\varepsilon + F_2(N)\varepsilon^2 + \dots\Big] \\
+&a_1(\varepsilon, N)\Big[F_0(N+1) + F_1(N+1)\varepsilon + F_2(N+1)\varepsilon^2 + \dots\Big] \\
+& \\
&\vdots \\
+&a_d(\varepsilon, N)\Big[F_0(N+d) + F_1(N+d)\varepsilon + F_2(N+d)\varepsilon^2 + \dots\Big] \\
&\qquad = h_0(N) + h_1(N)\varepsilon + h_1(N)\varepsilon^2 + \dots
\end{aligned}
$$

given (in terms of indefinite nested sums and products)

Ansatz (for power series)

$$
\begin{aligned}
&a_0(\varepsilon,N)\Big[F_0(N)+F_1(N)\varepsilon+F_2(N)\varepsilon^2+\dots\Big]\\
&+a_1(\varepsilon,N)\Big[F_0(N+1)+F_1(N+1)\varepsilon+F_2(N+1)\varepsilon^2+\dots\Big]\\
&+\\
&\ \vdots\\
&+a_d(\varepsilon,N)\Big[F_0(N+d)+F_1(N+d)\varepsilon+F_2(N+d)\varepsilon^2+\dots\Big]\\
&\qquad\qquad\qquad\qquad=h_0(N)+h_1(N)\varepsilon+h_1(N)\varepsilon^2+\dots
\end{aligned}
$$

$\Downarrow$ constant terms must agree

$$\boxed{a_0(0,N)F_0(N)+a_1(0,N)F_0(N+1)+\dots+a_d(0,N)F_0(N+d)=h_0(N)}$$

Ansatz (for power series)

$$
\begin{aligned}
&a_0(\varepsilon,N)\Big[F_0(N)+F_1(N)\varepsilon+F_2(N)\varepsilon^2+\dots\Big]\\
+&a_1(\varepsilon,N)\Big[F_0(N+1)+F_1(N+1)\varepsilon+F_2(N+1)\varepsilon^2+\dots\Big]\\
+&\\
\vdots&\\
+&a_d(\varepsilon,N)\Big[F_0(N+d)+F_1(N+d)\varepsilon+F_2(N+d)\varepsilon^2+\dots\Big]\\
&\qquad\qquad\qquad\qquad = h_0(N)+h_1(N)\varepsilon+h_1(N)\varepsilon^2+\dots
\end{aligned}
$$

$\Downarrow$ constant terms must agree

$$\boxed{a_0(0,N)F_0(N)+a_1(0,N)F_0(N+1)+\dots+a_d(0,N)F_0(N+d)=h_0(N)}$$

REC solver: Given the initial values $F_0(1), F_0(2), \dots, F_0(d)$, **decide** if $F_0(N)$ can be written in terms of indefinite nested sums and products.

Ansatz (for power series)

$$
\begin{aligned}
&a_0(\varepsilon, N)\Big[F_0(N) + F_1(N)\varepsilon + F_2(N)\varepsilon^2 + \dots\Big] \\
+&a_1(\varepsilon, N)\Big[F_0(N+1) + F_1(N+1)\varepsilon + F_2(N+1)\varepsilon^2 + \dots\Big] \\
+& \\
&\vdots \\
+&a_d(\varepsilon, N)\Big[F_0(N+d) + F_1(N+d)\varepsilon + F_2(N+d)\varepsilon^2 + \dots\Big] \\
&\qquad = h_0(N) + h_1(N)\varepsilon + h_1(N)\varepsilon^2 + \dots
\end{aligned}
$$

$\Downarrow$ constant terms must agree

$$
\boxed{a_0(0,N)F_0(N) + a_1(0,N)F_0(N+1) + \dots + a_d(0,N)F_0(N+d) = h_0(N)}
$$

Ansatz (for power series)

$$
\begin{aligned}
&a_0(\varepsilon,N)\Big[F_0(N)+F_1(N)\varepsilon+F_2(N)\varepsilon^2+\dots\Big]\\
&+a_1(\varepsilon,N)\Big[F_0(N+1)+F_1(N+1)\varepsilon+F_2(N+1)\varepsilon^2+\dots\Big]\\
&+\\
&\ \vdots\\
&+a_d(\varepsilon,N)\Big[F_0(N+d)+F_1(N+d)\varepsilon+F_2(N+d)\varepsilon^2+\dots\Big]\\
&\qquad\qquad\qquad\qquad=h_0(N)+h_1(N)\varepsilon+h_1(N)\varepsilon^2+\dots
\end{aligned}
$$

$\Downarrow$ constant terms must agree

$$
\boxed{a_0(0,N)F_0(N)+a_1(0,N)F_0(N+1)+\dots+a_d(0,N)F_0(N+d)=h_0(N)}
$$

$$
\begin{aligned}
&a_0(\varepsilon, N)\Big[F_1(N)\varepsilon + F_2(N)\varepsilon^2 + \dots\Big] \\
&+a_1(\varepsilon, N)\Big[F_1(N+1)\varepsilon + F_2(N+1)\varepsilon^2 + \dots\Big] \\
&+ \\
&\ \vdots \\
&+a_d(\varepsilon, N)\Big[F_1(N+d)\varepsilon + F_2(N+d)\varepsilon^2 + \dots\Big] \\
&\qquad\qquad\qquad\qquad = h_0'(N) + h_1'(N)\varepsilon + h_2'(N)\varepsilon^2 + \dots
\end{aligned}
$$

$$
\begin{aligned}
&a_0(\varepsilon,N)\Big[F_1(N)\varepsilon + F_2(N)\varepsilon^2 + \dots\Big]\\
&+a_1(\varepsilon,N)\Big[F_1(N+1)\varepsilon + F_2(N+1)\varepsilon^2 + \dots\Big]\\
&+\\
&\ \vdots\\
&+a_d(\varepsilon,N)\Big[F_1(N+d)\varepsilon + F_2(N+d)\varepsilon^2 + \dots\Big]\\
&\qquad = \underbrace{h_0'(N)}_{=0} + h_1'(N)\varepsilon + h_2'(N)\varepsilon^2 + \dots
\end{aligned}
$$

Devide by ε

$$
\begin{aligned}
&\quad a_0(\varepsilon, N)\Big[F_1(N) + F_2(N)\varepsilon + \dots\Big] \\
&+a_1(\varepsilon, N)\Big[F_1(N+1) + F_2(N+1)\varepsilon + \dots\Big] \\
&+ \\
&\ \vdots \\
&+a_d(\varepsilon,N)\Big[F_1(N+d) + F_2(N+d)\varepsilon + \dots\Big] = h_1'(N) + h_2'(N)\varepsilon + \dots
\end{aligned}
$$

Now repeat for $F_1(N), F_2(N), \dots$

Remark: Works the same for Laurent series.

Blümlein, Klein, CS, Stan, J. Symbol. Comput. 2012; arXiv:1011.2656[cs.SC]
Ablinger, Blümlein, Round, CS, LL2012, arXiv:1210.1685 [cs.SC]

$$F(N)=\sum_{k=1}^{N}(-1)^k e^{-\frac{3\varepsilon\gamma}{2}}\Gamma\big(-1-\frac{3\varepsilon}{2}\big)B\big(2+k,\frac{\varepsilon}{2}\big)B(-\varepsilon+k,-\varepsilon)B\big(1-\frac{\varepsilon}{2}+k,1+\frac{\varepsilon}{2}\big)\binom{N}{k}$$

↓ (summation package `Sigma.m`)

$$2(N+1)^2F(N)+\big(3\varepsilon^2+3\varepsilon N+9\varepsilon-4N^2-12N-8\big)F(N+1)$$
$$-(2\varepsilon-N-1)(\varepsilon+2N+6)F(N+2)=0\varepsilon^{-3}-\frac{16}{3}\varepsilon^{-2}+\frac{40}{3}\varepsilon^{-1}-\big(2\zeta_2-\frac{68}{3}\big)\varepsilon^0+\dots$$

$$F(1)=\frac{2}{3}\varepsilon^{-3}-\frac{11}{6}\varepsilon^{-2}+\big(\frac{\zeta_2}{4}+\frac{79}{24}\big)\varepsilon^{-1}+\dots$$
$$F(2)=\frac{8}{9}\varepsilon^{-3}-\frac{73}{27}\varepsilon^{-2}+\big(\frac{\zeta_2}{3}+\frac{1415}{324}\big)\varepsilon^{-1}+\dots$$

↓

$$\boxed{F(N)=F_{-3}(N)\,\varepsilon^{-3}+F_{-2}(N)\,\varepsilon^{-2}+F_{-1}(N)\,\varepsilon^{-1}+\dots}$$

$$F(N)=\sum_{k=1}^{N}(-1)^k e^{-\frac{3\varepsilon\gamma}{2}}\Gamma\big(-1-\frac{3\varepsilon}{2}\big)B\big(2+k,\frac{\varepsilon}{2}\big)B(-\varepsilon+k,-\varepsilon)B\big(1-\frac{\varepsilon}{2}+k,1+\frac{\varepsilon}{2}\big)\binom{N}{k}$$

↓ (summation package Sigma.m)

$$2(N+1)^2F(N)+\big(3\varepsilon^2+3\varepsilon N+9\varepsilon-4N^2-12N-8\big)F(N+1)$$
$$-(2\varepsilon-N-1)(\varepsilon+2N+6)F(N+2)=0\varepsilon^{-3}-\frac{16}{3}\varepsilon^{-2}+\frac{40}{3}\varepsilon^{-1}-\big(2\zeta_2-\frac{68}{3}\big)\varepsilon^0+\dots$$

$$F(1)=\frac{2}{3}\varepsilon^{-3}-\frac{11}{6}\varepsilon^{-2}+\big(\frac{\zeta_2}{4}+\frac{79}{24}\big)\varepsilon^{-1}+\dots$$
$$F(2)=\frac{8}{9}\varepsilon^{-3}-\frac{73}{27}\varepsilon^{-2}+\big(\frac{\zeta_2}{3}+\frac{1415}{324}\big)\varepsilon^{-1}+\dots$$

↓ (summation package Sigma.m)

$$F(N)=\tfrac{4N}{3(N+1)}\varepsilon^{-3}-\Big(\tfrac{2(2N+1)}{3(N+1)}S_1(N)+\tfrac{2N(2N+3)}{3(N+1)^2}\Big)\varepsilon^{-2}$$
$$\Big(\tfrac{(1-4N)}{6(N+1)}S_1(N)^2-\tfrac{N\left(N^2-2\right)}{3(N+1)^3}+\tfrac{(3N+2)(4N+5)}{3(N+1)^2}S_1(N)+\tfrac{(1-4N)}{6(N+1)}S_2(N)+\tfrac{N\zeta_2}{2(N+1)}\Big)\varepsilon^{-1}+\dots$$

Find a recurrence for the integral/sum

$$F(N) = \int_0^1 \cdots \int_0^1 \Phi(\varepsilon, N, x_1, x_2, \ldots, x_7) dx_1 dx_2 \ldots dx_7$$
$$\stackrel{?}{=} F_{-3}(N)\varepsilon^{-3} + F_{-2}(N)\varepsilon^{-2} + F_{-1}(N)\varepsilon^{-1} + \ldots$$

multivariate
Almquist/Zeilberger
(J. Ablinger)

$$a_0(\varepsilon, N)F(N) + \ldots + a_d(\varepsilon, N)F(N+d) = h(\varepsilon, N)$$

Find a recurrence for the integral/sum

$$F(N) = \int_0^1 \cdots \int_0^1 \Phi(\varepsilon, N, x_1, x_2, \ldots, x_7) dx_1 dx_2 \ldots dx_7$$
$$\stackrel{?}{=} F_{-3}(N)\varepsilon^{-3} + F_{-2}(N)\varepsilon^{-2} + F_{-1}(N)\varepsilon^{-1} + \ldots$$

ε-recurrence solver

multivariate
Almquist/Zeilberger
(J. Ablinger)

$$a_0(\varepsilon, N)F(N) + \ldots + a_d(\varepsilon, N)F(N+d) = h(\varepsilon, N)$$

Find a recurrence for the integral/sum

$$F(N) = \int_0^1 \cdots \int_0^1 \Phi(\varepsilon, N, x_1, x_2, \ldots, x_7) dx_1 dx_2 \ldots dx_7$$
$$\stackrel{?}{=} F_{-3}(N)\varepsilon^{-3} + F_{-2}(N)\varepsilon^{-2} + F_{-1}(N)\varepsilon^{-1} + \ldots$$

ε-recurrence solver

multivariate Almquist/Zeilberger (J. Ablinger)

$$\sum_{i_1} \cdots \sum_{i_7} f(\varepsilon, N, i_1, i_2, \ldots, i_7)$$

Wegschaider's MultiSum Package (F. Stan)

$$a_0(\varepsilon, N)F(N) + \ldots + a_d(\varepsilon, N)F(N+d) = h(\varepsilon, N)$$

Find a recurrence for the integral/sum

$$F(N) = \int_0^1 \cdots \int_0^1 \Phi(\varepsilon, N, x_1, x_2, \ldots, x_7) dx_1 dx_2 \ldots dx_7$$
$$\stackrel{?}{=} F_{-3}(N)\varepsilon^{-3} + F_{-2}(N)\varepsilon^{-2} + F_{-1}(N)\varepsilon^{-1} + \ldots$$

ε-recurrence solver

multivariate Almquist/Zeilberger (J. Ablinger)

$$\sum_{i_1} \cdots \sum_{i_7} f(\varepsilon, N, i_1, i_2, \ldots, i_7)$$

Wegschaider's MultiSum Package (F. Stan)

Holonomic/difference field approach (M. Round)

$$a_0(\varepsilon, N)F(N) + \ldots + a_d(\varepsilon, N)F(N+d) = h(\varepsilon, N)$$

The general tactic

Feynman integrals

$\downarrow$ non-trivial transformations (DESY)

multiple sums

$\downarrow$ symbolic summation

compact expression in terms of special functions

Tactic 2: Expand a recurrence in ε

Blümlein, Klein, CS, Stan, J. Symbol. Comput. 2012; arXiv:1011.2656 [cs.SC]
Ablinger, Blümlein, Round, CS, LL2012, arXiv:1210.1685 [cs.SC]

The general tactic

Feynman integrals

↓ non-trivial transformations (DESY)

multiple sums

↓ symbolic summation

compact expression in terms of special functions

Tactic 3: Guess a recurrence and solve it

J. Blümlein, M. Kauers, S. Klein, CS, Comput. Phys. Comm. 180, arXiv:0902.4091 [hep-ph]

In the non-singlet (3-loop, massless) case ~ 360 diagrams contribute. The integrals are of the form:

$$F(n,\varepsilon) = \int_0^1 dx_1 \cdots \int_0^1 dx_7 \sum_{i=1}^{K} \frac{p_i(x_1, x_2, \dots, x_7)^{n+\cdots+r_i\varepsilon+\dots}}{q_i(x_1, x_2, \dots, x_7)^{\cdots+s_i\varepsilon+\dots}}$$

where $K \in \mathbb{N}$, $r_i, s_i \in \mathbb{Q}$, and p_i, q_i are polynomials in $x_1, \dots, x_7$.

Vermaseren, Moch: 3-5 CPU years (2004)

In the non-singlet (3-loop, massless) case ~ 360 diagrams contribute. The integrals are of the form:

$$F(n,\varepsilon) = \int_0^1 dx_1 \cdots \int_0^1 dx_7 \sum_{i=1}^{K} \frac{p_i(x_1,x_2,\ldots,x_7)^{n+\cdots+r_i\varepsilon+\ldots}}{q_i(x_1,x_2,\ldots,x_7)^{\cdots+s_i\varepsilon+\ldots}}$$

$$= F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + F_{-1}(n)\varepsilon^{-1} + \boxed{F_0(n)}\varepsilon^{0} + \ldots$$

$$\downarrow$$

Initial values $F_0(i)$, $i = 1, \ldots, 5114$

In the non-singlet (3-loop, massless) case ~ 360 diagrams contribute. The integrals are of the form:

$$F(n,\varepsilon) = \int_0^1 dx_1 \cdots \int_0^1 dx_7 \sum_{i=1}^{K} \frac{p_i(x_1,x_2,\ldots,x_7)^{n+\cdots+r_i\varepsilon+\ldots}}{q_i(x_1,x_2,\ldots,x_7)^{\cdots+s_i\varepsilon+\ldots}}$$
$$= F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + F_{-1}(n)\varepsilon^{-1} + \boxed{F_0(n)}\varepsilon^0 + \ldots$$

$$\downarrow$$

Initial values $F_0(i)$, $i = 1, \ldots, 5114$

$\downarrow$ Recurrence guesser (M. Kauers)

$$a_0(n)F_0(n) + a_1(n)F_0(n+1) + \cdots + a_{35}(n)F_0(n+35) = 0$$

$$a_0(n)F_0(n) + a_1(n)F_0(n+1) + \cdots + \boxed{a_{35}(n)}\, F_0(n+35) = 0$$

$$a_0(n)F_0(n) + a_1(n)F_0(n+1) + \cdots + \boxed{a_{35}(n)}\,F_0(n+35) = 0$$

$$a_{35}(n) = \boxed{A_0} + A_1 n + A_2 n^2 + \cdots + A_{938} n^{983} \in \mathbb{Z}[n]$$

$$a_0(n)F_0(n) + a_1(n)F_0(n+1) + \cdots + \boxed{a_{35}(n)}\, F_0(n+35) = 0$$

$$a_{35}(n) = \boxed{A_0} + A_1 n + A_2 n^2 + \cdots + A_{938} n^{983} \in \mathbb{Z}[n]$$

$A_0 =$ 4640944309211313672503980223716264124200407085993854002412460315194
95765021269344971048446299722216293405285738333200767150194016391501666
27950213807356109710952045603966273388757782697588602201277983560532017
37487592671445911325765145271945214255462153147308420597210761595329365
51563452998613135384718911305253299053198893606401464021608911620974192
09001668029951620780182947258262939450801154511774527832503874341661898
89167522107378468797979810265385510643937043867557563467523740406094658
99100467933353731959645624977524424672990654427732309881685346483771128
69020837147452024401528169079406933665344476181260243344172097691636706
62803059675535809027169693064474147719610219849628486896079642312975136
20776876867741883488363846944854496482629372436829699055391369178850397
00381638011612302679580897488076647721311930634735316787779620757659951
5202809978299053753901432067359626151

(885 decimal digits)

In the non-singlet (3-loop, massless) case ~ 360 diagrams contribute. The integrals are of the form:

$$F(n,\varepsilon) = \int_0^1 dx_1 \cdots \int_0^1 dx_7 \sum_{i=1}^{K} \frac{p_i(x_1,x_2,\ldots,x_7)^{n+\cdots+r_i\varepsilon+\ldots}}{q_i(x_1,x_2,\ldots,x_7)^{\cdots+s_i\varepsilon+\ldots}}$$

$$= F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + F_{-1}(n)\varepsilon^{-1} + \boxed{F_0(n)}\varepsilon^0 + \ldots$$

$\downarrow$

Initial values $F_0(i)$, $i = 1, \ldots, 5114$

$\downarrow$ Recurrence guesser (M. Kauers)

$$a_0(n)F_0(n) + a_1(n)F_0(n+1) + \cdots + a_{35}(n)F_0(n+35) = 0$$

$\downarrow$ Sigma

CLOSED FORM

Blümlein, Kauers, Klein, CS, Comput. Phys. Comm. 180, arXiv:0902.4091 [hep-ph]

Tactic 4: Solve coupled systems of differential equations

A coupled differential system for $\hat{I}_1(x), \hat{I}_2(x), \hat{I}_3(x)$

(produced by IBP [extension of REDUZE_2, A.v. Manteuffel])

$$D_x \begin{pmatrix} \hat{I}_1(x) \\ \hat{I}_2(x) \\ \hat{I}_3(x) \end{pmatrix} = \begin{pmatrix} -\frac{-1-\varepsilon+x}{(x-1)x} & -\frac{2}{(x-1)x} & 0 \\ \frac{\varepsilon(3\varepsilon+2)}{4(x-1)} & -\frac{-2-\varepsilon+3x+3\varepsilon x}{2(x-1)x} & -\frac{\varepsilon+1}{2(x-1)} \\ -\frac{\varepsilon(3\varepsilon+2)(x-2)}{4(x-1)x} & \frac{-2-5\varepsilon+x+3\varepsilon x}{2(x-1)x} & \frac{(-2\varepsilon-x+\varepsilon x)}{2(x-1)x} \end{pmatrix} \begin{pmatrix} \hat{I}_1(x) \\ \hat{I}_2(x) \\ \hat{I}_3(x) \end{pmatrix} + \begin{pmatrix} \hat{R}_1(x) \\ \hat{R}_2(x) \\ -\hat{R}_2(x) \end{pmatrix}$$

where

$$\hat{R}_1(x) = \frac{\hat{B}_4(x)}{(x-1)x},$$

$$\hat{R}_2(x) = \frac{-(\varepsilon+2)^3}{16(\varepsilon+1)(x-1)x}\hat{B}_1(x) + \frac{(\varepsilon+2)(3\varepsilon+4)\left(19\varepsilon^2+36\varepsilon+16\right)}{16\varepsilon(5\varepsilon+6)(x-1)x}\hat{B}_2(x)$$
$$+ \frac{(\varepsilon+1)^2(3\varepsilon+4)^2}{2\varepsilon(5\varepsilon+6)x}\hat{B}_3(x) + \frac{-24-50\varepsilon-25\varepsilon^2+8x+14\varepsilon x+6\varepsilon^2 x}{4(5\varepsilon+6)(x-1)x}\hat{B}_4(x)$$

$\hat{B}_1(x), \hat{B}_2(x), \hat{B}_3(x)$ have been solved with symbolic summation.

Tactic 4: the DE-REC approach

DE system

$$D\hat{I}(x) = A\,\hat{I}(x) + \hat{R}(x)$$

Tactic 4: the DE-REC approach

DE system	OreSys package (S. Gerhold) → uncoupling algorithm	uncoupled DE system
$D\hat{I}(x) = A\,\hat{I}(x) + \hat{R}(x)$		$\sum_i a_i(x) D^i \hat{I}_1(x) = r(x)$ $\hat{I}_k(x) = \mathsf{expr}_k(\hat{I}_1(x)), k > 1$

Tactic 4: the DE-REC approach

DE system	OreSys package (S. Gerhold) → uncoupling algorithm	uncoupled DE system
$D\hat{I}(x) = A\,\hat{I}(x) + \hat{R}(x)$		$\sum_i a_i(x) D^i \hat{I}_1(x) = r(x)$ $\hat{I}_k(x) = \mathsf{expr}_k(\hat{I}_1(x)), k > 1$

$$\hat{I}_1(x) = \sum_{N=0}^{\infty} I_1(N) x^N$$

Tactic 4: the DE-REC approach

DE system

$$D\hat{I}(x) = A\,\hat{I}(x) + \hat{R}(x)$$

→ OreSys package (S. Gerhold) — uncoupling algorithm →

uncoupled DE system

$$\sum_i a_i(x) D^i \hat{I}_1(x) = r(x)$$

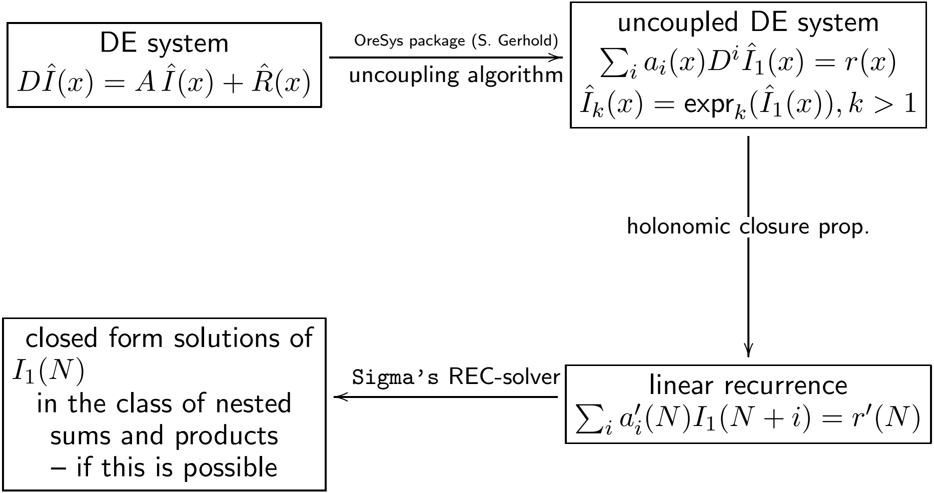
$$\hat{I}_k(x) = \mathsf{expr}_k(\hat{I}_1(x)), k > 1$$

↓ holonomic closure prop. ↓

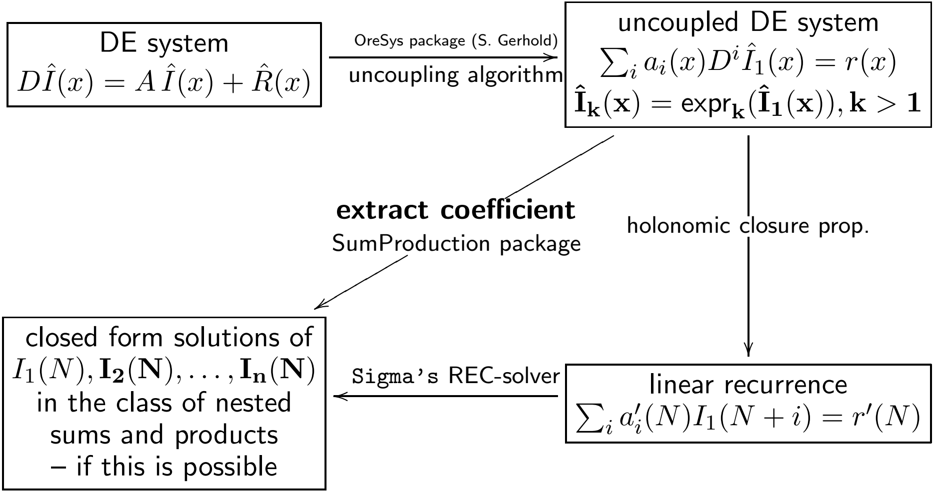
linear recurrence

$$\sum_i a_i'(N) I_1(N+i) = r'(N)$$

Tactic 4: the DE-REC approach



Tactic 4: the DE-REC approach (SolveCoupledSystem package)



J. Ablinger, A. Behring, J. Blmlein, A. De Freitas, A. von Manteuffel, CS. Comput. Phys. Comm. 202, pp. 33-112.

Solving a coupled differential system

In[4]:= $<<$ **OreSys.m**

OreSys by Stefan Gerhold (optimized by C. Schneider) © RISC-Linz

In[5]:= $<<$ **SolveCoupledSystem.m**

SolveCoupledSystem by Carsten Schneider © RISC-Linz

In[6]:= $\mathbf{coupledDESys = D[\{\hat{I}_1(x), \hat{I}_2(x), \hat{I}_3(x)\}, x] - A.\{\hat{I}_1(x), \hat{I}_2(x), \hat{I}_3(x)\};}$

In[7]:= $\mathbf{rhs = \{\hat{R}_1(x), \hat{R}_2(x), -\hat{R}_2(x)\}}$ in power series representation;

In[8]:= $\mathbf{SolveCoupledDESystem[coupledDESys, rhs, \{I_1[x], I_2[x], I_3[x]\}, \varepsilon, -3, \{-2, -2, -2\}, rhs, \dots]}$

Solving a coupled differential system

In[4]:= $<<$ **OreSys.m**

OreSys by Stefan Gerhold (optimized by C. Schneider) © RISC-Linz

In[5]:= $<<$ **SolveCoupledSystem.m**

SolveCoupledSystem by Carsten Schneider © RISC-Linz

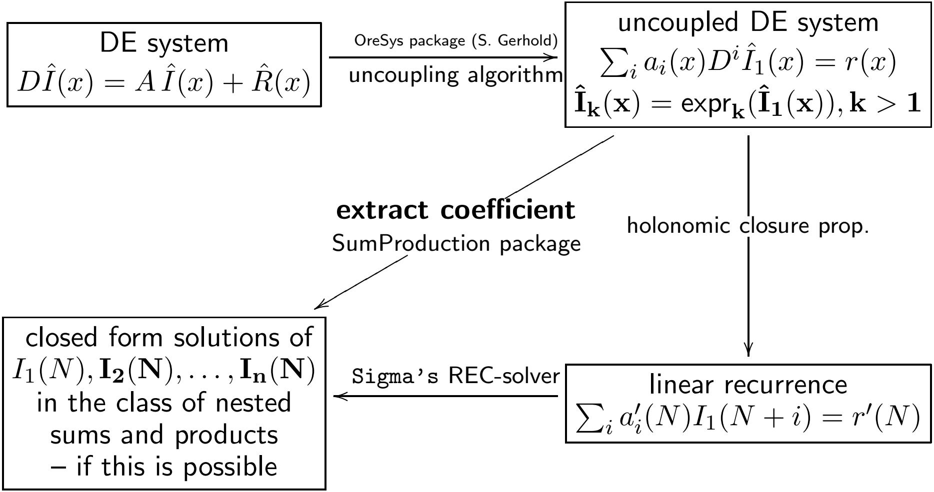
In[6]:= $\mathbf{coupledDESys = D[\{\hat{I}_1(x), \hat{I}_2(x), \hat{I}_3(x)\}, x] - A.\{\hat{I}_1(x), \hat{I}_2(x), \hat{I}_3(x)\};}$

In[7]:= $\mathbf{rhs = \{\hat{R}_1(x), \hat{R}_2(x), -\hat{R}_2(x)\}}$ in power series representation;

In[8]:= $\mathbf{SolveCoupledDESystem[coupledDESys, rhs, \{I_1[x], I_2[x], I_3[x]\}, \varepsilon, -3, \{-2, -2, -2\}, rhs, \ldots]}$

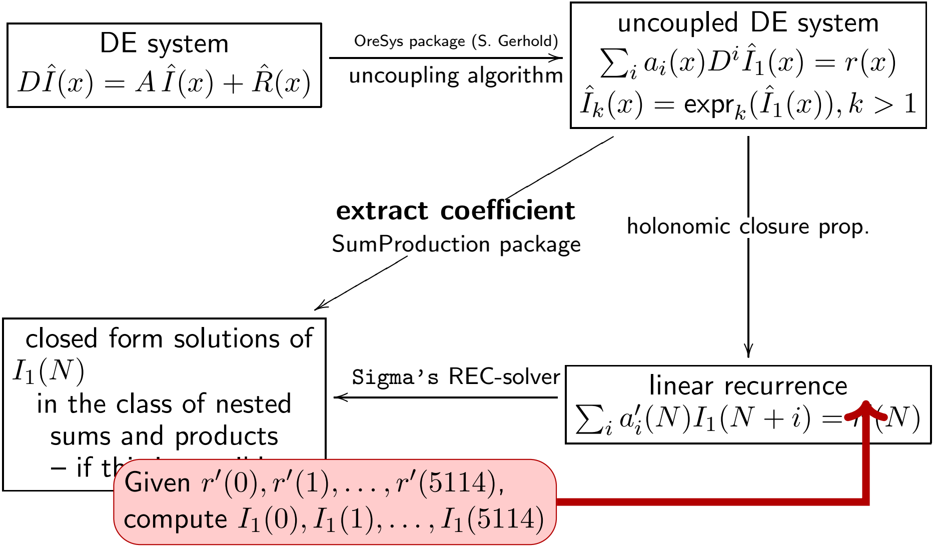
Out[8]= $$\Big\{\frac{1}{\varepsilon^3}\Big(\frac{4\big(3N^2+6N+4\big)}{3(N+1)^2}+\frac{4S_1[N]}{3(N+1)}\Big)+\frac{1}{\varepsilon^2}\Big(-\frac{2(20N^3+58N^2+57N+22)}{3(N+1)^3}+\frac{2(N+2)(2N-1)S_1[N]}{3(N+1)^2}-\frac{S_1[N]^2}{N+1}-\frac{S_2[N]}{N+1}\Big),$$
$$\frac{4}{3\varepsilon^3}-\frac{2}{\varepsilon^2}, \frac{8}{3\varepsilon^3}+\frac{1}{\varepsilon^2}\Big(-\frac{4\big(4N^2+7N+2\big)}{3(N+1)^2}+\frac{4(N+2)S_1[N]}{3(N+1)}\Big)\Big\}$$

Tactic 4: the DE-REC approach (SolveCoupledSystem package)



J. Ablinger, A. Behring, J. Blmlein, A. De Freitas, A. von Manteuffel, CS. Comput. Phys. Comm. 202, pp. 33-112.

Tactic 5: compute large moments (SolveCoupledSystem)



J. Blümlein, CS, Physics Letters B 771, pp. 31-36., 2017; arXiv:1701.04614 [hep-ph].

In the non-singlet (3-loop, massless) case ~ 360 diagrams contribute. The integrals are of the form:

$$F(n,\varepsilon) = \int_0^1 dx_1 \cdots \int_0^1 dx_7 \sum_{i=1}^{K} \frac{p_i(x_1,x_2,\ldots,x_7)^{n+\cdots+r_i\varepsilon+\ldots}}{q_i(x_1,x_2,\ldots,x_7)^{\cdots+s_i\varepsilon+\ldots}}$$

$$= F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + F_{-1}(n)\varepsilon^{-1} + \boxed{F_0(n)}\varepsilon^0 + \ldots$$

$\downarrow$

Initial values $F_0(i)$, $i = 1, \ldots, 5114$

$\downarrow$ Recurrence guesser (M. Kauers)

$$a_0(n)F_0(n) + a_1(n)F_0(n+1) + \cdots + a_{35}(n)F_0(n+35) = 0$$

$\downarrow$ Sigma

CLOSED FORM

Blümlein, Kauers, Klein, CS, Comput. Phys. Comm. 180, arXiv:0902.4091 [hep-ph]

In the non-singlet (3-loop, massless) case ~ 360 diagrams contribute. The integrals are of the form:

$$F(n,\varepsilon) = \int_0^1 dx_1 \cdots \int_0^1 dx_7 \sum_{i=1}^{K} \frac{p_i(x_1,x_2,\dots,x_7)^{n+\dots+r_i\varepsilon+\dots}}{q_i(x_1,x_2,\dots,x_7)^{\dots+s_i\varepsilon+\dots}}$$

$$= F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + F_{-1}(n)\varepsilon^{-1} + \boxed{F_0(n)}\varepsilon^{0} + \dots$$

$\downarrow$ a new method for large moments

Initial values $F_0(i)$, $i = 1, \dots, 5114$

$\downarrow$ Recurrence guesser (M. Kauers)

$$a_0(n)F_0(n) + a_1(n)F_0(n+1) + \dots + a_{35}(n)F_0(n+35) = 0$$

$\downarrow$ Sigma

CLOSED FORM

Blümlein, Kauers, Klein, CS, Comput. Phys. Comm. 180, arXiv:0902.4091 [hep-ph]

Blümlein, CS. Physics Letters B 771, arXiv:1701.04614

Concrete calculations of large moments:

- J. Ablinger, A. Behring, J. Blmlein, A. De Freitas, A. von Manteuffel, C. Schneider The Three-Loop Splitting Functions $P_{qg}^{(2)}$ and $P_{gg}^{(2,\mathrm{N_F})}$. Nucl. Phys. B. 922, pp. 1-40. 2017. ISSN 0550-3213.. arXiv:1705.01508 [hep-ph].

  1. computed ~ 2400 moments
  2. guessed all recurrences
  3. solved all recurrences in terms of harmonic sums.

Concrete calculations of large moments:

- J. Ablinger, A. Behring, J. Blmlein, A. De Freitas, A. von Manteuffel, C. Schneider The Three-Loop Splitting Functions $P_{qg}^{(2)}$ and $P_{gg}^{(2,\mathrm{N_F})}$. Nucl. Phys. B. 922, pp. 1-40. 2017. ISSN 0550-3213.. arXiv:1705.01508 [hep-ph].
  1. computed ~ 2400 moments
  2. guessed all recurrences
  3. solved all recurrences in terms of harmonic sums.

- massive Wilson coefficient A_{Qg}:
  1. computed 2000 moments
  2. guessed and solved some recurrences.

Concrete calculations of large moments:

- J. Ablinger, A. Behring, J. Blmlein, A. De Freitas, A. von Manteuffel, C. Schneider The Three-Loop Splitting Functions $P_{qg}^{(2)}$ and $P_{gg}^{(2,\mathrm{N_F})}$. Nucl. Phys. B. 922, pp. 1-40. 2017. ISSN 0550-3213.. arXiv:1705.01508 [hep-ph].
  1. computed ~ 2400 moments
  2. guessed all recurrences
  3. solved all recurrences in terms of harmonic sums.

- massive Wilson coefficient A_{Qg}:
  1. computed 2000 moments
  2. guessed and solved some recurrences.

- special case: $A_{Qg}^{T_F^2,(3)}$ (as case study)
  1. computed 8000 moments
  2. guessed all recurrences
  3. solved some recurrences

SUMMARY (of RISC packages)

Backbone: a new difference ring/field approach implemented in `Sigma.m`

SUMMARY (of RISC packages)

Backbone: a new difference ring/field approach implemented in `Sigma.m`

- symbolic summation
  - CS's `EvaluateMultiSum.m`
  - M. Round's `RhoSum.m` (difference ring/holonomic approach)
  - K. Wegschaider's `MultiSum.m` (enhanced Sister Celine/WZ approach, utilized by F. Stan)

SUMMARY (of RISC packages)

Backbone: a new difference ring/field approach implemented in `Sigma.m`

- symbolic summation
  - CS's `EvaluateMultiSum.m`
  - M. Round's `RhoSum.m` (difference ring/holonomic approach)
  - K. Wegschaider's `MultiSum.m` (enhanced Sister Celine/WZ approach, utilized by F. Stan)
- symbolic integration
  - J. Ablinger's `MultiIntegrate.m` (refined multivariate Almkvist-Zeilberger approach)

SUMMARY (of RISC packages)

Backbone: a new difference ring/field approach implemented in `Sigma.m`

- symbolic summation
  - CS's `EvaluateMultiSum.m`
  - M. Round's `RhoSum.m` (difference ring/holonomic approach)
  - K. Wegschaider's `MultiSum.m` (enhanced Sister Celine/WZ approach, utilized by F. Stan)
- symbolic integration
  - J. Ablinger's `MultiIntegrate.m` (refined multivariate Almkvist-Zeilberger approach)
- solving coupled systems
  - S. Gerhold's `OreSys.m`
  - CS's SolveCoupledSystem.m

SUMMARY (of RISC packages)

Backbone: a new difference ring/field approach implemented in `Sigma.m`

- symbolic summation
  - CS's `EvaluateMultiSum.m`
  - M. Round's `RhoSum.m` (difference ring/holonomic approach)
  - K. Wegschaider's `MultiSum.m` (enhanced Sister Celine/WZ approach, utilized by F. Stan)
- symbolic integration
  - J. Ablinger's `MultiIntegrate.m` (refined multivariate Almkvist-Zeilberger approach)
- solving coupled systems
  - S. Gerhold's `OreSys.m`
  - CS's SolveCoupledSystem.m
- special function algorithms
  - J. Ablinger's `HarmonicSums.m`

What comes next?

The underlying machinery: the (inverse) Mellin transform

$$F(n) \qquad = \int_0^1 x^n f(x)\, dx$$

The underlying machinery: the (inverse) Mellin transform

$$\sum_{i=1}^{n} \frac{\binom{2i}{i}\sum_{j=1}^{i}\frac{1}{j^2}}{i} = \int_0^1 x^n f(x)\,dx$$

The underlying machinery: the (inverse) Mellin transform

$$\sum_{i=1}^{n} \frac{\binom{2i}{i} \sum_{j=1}^{i} \frac{1}{j^2}}{i} = \int_0^1 x^n f(x)\, dx$$

$$\begin{aligned}
&(-16x + 48x^2) f(x) \\
&+4x\left(13 - 179x + 244x^2\right) f'(x) \\
&+\left(-6x + 578x^2 - 2524x^3 + 2240x^4\right) f''(x) \\
&+\left(-39x^2 + 776x^3 - 2052x^4 + 1360x^5\right) f^{(3)}(x) \\
&+\left(-28x^3 + 268x^4 - 512x^5 + 272x^6\right) f^{(4)}(x) \\
&+\left(-4x^4 + 24x^5 - 36x^6 + 16x^7\right) f^{(5)}(x) = 0
\end{aligned}$$

using `HarmonicSums`

The underlying machinery: the (inverse) Mellin transform

$$\sum_{i=1}^{n} \frac{\binom{2i}{i} \sum_{j=1}^{i} \frac{1}{j^2}}{i} = \int_0^1 ((4\,x)^n - 1) \left(\frac{-4\pi \int_0^x \frac{1}{\sqrt{1-\tau}\sqrt{\tau}} d\tau}{3(1-4x)} + \frac{2 \left(\int_0^x \frac{1}{\sqrt{1-\tau}\sqrt{\tau}} d\tau \right)^2}{1-4x} - \frac{2 \left(\int_0^x \frac{1}{\sqrt{1-\tau}\sqrt{\tau}} d\tau \right)^3}{3\pi(1-4x)} \right) dx$$

$$\begin{aligned}
&(-16x + 48x^2) f(x) \\
&+4x(13 - 179x + 244x^2) f'(x) \\
&+(-6x + 578x^2 - 2524x^3 + 2240x^4) f''(x) \\
&+(-39x^2 + 776x^3 - 2052x^4 + 1360x^5) f^{(3)}(x) \\
&+(-28x^3 + 268x^4 - 512x^5 + 272x^6) f^{(4)}(x) \\
&+(-4x^4 + 24x^5 - 36x^6 + 16x^7) f^{(5)}(x) = 0
\end{aligned}$$

using `HarmonicSums`

The underlying machinery: the (inverse) Mellin transform

$$
\begin{aligned}
F(n) \qquad &= \int_0^1 \left((4\,x)^n - 1\right) \left(\frac{-4\pi \ \int_0^x \frac{1}{\sqrt{1-\tau}\sqrt{\tau}} d\tau}{3(1-4x)} + \frac{2\left(\int_0^x \frac{1}{\sqrt{1-\tau}\sqrt{\tau}} d\tau\right)^2}{1-4x} \right. \\
&\qquad \left. - \frac{2\left(\int_0^x \frac{1}{\sqrt{1-\tau}\sqrt{\tau}} d\tau\right)^3}{3\pi(1-4x)} \right) dx
\end{aligned}
$$

The underlying machinery: the (inverse) Mellin transform

$$F(n) = \int_0^1 \left((4\,x)^n - 1\right)\left(\frac{-4\pi\ \int_0^x \frac{1}{\sqrt{1-\tau}\sqrt{\tau}}d\tau}{3(1-4x)} + \frac{2\left(\int_0^x \frac{1}{\sqrt{1-\tau}\sqrt{\tau}}d\tau\right)^2}{1-4x} - \frac{2\left(\int_0^x \frac{1}{\sqrt{1-\tau}\sqrt{\tau}}d\tau\right)^3}{3\pi(1-4x)}\right)dx$$

$$\begin{aligned}(1-n)n(-1+2n)(1+2n)F(-2+n)\\ +2n(1+2n)\left(3-4n+6n^2\right)F(-1+n)\\ -4\left(1+6n+14n^2+16n^3+9n^4\right)F(n)\\ +16(1+n)^4F(1+n)=0\end{aligned}$$

using `HarmonicSums/Sigma`

The underlying machinery: the (inverse) Mellin transform

$$\sum_{i=1}^{n} \frac{\binom{2i}{i}\sum_{j=1}^{i}\frac{1}{j^2}}{i} = \int_0^1 ((4\,x)^n - 1)\left(\frac{-4\pi\ \int_0^x \frac{1}{\sqrt{1-\tau}\sqrt{\tau}}d\tau}{3(1-4x)} + \frac{2\left(\int_0^x \frac{1}{\sqrt{1-\tau}\sqrt{\tau}}d\tau\right)^2}{1-4x} - \frac{2\left(\int_0^x \frac{1}{\sqrt{1-\tau}\sqrt{\tau}}d\tau\right)^3}{3\pi(1-4x)}\right)dx$$

$$\begin{aligned}(1-n)n(-1+2n)(1+2n)F(-2+n)\\ +2n(1+2n)\left(3-4n+6n^2\right)F(-1+n)\\ -4\left(1+6n+14n^2+16n^3+9n^4\right)F(n)\\ +16(1+n)^4F(1+n)=0\end{aligned}$$

using `HarmonicSums`/`Sigma`

The underlying machinery: the (inverse) Mellin transform

$$\sum_{i=1}^{n}\frac{\binom{2i}{i}\sum_{j=1}^{i}\frac{1}{j^2}}{i}=\int_0^1((4\,x)^n-1)\left(\frac{-4\pi\ \int_0^x\frac{1}{\sqrt{1-\tau}\sqrt{\tau}}d\tau}{3(1-4x)}+\frac{2\left(\int_0^x\frac{1}{\sqrt{1-\tau}\sqrt{\tau}}d\tau\right)^2}{1-4x}-\frac{2\left(\int_0^x\frac{1}{\sqrt{1-\tau}\sqrt{\tau}}d\tau\right)^3}{3\pi(1-4x)}\right)dx$$

The underlying machinery: the (inverse) Mellin transform

recurrence solver

$$\sum_{i=1}^{n} \frac{\binom{2i}{i}\sum_{j=1}^{i} \frac{1}{j^2}}{i} = \int_0^1 \left((4\,x)^n - 1\right)\left(\frac{-4\pi\ \int_0^x \frac{1}{\sqrt{1-\tau}\sqrt{\tau}}d\tau}{3(1-4x)} + \frac{2\left(\int_0^x \frac{1}{\sqrt{1-\tau}\sqrt{\tau}}d\tau\right)^2}{1-4x} - \frac{2\left(\int_0^x \frac{1}{\sqrt{1-\tau}\sqrt{\tau}}d\tau\right)^3}{3\pi(1-4x)}\right)dx$$

$\leftarrow$: most indefinite nested integrals cannot be expressed in terms of indefinite nested sums

The underlying machinery: the (inverse) Mellin transform

recurrence solver

$$\sum_{i=1}^{n} \frac{\binom{2i}{i}\sum_{j=1}^{i} \frac{1}{j^2}}{i} = \int_0^1 ((4\,x)^n - 1)\left(\frac{-4\pi\ \int_0^x \frac{1}{\sqrt{1-\tau}\sqrt{\tau}}d\tau}{3(1-4x)} + \frac{2\left(\int_0^x \frac{1}{\sqrt{1-\tau}\sqrt{\tau}}d\tau\right)^2}{1-4x} - \frac{2\left(\int_0^x \frac{1}{\sqrt{1-\tau}\sqrt{\tau}}d\tau\right)^3}{3\pi(1-4x)}\right)dx$$

differential equation solver

- $\leftarrow$: most indefinite nested integrals cannot be expressed in terms of indefinite nested sums
- $\rightarrow$: most indefinite nested sums cannot be expressed in terms of indefinite nested integrals

The underlying machinery: the (inverse) Mellin transform

recurrence solver

$$\sum_{i=1}^{n} \frac{\binom{2i}{i}\sum_{j=1}^{i}\frac{1}{j^2}}{i} = \int_0^1 \left((4\,x)^n - 1\right)\left(\frac{-4\pi\ \int_0^x \frac{1}{\sqrt{1-\tau}\sqrt{\tau}}d\tau}{3(1-4x)} + \frac{2\left(\int_0^x \frac{1}{\sqrt{1-\tau}\sqrt{\tau}}d\tau\right)^2}{1-4x} - \frac{2\left(\int_0^x \frac{1}{\sqrt{1-\tau}\sqrt{\tau}}d\tau\right)^3}{3\pi(1-4x)}\right)dx$$

differential equation solver

$\leftarrow$: most indefinite nested integrals cannot be expressed in terms of indefinite nested sums

$\rightarrow$: most indefinite nested sums cannot be expressed in terms of indefinite nested integrals

GOAL: extend the available recurrence (DE) solvers.

A very basic example:

$$\frac{1}{\binom{2n}{n}^2} = \int_0^1 f(x)x^n dx$$

A very basic example:

$$\frac{1}{\binom{2n}{n}^2} = \int_0^1 f(x) x^n dx$$

$$\downarrow$$

$$\left(4x^2 - 4x^3\right) f''(x) + \left(12x - 16x^2\right) f'(x) + (4 - 9x) f(x) = 0$$

A very basic example:

$$\frac{1}{\binom{2n}{n}^2} = \int_0^1 f(x) x^n dx$$

$$\downarrow$$

$$\left(4x^2 - 4x^3\right) f''(x) + \left(12x - 16x^2\right) f'(x) + (4 - 9x) f(x) = 0$$

$$\downarrow$$

$$c_1 \frac{K(1-x)}{x} + c_2 \frac{Q_{-\frac{1}{2}}(2x-1)}{x}$$

$K(x) = \int_0^1 \frac{1}{(1-t^2)(1-x^2t^2)} dt = \frac{\pi}{2} \sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{16^n} x^{2n}$: compl. elliptic integral (1st kind)

$Q_n(x)$: Legendre polynomial (2nd kind)

A very basic example:

$$\frac{1}{\binom{2n}{n}^2} = \int_0^1 \left(\frac{3}{16}\frac{K(1-x)}{x} - \frac{1}{8}\frac{Q_{-\frac{1}{2}}(2x-1)}{x}\right) x^n dx$$

$$\downarrow$$

$$\left(4x^2 - 4x^3\right) f''(x) + \left(12x - 16x^2\right) f'(x) + (4-9x) f(x) = 0$$

$$\downarrow$$

$$c_1 \frac{K(1-x)}{x} + c_2 \frac{Q_{-\frac{1}{2}}(2x-1)}{x}$$

$$\downarrow$$

$$c_1 = \frac{3}{16}, \quad c_2 = -\frac{1}{8}$$

$K(x) = \int_0^1 \frac{1}{(1-t^2)(1-x^2t^2)} dt = \frac{\pi}{2} \sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{16^n} x^{2n}$: compl. elliptic integral (1st kind)

$Q_n(x)$: Legendre polynomial (2nd kind)

Concrete calculations of large moments:

- J. Ablinger, A. Behring, J. Blmlein, A. De Freitas, A. von Manteuffel, C. Schneider The Three-Loop Splitting Functions $P_{qg}^{(2)}$ and $P_{gg}^{(2,\mathrm{N_F})}$. Nucl. Phys. B. 922, pp. 1-40. 2017. ISSN 0550-3213.. arXiv:1705.01508 [hep-ph].
    1. computed ~ 2400 moments
    2. guessed all recurrences
    3. solved all recurrences in terms of harmonic sums.

- massive Wilson coefficient A_{Qg}:
    1. computed 2000 moments
    2. guessed and solved some recurrences.

- special case: $A_{Qg}^{T_F^2,(3)}$ (as case study)
    1. computed 8000 moments
    2. guessed all recurrences
    3. solved some recurrences

For more details on our concrete calculations see:

[J.Ablinger, J. Blümlein, A. De Freitas, M. van Hoeij, E. Imamoglu, C.G. Raab, C.-S. Radu, CS. Iterated Elliptic and Hypergeometric Integrals for Feynman Diagrams. submitted, pp. 1-68. 2017. arXiv:1706.01299 [hep-th]

For more details on our concrete calculations see:

[J.Ablinger, J. Blümlein, A. De Freitas, M. van Hoeij, E. Imamoglu, C.G. Raab, C.-S. Radu, CS. Iterated Elliptic and Hypergeometric Integrals for Feynman Diagrams. submitted, pp. 1-68. 2017. arXiv:1706.01299 [hep-th]

Note: q-summation enters the game which is covered by our difference ring theory!

Example

$$\sum_{j=1}^{\infty}\sum_{k=1}^{j}\frac{1-q^k}{(q^j-1)^3(q^{1+j-k}-1)}$$

Example

$$\sum_{j=1}^{a} \sum_{k=1}^{j} \frac{1-q^k}{(q^j-1)^3(q^{1+j-k}-1)}$$

Example

$$\sum_{j=1}^{a}\sum_{k=1}^{j}\frac{1-q^k}{(q^j-1)^3(q^{1+j-k}-1)}$$

$$=(q-1)\sum_{i=1}^{a}\frac{\sum_{j=1}^{i}\frac{1}{1-q^j}}{(-1+q^i)^3}+q\sum_{i=1}^{a}\frac{\sum_{j=1}^{i}\frac{1}{1-q^j}}{(-1+q^i)^2}+\frac{q\sum_{i=1}^{a}\frac{1}{(1-q^i)^2}}{q-1}$$

exploiting our summation tools

Example

$$\sum_{j=1}^{\infty}\sum_{k=1}^{j}\frac{1-q^k}{(q^j-1)^3(q^{1+j-k}-1)} \qquad q>1$$

$$=(q-1)\sum_{i=1}^{\infty}\frac{\sum_{j=1}^{i}\frac{1}{1-q^j}}{(-1+q^i)^3}+q\sum_{i=1}^{\infty}\frac{\sum_{j=1}^{i}\frac{1}{1-q^j}}{(-1+q^i)^2}+\frac{q\sum_{i=1}^{\infty}\frac{1}{(1-q^i)^2}}{q-1}$$

exploiting our summation tools

Example

$$\sum_{j=1}^{\infty}\sum_{k=1}^{j}\frac{1-q^k}{(q^j-1)^3(q^{1+j-k}-1)} \qquad q>1$$

$$= -\tfrac{3q-1}{2(q-1)}\sum_{i=1}^{\infty}\tfrac{1}{1-q^i} + \tfrac{2q^2-9q+5}{2(q-1)}\sum_{i=1}^{\infty}\tfrac{q^i}{(1-q^i)^2} + \tfrac{1-q}{2}\Big(\sum_{i=1}^{\infty}\tfrac{q^i}{(1-q^i)^2}\Big)^2$$
$$+ (q-2)\sum_{i=1}^{\infty}\tfrac{i\,q^i}{(1-q^i)^2} - \tfrac{1}{2}\Big(\sum_{i=1}^{\infty}\tfrac{1}{1-q^i}\Big)^2 + \tfrac{5(q-1)}{2}\sum_{i=1}^{\infty}\tfrac{q^{3i}}{(1-q^i)^4}$$
$$+ \tfrac{7q-11}{2}\sum_{i=1}^{\infty}\tfrac{q^{2i}}{(1-q^i)^3} + (q-1)\sum_{i=1}^{\infty}\tfrac{i\,q^{2i}}{(1-q^i)^3}$$

by human insight and experiments

Example

$$\sum_{j=1}^{\infty}\sum_{k=1}^{j}\frac{1-q^k}{(q^j-1)^3(q^{1+j-k}-1)} \qquad\qquad q>1$$

$$= -\tfrac{3q-1}{2(q-1)}\sum_{i=1}^{\infty}\tfrac{1}{1-q^i} + \tfrac{2q^2-9q+5}{2(q-1)}\sum_{i=1}^{\infty}\tfrac{q^i}{(1-q^i)^2} + \tfrac{1-q}{2}\Big(\sum_{i=1}^{\infty}\tfrac{q^i}{(1-q^i)^2}\Big)^2$$
$$+ (q-2)\sum_{i=1}^{\infty}\tfrac{i\,q^i}{(1-q^i)^2} - \tfrac{1}{2}\Big(\sum_{i=1}^{\infty}\tfrac{1}{1-q^i}\Big)^2 + \tfrac{5(q-1)}{2}\sum_{i=1}^{\infty}\tfrac{q^{3i}}{(1-q^i)^4}$$
$$+ \tfrac{7q-11}{2}\sum_{i=1}^{\infty}\tfrac{q^{2i}}{(1-q^i)^3} + (q-1)\sum_{i=1}^{\infty}\tfrac{i\,q^{2i}}{(1-q^i)^3}$$

$$\Downarrow \quad (q\to 1)$$

$$\sum_{j=1}^{\infty}\sum_{k=1}^{j}\frac{k}{j^3(j-k+1)} = -\zeta(2)+\frac{1}{2}\zeta(2)^2+2\zeta(3)$$

Example

$$\sum_{j=1}^{\infty}\sum_{k=1}^{j}\frac{1-q^k}{(q^j-1)^3(q^{1+j-k}-1)} \qquad q>1$$

$$= -\frac{3q-1}{2(q-1)}\sum_{i=1}^{\infty}\frac{1}{1-q^i} + \frac{2q^2-9q+5}{2(q-1)}\sum_{i=1}^{\infty}\frac{q^i}{(1-q^i)^2} + \frac{1-q}{2}\Big(\sum_{i=1}^{\infty}\frac{q^i}{(1-q^i)^2}\Big)^2$$
$$+ (q-2)\sum_{i=1}^{\infty}\frac{i\,q^i}{(1-q^i)^2} - \frac{1}{2}\Big(\sum_{i=1}^{\infty}\frac{1}{1-q^i}\Big)^2 + \frac{5(q-1)}{2}\sum_{i=1}^{\infty}\frac{q^{3i}}{(1-q^i)^4}$$
$$+ \frac{7q-11}{2}\sum_{i=1}^{\infty}\frac{q^{2i}}{(1-q^i)^3} + (q-1)\sum_{i=1}^{\infty}\frac{i\,q^{2i}}{(1-q^i)^3}$$

$$\Downarrow \quad (q\to 1)$$

$$\sum_{j=1}^{\infty}\sum_{k=1}^{j}\frac{k}{j^3(j-k+1)} = -\zeta(2)+\frac{1}{2}\zeta(2)^2+2\zeta(3)$$

Goal

1. generalize our tools from $q=1$ to the general q-case for automatic simplification

Example

$$\sum_{j=1}^{\infty}\sum_{k=1}^{j}\frac{1-q^k}{(q^j-1)^3(q^{1+j-k}-1)} \qquad q>1$$

$$= -\frac{3q-1}{2(q-1)}\sum_{i=1}^{\infty}\frac{1}{1-q^i} + \frac{2q^2-9q+5}{2(q-1)}\sum_{i=1}^{\infty}\frac{q^i}{(1-q^i)^2} + \frac{1-q}{2}\Big(\sum_{i=1}^{\infty}\frac{q^i}{(1-q^i)^2}\Big)^2$$
$$+ (q-2)\sum_{i=1}^{\infty}\frac{i\,q^i}{(1-q^i)^2} - \frac{1}{2}\Big(\sum_{i=1}^{\infty}\frac{1}{1-q^i}\Big)^2 + \frac{5(q-1)}{2}\sum_{i=1}^{\infty}\frac{q^{3i}}{(1-q^i)^4}$$
$$+ \frac{7q-11}{2}\sum_{i=1}^{\infty}\frac{q^{2i}}{(1-q^i)^3} + (q-1)\sum_{i=1}^{\infty}\frac{i\,q^{2i}}{(1-q^i)^3}$$

$$\Downarrow \quad (q\to 1)$$

$$\sum_{j=1}^{\infty}\sum_{k=1}^{j}\frac{k}{j^3(j-k+1)} = -\zeta(2)+\frac{1}{2}\zeta(2)^2+2\zeta(3)$$

Goal

1. generalize our tools from $q=1$ to the general q-case for automatic simplification
2. explore more general cases (like elliptic nested sums)