

# Towards a symbolic summation theory for unspecified sequences

Peter Paule and Carsten Schneider

**Abstract** The article addresses the problem whether indefinite double sums involving a generic sequence can be simplified in terms of indefinite single sums. Depending on the structure of the double sum, the proposed summation machinery may provide such a simplification without exceptions. If it fails, it may suggest a more advanced simplification introducing in addition a single nested sum where the summand has to satisfy a particular constraint. More precisely, an explicitly given parameterized telescoping equation must hold. Restricting to the case that the arising unspecified sequences are specialized to the class of indefinite nested sums defined over hypergeometric, multi-basic or mixed hypergeometric products, it can be shown that this constraint is not only sufficient but also necessary.

## 1 Introduction

Over recent years the second named author succeeded in developing a difference field (resp. ring) theory which allows to treat within a common algorithmic framework summation problems with elements from algebraically specified domains as well as problems involving concrete sequences which are analytically specified (e.g., from quantum field theory, combinatorics, number theory, and special functions). In this article we establish a new algebraic/algorithmic connection between this setting and summation problems involving generic sequences. We feel there is a high application potential for this connection. One future domain for algorithmic discovery (as described below) might be identities involving elliptic functions and modular forms.

In the course of a project devoted to an algorithmic revival of MacMahon's partition analysis, Andrews and Paule showed in [5] that a variant of partition analysis

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can be applied also for simplification of multiple combinatorial sums. Starting with the pioneering work of Abramov [3, 4], Gosper [7], Karr [8, 9], and Zeilberger [24], significant progress has been made. In particular, in the context of summation in difference fields and, more generally, difference rings [19, 21, 22] Schneider has developed substantial extensions and generalizations [15, 17, 18, 20] of Karr's seminal work. Owing to such an algorithmic machinery, the summation problems treated in [5] can nowadays be done in a jiffy with Schneider's `Sigma` package [16].

Nevertheless, the present article connects to [5] in various ways. First, it also considers a class of summation identities related to the celebrated Calkin sum which is the case  $\ell = 3$  of

$$C_\ell(n) := \sum_{k=0}^n \left( \sum_{j=0}^k \binom{n}{j} \right)^\ell.$$

More generally, we will focus also on the truncated versions

$$C_\ell(a, n) := \sum_{k=0}^a \left( \sum_{j=0}^k \binom{n}{j} \right)^\ell.$$

And second, similarly to [5] presenting a “non-standard” variation of the method of partition analysis, we present “non-standard” variations of difference field summation techniques.

The first “non-standard” ingredient is the aspect of “generic” summation in difference fields and rings. First pioneering steps in this direction were made by Kauers and Schneider; see [10, 11].

To illustrate the generic aspect, consider the problem of simplifying the sums

$$C_1(a, n) = \sum_{k=0}^a \sum_{j=0}^k \binom{n}{j} \quad \text{and} \quad C_1(n) = \sum_{k=0}^n \sum_{j=0}^k \binom{n}{j}.$$

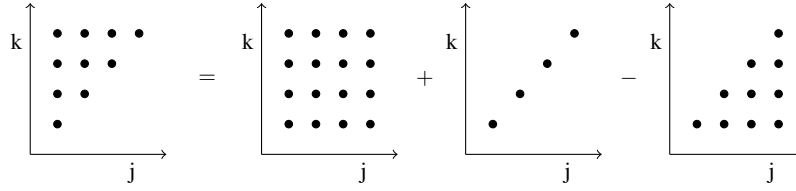
A rewriting of  $C_1(a, n)$  is obtained by specializing  $Y_k = 1$  and  $X_j = \binom{n}{j}$  in the generic summation relation

$$\sum_{k=0}^a \left( \sum_{j=0}^k X_j \right) Y_k = \left( \sum_{k=0}^a Y_k \right) \left( \sum_{j=0}^a X_j \right) + \sum_{k=0}^a Y_k X_k - \sum_{k=0}^a X_k \left( \sum_{j=0}^k Y_j \right). \quad (1)$$

Pictorially, (1) corresponds to summing over a square shaped grid in two different ways; see Fig 1.

Specializing (1) as proposed results in

$$\begin{aligned} C_1(a, n) &= (a+1) \sum_{j=0}^a \binom{n}{j} + \sum_{k=0}^a \binom{n}{k} - \sum_{k=0}^a \binom{n}{k} (k+1) \\ &= (a+1) \sum_{k=0}^a \binom{n}{k} - \sum_{k=0}^a k \binom{n}{k}. \end{aligned}$$



**Fig. 1** Summing over a rectangular grid in two different ways.

This means that the application of (1) indeed results in a simplification: the original double sum is expressed in terms of single sums. Specializing  $a = n$  the single sums in turn simplify further by the binomial theorem:

$$\sum_{k=0}^n k \binom{n}{k} = n \sum_{k=1}^n \binom{n-1}{k-1} = n \sum_{k=0}^{n-1} \binom{n-1}{k} = n 2^{n-1}.$$

This yields

$$C_1(n) = C_1(n, n) = (n+1)2^n - n2^{n-1} = 2^{n-1}(n+2).$$

We remark that the generic formula (1) can be obtained with the `Sigma` package<sup>1</sup>:

```
In[1]:= << Sigma.m
```

Sigma - A summation package by Carsten Schneider © RISC-JKU

```
In[2]:= mySum1 = SigmaSum[Y[k]SigmaSum[X[j],j,0,k],k,0,a]
```

$$\text{Out[2]} = \sum_{k=0}^a \left( \sum_{j=0}^k x[j] \right) Y[k]$$

```
In[3]:= res1 = SigmaReduce[mySum1,XList -> {X,Y},XWeight -> {2,1},
```

```
SimplifyByExt -> MinDepth,SimpleSumRepresentation -> True]
```

$$\text{Out[3]} = - \sum_{i=0}^a \left( \sum_{j=0}^i Y[j] \right) X[i] + \left( \sum_{i=0}^a X[i] \right) \left( \sum_{i=0}^a Y[i] \right) + \sum_{i=0}^a X[i] Y[i]$$

**Remark 1.1.** Applying `SigmaReduce` with the option `XList -> {X,Y}` one activates the summation algorithms given in [11, 18] by telling `Sigma` that  $X[j](= X_j)$  and  $Y[k](= Y_k)$  are generic sequences. With the option `SimplifyByExt -> MinDepth` the underlying algorithms try to simplify the sum `In[2]` so that the nested depth (i.e., the number of nested sum quantifiers) is minimized. Moreover, the option `SimpleSumRepresentation -> True` implies that the found sum representations have only denominators, if possible, that are linear. For this particular instance, the underlying algorithm would detect that the input expression cannot be simplified further if  $X$  and  $Y$  are considered as equally complicated. However, using in

<sup>1</sup> Freely available with password request at

<http://www.risc.jku.at/research/combinat/software/Sigma/>.

addition the option `xweight → {2,1}` one tells `Sigma` that  $X[k]$  is counted as a more nested expression than  $Y[k]$ . This extra information will finally produce the output given in `Out[3]` by introducing the sum  $\sum_{i=0}^a (\sum_{j=0}^i Y[j])X[i]$  which is considered as simpler than the sum `ln[2]`.

Next we apply the same strategy to

$$C_2(a, n) = \sum_{k=0}^a \left( \sum_{j=0}^k \binom{n}{j} \right)^2 \quad \text{and} \quad C_2(n) = \sum_{k=0}^n \left( \sum_{j=0}^k \binom{n}{j} \right)^2.$$

A generic formula for this situation is obtained from (1) by replacing  $Y_k$  with  $Y_k \sum_{j=0}^k X_j$ , and by rewriting the resulting right-hand side by using (1) together with some manipulation. Doing this by hand already becomes quite tedious; so we use `Sigma` to carry out this task automatically:

$$\begin{aligned} \text{ln[4]:= mySum2} &= \sum_{k=0}^a \left( \sum_{j=0}^k X[j] \right)^2 Y[k]; \\ \text{ln[5]:= res2} &= \text{SigmaReduce[mySum2, XList} \rightarrow \{X, Y\}, \text{XWeight} \rightarrow \{2, 1\}, \\ &\quad \text{SimplifyByExt} \rightarrow \text{DepthNumberDegree, SimpleSumRepresentation} \rightarrow \text{True}] \\ \text{Out[5]=} & -2 \sum_{i=0}^a \left( \sum_{j=0}^i X[j] \right) \left( \sum_{j=0}^i Y[j] \right) X[i] + 2 \sum_{i=0}^a \left( \sum_{j=0}^i X[j] \right) X[i] Y[i] \\ & + \sum_{i=0}^a \left( \sum_{j=0}^i Y[j] \right) X[i]^2 + \left( \sum_{i=0}^a X[i] \right)^2 \left( \sum_{i=0}^a Y[i] \right) - \sum_{i=0}^a X[i]^2 Y[i] \end{aligned}$$

**Remark 1.2.** If we execute `SigmaReduce` with the same options as described in Remark 1.1, we would fail for this input sum: there is no alternative expression in terms of nested sums where the nesting depth is simpler – even with the assumption that  $X[k]$  is considered as more nested than  $Y[k]$ <sup>2</sup>. However, inserting the extra option `SimplifyByExt → DepthNumberDegree` one aims at a simplification where the degree of the most complicated sum  $\sum_{j=0}^k X[j]$  in `ln[4]` is minimized; in addition, extra sums with lower nesting depth will be used (exploiting the fact that  $Y[j]$  is less nested than  $X[j]$ ) whenever such a degree reduction can be performed. This simplification strategy can be set up by combining the enhanced telescoping algorithms from [15, Section 5] with [17] to make `Sigma` compute `Out[5]` as an alternative presentation of

$$\sum_{k=0}^a Y_k \left( \sum_{j=0}^k X_j \right)^2. \quad (2)$$

Specializing  $Y_k = 1$  and  $X_j = \binom{n}{j}$  in this generic relation `Out[5]` gives

$$C_2(a, n) = (a+1) \left( \sum_{k=0}^a \binom{n}{k} \right)^2 - 2 \sum_{k=0}^a k \binom{n}{k} \sum_{j=0}^k \binom{n}{j} + \sum_{k=0}^a k \binom{n}{k}^2. \quad (3)$$

<sup>2</sup> If a simpler expression exists, `Sigma` would find it with the same options as described in Remark 1.1.

The specialization  $a = n$  is treated algorithmically in Subsection 3.2 resulting in the presentation (35) for  $C_2(n)$ .

The paper is organized as follows. After introducing the basic notions and constructions for setting up summation problems in terms of generic sequences in Section 2, in Section 3 we explain the basic simplification machinery to reduce double sums to expressions in terms of single nested sums. In Section 4 we reformulate this simplification methodology in the setting of abstract difference rings, and in Section 5 we connect these ideas with the ring of sequences utilizing an advanced difference ring theory; further supporting tools and notions (like  $R\Pi\Sigma$ -rings) can be found in Section 8 of the Appendix. Putting everything together will enable us to show that the suggested simplification strategy forms a complete algorithm for inputs that are given in terms of indefinite nested sums defined over hypergeometric products, multibasic products and their mixed versions. In Section 6 we give further details how this simplification engine is implemented in the package `Sigma` and elaborate various concrete examples. In Section 7 the paper concludes by giving some pointers to future research.

## 2 Generic sequences and sums

We want to model sequences and sums generically. To this end we introduce a set  $X$  of indeterminates indexed over  $\mathbb{Z}$  together with the ring of multivariate polynomials in these symbols over  $\mathbb{K}$ <sup>3</sup>,

$$X := \{X_j\}_{j \in \mathbb{Z}} \text{ and } \mathbb{K}_X := \mathbb{K}[X]. \quad (4)$$

It will be convenient to consider bilateral sequences  $f : \mathbb{Z} \rightarrow \mathbb{K}_X, j \mapsto f(j)$ . The set of bilateral sequences is denoted by  $\mathbb{K}_X^{\mathbb{Z}}$ . In the following we only speak about “sequences”; whether a sequence is bilateral or not will be always clear from the context.

**Convention 2.1.** We fix  $k$  as a “generic” symbol which in this article we overload with three different meanings which will be always clear from the context:

- As in Section 1,  $k$  can stand for an integer; i.e.,  $k \in \mathbb{Z}$ .
- It stands for the bilateral sequence  $k : \mathbb{Z} \rightarrow \mathbb{K}_X, j \mapsto j$ .
- More generally,  $k$  stands for a generic variable, respectively index; i.e., for a sequence  $P = (P(j))_{j \in \mathbb{Z}} \in \mathbb{K}_X^{\mathbb{Z}}$  we alternatively write  $P(k)$  ( $= P$ ); see Example 2.6.

In particular, the latter meaning arises in generic sequences and sums defined in Definitions 2.2 and 2.5, respectively.

**Definition 2.2 (generic sequences).** The symbol  $X_k$  with *generic index*  $k$  and its shifted versions  $X_{k+l}$ ,  $l \in \mathbb{Z}$ , denote bilateral sequences in  $\mathbb{K}_X^{\mathbb{Z}}$  defined as  $X_{k+l} :$

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<sup>3</sup>  $\mathbb{K}$  is a field of characteristic 0.

$\mathbb{Z} \rightarrow \mathbb{K}_X, j \mapsto X_{j+1}$ . The set of all such generic sequences is denoted by the symbol “ $\{X_k\}$ ”; i.e.,  $\{X_k\} := \{X_{k+l}\}_{l \in \mathbb{Z}}$ .

The ring  $\mathbb{K}_X[k, \{X_k\}]$  of polynomials in  $k$  and in generic sequences from  $\{X_k\}$  is a subring of the ring of sequences  $\mathbb{K}_X^{\mathbb{Z}}$  with the usual (component-wise) plus and times.

**Example 2.3.**  $P(k) = k^2 X_0 X_{k-1} X_{k+1} - k X_{-3} X_k^2 + X_3 - 2 \in \mathbb{K}_X[k, \{X_k\}]$  represents the sequence  $(p(j))_{j \in \mathbb{Z}}$ ,

$$P(k) : \mathbb{Z} \rightarrow \mathbb{K}_X, j \mapsto p(j) = j^2 X_0 X_{j-1} X_{j+1} - j X_{-3} X_j^2 + X_3 - 2.$$

**Lemma 2.4.** Let  $P(k) \in \mathbb{K}_X[k, \{X_k\}]$  be such that

$$P(j) = 0 \text{ for all } j \geq \mu$$

for some  $\mu \in \mathbb{Z}_{\geq 0}$ . Then  $P(k) = 0$ , the zero sequence.

*Proof.* The statement is obvious if one views  $P(k)$  as a polynomial in  $k$  over the integral domain  $\mathbb{K}_X[\{X_k\}]$ .  $\square$

**Definition 2.5 (generic sums).** Given  $P(k) \in \mathbb{K}_X^{\mathbb{Z}}$ , for  $a, b \in \mathbb{Z}$  the generic sum  $\sum_{l=a}^{k+b} P(l)$  denotes a sequence in  $\mathbb{K}_X^{\mathbb{Z}}$  defined as

$$\sum_{l=a}^{k+b} P(l) : \mathbb{Z} \rightarrow \mathbb{K}_X, j \mapsto \begin{cases} \sum_{l=a}^{j+b} P(l), & \text{if } a \leq j+b \\ 0, & \text{otherwise.} \end{cases} \quad (5)$$

**Example 2.6.** For any  $P(k) \in \mathbb{K}_X^{\mathbb{Z}}$  and

$$(f_P(k))_{k \in \mathbb{Z}} := \sum_{l=0}^k P(l) - \sum_{l=0}^{k-1} P(l)$$

one has

$$f_P(j) = \begin{cases} P(j), & \text{if } j \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

In other words, in the context of generic sequences and sums,

$$\sum_{l=0}^k P(l) - \sum_{l=0}^{k-1} P(l) \neq P(k). \quad (6)$$

This leads us to introducing an equivalence relation “ $\equiv$ ” such that in situations as in Example 2.6,

$$\left[ \sum_{l=0}^k P(l) \right] - \left[ \sum_{l=0}^{k-1} P(l) \right] \equiv [P(k)], \quad (7)$$

where we write  $[f]$  for the equivalence class of a sequence  $f \in \mathbb{K}_X^{\mathbb{Z}}$ .

**Definition 2.7.** For  $f = (f(j))_{j \in \mathbb{Z}}, g = (g(j))_{j \in \mathbb{Z}} \in \mathbb{K}_X^{\mathbb{Z}}$  define

$$f \equiv g : \Leftrightarrow \exists \lambda \in \mathbb{Z} : f(j) = g(j) \text{ for all } j \geq \lambda.$$

Obviously this introduces an equivalence relation on  $\mathbb{K}_X^{\mathbb{Z}}$ . Equivalence classes are denoted by  $[f]$ , the set of equivalence classes by  $\text{Seq}(\mathbb{K}_X)$ ; i.e.,

$$\text{Seq}(\mathbb{K}_X) = \{[f] : f \in \mathbb{K}_X^{\mathbb{Z}}\}.$$

Clearly,  $\text{Seq}(\mathbb{K}_X)$  forms a commutative ring with 1, which is defined by extending the usual (componentwise) sequence operations plus and times in an obvious way by  $[f] + [g] := [f + g]$  and  $[f][g] := [fg]$ .

The shift operator

$$S : \text{Seq}(\mathbb{K}_X) \rightarrow \text{Seq}(\mathbb{K}_X), [f] \mapsto S[f] := [Sf] \quad (8)$$

where  $Sf = (f(j+1))_{j \in \mathbb{Z}}$  if  $f = (f(j))_{j \in \mathbb{Z}}$ , is a ring automorphism, a property which is inherited from the shift operator on sequences from  $\mathbb{K}_X^{\mathbb{Z}}$ . For  $f(k) = (f(j))_{j \in \mathbb{Z}} \in \mathbb{K}_X^{\mathbb{Z}}$  and  $m \in \mathbb{Z}$  we often write  $f(k+m)$  instead of  $S^m f(k) = (f(j+m))_{j \in \mathbb{Z}}$ .

*Convention.* If things are clear from the context, for equivalence classes from  $\text{Seq}(\mathbb{K}_X)$  we will simply write  $f$  instead of  $[f]$ . Nevertheless, we will continue to use “ $\equiv$ ” to express equality between equivalence classes. For example, instead of (7) we write,

$$\sum_{l=0}^k P(l) - \sum_{l=0}^{k-1} P(l) \equiv P(k). \quad (9)$$

In the same spirit, given  $f(k) \in \mathbb{K}_X^{\mathbb{Z}}$  and  $m \in \mathbb{Z}$ , we will write

$$f(k+m) \text{ instead of } [f(k+m)],$$

provided that the meaning  $f(k+m) \in \text{Seq}(\mathbb{K}_X)$  is clear from the context.

Summation methods often rely on coefficient comparison. To apply this technique one usually exploits algebraic independence; for instance, equivalence classes  $[f]$  of generic sums like  $f = \sum_{l=0}^k X_l \in \mathbb{K}_X^{\mathbb{Z}}$  are algebraically independent over  $(\mathbb{K}_X[k, \{X_k\}], \equiv)$ .<sup>4</sup> Slightly more generally, we prove the following

**Lemma 2.8.** Let  $P(k) \in \mathbb{K}_X[k]$ . Then

$$\left[ \sum_{l=0}^k P(l) X_l \right] \text{ is transcendental over } (\mathbb{K}_X[k, \{X_k\}], \equiv).$$

*Proof.* For  $F(k) := \sum_{l=0}^k P(l) X_l \in \mathbb{K}_X^{\mathbb{Z}}$  suppose that

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<sup>4</sup> The quotient ring of  $\mathbb{K}_X[k, \{X_k\}]$  subject to the equivalence relation  $\equiv$ ; this ring is a subring of  $\text{Seq}(\mathbb{K}_X)$ .

$$0 \equiv q_0(k) + q_1(k)F(k) + \cdots + q_d(k)F(k)^d \quad (10)$$

for polynomials  $q_i(k) \in \mathbb{K}_X[k, \{X_k\}]$  with  $q_d(k) \neq 0$ .<sup>5</sup> Let  $d \geq 1$  be the minimal degree such that a relation like (10) holds. Denoting the sequence on the right side of (10) by  $(f(j))_{j \in \mathbb{Z}}$ , we have that there is a  $k_0 \in \mathbb{Z}$  such that

$$f(j) = 0 \text{ for all } j \geq k_0.$$

Define

$$l_0 := \max\{l \in \mathbb{Z} : X_l \text{ divides some monomial of some } q_i(k)\},$$

and set

$$j_0 := \max\{0, k_0, l_0 + 1\}.$$

Then

$$\begin{aligned} 0 &= \text{coefficient of } X_{j_0}^d \text{ in } f(j_0) = q_d(j_0)P(j_0)^d, \\ 0 &= \text{coefficient of } X_{j_0}^d \text{ in } f(j_0 + 1) = q_d(j_0 + 1)P(j_0 + 1)^d, \\ &\text{etc.} \end{aligned}$$

Since  $P(k) \in \mathbb{K}_X[k]$  has at most finitely many integer roots (if any), there is a  $\mu \in \mathbb{Z}_{\geq 0}$  such that

$$q_d(j) = 0 \text{ for all } j \geq \mu.$$

Consequently,  $q_d(k) \equiv 0$ , a contradiction to  $q_d(k) \neq 0$ . Therefore  $d = 0$ , and the statement follows from Lemma 2.4.  $\square$

### 3 The basic simplification

In the following, instead of considering sums like (2), we will restrict to a slightly less general class of sums by setting  $Y_j = 1$  for all  $j \geq 0$ , i.e., we will explore for  $p = 1, 2$  the sums

$$\sum_{j=0}^a \left( \sum_{l=0}^j X_l \right)^p \quad (11)$$

involving the generic sequence  $X_k$ . Obviously, for fixed  $p$  this sum can be viewed as a sequence  $s(a) = (s(a))_{a \in \mathbb{Z}} \in \mathbb{K}_X^{\mathbb{Z}}$ .<sup>6</sup> So, more precisely, we will investigate if and how sequences from  $\mathbb{K}_X^{\mathbb{Z}}$  given by such sum expressions can be simplified in terms of “simpler” generic sums.

<sup>5</sup> This means that  $q_d(k)$  is not equivalent to the 0-sequence  $(\dots, 0, 0, 0, \dots) \in \mathbb{K}_X^{\mathbb{Z}}$ .

<sup>6</sup> Note that  $s(a) = 0$  if  $a < 0$ .



### 3.1 Simplifications by sum extensions

We start to look at the case  $p = 1$  of (11), respectively  $C_1(a, n)$ , by considering the following problem.

Given a generic sum  $F(k) = \sum_{l=0}^k X_l \in \mathbb{K}_X^{\mathbb{Z}}$ ;  
find  $G(k) \in \mathbb{K}_X^{\mathbb{Z}}$ , “as simple as possible”, such that

$$G(k+1) - G(k) \equiv F(k+1). \quad (12)$$

Trivially,

$$G(k) = \sum_{j=0}^k F(j) \in \mathbb{K}_X^{\mathbb{Z}} \quad (13)$$

is always a solution to (12). So the problem splits into two parts: (a) to specify a concrete meaning of “as simple as possible”, and (b) to compute solutions which meet this specification.

For part (a), for the given problem we start by considering solutions of the form

$$G(k) = G_0(k) + G_1(k)F(k) \quad (14)$$

with  $G_j(k) \in \mathbb{K}_X[k, \{X_k\}]$  to be determined, the latter task being part (b) of the problem.

In practice the specifications given to settle part (a) of the problem are motivated by the context of the problem, but also driven by theory. For instance, here Lemma 2.8 implies that there is no solution  $G(k) \in \mathbb{K}_X[k, \{X_k\}]$  to the telescoping equation (12). In this sense<sup>7</sup>, the ansatz in (14) is the best possible we can achieve.

To execute part (b) of the problem we proceed by coefficient comparison. To this end, we substitute the ansatz (14) into (12) to obtain:

$$\begin{aligned} (G_1(k+1) - G_1(k))F(k) + G_0(k+1) - G_0(k) + G_1(k+1)X_{k+1} \\ \equiv F(k) + X_{k+1}. \end{aligned} \quad (15)$$

Owing to Lemma 2.8 we can do coefficient comparison with respect to powers of  $F(k)$  and obtain,

$$G_1(k+1) - G_1(k) \equiv 1.$$

It is straightforward to verify that

$$G_1(k) = k + d, \text{ with } d \in \mathbb{K}_X \text{ arbitrary,}$$

describes all the solutions in  $\mathbb{K}_X[k, \{X_k\}] = \mathbb{K}_X[\{X_k\}][k]$ . To keep things simple we set  $d = 0$ , and substituting  $G_1(k) = k$  into (15) yields

$$G_0(k+1) - G_0(k) \equiv -kX_{k+1}. \quad (16)$$

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<sup>7</sup> By difference ring theory (see Lemma 4.13 below) the exponent with which  $F(k)$  can appear in  $G(k)$  is at most 2. As it turns out, exponent 1 suffices here to obtain a solution of the desired form.

Using a similar idea as used in the proof of Lemma 2.8 reveals that (16) admits no solution  $G_0(k) \in \mathbb{K}_X[k, \{X_k\}]$ . So we are led to relax our specification of “simple” and—in view of (13)—set  $G_0$  to the trivial solution of (16); i.e., to the generic sum

$$G_0(k) = - \sum_{j=0}^k jX_j + F(k) \left( \equiv - \sum_{j=0}^k (j-1)X_j \right).$$

Putting things together,

$$G(k) = G_0(k) + G_1(k)F(k) = - \sum_{j=0}^k jX_j + (k+1)F(k) \in \mathbb{K}_X^{\mathbb{Z}} \quad (17)$$

is a solution of (12).

Finally, we convert (12) into the form of a summation identity. Passing from the generic sequence variable  $k$  to concrete integers  $k \in \mathbb{Z}$ , using (17) we can easily verify that for all  $k \geq 0$ ,

$$\begin{aligned} G(k) - G(k-1) &= -kX_k + (k+1)F(k) - kF(k-1) \\ &= -kX_k + (k+1)(F(k-1) + X_k) - kF(k-1) \\ &= X_k + F(k-1) = F(k).^8 \end{aligned}$$

Summing this telescoping relation over  $k$  from 0 to  $a \in \mathbb{Z}$ ,  $a \geq 0$ , produces<sup>9</sup>

$$\begin{aligned} \sum_{k=0}^a \sum_{j=0}^k X_j &= \sum_{k=0}^a F(k) = G(a) - G(-1) = G(a) \\ &= - \sum_{j=0}^a jX_j + (a+1)F(a) = - \sum_{j=0}^a jX_j + (a+1) \sum_{j=0}^a X_j. \end{aligned}$$

Finally, observe that the generic sequence  $X_k$  can be replaced by any concrete sequence  $(\bar{X}_k)_{k \geq 0}$  with  $\bar{X}_k \in \mathbb{K}$  yielding the identity

$$\sum_{k=0}^a \sum_{j=0}^k \bar{X}_j = - \sum_{j=0}^a j\bar{X}_j + (a+1) \sum_{j=0}^a \bar{X}_j. \quad (18)$$

With `Sigma` this can be obtained automatically. Namely, the package allows one to activate the desired mechanism by entering the sum

$$\text{In[6]:= mySum} = \sum_{k=0}^a \sum_{j=0}^k X[j];$$

and executing the function call

$$\text{In[7]:= SigmaReduce[mySum, XList} \rightarrow \{X\}, \text{SimpleSumRepresentation} \rightarrow \text{True}]$$

$$\text{Out[7]:= } (a+1) \sum_{i=0}^a X_i - \sum_{i=0}^a iX_i$$

<sup>8</sup> Note that  $F(-1) = 0$  by definition of a generic sum.

<sup>9</sup> According to (17):  $G(-1) = 0$ .

### 3.2 Simplifications by introducing constraints and sum extensions

Next, in view of the sum

$$\sum_{k=0}^a k \binom{n}{k} \sum_{j=0}^k \binom{n}{j},$$

arising in the presentation (3) for  $C_2(a, n)$ , we look at the following problem.

Given a generic sum  $F(k) = k X_k \sum_{j=0}^k X_j \in \mathbb{K}_X^{\mathbb{Z}}$ ;  
find  $G(k) \in \mathbb{K}_X^{\mathbb{Z}}$ , as simple as possible, such that

$$G(k+1) - G(k) \equiv F(k+1). \quad (19)$$

This time we start by considering solutions of the form

$$G(k) = G_0(k) + G_1(k)S(k) + G_2(k)S(k)^2 \quad (20)$$

with  $S(k) := \sum_{j=0}^k X_j$ , and where we again try to find the coefficients  $G_j(k)$  of polynomial form such that  $G_j(k) \in \mathbb{K}_X[k, \{X_k\}]$ .

To this end, we again proceed by coefficient comparison; i.e., we substitute the ansatz (20) into (19) to obtain:

$$\begin{aligned} (G_2(k+1) - G_2(k))S(k)^2 + (G_1(k+1) - G_1(k) + 2G_2(k+1)X_{k+1})S(k) \\ + G_0(k+1) - G_0(k) + G_1(k+1)X_{k+1} + G_2(k+1)X_{k+1}^2 \\ \equiv (k+1)X_{k+1}S(k) + (k+1)X_{k+1}^2. \end{aligned} \quad (21)$$

Owing to Lemma 2.8 we again can do coefficient comparison. With respect to  $S(k)^2$  we obtain,

$$G_2(k+1) - G_2(k) \equiv 0. \quad (23)$$

This has  $G_2(k) = c$ ,  $c \in \mathbb{K}_X$  arbitrary, as the general solution in  $\mathbb{K}_X[k, \{X_k\}] = \mathbb{K}_X[\{X_k\}][k]$ .

Coefficient comparison with respect to  $S(k)$  in (21) gives

$$G_1(k+1) - G_1(k) \equiv (k+1-2c)X_{k+1}. \quad (24)$$

In order to proceed, we suppose that the generic sequence  $Y_k \in \mathbb{K}_X^{\mathbb{Z}}$  is a solution to (24) and set  $G_1(k) := Y_k$ .

Finally, coefficient comparison with respect to  $S(k)^0$  in (21) gives

$$G_0(k+1) - G_0(k) \equiv (k+1-c)X_{k+1}^2 - Y_{k+1}X_{k+1}. \quad (25)$$

Similarly to the situation in equation (16) we relax our specification of “simple” and set  $G_0$  to the trivial solution of (25); i.e., to the generic sum

$$G_0(k) = \sum_{j=0}^k (j-c)X_j^2 - \sum_{j=0}^k X_j Y_j.$$

Combining all these ingredients yields the solution

$$G(k) = c \left( \sum_{j=0}^k X_j \right)^2 + Y_k \sum_{j=0}^k X_j + \sum_{j=0}^k (-cX_j^2 + jX_j^2 - X_j Y_j) \in \mathbb{K}_X^{\mathbb{Z}}, \quad (26)$$

under the assumption that

$$Y_k \in \mathbb{K}_X^{\mathbb{Z}} \text{ and } c \in \mathbb{K}_X \text{ are chosen so that (24) holds.} \quad (27)$$

Finally, as in Subsection 3.1 we convert (19) into a summation identity. Passing from the generic sequence variable  $k$  to concrete integers  $k \in \mathbb{Z}$ , using (26) we can easily verify that telescoping yields for all integers  $a \geq 0$ ,

$$\sum_{k=0}^a k X_k \sum_{j=0}^k X_j = c \left( \sum_{j=0}^a X_j \right)^2 - c \sum_{j=0}^a X_j^2 - \sum_{j=0}^a X_j Y_j + Y_a \sum_{j=0}^a X_j + \sum_{j=0}^a j X_j^2 \quad (28)$$

under the constraint that the sequence values  $Y_k \in \mathbb{K}_X$  and  $c \in \mathbb{K}_X$  are chosen such

$$Y_{k+1} - Y_k = (k+1-2c)X_{k+1} \text{ for all } k \geq 0. \quad (29)$$

Using `Sigma` this solution strategy can be automatically applied to the sum

$$\text{In[8]:= mySum} = \sum_{k=0}^a k X[k] \sum_{j=0}^k X[j];$$

with the procedure call<sup>10</sup>

$$\text{In[9]:= \{closedForm, constraint\} = SigmaReduce[mySum, XList} \rightarrow \{X\}, \text{ExtractConstraints} \rightarrow \{Y\}, \\ \text{SimpleSumRepresentation} \rightarrow \text{False}, \text{RefinedForwardShift} \rightarrow \text{False}]$$

$$\text{Out[9]=} \{c \left( \sum_{i=0}^a X[i] \right)^2 + Y[a] \sum_{i=0}^a X[i] + \sum_{i=0}^a (-cX[i]^2 + iX[i]^2 - X[i]Y[i]), \\ \{Y[a+1] - Y[a] = (1+a)X[a+1] - 2cX[a+1]\}\}$$

This yields the identity (26) with the constraint (29).

To produce the output in exactly the same form as in identity (28), one can use the option `SimpleSumRepresentation`  $\rightarrow$  `True` to the derived result:

$$\text{In[10]:= SigmaReduce[closedForm, a, XList} \rightarrow \{X, Y\}, \text{SimpleSumRepresentation} \rightarrow \text{True}]$$

$$\text{Out[10]=} c \left( \sum_{i=0}^a X[i] \right)^2 - c \sum_{i=0}^a X[i]^2 - \sum_{i=0}^a X[i]Y[i] + \left( \sum_{i=0}^a X[i] \right) Y[a] + \sum_{i=0}^a iX[i]^2$$

Further details on the calculation steps in the setting of difference rings will be given in Subsection 6.1.

---

<sup>10</sup> By using the option `RefinedForwardShift`  $\rightarrow$  `False`, `Sigma` follows the calculation steps carried out above. Without this option a more complicated (but more efficient) strategy is used that produces a slight variation of the output.

As a consequence, one can now fabricate specialized identities with the following strategy. Choose a concrete sequence  $\bar{X}_k \in \mathbb{K}$  such that one finds a “nice” solution  $\bar{Y}_k \in \mathbb{K}$  and  $c \in \mathbb{K}$  for

$$\bar{Y}_{k+1} - \bar{Y}_k = (1+k)\bar{X}_{k+1} - c2\bar{X}_{k+1}. \quad (30)$$

This will yield the specialized identity

$$\sum_{k=0}^a k \bar{X}_k \sum_{j=0}^k \bar{X}_j = c \left( \sum_{j=0}^a \bar{X}_j \right)^2 - c \sum_{j=0}^a \bar{X}_j^2 - \sum_{j=0}^a \bar{X}_j \bar{Y}_j + \bar{Y}_a \sum_{j=0}^a \bar{X}_j + \sum_{j=0}^a j \bar{X}_j^2. \quad (31)$$

**Example 3.1.** Taking  $\bar{X}_k = \binom{n}{k}$  in (31) leads to solving

$$\bar{Y}_{k+1} - \bar{Y}_k = (k+1-2c) \binom{n}{k+1} \text{ for all } k \geq 0. \quad (32)$$

which can be done by Sigma as follows:

```
In[11]:= ParameterizedTelescoping[{{(k+1)SigmaBinomial[n,k+1],-2SigmaBinomial[n,k+1]},k]
Out[11]:= {{1, n/4, -1/2(k+1)binom(n,k+1)}}
```

The output Out[11] means that as a solution to (32) we have

$$\bar{Y}_k = -\frac{1}{2}(k+1) \binom{n}{k+1} = -\frac{1}{2} \binom{n}{k} (n-k) \text{ and } c = \frac{n}{4}.$$

*Remark.* Alternatively, one can use the RISC package `fastZeil` [13] by

```
In[12]:= << RISC'fastZeil'
```

Fast Zeilberger Package version 3.61 written by Peter Paule, Markus Schorn, and Axel Riese ©RISC-JKU

```
In[13]:= Gosper[Binomial[n,k+1],k,1]
```

```
Out[13]:= (-2-2k+n)Binomial[n,1+k]==Delta[(1+k)Binomial[n,1+k]]
```

In[13] calls an extended version of Gosper’s algorithm. In the given example the last entry “1” asks the procedure to compute - in case it exists - a polynomial  $p_1(n)k + p_0(n)$  of order 1 in  $k$  such that the polynomial times the summand  $\binom{n}{k+1}$  telescopes. In Out[13] this polynomial is determined to be  $(-2)k + n - 2$ ;  $(\Delta_k f)(k) = f(k+1) - f(k)$  is the forward difference operator.

This turns (31) into

$$\begin{aligned} \sum_{k=0}^a k \binom{n}{k} \sum_{j=0}^k \binom{n}{j} &= \frac{n}{4} \left( \sum_{j=0}^a \binom{n}{j} \right)^2 + \frac{n}{4} \sum_{j=0}^a \binom{n}{j}^2 \\ &+ \frac{1}{2} \sum_{j=0}^a j \binom{n}{j}^2 - \frac{n-a}{2} \binom{n}{a} \sum_{j=0}^a \binom{n}{j}. \end{aligned} \quad (33)$$

For  $a = n$  we have, using  $\sum_{j=0}^m \binom{a}{j} \binom{b}{m-j} = \binom{a+b}{m}$  and  $\binom{n}{j} = \frac{n}{j} \binom{n-1}{j-1} = \frac{n}{j} \binom{n-1}{n-j}$ ,

$$\sum_{k=0}^n k \binom{n}{k} \sum_{j=0}^k \binom{n}{j} = \frac{n}{4} 2^{2n} + \frac{n}{4} \binom{2n}{n} + \frac{n}{2} \binom{2n-1}{n} = n 4^{n-1} + n \binom{2n-1}{n}.$$

Finally, substituting (33) into equation (3) yields,

$$C_2(a, n) = \left(a + 1 - \frac{n}{2}\right) \left(\sum_{j=0}^a \binom{n}{j}\right)^2 - \frac{n}{2} \sum_{j=0}^a \binom{n}{j}^2 + (n-a) \binom{n}{a} \sum_{j=0}^a \binom{n}{j}. \quad (34)$$

Similarly to before, for  $a = n$  this simplifies to

$$C_2(n) = C_2(n, n) = \left(\frac{n}{2} + 1\right) 2^{2n} - \frac{n}{2} \binom{2n}{n} = (n+2) 2^{2n-1} - n \binom{2n-1}{n}. \quad (35)$$

**Example 3.2.** Taking  $\bar{X}_k = H_k := \sum_{i=1}^k \frac{1}{i}$  in (31) leads to solving

$$\bar{Y}_{k+1} - \bar{Y}_k = (k+1 - 2c) H_{k+1} \text{ for all } k \geq 0.$$

The solution

$$\bar{Y}_k = \frac{1}{4} (-k^2 + 2k(k+1)H_k + k - 5) \text{ and } c = 0$$

turns (31) into

$$\begin{aligned} \sum_{k=0}^a k H_k \sum_{j=0}^k H_j &= \frac{1}{4} (-5 + a - a^2 + 2a(a+1)H_a) \sum_{j=0}^a H_j + \sum_{j=0}^a j H_j^2 \\ &\quad - \sum_{j=0}^a \frac{1}{4} (-5 + j - j^2 + 2j(1+j)H_j) H_j \\ &\stackrel{\text{Sigma}}{=} -\frac{(2a+1)(5a^2+5a-6)}{18} H_a + \frac{a(20a^2+3a-59)}{108} + \frac{a(a+1)(a+2)}{3} H_a^2. \end{aligned}$$

The second equality is obtained by applying `SigmaReduce` to the specialized expression. Here the underlying difference ring theory [22] is utilized in order to return an expression in terms of sums which are algebraically independent among each other.

**Example 3.3.** Taking  $\bar{X}_k = \binom{n}{k}^2$  in (31) leads to solving

$$\bar{Y}_{k+1} - \bar{Y}_k = (k+1 - 2c) \binom{n}{k+1}^2 \text{ for all } k \geq 0.$$

The solution

$$\bar{Y}_k = -\frac{(n-k)^2}{2n} \binom{n}{k}^2 \text{ and } c = \frac{n}{4}$$

turns (31) into

$$\begin{aligned}
\sum_{k=0}^a k \binom{n}{k}^2 \sum_{j=0}^k \binom{n}{j}^2 &= -\frac{\binom{n}{a}^2}{n} \frac{1}{2} (-a+n)^2 \sum_{j=0}^a \binom{n}{j}^2 + \frac{1}{4} n \left( \sum_{j=0}^a \binom{n}{j}^2 \right)^2 \\
&\quad - \frac{1}{4} n \sum_{j=0}^a \binom{n}{j}^4 + \sum_{j=0}^a j \binom{n}{j}^4 - \sum_{j=0}^a -\frac{\binom{n}{j}^4 (-j+n)^2}{2n} \\
&\stackrel{\text{sigma}}{=} \frac{-a^2 + 2an - n^2}{2n} \binom{n}{a}^2 \sum_{i=0}^a \binom{n}{i}^2 \\
&\quad + \frac{1}{2n} \sum_{i=0}^a i^2 \binom{n}{i}^4 + \frac{n}{4} \left( \sum_{i=0}^a \binom{n}{i}^2 \right)^2 + \frac{n}{4} \sum_{i=0}^a \binom{n}{i}^4
\end{aligned}$$

which holds for all  $a, n \in \mathbb{Z}_{\geq 0}$  with  $n \neq 0$ .

## 4 A reformulation in abstract difference rings

In the following we plan to gain more insight into when the double sums under consideration can be simplified to single sums. So far, we showed that the double sum on the left-hand side of (31) in terms of a sequence  $(\bar{X}_k)_{k \geq 0}$  with  $\bar{X}_k \in \mathbb{K}$  can be simplified to the right-hand side of (31) in terms of single nested sums provided that for  $c \in \mathbb{K}$  and  $\bar{Y}_k \in \mathbb{K}$  the parameterized telescoping equation (30) holds. In the following we will show that for certain classes of sequences  $\bar{X}_k$  and  $\bar{Y}_k$  the constraint (30) is not only sufficient but also necessary; see Theorem 5.7 below. In order to accomplish this task, we will utilize new results of difference ring theory [12, 19, 21, 22]; compare also [23]. To warm up, we first rephrase the constructions of the previous sections in the difference ring setting.

**Definition 4.1.** A *difference ring (resp. field)*  $(\mathbb{A}, \sigma)$  is a ring (resp. field)  $\mathbb{A}$  equipped with a ring (resp. field) automorphism  $\sigma : \mathbb{A} \rightarrow \mathbb{A}$ .

In fact, in Section 2 we introduced the difference ring  $(\text{Seq}(\mathbb{K}_X), S)$  where  $\text{Seq}(\mathbb{K}_X)$  is the ring of (equivalent) sequences equipped with the ring automorphism defined in (8). In addition, we considered the subring  $\mathbb{A}_1 := (\mathbb{K}_X[k, \{X_k\}], \equiv)$  of  $\text{Seq}(\mathbb{K}_X)$ . Since  $\mathbb{A}_1$  is closed under  $S$ , the restricted version of  $S$  to  $\mathbb{A}_1$  forms a ring automorphism. In short, we obtain the difference ring  $(\mathbb{A}_1, S)$  which is a subdifference ring of  $(\text{Seq}(\mathbb{K}_X), S)$ .

**Definition 4.2.** A difference ring  $(\mathbb{A}', \sigma')$  is called a *subdifference ring* of  $(\mathbb{A}, \sigma)$  if  $\mathbb{A}'$  is a subring of  $\mathbb{A}$  and  $\sigma'(a) = \sigma(a)$  for all  $a \in \mathbb{A}'$ . Conversely,  $(\mathbb{A}, \sigma)$  is called a *difference ring extension* of  $(\mathbb{A}', \sigma')$ . Since  $\sigma'$  agrees with  $\sigma$  on  $\mathbb{A}'$ , we usually do not distinguish anymore between them.

Further, by Lemma 2.8 the sequence  $\sum_{l=0}^k X_l \in \text{Seq}(\mathbb{K}_X)$  is transcendental over  $\mathbb{A}_1$ . Thus the smallest subring of  $\text{Seq}(\mathbb{K}_X)$  that contains  $\mathbb{A}_1$  and  $\sum_{l=0}^k X_l$  forms a polynomial ring which we denote by

$$\mathbb{A}_2 := \mathbb{K}_X[k, \{X_k\}] \left[ \sum_{l=0}^k X_l \right]. \quad (36)$$

Then using the fact that

$$S \sum_{l=0}^k X_l \equiv \sum_{l=0}^{k+1} X_l \equiv \sum_{l=0}^k X_l + X_{l+1} \quad (37)$$

holds with  $X_{l+1} \in \mathbb{K}_X[k, \{X_k\}]$  it follows that  $\mathbb{A}_2$  is closed under  $S$  and thus  $(\mathbb{A}_2, S)$  is a subdifference ring of  $(\text{Seq}(\mathbb{K}_X), S)$ . Summarizing, we obtain the following chain of difference ring extensions:

$$(\mathbb{K}_X, S) \leq (\mathbb{A}_1, S) \leq (\mathbb{A}_2, S) \leq (\text{Seq}(\mathbb{K}_X), S)$$

where  $(\mathbb{K}_X, S)$  is the trivial difference ring with  $S(f) \equiv f$  for all  $f \in \mathbb{K}_X$ , i.e., the elements in  $\mathbb{K}_X$  are precisely the constant sequences.

In the light of these constructions, we can reformulate the problem in Subsection 3.2 within the difference ring  $(\mathbb{A}_2, S)$  as follows: Given the sequence  $F(k) = kX_k \sum_{j=0}^k X_j \in \mathbb{A}_2$ , find a sequence  $G(k) \in \mathbb{A}_2$  or in a suitable subring of  $\text{Seq}(\mathbb{K}_X)$  such that

$$G(k+1) - G(k) \equiv F(k).$$

Here we found out that we can choose (26) with  $Y_k \in \text{Seq}(\mathbb{K}_X)$  and  $c \in \mathbb{K}_X$  which satisfies the constraint (27). Thus specializing  $X_k$  to concrete sequences  $(\bar{X}_k)_{k \geq 0}$  with  $\bar{X}_k \in \mathbb{K}$  such that there is a nice sequence  $(\bar{Y}_k)_{k \geq 0}$  with  $\bar{Y}_k \in \mathbb{K}$  that satisfies property (30) for some  $c \in \mathbb{K}$  will lead to the simplification (31).

In the following we denote by  $\text{Seq}(\mathbb{K})$  the subset of all sequences of  $\text{Seq}(\mathbb{K}_X)$  whose entries are from  $\mathbb{K}$ . Then it follows that  $\text{Seq}(\mathbb{K})$  is a subring of  $\text{Seq}(\mathbb{K}_X)$  and that  $S : \text{Seq}(\mathbb{K}_X) \rightarrow \text{Seq}(\mathbb{K}_X)$  restricted to  $\text{Seq}(\mathbb{K})$  forms a ring automorphism. Thus  $(\text{Seq}(\mathbb{K}), S)$  forms a subdifference ring of  $(\text{Seq}(\mathbb{K}_X), S)$ . Sometimes  $(\text{Seq}(\mathbb{K}_X), S)$  is also called the *difference ring of sequences*.

**Remark 4.3.** Usually, the difference ring  $(\text{Seq}(\mathbb{K}), S)$  is defined by starting with the commutative ring  $\mathbb{K}^{\mathbb{Z}_{\geq 0}}$  with 1 and defining the equivalence relation

$$f \equiv g \Leftrightarrow \exists \lambda \in \mathbb{Z}_{\geq 0} : f(j) = g(j) \text{ for all } j \geq \lambda$$

for  $f = (f(j))_{j \geq 0}, g = (g(j))_{j \geq 0} \in \mathbb{K}^{\mathbb{Z}_{\geq 0}}$ ; compare [14]. It is easily seen that the set of equivalence classes  $[f]$  with  $f \in \mathbb{K}^{\mathbb{Z}_{\geq 0}}$  forms a commutative ring with 1 which is isomorphic to  $\text{Seq}(\mathbb{K})$ . In a nutshell, we can either choose  $(a_n)_{n \in \mathbb{Z}_{\geq 0}}$  or  $(a_n)_{n \in \mathbb{Z}}$  in order to describe the equivalence classes of  $\text{Seq}(\mathbb{K})$ .

Subsequently, we will pursue a more general and ambitious goal. Namely, we will show that our new method produces constraints given in terms of parameterized telescoping equations that provide not only sufficient but also necessary conditions in order to simplify a nested sum in terms of generic sequences to an expression in terms of single nested sums over the given summand objects. In order to derive



this extra insight, we will consider not an arbitrary specialization of  $X_k, Y_k$  to general sequences  $(\tilde{X}_k)_{k \geq 0}, (\tilde{Y}_k)_{k \geq 0} \in \text{Seq}(\mathbb{K})$  but only to those sequences that can be generated by expressions in terms of indefinite nested sums defined over products. Typical examples are, e.g., the left- and right-hand sides of (33), and (34); for a more precise definition we refer to Definition 5.3 below. With this restriction, we will then utilize Schneider's newly established difference ring results [12, 19, 21, 22] to show that (31) is the only possible simplification of a double sum in terms of single sums.

In Schneider's difference ring approach sequences are represented by elements from a ring  $\mathbb{A}$  which is given either by certain rational function field extensions, polynomial ring extensions or by polynomial ring extensions factored out by certain ideals. In addition, a so-called evaluation function  $\text{ev} : \mathbb{A} \times \mathbb{Z}_{\geq 0} \rightarrow \mathbb{K}$  accompanies this ring construction that links the generators (variables) of the ring to the sequence interpretation. We will not give a full account on all the construction aspects [21, 22], but will emphasize only the key steps that are relevant for our considerations below. Further details can be found in the Appendix 8 below.

**Example 4.4.** Consider the rational function field  $\mathbb{A} = \mathbb{K}(k)$  in the variable  $k$ . Then we define the *evaluation function*  $\text{ev} : \mathbb{A} \times \mathbb{Z}_{\geq 0} \rightarrow \mathbb{K}$  by

$$\text{ev}\left(\frac{p}{q}, i\right) = \begin{cases} 0 & \text{if } q(i) = 0 \\ \frac{p(i)}{q(i)} & \text{if } q(i) \neq 0; \end{cases} \quad (38)$$

where  $p, q \in \mathbb{K}[k]$  are polynomials with  $q \neq 0$ ; here  $p(i), q(i)$  are the usual evaluations of polynomials at  $i \in \mathbb{Z}_{\geq 0}$ . Note that here we introduce yet another meaning of  $k$ , different from those introduced in Convention 2.1:  $k$  is an algebraic variable (indeterminate) that produces the rational function field  $\mathbb{K}(k)$ . E.g.,  $f = 1 + k + k^2$  in this context is considered as a polynomial in the variable  $k$  with integer coefficients and  $s = (\text{ev}(f, i))_{i \geq 0} \in \text{Seq}(\mathbb{K})$  provides us with the corresponding sequence interpretation. With our earlier notations from Convention 2.1 we could simply write  $P(k) = 1 + k + k^2$  to abbreviate the same sequence  $s$ .

Besides such a ring  $\mathbb{A}$ , also a ring automorphism  $\sigma : \mathbb{A} \rightarrow \mathbb{A}$  is introduced which scopes the shift behavior accordingly: for any  $x \in \mathbb{A}$  we will take care that

$$(\text{ev}(\sigma(x), i))_{i \geq 0} \equiv (\text{ev}(x, i+1))_{i \geq 0} = (\text{ev}(x, i))_{i \geq 1} \quad (39)$$

holds. In addition, the construction is carried out so that the *set of constants*<sup>11</sup>

$$\text{const}(\mathbb{A}, \sigma) = \{c \in \mathbb{A} \mid \sigma(c) = c\}$$

of the difference ring  $(\mathbb{A}, \sigma)$  equals precisely the field  $\mathbb{K}$  in which the sequences are evaluated. All these properties hold, for instance, for the ground field  $\mathbb{A} = \mathbb{K}(k)$  given in Example 4.4.

---

<sup>11</sup> Note that  $\text{const}(\mathbb{A}, \sigma)$  in general is a subring of  $\mathbb{A}$ .

**Example 4.5.** Consider for instance the sequence  $(\bar{X}_i)_{i \geq 0}$  with  $\bar{X}_0 = 0$  and  $\bar{X}_i = \frac{1}{i}$  for  $i \geq 1$ . Then we can choose the rational function  $x := \frac{1}{k} \in \mathbb{A}$ . In particular, we get (39). Further, we have  $\mathbb{K} = \text{const}(\mathbb{K}(k), \sigma)$ .

In the following we will reconsider the calculation steps of Section 3 within such abstract difference rings. In this context we will consider  $X_k$  not as a generic sequence, but as a sequence  $(\bar{X}_i)_{i \geq 0} \in \text{Seq}(\mathbb{K})$  which can be modeled by an element  $x \in \mathbb{A}$  of a given difference ring  $(\mathbb{A}, \sigma)$  with  $\mathbb{K} = \text{const}(\mathbb{A}, \sigma)$ .

**Definition 4.6.** Let  $(\mathbb{A}, \sigma)$  be a difference ring with constant field  $\mathbb{K}$  and equipped with an *evaluation function*  $\text{ev}$  satisfying (39). We say that a *sequence*  $\bar{X}_k \in \mathbb{K}$  is *modeled by*  $x \in \mathbb{A}$  if  $\bar{X}_k = \text{ev}(x, k)$  for all  $k$  from a certain point on.

In particular,  $\bar{X}_{k+i}$  with  $i \in \mathbb{Z}$  is then modeled by  $\sigma^i(x) \in \mathbb{A}$ . What we understand by “modeled by” has been illustrated also in the Example 4.5.

**Remark 4.7.** Note that the generic aspect is moved from a generic sequence  $X_k$  to a “generic” difference ring  $(\mathbb{A}, \sigma)$  and choosing an  $x \in \mathbb{A}$  from this ring  $\mathbb{A}$ . This change of paradigm will be very useful in Section 5 in order to show that the found simplifications are optimal in the sequence world.

Next we explain how to adjoin the formal sum<sup>12</sup>

$$\sum_{i=0}^k \bar{X}_i \quad (40)$$

to such an arbitrary ring  $\mathbb{A}$  with the shift behavior

$$\sum_{i=0}^{k+1} \bar{X}_i \equiv \sum_{i=0}^k \bar{X}_i + \bar{X}_{k+1}. \quad (41)$$

To this end, we introduce a new variable  $s$  being transcendental over  $\mathbb{A}$  and consider the polynomial ring  $\mathbb{A}[s]$ . More precisely, using the fixed element  $x \in \mathbb{A}$ , we define

$$\text{ev}(s, i) := \sum_{j=1}^i \text{ev}(x, j) = \sum_{j=1}^i \bar{X}_j \quad (42)$$

in order to give  $s$  the sequence meaning of our sum (40). More precisely, we extend this definition of  $s$  to  $\mathbb{A}[s]$  by

$$\text{ev}\left(\sum_{l=0}^d f_l s^l, i\right) = \sum_{l=0}^d \text{ev}(f_l, i) \text{ev}(s, i)^l \quad (43)$$

for any polynomial  $\sum_{l=0}^d f_l s^l \in \mathbb{A}[s]$  with  $f_l \in \mathbb{A}$ .

Finally, we extend also the automorphism  $\sigma : \mathbb{A} \rightarrow \mathbb{A}$  to  $\sigma' : \mathbb{A}[s] \rightarrow \mathbb{A}[s]$  with  $\sigma'(h) = \sigma(h)$  for all  $h \in \mathbb{A}$  and

<sup>12</sup> Note that  $\mathbb{K} \subseteq \mathbb{K}_X$  and thus the evaluation of a sum has been defined already in (5).

$$\sigma'(s) = s + \sigma(x). \quad (44)$$

Note that to define the shift operator, we again used the fixed element  $x \in \mathbb{A}$ . More precisely, there is exactly one such automorphism where for  $f = \sum_{l=0}^d f_l s^l$  we obtain the map

$$\sigma'(f) = \sum_{l=0}^d \sigma(f_l)(s + \sigma(x))^l;$$

since  $\sigma$  and  $\sigma'$  agree on  $\mathbb{A}$ , we do not distinguish them anymore. In particular, by our construction it follows that

$$(\text{ev}(\sigma(f), i))_{i \geq 0} \equiv (\text{ev}(f, i+1))_{i \geq 0} = (\text{ev}(f, i))_{i \geq 1}$$

for all  $f \in \mathbb{A}[s]$ .

Summarizing, we constructed a difference ring extension  $(\mathbb{A}[s], \sigma)$  of  $(\mathbb{A}, \sigma)$  where  $s$  models the sum (40):  $\text{ev}$  provides the sequence representation and  $\sigma$  describes the corresponding shift behavior.

Note that this abstract construction can be turned to concrete applications.

**Example 4.8.** We specialize  $(\mathbb{A}, \sigma)$  to  $\mathbb{A} = \mathbb{K}(k)$  and  $\sigma(k) = k + 1$ . Starting with this ring, we want to model the harmonic numbers  $H_k = \sum_{i=1}^k \bar{X}_i$  with  $\bar{X}_i = \frac{1}{i}$ . Thus we set  $x := \frac{1}{k}$  and follow the above construction, i.e., we take the difference ring extension  $(\mathbb{A}[s], \sigma)$  of  $(\mathbb{A}, \sigma)$  with  $s$  being transcendental over  $\mathbb{A}$  and with  $\sigma(s) = s + \beta$  where  $\beta := \sigma(x) = \frac{1}{k+1}$ . Further, we extend  $\text{ev}$  from  $\mathbb{A}$  to  $\mathbb{A}[s]$  by (42) and (43). For  $f = ks$  this yields, e.g.,  $\text{ev}(f, i) = iH_i$  for  $i \geq 0$ . Moreover, we obtain  $\text{ev}(\sigma(f), i) = \text{ev}((i+1)H_{i+1}, i) = \text{ev}(iH_i, i+1)$  for all  $i \geq 0$ . In a nutshell, we have rephrased the sequence of harmonic numbers  $H_k$  by  $s$  in  $\mathbb{A}[s]$  where  $\text{ev}$  provides the sequence representation and  $\sigma$  describes the corresponding shift behavior.

We emphasize that this elementary construction is still too naive for our subsequent considerations. Namely, a key feature will be that

$$\text{const}(\mathbb{A}[s], \sigma) = \text{const}(\mathbb{A}, \sigma) \quad (45)$$

holds. Together with our earlier assumption that  $\text{const}(\mathbb{A}, \sigma) = \mathbb{K}$  holds, this will imply that in  $(\mathbb{A}[s], \sigma)$  the set of constants is precisely  $\mathbb{K}$ . We install this special construction in the form of a definition.

**Definition 4.9.** Let  $(\mathbb{A}[s], \sigma)$  be a difference ring extension of  $(\mathbb{A}, \sigma)$  with  $s$  being transcendental over  $\mathbb{A}$  and  $\sigma(s) = s + \beta$  for some  $\beta \in \mathbb{A}$ . Then this extension is called a  $\Sigma$ -extension if (45) holds.

In the following we will rely heavily on the following result [21, Thm. 2.12]; for the field version see [8].

**Theorem 4.10.** Let  $(\mathbb{A}, \sigma)$  be a difference ring with constant field  $\mathbb{K}$  and let  $(\mathbb{A}[s], \sigma)$  be a difference ring extension of  $(\mathbb{A}, \sigma)$  with  $s$  being transcendental over  $\mathbb{A}$  and with  $\sigma(s) = s + \beta$  where  $\beta \in \mathbb{A}$ . Then this is a  $\Sigma$ -extension (i.e.,  $\text{const}(\mathbb{A}[s], \sigma) = \text{const}(\mathbb{A}, \sigma)$ ) iff there is no  $g \in \mathbb{A}$  with  $\sigma(g) = g + \beta$ .

**Remark 4.11.** Consider the difference ring extension  $(\mathbb{A}_2, S)$  of  $(\mathbb{A}_1, S)$  with (36) and (37). By Lemma 2.8  $\mathbb{A}_2$  is a polynomial ring over the coefficient domain  $\mathbb{A}_1$ . One can show that  $\text{const}(\mathbb{A}_2, S) = \text{const}(\mathbb{A}_1, S) = \mathbb{K}_X$  which implies that  $(\mathbb{A}_2, S)$  is a  $\Sigma$ -extension of  $(\mathbb{A}_1, S)$ . By Theorem 4.10<sup>13</sup> this implies that the generic sum  $\sum_{i=0}^k X_k$  cannot be simplified via telescoping in the difference ring  $(\mathbb{A}_1, S)$ . However, specializing  $X_k$  to a particular sequence  $(\bar{X}_k)_{k \geq 0}$ , the situation might be different.

Let us turn back to our generic construction: we are given an arbitrary difference ring  $(\mathbb{A}, \sigma)$  in which we choose  $x \in \mathbb{A}$  which models the desired sequence  $\bar{X}_k$ . Suppose that there exists<sup>14</sup> a  $g \in \mathbb{A}$  such that  $\sigma(g) = g + \sigma(x)$  holds. In this case one can model the sum (40) having the shift-behavior as in (41) by  $g$  with  $\sigma(g) = g + \beta$ . In other words, the double sum on the left-hand side of (18) turns into a single sum in  $(\mathbb{A}, \sigma)$ . In the following we will ignore this degenerated case and assume that such a  $g$  does not exist.

More precisely, we suppose that we are given a difference ring  $(\mathbb{A}, \sigma)$  with constant field  $\mathbb{K}$  with the following properties:

1.  $\text{const}(\mathbb{A}, \sigma) = \mathbb{K}$ ;
2. there is a  $k \in \mathbb{A}$  with  $\sigma(k) = k + 1$ ;
3. the sequence  $\bar{X}_k \in \mathbb{K}$  for  $k \geq 0$  can be modeled by an  $x \in \mathbb{A}$ ;
4. there is no  $g \in \mathbb{A}$  with  $\sigma(g) = g + \sigma(x)$ , i.e., we cannot represent the sum (40) in  $(\mathbb{A}, \sigma)$ .

The third assumption together with Theorem 4.10 implies that one can construct the  $\Sigma$ -extension  $(\mathbb{A}[s], \sigma)$  of  $(\mathbb{A}, \sigma)$  with  $\sigma(s) = s + \sigma(x)$ . This means that  $\mathbb{A}[s]$  is a polynomial ring and  $\text{const}(\mathbb{A}[s], \sigma) = \mathbb{K}$ .

**Example 4.12.** Consider our concrete difference ring extension  $(\mathbb{A}[s], \sigma)$  of  $(\mathbb{A}, \sigma)$  from Ex. 4.5 with  $\mathbb{A} = \mathbb{K}(k)$  and  $\sigma(s) = s + \beta$  with  $\beta = \frac{1}{k+1}$ . Using *Sigma* (or, e.g., Abramov's or Gosper's algorithms [3, 7, 13]), one can verify that there is no  $g \in \mathbb{K}(k)$  with  $\sigma(g) = g + \beta$ . Hence by Theorem 4.10 our extension is a  $\Sigma$ -extension.

Within such a difference ring setting the telescoping problem in Subsection 3.2 can be rephrased as follows.

Given  $(\mathbb{A}[s], \sigma)$  with the properties (1)–(4) from above and  $f = kxs \in \mathbb{A}[s]$ .  
Find a  $g \in \mathbb{A}[s]$  such that

$$\sigma(g) - g = \sigma(f) \tag{46}$$

holds (note:  $\sigma(f) = (k+1)\sigma(x)(s + \sigma(x))$ ).

Now we repeat the calculation steps of Subsection 3.2 within this (more abstract) difference ring exploiting the following extra insight [21, Lemma 7.2].

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<sup>13</sup> In the theorem we require that the set of constants form a field. However, if  $\text{const}(\mathbb{A}[s], \sigma) = \text{const}(\mathbb{A}, \sigma)$ , to prove the non-existence of a telescoping solution one does not need to assume that  $\text{const}(\mathbb{A}, \sigma)$  is a field.

<sup>14</sup> In *Sigma* the existence can be decided constructively by efficient telescoping algorithms [17,20] provided that  $(\mathbb{A}, \sigma)$  is a simple *RIT* $\Sigma$ -ring; see Appendix 8.

**Lemma 4.13.** Let  $(\mathbb{A}[s], \sigma)$  be a  $\Sigma$ -extension of  $(\mathbb{A}, \sigma)$  and  $f, g \in \mathbb{A}[s]$  with  $\sigma(g) - g = f$ . Then  $\deg(g) \leq \deg(f) + 1$ .

Thus any solution  $g \in \mathbb{A}[s]$  of (46) must have the form

$$g = g_0 + g_1 s + g_2 s^2;$$

compare (14). Plugging  $g$  into (46) we get

$$\sigma(g_2)(s + \sigma(x))^2 + \sigma(g_1 s + g_0) - [g_2 s^2 + g_1 s + g_0] = (k+1)\sigma(x)(s + \sigma(x)).$$

The polynomials on the left- and right-hand sides agree if they agree coefficient-wise. Thus comparing coefficients with respect to  $s^2$ , it follows that  $\sigma(g_2) = g_2$  which implies that  $g_2 \in \mathbb{K}$ . Thus we take an undetermined parameter  $c \in \mathbb{K}$  and set  $g_2 := c$ . Using this information we get

$$\begin{aligned} & [\sigma(g_1)(s + \sigma(x)) + \sigma(g_0)] - [g_1 s + g_0] \\ &= (k+1)\sigma(x)(s + \sigma(x)) + c[-\sigma(x)^2 - 2\sigma(x)s]. \end{aligned} \quad (47)$$

Again by coefficient comparison with respect to  $s$  we obtain the constraint

$$\sigma(g_1) - g_1 = (1+k-2c)\sigma(x); \quad (48)$$

compare with (24). Now suppose we find a  $c \in \mathbb{K}$  and a  $y \in \mathbb{A}$  such that

$$\sigma(y) - y = (1+k)\sigma(x) - 2c\sigma(x) \quad (49)$$

holds. Consequently, we get the general solution  $g_1 = y + d$  of (48) for some undetermined constant  $d \in \mathbb{K}$ . Plugging the solution into (47) yields

$$\sigma(g_0) - g_0 = (k+1-c)\sigma(x)^2 - \sigma(x)\sigma(y) - d\sigma(x); \quad (50)$$

this is equivalent to (25) when  $d = 0$ . At this point two scenarios may happen.

*Case 1.* We find a  $g_0 \in \mathbb{A}$  and  $d \in \mathbb{K}$  such that (50) holds. Then combining the derived sub-results provides the solution

$$g = c s^2 + (y+d)s + g_0. \quad (51)$$

*Case 2.* We do not find a  $g_0 \in \mathbb{A}$  and  $d \in \mathbb{K}$  such that (50) holds. Then we can construct the polynomial ring  $\mathbb{A}[s][t]$  and extend the automorphism  $\sigma$  from  $\mathbb{A}[s]$  to  $\mathbb{A}[s][t]$  subject to the relation

$$\sigma(t) = t + \left( \sigma(x)^2 - c\sigma(x)^2 + k\sigma(x)^2 - \sigma(x)\sigma(y) \right). \quad (52)$$

By Theorem 4.10 it follows that this extension is a  $\Sigma$ -extension. Namely, we have  $\text{const}(\mathbb{A}[s][t], \sigma) = \mathbb{K}$ . This, in particular, implies the solution  $g_0 = t$  and  $d = 0$  for (50). Finally, in this case, combining the obtained representations of the coeffi-

icients produces the solution

$$g = cs^2 + ys + t \quad (53)$$

within the difference ring  $(\mathbb{A}[s][t], \sigma)$  where  $c \in \mathbb{K}$  and  $y$  are a solution of (49); compare with (26).

The previous considerations can be summarized as follows.

**Theorem 4.14.** Let  $(\mathbb{A}, \sigma)$  be a difference ring with constant field  $\mathbb{K}$  and with  $k \in \mathbb{A}$  where  $\sigma(k) = k + 1$ . Let  $(\mathbb{A}[s], \sigma)$  be a  $\Sigma$ -extension of  $(\mathbb{A}, \sigma)$  with  $\sigma(s) = s + \sigma(x)$  for some  $x \in \mathbb{A}$ . Then the following holds.

(1) There is a  $g \in \mathbb{A}[s]$  with  $\sigma(g) - g = \sigma(kxs)$  iff the following two statements hold:

- (a) there is a  $y \in \mathbb{A}$  and  $c \in \mathbb{K}$  with (49),
- (b) and there is a  $g_0 \in \mathbb{A}$  and  $d \in \mathbb{K}$  with (50) (where  $c$  is the one from part (a)).

If (a) and (b) hold, we get the solution  $g$  as given in (51).

(2) There is a  $\Sigma$ -extension  $(\mathbb{A}[s][t], \sigma)$  of  $(\mathbb{A}[s], \sigma)$  with  $\sigma(t) - t \in \mathbb{A}$  together with a  $g \in \mathbb{A}[s][t] \setminus \mathbb{A}[s]$  with  $\sigma(g) - g = \sigma(kxs)$  iff the following two statements hold:

- (a) there is a  $y \in \mathbb{A}$  and  $c \in \mathbb{K}$  with (49),
- (b) there is no  $g_0 \in \mathbb{A}$  and  $d \in \mathbb{K}$  with (50) (where  $c$  is the one from part (a)).

If (a) and (b) hold, we get the solution  $g$  as given in (53) with (52).

Part 2 of the theorem describes the situation where one can adjoin a  $\Sigma$ -extension with the generator  $t$  in order to gain a parameterized telescoping solution for (50). Using the following extra insight from difference ring theory, we can generalize this situation if one allows a tower of single nested  $\Sigma$ -extensions.

**Theorem 4.15.** Let  $(\mathbb{A}, \sigma)$  be a difference ring with constant field  $\mathbb{K}$  and with  $k \in \mathbb{A}$  where  $\sigma(k) = k + 1$ . Let  $(\mathbb{A}[s], \sigma)$  be a  $\Sigma$ -extension of  $(\mathbb{A}, \sigma)$  such that  $\sigma(s) = s + \sigma(x)$  for some  $x \in \mathbb{A}$ . Then there is a tower of  $\Sigma$ -extensions  $(\mathbb{A}[s][t_1] \dots [t_e], \sigma)$  of  $(\mathbb{A}[s], \sigma)$  with  $\sigma(t_i) - t_i \in \mathbb{A}$  for  $1 \leq i \leq e$  together with a  $g \in \mathbb{A}[s][t_1, \dots, t_e] \setminus \mathbb{A}[s]$  with  $\sigma(g) - g = \sigma(kxs)$  iff the following two statements hold:

- (a) there is a  $y \in \mathbb{A}$  and  $c \in \mathbb{K}$  with (49),
- (b) there is no  $g_0$  and  $d \in \mathbb{K}$  with (50) (where  $c$  is the one from part (a)).

If (a) and (b) hold, we obtain the solution  $g$  as given in (53) with (52) (i.e.,  $e := 1$  and  $t_1 := t$ ).

*Proof.* If statements (a) and (b) hold, we can take (52) and get the solution  $g$  as given in (53). What remains to show is the other direction. Suppose that there is a tower of  $\Sigma$ -extensions  $(\mathbb{A}[s][t_1] \dots [t_e], \sigma)$  of  $(\mathbb{A}, \sigma)$  with  $\beta_i = \sigma(t_i) - t_i \in \mathbb{A}$  for  $1 \leq i \leq e$ . Assume further that there is a  $g \in \mathbb{A}[s][t_1, \dots, t_e] \setminus \mathbb{A}[s]$  with  $\sigma(g) - g = \sigma(kxs)$ . By [2, Prop. 1] it follows that

$$g = g' + \kappa_1 t_1 + \dots + \kappa_e t_e \quad (54)$$

for some  $g' \in \mathbb{A}[s]$  and  $(\kappa_1, \dots, \kappa_e) \in \mathbb{K}^e \setminus \{(0, \dots, 0)\}$ . Take the polynomial ring  $\mathbb{A}[s][t]$  and extend  $\sigma$  from  $\mathbb{A}[s]$  to  $\mathbb{A}[s][t]$  subject to the relation  $\sigma(t) = t + h$  with  $h := \kappa_1 \beta_1 + \dots + \kappa_e \beta_e$ . By construction we have that

$$\sigma(g' + t) - (g' + t) = \sigma(g') - g' + \kappa_1 \beta_1 + \dots + \kappa_e \beta_e = \sigma(g) - g = \sigma(kxs). \quad (55)$$

Now suppose that  $(\mathbb{A}[s][t], \sigma)$  is not a  $\Sigma$ -extension of  $(\mathbb{A}[s], \sigma)$ . Then there is a  $\gamma \in \mathbb{A}[s]$  with  $\sigma(\gamma) - \gamma = \kappa_1 \beta_1 + \dots + \kappa_e \beta_e$ . Let  $j$  be maximal such that  $\kappa_j$  is non-zero. Then we conclude that  $\sigma(\gamma') - \gamma' = \beta_j$  with

$$\gamma' := \frac{1}{\kappa_j}(\gamma - \kappa_1 t_1 - \dots - \kappa_{j-1} t_{j-1}) \in \mathbb{A}[s][t_1] \dots [t_{j-1}]$$

which implies that  $(\mathbb{A}[s][t_1] \dots [t_j], \sigma)$  is not a  $\Sigma$ -extension of  $(\mathbb{A}[s][t_1] \dots [t_{j-1}], \sigma)$  by Theorem 4.10; a contradiction. Thus  $(\mathbb{A}[s][t], \sigma)$  is a  $\Sigma$ -extension of  $(\mathbb{A}[s], \sigma)$ . Together with (55) we can apply part 2 of Theorem 4.14. This concludes the proof.  $\square$

## 5 A refinement to the class of indefinite nested sums over mixed $(q-)$ hypergeometric products

In Theorems 4.14 and 4.15 we established criteria for the simplification of our double sum in the setting of difference rings. More precisely, we assumed that we are given a  $\Sigma$ -extension  $(\mathbb{A}[s], \sigma)$  of  $(\mathbb{A}, \sigma)$  with  $\sigma(s) = s + \sigma(x)$  for some fixed  $x \in \mathbb{A}$  and derived criteria when one can find a  $g \in \mathbb{A}[s]$  or in an appropriate  $\Sigma$ -extension such that  $g$  solves the telescoping equation (46) with  $f = kxs$ . In the following we will transfer this result from the difference ring  $(\mathbb{A}[s], \sigma)$  to the ring of sequences  $(\text{Seq}(\mathbb{K}), S)$ . To this end, we assume that we are given a ring embedding, i.e., an injective ring homomorphism  $\tau$  from  $\mathbb{A}$  into  $\text{Seq}(\mathbb{K})$  with the additional property that  $\tau(\sigma(f)) \equiv S(\tau(f))$  holds for all  $f \in \mathbb{A}$ , i.e., we require that the diagram

$$\begin{array}{ccc} \mathbb{A} & \xrightarrow{\sigma} & \mathbb{A} \\ \downarrow \tau & & \downarrow \tau \\ \text{Seq}(\mathbb{K}) & \xrightarrow{S} & \text{Seq}(\mathbb{K}) \end{array}$$

commutes. In addition, we assume naturally that  $\tau(c) \equiv (c)_{n \geq 0}$  holds for all  $c \in \mathbb{K}$ . Such a map  $\tau$  is also called a  $\mathbb{K}$ -embedding (it is called a  $\mathbb{K}$ -homomorphism if the injectivity of  $\tau$  is dropped). Note that for such a  $\mathbb{K}$ -embedding it follows that  $\tau(\mathbb{A})$  is a subring of  $\text{Seq}(\mathbb{K})$  and  $S$  restricted to  $\tau(\mathbb{A})$  forms a ring automorphism. Note that  $(\mathbb{A}, \sigma)$  and  $(\tau(\mathbb{A}), S)$  are the same up to renaming of the elements by  $\tau$ .

**Example 5.1.** Consider the difference field  $(\mathbb{K}(k), \sigma)$  from Example 4.4 with the evaluation function  $\text{ev} : \mathbb{K}(k) \times \mathbb{Z}_{\geq 0} \rightarrow \mathbb{K}$  as in (38). Then we can define the map  $\tau : \mathbb{K}(k) \rightarrow \text{Seq}(\mathbb{K})$  with  $\tau(f) = (\text{ev}(f, i))_{i \geq 0}$  for  $f \in \mathbb{K}(k)$ . One can easily see that  $\tau$  is a

ring homomorphism and with (39) it follows that  $\tau$  is a  $\mathbb{K}$ -homomorphism. Finally,  $\tau(f) \equiv 0$  implies that  $f = 0$  since the numerator and denominator of  $f$  can have only finitely many roots. Consequently,  $\tau$  is a  $\mathbb{K}$ -embedding. The subdifference ring  $(\tau(\mathbb{K}(k)), S)$  of  $(\text{Seq}(\mathbb{K}), S)$  is also called the *difference ring of rational sequences*.

**Example 5.2.** Consider the  $\Sigma$ -extension  $(\mathbb{K}(k)[s], \sigma)$  of  $(\mathbb{K}(k), \sigma)$  from Ex. 4.12 (see also Ex. 4.4) with the corresponding evaluation function  $\text{ev} : \mathbb{K}(k)[s] \times \mathbb{Z}_{\geq 0} \rightarrow \mathbb{K}$  that models the harmonic numbers  $H_k$  with  $s$ . Then using similar arguments as in Example 5.1 we conclude that  $\tau : \mathbb{K}(k)[s] \rightarrow \text{Seq}(\mathbb{K})$  defined by  $\tau(f) = (\text{ev}(f, i))_{i \geq 0}$  for  $f \in \mathbb{K}(k)[s]$  is a  $\mathbb{K}$ -homomorphism. By difference ring theory [22] it follows that  $\tau$  is injective, and thus  $\tau$  is a  $\mathbb{K}$ -embedding.

More generally, we succeeded in such a construction in [22] not only for the harmonic numbers  $H_k$  as elaborated in Example 5.2 but for the general class of sequences that can be given in terms of nested sums over hypergeometric/ $q$ -hypergeometric/mixed-hypergeometric products.

**Definition 5.3.** Let  $\mathbb{K} = \mathbb{K}'(q_1, \dots, q_v)$  be a rational function field where  $\mathbb{K}'$  is a field of characteristic 0. A product  $\prod_{j=l}^k f(j, q_1^j, \dots, q_v^j)$ ,  $l \in \mathbb{Z}_{\geq 0}$ , is called *mixed-multibasic hypergeometric* [6] (in short *mixed hypergeometric*) in  $k$  over  $\mathbb{K}$  if  $f(y, z_1, \dots, z_v)$  is an element from the rational function field  $\mathbb{K}(y, z_1, \dots, z_v)$  where the numerator and denominator of  $f(j, q_1^j, \dots, q_v^j)$  are nonzero for all  $j \in \mathbb{Z}$  with  $j \geq l$ . Such a product is evaluated to a sequence following the rule

$$\prod_{j=l}^k f(j, q_1^j, \dots, q_v^j) : \mathbb{Z} \rightarrow \mathbb{K}, m \mapsto \begin{cases} \prod_{j=l}^m f(j, q_1^j, \dots, q_v^j), & \text{if } l \leq m \\ 1, & \text{otherwise.} \end{cases}$$

Further, such a product is called  *$q$ -hypergeometric* if  $f$  is free of  $y$ ,  $v = 1$  and  $q_1 = q$ , i.e.,  $f \in \mathbb{K}(z_1)$  with  $\mathbb{K} = \mathbb{K}'(q)$ . It is called *hypergeometric* if  $v = 0$ , i.e.,  $f \in \mathbb{K}(y)$  with  $\mathbb{K} = \mathbb{K}'$ .

An *expression in terms of nested sums over hypergeometric/ $q$ -hypergeometric/mixed hypergeometric products in  $k$  over  $\mathbb{K}$*  is composed recursively by the three operations  $(+, -, \cdot)$  with

- elements from the rational function field  $\mathbb{K}(k)$ ,
- hypergeometric/ $q$ -hypergeometric/mixed hypergeometric products in  $k$  over  $\mathbb{K}$ ,
- and sums of the form  $\sum_{j=l}^k f(j)$  with  $l \in \mathbb{Z}_{\geq 0}$  where  $f(j)$  is an expression in terms of nested sums over hypergeometric/ $q$ -hypergeometric/mixed hypergeometric products in  $j$  over  $\mathbb{K}$ ; here it is assumed that the evaluation<sup>15</sup> of  $f(j)|_{j \rightarrow \lambda}$  for all  $\lambda \in \mathbb{Z}$  with  $\lambda \geq l$  does not introduce any poles.

Given such an expression  $F(k)$  the evaluation  $F(k)|_{k \rightarrow \lambda}$  might be only defined for all  $\lambda \geq l$  for some  $l \in \mathbb{Z}_{\geq 0}$ . In order to obtain an evaluation for all  $\lambda \in \mathbb{Z}_{\geq 0}$ , we set  $F(k)|_{k \rightarrow \lambda} = 0$  for  $\lambda = 0, \dots, l-1$ . Similarly to Definition 2.5 we will give such products and sums defined over such products two different meanings. They form

<sup>15</sup> Note that  $\mathbb{K} \subseteq \mathbb{K}_X$  and thus the evaluation of a sum has been defined already in (5).



expressions that evaluate to sequences as introduced above, or they are just shorthand notations for the underlying sequences  $(F(k)|_{k \rightarrow \lambda})_{\lambda \geq 0}$ . The meaning (expression or sequence) of such a sums or products will be always clear from the context. E.g., the harmonic numbers  $H_n$  or the left- and right-hand sides of (33) and (34) are either expressions in terms of indefinite nested sums over hypergeometric products in  $a$  over  $\mathbb{K} = \mathbb{Q}(n)$  or they are shorthand notations for sequences in  $\mathbb{K}$ .

In general, as the sum  $H_k \in \text{Seq}(\mathbb{K})$  can be rephrased in the difference ring  $(\mathbb{K}(k)[s], \sigma)$  given in Example 5.2, we can represent nested sums as defined in Definition 5.3 in a particular class of difference rings called *simple RIIΣ-rings*; for their definition we refer to the Appendix 8. At this point we want to emphasize only the following crucial properties [12, 22] of simple RIIΣ-rings that enable one to treat the above class of nested sums in full generality.

**Theorem 5.4.** Let  $\bar{X}_k (= \bar{X}(k)) \in \text{Seq}(\mathbb{K})$  be a sequence given in terms of nested sums over hypergeometric (resp.  $q$ -hypergeometric or mixed hypergeometric) products where  $\mathbb{K}$  is algebraically closed<sup>16</sup>. Then the following holds.

- (1) There is a simple RIIΣ-ring  $(\mathbb{A}, \sigma)$  with constant field  $\mathbb{K}$  equipped with a  $\mathbb{K}$ -embedding  $\tau : \mathbb{A} \rightarrow \text{Seq}(\mathbb{K})$  and with  $x \in \mathbb{A}$  such that  $\tau(x) \equiv \bar{X}_k$  holds.

Moreover, for this  $\tau$  one has:

- (2a) For any  $h \in \mathbb{A}$  there is a sequence  $H(k)$  expressible in terms of nested sums over hypergeometric (resp.  $q$ -hypergeometric or mixed hypergeometric) products with  $\tau(h) \equiv H(k)$ .
- (2b) If the difference ring extension  $(\mathbb{A}[s], \sigma)$  of  $(\mathbb{A}, \sigma)$  with  $s$  being transcendental over  $\mathbb{A}$  and  $\sigma(s) = s + \sigma(x)$ ,  $x$  as in part (1), forms a Σ-extension, then the difference ring homomorphism  $\tau' : \mathbb{A}[s] \rightarrow \text{Seq}(\mathbb{K})$  defined by  $\tau'|_{\mathbb{A}} = \tau$  and  $\tau'(s) \equiv \sum_{k=0}^n \bar{X}_k$  forms a  $\mathbb{K}$ -embedding<sup>17</sup>.

In particular, the simple RIIΣ-ring  $(\mathbb{A}, \sigma)$  with  $f$  and the embedding  $\tau$  can be computed explicitly; for further details see Appendix 8.

Note that part (1) implies that a finite number of nested sums over hypergeometric,  $q$ -hypergeometric or mixed hypergeometric can be always formalized in a simple RIIΣ-ring, and part (2a) states that any element in such a ring can be reinterpreted as such a sum or product. This representation justifies the following definition.

**Definition 5.5.** A sub-difference ring  $(\mathbb{S}, S)$  of  $(\text{Seq}(\mathbb{K}), S)$  is called a *product-sum sequence ring*, if there is a simple RIIΣ-ring  $(\mathbb{A}, \sigma)$  with constant field  $\mathbb{K}$  together with a  $\mathbb{K}$ -embedding  $\tau : \mathbb{A} \rightarrow \text{Seq}(\mathbb{K})$  with  $\tau(\mathbb{A}) = \mathbb{S}$ .

Now let us reconsider our difference ring calculations of Subsection 4 within such a product-sum sequence ring  $(\mathbb{S}, S)$  where  $\bar{X}_k$  stands for a sequence that is given in terms of nested sums over products. According to Theorem 5.4, this means

<sup>16</sup> Algorithmically, one starts with a base field  $K$  (like  $\mathbb{Q}$  or  $\mathbb{Q}(n)$ ) and constructs —if necessary— a finite algebraic extension of it such that statement (1) is true.

<sup>17</sup> This means that  $\tau(\sum_{i=0}^r f_i s^i) \equiv \sum_{i=0}^r \tau(f_i) \left( \left( \sum_{k=0}^n \bar{X}_k \right)^i \right)_{n \geq 0}$  for  $f_0, \dots, f_r \in \mathbb{A}$ .

that there is a simple  $RIT\Sigma$ -ring  $(\mathbb{A}, \sigma)$  with constant field  $\mathbb{K}$  equipped with a  $\mathbb{K}$ -embedding  $\tau : \mathbb{A} \rightarrow \text{Seq}(\mathbb{K})$  and with an  $x \in \mathbb{A}$  such that  $\tau(x) \equiv \bar{X}_k$  holds. Suppose the decision procedure implemented in `Sigma` tells us (as above in Example 4.12) that there is no  $g \in \mathbb{A}$  such that  $\sigma(g) = g + \sigma(x)$  holds. Note that this implies that there is no sequence  $G(k) \in \tau(\mathbb{A})$  expressible in terms of nested sums with  $G(k+1) - G(k) \equiv \bar{X}_{k+1}$  or equivalently it follows that

$$\sum_{i=0}^k \bar{X}_i \notin \tau(\mathbb{A}).$$

Furthermore, we conclude by part (2b) of Theorem 5.4 that we can extend the  $\mathbb{K}$ -embedding  $\tau$  from  $\mathbb{A}$  to  $\mathbb{A}[s]$  with  $\tau(s) \equiv \sum_{i=0}^k \bar{X}_i$ . From this it can be derived that  $(\mathbb{A}[s], \sigma)$  and  $(\tau(\mathbb{A}[s]), S)$  are isomorphic, i.e., the difference rings are the same up to renaming of the objects using  $\tau$ .

With this background we restart our calculations to obtain a solution  $g$  of the telescoping equation

$$\sigma(g) - g = (k+1) \sigma(xs) = (k+1) \sigma(x)(s + \sigma(x)). \quad (56)$$

In the first major step we assumed that we can find a  $c \in \mathbb{K}$  and a  $y \in \mathbb{A}$  such that (49) holds. Now let  $\bar{Y}_k$  be the sequence in terms of nested sums with  $\tau(y) \equiv \bar{Y}_k \in \tau(\mathbb{A})$ . Then by construction it follows that (30) holds for  $\bar{Y}_k$  and  $c$ .

We proceed with our calculations by entering in the already worked out case distinction.

*Case I.* We can compute a  $d \in \mathbb{K}$  and  $g_0 \in \mathbb{A}$  with (50). Then for the sequence  $G_0(k)$  with  $\tau(g_0) = G_0(k)$  in terms of nested sums we obtain

$$G_0(k+1) - G_0(k) \equiv \bar{X}_{k+1}^2 - c\bar{X}_{k+1}^2 + k\bar{X}_{k+1}^2 - \bar{X}_{k+1}\bar{Y}_{k+1} - d\bar{X}_{k+1}. \quad (57)$$

Further, the  $g \in \mathbb{A}[s]$  with (51) is a solution of (56) under the assumption that  $c \in \mathbb{K}$  and  $y$  are a solution of (49). This implies that

$$S(\tau(g)) - \tau(g) \equiv \tau((k+1) \sigma(x)(s + \sigma(x))) \equiv ((k+1) \bar{X}_{k+1} (\sum_{i=0}^k \bar{X}_i + \bar{X}_{k+1}))_{k \geq 0}.$$

By construction, we obtain  $\tau(g) \equiv G(k) \in \tau(\mathbb{A}[s])$  with  $G(k) = c \left( \sum_{i=0}^k \bar{X}_i \right)^2 + (\bar{Y}_k + d) \sum_{i=0}^k \bar{X}_i + G_0(k)$ , and thus  $G(k)$  is a solution of

$$G(k+1) - G(k) \equiv (k+1) \bar{X}_{k+1} \left( \sum_{j=0}^k \bar{X}_j + \bar{X}_{k+1} \right) \quad (58)$$

under the constraint that (30) holds for  $\bar{Y}_k$  and  $c \in \mathbb{K}$ . Passing from the generic sequence variable  $k$  to concrete integers  $k \in \mathbb{Z}$ , using (58) we can check that telescoping yields

$$\sum_{k=0}^a k \bar{X}_k \sum_{j=0}^k \bar{X}_j = G(a) - G(-1) = c \left( \sum_{i=0}^a \bar{X}_i \right)^2 + (\bar{Y}_a + d) \sum_{i=0}^a \bar{X}_i + G_0(a) - G_0(-1). \quad (59)$$

*Case 2.* There does not exist a  $d \in \mathbb{K}$  and  $g_0 \in \mathbb{A}$  with (50). By Theorem 5.4 we can extend the  $\mathbb{K}$ -embedding from  $\mathbb{A}[s]$  to  $\mathbb{A}[s][t]$  with  $\tau(t) \equiv G_0(k)$  where

$$G_0(k) = \sum_{i=0}^k (-c\bar{X}_i^2 + i\bar{X}_i^2 - \bar{X}_i\bar{Y}_i). \quad (60)$$

In particular, we conclude that  $G_0(k) \notin \tau(\mathbb{A})$ . Moreover, the solution (53) of (56) yields the solution (26) of (58) under the constraint that (30) holds for  $\bar{Y}_k$  and  $c \in \mathbb{K}$ . Finally, we arrive at our simplification given in (31).

In Theorem 4.14 of Section 4 we summarized the considerations leading to cases (1) and (2). Before we can reformulate these cases in the context of sequences, we collect some key properties indicated already above.

**Lemma 5.6.** Let  $(\mathbb{A}, \sigma)$  be a simple  $R\Pi\Sigma$ -ring (see Definition 8.2) with constant field  $\mathbb{K}$ , and let  $\tau : \mathbb{A} \rightarrow \text{Seq}(\mathbb{K})$  be a  $\mathbb{K}$ -embedding. Set  $\mathbb{S} = \tau(\mathbb{A})$  and let  $f \in \mathbb{A}$  with  $\tau(f) \equiv F = (F(k))_{k \geq 0} \in \mathbb{S}$  and define  $\bar{S} := (\sum_{j=0}^k F(j))_{k \geq 0} \in \text{Seq}(\mathbb{K})$ . Then the following statements are equivalent.

- (1) There is a  $\Sigma$ -extension  $(\mathbb{A}[s], \sigma)$  of  $(\mathbb{A}, \sigma)$  with  $\sigma(s) = s + \sigma(f)$ .
- (2) There is no  $G \in \mathbb{S}$  with  $S(G) - G \equiv S(F)$ .
- (3)  $\mathbb{S}[\bar{S}]$  forms a polynomial ring.
- (4)  $\bar{S} \notin \mathbb{S}$ .

*Proof.* (1)  $\Leftrightarrow$  (2): There is a  $\Sigma$ -extension  $(\mathbb{A}[s], \sigma)$  of  $(\mathbb{A}, \sigma)$  iff there is no  $g \in \mathbb{A}$  with  $\sigma(g) = g + \sigma(f)$  by Theorem 4.10. Since  $\tau$  is a  $\mathbb{K}$ -embedding, the latter condition is equivalent to saying that there is no  $G \in \tau(\mathbb{A})$  with  $S(G) - G \equiv \tau(\sigma(f)) \equiv S(\tau(f)) \equiv S(F)$ .

(1)  $\Rightarrow$  (3): By part (2b) of Theorem 5.4 one can extend  $\tau$  from  $\mathbb{A}$  to  $\mathbb{A}[s]$  by  $\tau(s) \equiv S$ . Since  $\mathbb{A}[s]$  is a polynomial ring,  $\mathbb{S}[\bar{S}]$  forms a polynomial ring.

(3)  $\Rightarrow$  (4) holds trivially.

(4)  $\Rightarrow$  (2): Suppose that there is a  $G \in \mathbb{S}$  with  $S(G) - G \equiv \tau(\sigma(f))$ . Since  $S(\bar{S}) \equiv \bar{S} + (F(k+1))_{k \geq 0} \equiv \bar{S} + (F(k))_{k \geq 1} \equiv \bar{S} + S(\tau(f)) \equiv \bar{S} + \tau(\sigma(f))$ , we conclude that  $S(\bar{S} - G) \equiv \bar{S} - G$  and thus  $\bar{S} \equiv G + (c, c, c, \dots)$  for some  $c \in \mathbb{K}$ . Hence  $\bar{S} \in \mathbb{S}$ .  $\square$

With Lemma 5.6 and the above considerations the statements of part 1 of Theorem 4.14 and Theorem 4.15 (which is a slightly more general version of part 2 of Theorem 4.14) translate directly to the corresponding statements of the following Theorem 5.7.

**Theorem 5.7.** Let  $(\mathbb{S}, S)$  be a product-sum sequence ring containing the sequence  $k$  with  $S(k) = k + 1$ . Let  $\bar{X}_k \in \mathbb{S}$  and suppose that  $\sum_{i=0}^k \bar{X}_i \notin \mathbb{S}$ . Then within the polynomial ring  $\mathbb{S}' := \mathbb{S}[\sum_{i=0}^k \bar{X}_i]$  the following two statements hold:

- (1)  $\sum_{k=0}^a k \bar{X}_k \sum_{i=0}^k \bar{X}_i \in \mathbb{S}'$  iff
  - (a) there is a  $\bar{Y}_k \in \mathbb{S}$  and  $c \in \mathbb{K}$  with (30),
  - (b) and there is a  $G_0(k) \in \mathbb{S}$  and  $d \in \mathbb{K}$  with (57) (where  $c$  is the one from part (a)).

If (a) and (b) hold, we get the simplification given in (59).

(2) Suppose that  $Z_a := \sum_{k=0}^a k \bar{X}_k \sum_{i=0}^k \bar{X}_i \notin \mathbb{S}'$ . Then the sequence  $Z_a$  can be given in terms of single nested sums whose summands are from  $\mathbb{S}$  iff the following two statements hold:

- (a) there is a  $\bar{Y}_k \in \mathbb{S}$  and  $c \in \mathbb{K}$  with (30),
- (b) there is no  $G_0(k) \in \mathbb{S}$  and  $d \in \mathbb{K}$  with (57) (where  $c$  is the one from part (a)).

If (a) and (b) hold, we obtain the simplification (28).

## 6 Using the Sigma package

### 6.1 The symbolic approach with Sigma

As already demonstrated in ln[7] the difference ring machinery is activated in Sigma by executing the function call `SigmaReduce` to the given summation problem. If a generic sequence  $X_k$  arises within the summation problem, this information has to be passed to `SigmaReduce` with the option `XList`  $\rightarrow \{X\}$ . Then the generic sequence  $X_k$  and its shifted versions  $\dots, X_{k-2}, X_{k-1}, X_k, X_{k+1}, X_{k+2}, \dots$  are represented by the variables  $\dots, x_{-2}, x_{-1}, x_0, x_1, x_2, \dots$ , respectively. Namely, as worked out in [10, 11] Sigma takes the field  $\mathbb{G} = \mathbb{K}(\dots, x_{-2}, x_{-1}, x_0, x_1, x_2, \dots)$  with infinitely many variables and uses the field automorphism  $\sigma : \mathbb{G} \rightarrow \mathbb{G}$  with  $\sigma(x_i) = x_{i+1}$  for all  $i \in \mathbb{Z}$  and  $\sigma(c) = c$  for all  $c \in \mathbb{K}$ . The obtained difference field  $(\mathbb{G}, \sigma)$  with  $\text{const}(\mathbb{G}, \sigma) = \mathbb{K}$  is also called the *difference field of free sequences*. In order to define the underlying evaluation function for  $\mathbb{G}$ , the constant field  $\mathbb{K}$  has to be constructed accordingly. Here one takes the rational function field  $\mathbb{K} = \mathbb{K}'(\dots, X_{-2}, X_{-1}, X_0, X_1, X_2, \dots)$  again with infinitely many variables where  $\mathbb{K}'$  is a field of characteristic 0; note that  $\mathbb{K}'_X$  (see our earlier Definition 4) and  $\mathbb{K}$  are closely related:  $\mathbb{K}'_X$  is the polynomial ring in the variables  $X_i$  with  $i \in \mathbb{Z}$  and  $\mathbb{K}$  is simply its quotient field. The evaluation function  $\text{ev}$  for  $\mathbb{G}$  is provided with  $\text{ev}(x_i, j) = X_{i+j}$  for  $i, j \in \mathbb{Z}$ .

Usually, in generic summation problems as considered in this article, the summation input of `SigmaReduce` depends not only on generic sequences, but on generic sums (see Definition 2.5) and more generally, on nested sums and products defined over generic sequences. In this case, the input expression is represented accordingly with a tower of *RPIE*-extensions over  $(\mathbb{G}, \sigma)$ , see the Appendix 8, which leads to a difference ring  $(\mathbb{A}, \sigma)$ . This construction can be carried out automatically by the tools given in [19, 21, 22] in combination with the machinery described in [10, 11]. Finally, Sigma tries to simplify the given summation problem using the different telescoping algorithms from [17, 18, 20].

*Calculation steps for Subsection 3.1:* In order to tackle the sum on the left-hand side of (18) Sigma represents  $X_j$  by  $x_0 \in \mathbb{G}$ . By default the difference field extension  $(\mathbb{G}(k), \sigma)$  of  $(\mathbb{G}, \sigma)$  with  $\sigma(k) = k + 1$  and  $\text{const}(\mathbb{G}(k), \sigma) = \mathbb{K}$  is adjoined automatically. Furthermore, the  $\Sigma$ -extension  $(\mathbb{G}(k)[s], \sigma)$  of  $(\mathbb{G}(k), \sigma)$  with  $\sigma(s) = s + x_1$  is constructed to model the generic sum  $\sum_{j=0}^k X_j$  with  $\sum_{j=0}^{k+1} X_j = \sum_{j=0}^k X_j + X_{k+1}$ ;

internally Theorem 4.10 is applied to check that this is indeed a  $\Sigma$ -extension. As a consequence, we have that  $\text{const}(\mathbb{G}(k)[s], \sigma) = \mathbb{K}$ . Now exactly the steps from Subsection 3.1 with  $f = \sigma(s) = s + x_1$  are carried out in this difference ring, and the expression (18) (with the options `SimpleSumRepresentation`→`True` and `SimplifyByExt`→`MinDepth` activated; see Remark 1.1 for further explanations) is returned.

*Calculation steps for Subsection 3.2:* The tactic of Subsection 3.1 fails for the double sum on the left-hand side of (28). But, using in addition the `Sigma`-option `ExtractConstraints`→ $\{Y\}$ , as demonstrated in In[9], the new machinery introduced in Section 4 is activated. Internally, again the difference ring  $(\mathbb{G}(k)[s], \sigma)$  with constant field  $\mathbb{K}$  is constructed, and the computation steps are carried out with  $\sigma(f) = (k+1)x_1(s+x_1)$  (instead of  $\sigma(f) = (k+1)\sigma(x)(s+\sigma(x))$ ). They are precisely the same as in Section 4. In this process we produce the constraint

$$\sigma(g_1) - g_1 = (1+k)x_1 - 2cx_1;$$

compare with (48). Since `Sigma` does not find a solution  $g_1 \in \mathbb{G}(k)[s]$ , it extends the underlying difference field  $\mathbb{G}$  by the new variables  $\dots, y_{-2}, y_{-1}, y_0, y_1, y_2, \dots$  and extends the automorphism  $\sigma$  with  $\sigma(y_i) = y_{i+1}$  for all  $i \in \mathbb{Z}$ . Now we continue our calculation with  $g_1 = y_i + d$  and a new variable  $c$  (i.e., we extend the constant field  $\mathbb{K}$  by  $c$ ) and obtain the constraint

$$\sigma(g_0) - g_0 = x_1^2 - cx_1^2 + kx_1^2 - x_1y_1 - dx_1$$

of  $g_0$ ; compare with (50). Since we do not find a  $g_0 \in \mathbb{G}(k)(s)$  (with the updated  $\mathbb{G}$  containing now also the variables  $y_i$  with  $i \in \mathbb{Z}$  and the new constant  $c$ ) and  $d \in \mathbb{K}(c)$ , we construct the  $\Sigma$ -extension  $(\mathbb{G}(k)[s][t], \sigma)$  of  $(\mathbb{G}(k)[s], \sigma)$  with

$$\sigma(t) = t + (x_1^2 - cx_1^2 + kx_1^2 - x_1y_1).$$

This finally produces the solution  $g = cs^2 + y_0s + t$ . Reinterpreting this result in terms of the generic sequences  $X_k$  and  $Y_k$  produces the output Out[9].

Concerning this concrete summation problem the following remarks are relevant.

1. The output Out[9] provides the full information that is needed to apply Theorem 5.7 taking care of the two possible scenarios. Specializing  $X_k$  and  $Y_k$  (where  $Y_k$  and  $c$  are solutions of the constraint (30)) to concrete sequences in  $(\mathbb{S}, \mathbb{S})$ , it might happen that the found sum extension simplifies further in the given ring  $\mathbb{S}$ . This situation is covered by part (1) of Theorem 5.7. Otherwise, if the sum cannot be simplified in  $\mathbb{S}$ , part (2) of the Theorem 5.7 can be applied.
2. Fix a product-sum sequence ring  $(\mathbb{S}, \mathbb{S})$ . If  $\sum_{j=0}^k \bar{X}_j \notin \mathbb{S}$ , the output gives a full characterization when the sum  $\sum_{k=0}^a \bar{X}_k \sum_{j=0}^k \bar{X}_j$  can be written as an expression in terms of single nested sums; see Theorem 5.7 for further details. However, if we enter the special case  $\sum_{j=0}^k \bar{X}_j \in \mathbb{S}$ , then the result provides only a sufficient criterion to get such a simplification. Still the toolbox can be applied also in such

a case as worked out in Example 3.2; there we chose  $X_j = H_j$  for which the simplification  $\sum_{j=0}^k \bar{X}_j = -n + (1+n)H_n$  is possible.

3. Specializing the identities in (18) to concrete sequences  $\bar{X}_k$  often leads to further simplifications.

We considered the very special case of the input expression  $\sum_{k=0}^a k X_k \sum_{i=0}^k X_i$ . However, the proposed method works for any input sum  $\sum_{k=0}^a f(k)$  where the summand  $f(k)$  is built by a finite number of generic sequences, say  $X, Y, \dots, Z$ , and over nested sums over hypergeometric/ $q$ -hypergeometric/mixed hypergeometric products. A typical function call, for instance, is `ln[3]`. Here the same ideas are applied as in Section 3 where instead of  $\sum_{i=0}^k X_i$  the most nested sum (and among the most nested sums the one with highest degree) of the summand  $f(k)$  is chosen. In particular, the following refinements can be activated.

1. In Subsection 3.2 we combined the telescoping algorithm from [20] with our new idea to extract constraints in form of parameterized telescoping equations and to encode these constraints in the output expression by using new generic sequences. Within `Sigma` also other enhanced telescoping strategies for simplification [15, 17, 20] can be combined with this new feature. For further details on the possible options we refer also to Remarks 1.1 and 1.2.
2. In Subsection 3.2 the most complicated sum occurs only linearly. As a consequence we run into three constraints given by step-wise coefficient comparison. Namely, for our ansatz (20) we get the constraint (23), which can always be treated, the constraint (24) where we introduced a generic sequence  $Y_k$  subject to the parameterized telescoping relation (29), and the constraint (25) which we could handle by the sum extension (60). More generally, if the most complicated sum occurs with degree  $d > 1$ , one ends up with  $d+2$  constraints. Some of them can be solved directly by `Sigma` within the given difference ring, but in general there will remain constraints which can only be treated by introducing a new generic sequence that must satisfy a certain parameterized telescoping equation. Activating the option `ExtractConstraints`  $\rightarrow \{Y^{(1)}, \dots, Y^{(l)}\}$ , `SigmaReduce` is allowed to provide (if necessary) up to  $l$  constraints in form of parameterized telescoping equations, each one with a different generic sequence from  $Y^{(1)}, \dots, Y^{(l)}$ . If not successful, i.e., if more than  $l$  generic sequences are needed, `Sigma` gives up and returns the input expression.

## 6.2 Discovery of identities

We illustrate how the presented techniques can support the (re)discovery of numerous identities. We start with the generic sum

$$\text{ln[14]:= mySum} = \sum_{k=0}^n \left( \sum_{j=0}^k X[j] \right)^2;$$

and obtain the following general simplification formula

$\text{In[15]} := \{\text{closedForm, constraint}\} = \text{SigmaReduce}[\text{mySum, XList} \rightarrow \{\mathbf{X}\}, \text{ExtractConstraints} \rightarrow \{\mathbf{Y}\},$   
 $\text{SimpleSumRepresentation} \rightarrow \text{False}, \text{RefinedForwardShift} \rightarrow \text{False}]$

$$\text{Out[15]} = \{(a+c) \left( \sum_{i=0}^a x[i]^2 \right) + \sum_{i=0}^a (x[i]^2 - cx[i]^2 - ix[i]^2 - x[i]y[i]) + y[a] \sum_{i=0}^a x[i],$$

$$\{y[a+1] - y[a] == -2ax[a+1] - 2cx[a+1]\}$$

The result can be simplified further to the form

$\text{In[16]} := \text{SigmaReduce}[\text{closedForm, a, XList} \rightarrow \{\mathbf{X}, \mathbf{Y}\}, \text{SimpleSumRepresentation} \rightarrow \text{True}]$

$$\text{Out[16]} = (a+c) \left( \sum_{i=0}^a x[i]^2 \right) - c \sum_{i=0}^a x[i]^2 - \sum_{i=0}^a x[i]y[i] + y[a] \sum_{i=0}^a x[i] + \sum_{i=0}^a x[i]^2 - \sum_{i=0}^a ix[i]^2$$

This means that the identity

$$\sum_{k=0}^a \left( \sum_{j=0}^k \bar{X}_j \right)^2 = (a+c) \left( \sum_{k=0}^a \bar{X}_k \right)^2 - c \sum_{k=0}^a \bar{X}_k^2 - \sum_{k=0}^a \bar{X}_k \bar{Y}_k + \bar{Y}_a \sum_{k=0}^a \bar{X}_k + \sum_{k=0}^a \bar{X}_k^2 - \sum_{k=0}^a k \bar{X}_k^2 \quad (61)$$

holds for any sequences  $(\bar{X}_k)_{k \geq 0}$ ,  $(\bar{Y}_k)_{k \geq 0}$  with  $\bar{X}_k, \bar{Y}_k \in \mathbb{K}$  and  $c \in \mathbb{K}$  if  $c$  and  $\bar{Y}_k$  are a solution of the parameterized telescoping equation

$$\bar{Y}_{k+1} - \bar{Y}_k = -2k\bar{X}_{k+1} - 2c\bar{X}_{k+1}. \quad (62)$$

Even more holds by a straightforward variant of Theorem 5.7: if one takes a product-sum sequence ring  $(\mathbb{S}, \mathcal{S})$  and takes a sequence  $\bar{X}_k$  which is in  $\mathbb{S}$  but where the sequence of  $\sum_{j=0}^k \bar{X}_j$  is not in  $\mathbb{S}$ , then the double sum on the left-hand side of (61) can be simplified to single nested sums defined over  $\mathbb{S}$  if and only if there is a solution  $c \in \mathbb{K}$  and  $\bar{Y}_k$  in  $\mathbb{S}$  of (62). In this case the right-hand side of (62) with the explicitly given  $c$  and  $\bar{Y}_k$  produces such a simplification.

**Example 6.1.**  $\bar{X}_k = \binom{n}{k}$ : Plugging the solution  $c = \frac{2-n}{2}$  and  $\bar{Y}_k = \binom{n}{k}(-k+n)$  of (62) into (61) yields

$$\sum_{k=0}^a \left( \sum_{j=0}^k \binom{n}{j} \right)^2 = (-a+n) \binom{n}{a} \sum_{k=0}^a \binom{n}{k} + \left( a + \frac{2-n}{2} \right) \left( \sum_{k=0}^a \binom{n}{k} \right)^2$$

$$+ \sum_{k=0}^a \binom{n}{k}^2 - \frac{2-n}{2} \sum_{k=0}^a \binom{n}{k}^2 - \sum_{k=0}^a k \binom{n}{k}^2 - \sum_{k=0}^a \binom{n}{k}^2 (-k+n)$$

$$\stackrel{\text{Sigma}}{=} \binom{n}{a} (-a+n) \sum_{k=0}^a \binom{n}{k} + \frac{1}{2} (2+2a-n) \left( \sum_{k=0}^a \binom{n}{k} \right)^2 - \frac{1}{2} n \sum_{k=0}^a \binom{n}{k}^2$$

which is valid for all  $a, n \in \mathbb{Z}_{\geq 0}$ . Following the same tactic, we “discover” the identities

$$\sum_{k=0}^a \left( \sum_{j=0}^k x^j \binom{n}{j} \right)^2 = -\frac{nx}{x+1} \sum_{k=0}^a x^{2k} \binom{n}{k}^2 + \frac{(1+a+x+ax-nx)}{x+1} \left( \sum_{k=0}^a x^k \binom{n}{k} \right)^2$$

$$\begin{aligned}
& + \frac{x-1}{x+1} \sum_{k=0}^a kx^{2k} \binom{n}{k}^2 - \frac{2x^{a+1}(a-n)}{x+1} \binom{n}{a} \sum_{k=0}^a x^k \binom{n}{k}, \\
\sum_{k=0}^a \left( \sum_{j=0}^k (-1)^j \binom{n}{j} \right)^2 & = \frac{n}{2(2n-1)} \sum_{k=0}^a \binom{n}{k}^2 - \frac{(2a-3n+2)(a-n)^2}{2n^2(2n-1)} \binom{n}{a}^2;
\end{aligned}$$

the first identity holds for  $x \in \mathbb{K} \setminus \{-1\}$  and  $a, n \in \mathbb{Z}_{\geq 0}$  and the second holds for  $a, n \in \mathbb{Z}_{\geq 0}$  with  $n \neq 0$ . Furthermore we obtain

$$\begin{aligned}
\sum_{k=0}^a \left( \sum_{j=0}^k \frac{x^j}{\binom{n}{j}} \right)^2 & = \frac{1+n+x}{x+1} \sum_{k=0}^a \frac{x^{2k}}{\binom{n}{k}^2} + \frac{x-1}{x+1} \sum_{k=0}^a \frac{kx^{2k}}{\binom{n}{k}^2} \\
& + \frac{a-n+2x+ax}{x+1} \left( \sum_{k=0}^a \frac{x^k}{\binom{n}{k}} \right)^2 - \frac{2(a+1)x^{a+1}}{(x+1)\binom{n}{a}} \sum_{k=0}^a \frac{x^k}{\binom{n}{k}}, \\
\sum_{k=0}^a \left( \sum_{j=0}^k \frac{(-1)^j}{\binom{n}{j}} \right)^2 & = \frac{(n+1)^2(4an^2+22an+30a+3n^2+23n+38)}{2(n+2)^2(n+3)(2n+5)} + \frac{2(-1)^a(a+1)(a+2)(n+1)}{(n+2)^2(n+3)} \frac{1}{\binom{n}{a}} \\
& + \frac{(a+1)^2(6+2a+n)}{2(n+2)^2(2n+5)} \frac{1}{\binom{n}{a}^2} + \frac{n+2}{2(2n+5)} \sum_{k=0}^a \frac{1}{\binom{n}{k}^2}
\end{aligned}$$

for all  $x \in \mathbb{K} \setminus \{-1\}$  and  $a, n \in \mathbb{Z}_{\geq 0}$  with  $a \leq n$ .

Similarly, for the generic double sum

$$\text{In[17]:= mySum} = \sum_{k=0}^a (-1)^k \left( \sum_{j=0}^k \mathbf{X}[j] \right)^2;$$

Sigma finds the general simplification

$$\text{In[18]:= {closedForm, constraint} = SigmaReduce[mySum, XList \to \{X\}, ExtractConstraints \to \{Y\}, \\ \text{SimpleSumRepresentation} \to \text{False}, RefinedForwardShift \to \text{False}]$$

$$\text{Out[18]=} \left\{ -\frac{1}{2}c \left( \sum_{i=0}^a \mathbf{X}[i] \right)^2 + \frac{1}{2}(-1)^a \left( \sum_{i=0}^a \mathbf{X}[i] \right)^2 + \frac{1}{2} \sum_{i=0}^a \left( (-1)^i \mathbf{X}[i] + c\mathbf{X}[i] + \mathbf{Y}[i] \right) \mathbf{X}[i] - \frac{1}{2} \mathbf{Y}[a] \sum_{i=0}^a \mathbf{X}[i], \right. \\ \left. \{ \mathbf{Y}[a+1] - \mathbf{Y}[a] == 2(-1)^a \mathbf{X}[a+1] - 2c\mathbf{X}[a+1] \} \right\}.$$

where the result can be simplified further to

$$\text{In[19]:= SigmaReduce[closedForm, a, XList \to \{X, Y\}, SimpleSumRepresentation \to \text{True}]$$

$$\text{Out[19]=} \left( -\frac{c}{2} + \frac{1}{2}(-1)^a \right) \left( \sum_{i=0}^a \mathbf{X}[i] \right)^2 + \frac{1}{2}c \sum_{i=0}^a \mathbf{X}[i]^2 + \frac{1}{2} \sum_{i=0}^a (-1)^i \mathbf{X}[i]^2 + \frac{1}{2} \sum_{i=0}^a \mathbf{X}[i] \mathbf{Y}[i] - \frac{1}{2} \mathbf{Y}[a] \sum_{i=0}^a \mathbf{X}[i]$$

This means that for any sequences  $\bar{X}_k \in \mathbb{K}$ ,  $\bar{Y}_k \in \mathbb{K}$  and  $c \in \mathbb{K}$  with

$$\bar{Y}_{k+1} - \bar{Y}_k = 2(-1)^k \bar{X}_{k+1} - 2c \bar{X}_{k+1}, \quad (63)$$

we obtain the simplification

$$\sum_{k=0}^a (-1)^k \left( \sum_{j=0}^k \bar{X}_j \right)^2 = \left( -\frac{c}{2} + \frac{1}{2}(-1)^a \right) \left( \sum_{k=0}^a \bar{X}_k \right)^2$$



$$+ \frac{1}{2}c \sum_{k=0}^a \bar{X}_k^2 + \frac{1}{2} \sum_{k=0}^a (-1)^k \bar{X}_k^2 + \frac{1}{2} \sum_{k=0}^a \bar{X}_k \bar{Y}_k - \frac{1}{2} \bar{Y}_a \sum_{k=0}^a \bar{X}_k. \quad (64)$$

In addition, by a slight modification of Theorem 5.7 we obtain the following stronger statement for any product-sum sequence ring  $(\mathbb{S}, S)$  under the assumption that  $\bar{X}_k$  is in  $\mathbb{S}$ , but  $\sum_{j=0}^k \bar{X}_j$  is not in  $\mathbb{S}$ : the double sum can be simplified to single nested sums defined over  $\mathbb{S}$  if and only if (64) holds and there are  $\bar{Y}_k \in \mathbb{S}$  and  $c \in \mathbb{K}$  with (63). Again proceeding as above one can find, for instance, the following identities:

$$\begin{aligned} \sum_{k=0}^a (-1)^k \left( \sum_{j=0}^k \binom{n}{j} \right)^2 &= \frac{(-a+n)(-1)^a \binom{n}{a}}{n} \sum_{k=0}^a \binom{n}{k} + \frac{(-1)^a}{2} \left( \sum_{k=0}^a \binom{n}{k} \right)^2 \\ &\quad - \frac{1}{2} \sum_{k=0}^a (-1)^k \binom{n}{k}^2 + \frac{1}{n} \sum_{k=0}^a (-1)^k k \binom{n}{k}^2, \\ \sum_{k=0}^a (-1)^k \left( \sum_{j=0}^k (-1)^j \binom{n}{j} \right)^2 &= \frac{1}{2} \sum_{k=0}^a (-1)^k \binom{n}{k}^2 \\ &\quad - \frac{1}{n} \sum_{k=0}^a (-1)^k k \binom{n}{k}^2 + \frac{(-1)^a \binom{n}{a}^2 (-a+n)^2}{2n^2}, \\ \sum_{k=0}^a (-1)^k \left( \sum_{j=0}^k \frac{1}{\binom{n}{j}} \right)^2 &= \frac{(a+1)(-1)^a}{(n+2) \binom{n}{a}} \sum_{k=0}^a \frac{1}{\binom{n}{k}} + \frac{(-1)^a}{2} \left( \sum_{k=0}^a \frac{1}{\binom{n}{k}} \right)^2 \\ &\quad + \frac{n}{2(n+2)} \sum_{k=0}^a \frac{(-1)^k}{\binom{n}{k}^2} - \frac{1}{n+2} \sum_{k=0}^a \frac{(-1)^k k}{\binom{n}{k}^2}, \\ \sum_{k=0}^a (-1)^k \left( \sum_{j=0}^k \frac{(-1)^j}{\binom{n}{j}} \right)^2 &= -\frac{n}{2(n+2)} \sum_{k=0}^a \frac{(-1)^k}{\binom{n}{k}^2} + \frac{1}{n+2} \sum_{k=0}^a \frac{(-1)^k k}{\binom{n}{k}^2} \\ &\quad + \frac{n+1}{n+2} \sum_{k=0}^a \frac{1}{\binom{n}{k}} + \frac{(a+1)(n+1)}{(n+2)^2 \binom{n}{a}} \\ &\quad + \frac{(n+1)^2 (-1)^a}{2(n+2)^2} + \frac{(a+1)^2 (-1)^a}{2(n+2)^2 \binom{n}{a}^2}, \end{aligned}$$

where the first two identities are valid for  $a, n \in \mathbb{Z}$  and  $n \neq 0$  and the last two identities are valid for  $a, n \in \mathbb{Z}$  with  $a \leq n$ .

## 7 Conclusion

In this article, under the umbrella of algorithmic symbolic summation, we established new algebraic connections between summation problems involving generic sequences and difference field/ring theory taking special care of concrete sequences arising in contexts like analysis, combinatorics, number theory and special func-

tions. We feel this is only the “first word” in view of the high potential for applications of various kinds. One future application domain is summation identities involving elliptic functions or modular forms. This will be especially interesting in upcoming calculations [1] emerging in renormalizable Quantum Field Theories. Another more concrete application domain is the area of  $q$ -identities involving  $q$ -hypergeometric series and sums. But already for  $q = 1$  one can study aspects of *definite* summation. We plan to investigate these questions in forthcoming articles. For example, if we specialize our sums to definite versions by setting  $a = n$  (and possibly consider the even or odd case), further simplifications can be achieved by Sigma. Typical examples are

$$\begin{aligned} \sum_{k=0}^n \left( \sum_{j=0}^k \frac{1}{\binom{n}{j}} \right)^2 &= \frac{3(n+1)^3(n+2)}{4(2n+1)(2n+3)\binom{2n}{n}} \sum_{k=1}^n \frac{\binom{2k}{k}}{k} + 2^{-n-1}(n+1) \sum_{k=1}^n \frac{2^k}{k} \\ &\quad + 2^{-2n-3}(n+1)^2(n+2) \left( \sum_{k=1}^n \frac{2^k}{k} \right)^2 + \frac{n^2+6n+6}{2(2n+3)}, \\ \sum_{k=0}^{2n} (-1)^k \left( \sum_{j=0}^k \frac{1}{\binom{2n}{j}} \right)^2 &= \frac{2^{-2n-2}(2n+1)(4n+3)}{n+1} \sum_{k=1}^{2n} \frac{2^k}{k} \\ &\quad + 2^{-4n-3}(2n+1)^2 \left( \sum_{k=1}^{2n} \frac{2^k}{k} \right)^2 + \frac{3n+2}{2(n+1)}, \\ \sum_{k=0}^{2n} (-1)^k \left( \sum_{j=0}^k \binom{2n}{j} \right)^2 &= 2^{4n-1}, \end{aligned}$$

where the first two identities are valid for  $n \geq 0$  and the last identity holds for  $n \geq 1$ .

## 8 Appendix: Simple $R\Pi\Sigma$ -rings and algorithmic properties

For a given difference ring (resp. field)  $(\mathbb{A}, \sigma)$ , i.e., a ring (resp. field)  $\mathbb{A}$  equipped with a ring (resp. field) automorphism  $\sigma : \mathbb{A} \rightarrow \mathbb{A}$  the set of constants  $\mathbb{K} := \text{const}(\mathbb{A}, \sigma) = \{c \in \mathbb{A} \mid \sigma(c) = c\}$  forms a subring (resp. subfield) of  $\mathbb{A}$ . In this article we suppose that  $\mathbb{A}$  contains the rational numbers  $\mathbb{Q}$  as a subfield. Since  $\sigma(1) = 1$ , this implies that  $\mathbb{Q} \subseteq \mathbb{K}$  always holds. Moreover, by construction we will take care that  $\mathbb{K}$  will be always a field which will be called the constant field of  $(\mathbb{A}, \sigma)$ .

In the following we introduce the class of simple  $R\Pi\Sigma$ -rings that forms the fundament of Sigma’s difference ring engine. Depending on the given input problem, the ground field is chosen accordingly among one of the following three difference fields.

**Definition 8.1.** We consider the following three difference fields  $(\mathbb{F}, \sigma)$  with constant field  $\mathbb{K}$ .

- (1) The *rational case*:  $\mathbb{F} = \mathbb{K}(k)$  where  $\mathbb{K}(k)$  is a rational function field and  $\sigma(k) = k + 1$ .
- (2) The *q-rational case*:  $\mathbb{F} = \mathbb{K}(z)$  where  $\mathbb{K}(z)$  is a rational function field,  $\mathbb{K} = \mathbb{K}'(q)$  is a rational function field ( $\mathbb{K}'$  is a field) and  $\sigma(z) = qz$ .
- (3) The *mixed case*:  $(\mathbb{K}(k)(z_1, \dots, z_v), \sigma)$  where  $\mathbb{K}(k)(z_1, \dots, z_v)$  is a rational function field,  $\mathbb{K} = \mathbb{K}'(q_1, \dots, q_v)$  is a rational function field ( $\mathbb{K}'$  is a field),  $\sigma(k) = k + 1$ , and  $\sigma(z_i) = q_i z_i$  for  $1 \leq i \leq v$ .

We remark that these difference fields can be embedded into the ring of sequences  $(\text{Seq}(\mathbb{K}), S)$  as expected. For the rational case see Example 5.1, and for the other two cases we refer to [22, Ex. 5.3]. Further aspects can be found in [6].

On top of such a ground field, a tower of extensions is built recursively depending on the input that is passed to `Sigma`. Let  $(\mathbb{A}, \sigma)$  be the already constructed difference ring with constant field  $\mathbb{K}$ . Then the tower can be extended by one of the following three types of extensions [8, 21]; compare Definition 4.9.

- (1)  **$\Sigma$ -extension**: Given  $\beta \in \mathbb{A}$ , take the polynomial ring  $\mathbb{A}[t]$  ( $t$  is transcendental over  $\mathbb{A}$ ) and extend the automorphism  $\sigma$  from  $\mathbb{A}$  to  $\mathbb{A}[t]$  subject to the relation  $\sigma(t) = t + \beta$ . If  $\text{const}(\mathbb{A}[t], \sigma) = \text{const}(\mathbb{A}, \sigma)$ , the difference ring  $(\mathbb{A}[t], \sigma)$  is called a  $\Sigma$ -extension of  $(\mathbb{A}, \sigma)$ .
- (2)  **$\Pi$ -extension**: Given a unit  $\alpha \in \mathbb{A}^*$ , take the Laurent polynomial ring  $\mathbb{A}[t, t^{-1}]$  ( $t$  is transcendental over  $\mathbb{A}$ ) and extend the automorphism  $\sigma$  from  $\mathbb{A}$  to  $\mathbb{A}[t, t^{-1}]$  subject to the relation  $\sigma(t) = \alpha t$  (and  $\sigma(t^{-1}) = \frac{1}{\alpha} t^{-1}$ ). If  $\text{const}(\mathbb{A}[t, t^{-1}], \sigma) = \text{const}(\mathbb{A}, \sigma)$ , the difference ring  $(\mathbb{A}[t, t^{-1}], \sigma)$  is called a  $\Pi$ -extension of  $(\mathbb{A}, \sigma)$ .
- (3)  **$R$ -extension**: Given a primitive  $\lambda$ th root of unity  $\alpha \in \mathbb{K}$  with  $\lambda \geq 2$ , take the algebraic ring  $\mathbb{A}[t]$  subject to the relation  $t^\lambda = 1$  and extend the automorphism  $\sigma$  from  $\mathbb{A}$  to  $\mathbb{A}[t]$  subject to the relation  $\sigma(t) = \alpha t$ . If  $\text{const}(\mathbb{A}[t], \sigma) = \text{const}(\mathbb{A}, \sigma)$ , the difference ring  $(\mathbb{A}[t], \sigma)$  is called an  $R$ -extension of  $(\mathbb{A}, \sigma)$ .

More generally, we call a difference ring  $(\mathbb{E}, \sigma)$  an *RPII $\Sigma$ -extension* of a difference ring  $(\mathbb{A}, \sigma)$  if it is built by a tower

$$\mathbb{A} = \mathbb{E}_0 \leq \mathbb{E}_1 \leq \dots \leq \mathbb{E}_e = \mathbb{E} \quad (65)$$

of  $R$ -,  $\Pi$ -, and  $\Sigma$ -extensions starting from the difference ring  $(\mathbb{A}, \sigma)$ . Note that by construction we have that  $\text{const}(\mathbb{E}, \sigma) = \text{const}(\mathbb{A}, \sigma) = \mathbb{K}$ . Finally, we restrict to the following case that is relevant for this article.

**Definition 8.2.** We call a difference ring  $(\mathbb{E}, \sigma)$  a *simple RPII $\Sigma$ -ring with constant field  $\mathbb{K}$*  if it is an *RPII $\Sigma$ -extension* of a difference ring  $(\mathbb{A}, \sigma)$  built by the tower (65) with the following properties:

- (1)  $(\mathbb{A}, \sigma)$  is one of the three difference fields from Definition 8.1;
- (2) for  $i$  with  $1 \leq i \leq e$  the following holds: if  $(\mathbb{E}_i, \sigma)$  is a  $\Pi$ -extension of  $(\mathbb{E}_{i-1}, \sigma)$  with  $\mathbb{E}_i = \mathbb{E}_{i-1}[t_i, t_i^{-1}]$ , then  $\sigma(t_i)/t_i \in \mathbb{A}^*$ .

Note that within such a simple *RPII $\Sigma$ -ring* the generators of

- (a)  $R$ -extensions model algebraic products of the form  $\alpha^k$  where  $\alpha$  is a primitive root of unity;
- (b)  $\Pi$ -extensions model ( $q$ -)hypergeometric/mixed hypergeometric products depending on the chosen base field  $(\mathbb{A}, \sigma)$ ;
- (c)  $\Sigma$ -extensions represent nested sums whose summands are built recursively by polynomial expressions in terms of objects that are introduced in (a), (b) and (c).

Given such a simple  $R\Pi\Sigma$ -ring with constant field  $\mathbb{K}$ , we can exploit the algorithmic properties summarized in Theorem 5.4 that are incorporated within the summation package `Sigma`. For a detailed description of parts (1) and (2a) of Theorem 5.4 we refer to [22, Section 7.2]; for part (2b) of Theorem 5.4 we refer to [22, Section 5].

In the following we sketch some further aspects. Namely, given an expression  $X(k)(= X_k)$  in terms of nested sums over hypergeometric (resp.  $q$ -hypergeometric or mixed hypergeometric) products, one can always construct algorithmically an  $R\Pi\Sigma$ -ring  $(\mathbb{E}, \sigma)$  together with an evaluation function  $\text{ev} : \mathbb{E} \times \mathbb{Z}_{\geq 0} \rightarrow \mathbb{K}$  with the following two properties (A) and (B).

(A)  $(\mathbb{E}, \sigma)$  is constructed explicitly by the tower of extensions (65) with the generators  $t_i$  ( $\mathbb{E}_i = \mathbb{E}_{i-1}[t_i]$  for  $R$ - or  $\Sigma$ -extensions and  $\mathbb{E}_i = \mathbb{E}_{i-1}[t_i, t_i^{-1}]$  for a  $\Pi$ -extension) where for  $1 \leq i \leq e$ , there is an explicitly given product or a nested sum over products, say  $F_i(k)$ , and a  $\lambda_i \in \mathbb{Z}_{\geq 0}$  such that  $\text{ev}(t_i, k) = F_i(k)$  holds for all  $k \geq \lambda_i$ . In particular, the resulting map  $\tau : \mathbb{E} \rightarrow \text{Seq}(\mathbb{K})$  with  $\tau(f) \equiv (\text{ev}(f, k))_{k \geq 0}$  yields a  $\mathbb{K}$ -embedding.

**Example 8.3.** Consider the  $R\Pi\Sigma$ -ring  $(\mathbb{K}(k)[s], \sigma)$  from Example 4.8. There we obtained  $\text{ev}$  with  $\text{ev}(s, k) = H_k$  for all  $k \geq \lambda$  with  $\lambda = 0$ .

(B) One can construct an element  $x \in \mathbb{E}$  and a  $\lambda \in \mathbb{Z}_{\geq 0}$  such that  $X(i) = \text{ev}(x, i)$  holds for all  $i \geq \lambda$ . In particular, this  $x \in \mathbb{E}$  can be rephrased again as an expression in terms of products or sums defined over such products in the following way: replacing the generators  $t_i$  in  $f$  by the attached sums or products<sup>18</sup> one gets an expression  $X'(k)$  in terms of nested sums over products such that  $X(k) = \text{ev}(x, k) = X'(k)$  holds for all  $k \in \mathbb{Z}_{\geq 0}$  with  $k \geq \lambda$ .

In addition, the summation paradigms of refined parameterized telescoping [17–22] and recurrence solving can be carried out in such simple  $R\Pi\Sigma$ -rings. In a nutshell, we can solve the telescoping problem and enhanced versions of it in the  $R\Pi\Sigma$ -ring  $(\mathbb{E}, \sigma)$  or equivalently in the product-sum sequence ring  $(\mathbb{S}, S)$ . This enables one to discover, e.g., the identities given in Section 7.

Furthermore, the difference ring algorithms combined with the algorithms given in [11] work also for difference rings where one starts with the free difference field  $(\mathbb{G}, \sigma)$  introduced in Subsection 6.1 as base field, adjoins the generators given in Definition 8.1, and puts a tower of  $R\Pi\Sigma$ -extensions on top; compare Subsection 6.1.

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<sup>18</sup> In the  $q$ -case (resp. in the mixed case) we also have to replace  $z$  by  $q^k$  (resp.  $z_i$  by  $q_i^k$  for  $1 \leq i \leq v$ ).

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