Resultants: Algebraic and Differential

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1 Introduction

This report summarises ongoing discussions of the authors on the topic of differential resultants which have three goals in mind. First, we aim to try to understand existing literature on the topic. Second, we wish to formulate some interesting questions and research goals based on our understanding of the literature. Third, we would like to advance the subject in one or more directions, by pursuing some of these questions and research goals. Both authors have somewhat more background in nondifferential, as distinct from differential, computational algebra. For this reason, our approach to learning about differential resultants has started with a careful review of the corresponding theory of resultants in the purely algebraic (polynomial) case. We try, as far as possible, to adapt and extend our knowledge of purely algebraic resultants to the differential case. Overall, we have the hope of helping to clarify, unify and further develop the computational theory of differential resultants.

There are interesting notions of a differential polynomial resultant in the literature. At first glance it could appear that these notions differ in essential ways. For example, Zwillinger [30] suggested that the concept of a differential resultant of a system of two coupled algebraic ordinary differential equations (AODEs) for $(y(x), z(x))$ (where x is the independent variable and y and z are the dependent variables) could be developed. Such a differential resultant would be a single AODE for $z(x)$ only. While that author sketches how such differential elimination could work for a specific example, no general method is presented. Chardin [5] presented an elegant treatment of resultants and subresultants of (noncommutative) ordinary differential operators. Carra'-Ferro (see for example [1, 2]) published several works on differential resultants of various kinds, with firm algebraic foundations, but the relations to Zwillinger's suggested notion and Chardin's theory might not be immediately clear from glancing through these works.

In fact our study of the subject has revealed to us that the approaches of all three authors mentioned above are intimately related. It would appear that the common source for the essential basic notion of differential resultant can be traced to work of Ritt [17] in the 1930s. After reviewing relevant background material on algebra, both classical and differential, in Section 2, we will present in Section 3 the simplest case of the differential resultant originally proposed by Ritt: namely, that of two linear homogeneous ordinary differential polynomials over a differential ring or field. Chardin's theory is most closely associated with this simple special case. In Section 4 we will review the algebraic theory of the multipolynomial resultant of Macaulay. In Section 5, using the concepts and results of Section 4, we extend the concept of Section 3 to that of two arbitrary ordinary differential polynomials over a differential field or ring. This could be viewed as a simpler and more streamlined account of Carra'-Ferro's theory. We will see that this theory can be applied to the problem of differential elimination, thereby providing a systematic treatment of the approach suggested by Zwillinger. In Section 6 we survey briefly some of the work in this area post that of Carra'-Ferro, and in the final section we pose questions for investigation.

2 Elementary background material

Let R be a commutative ring with identity element 1. In the first subsection we shall review the definition of the classical Sylvester resultant $\mathrm{res}(f,g)$ of $f(x),g(x)\in R[x]$. We shall state the requirements on R so that $\mathrm{res}(f,g)=0$ is a necessary and sufficient condition for the existence in some extension of R of a solution α to the system

$$f(x)=g(x)=0\ .$$

In the second subsection we review elementary differential algebra on R. In particular we define the notion of a derivation on R, and introduce the ring of differential polynomials over R. The elementary background concepts from this section will provide the foundation for the theory of the differential Sylvester resultant, developed in the next section.

2.1 Sylvester resultant

In this subsection we review the basic theory of the Sylvester resultant for algebraic polyomials, with an emphasis on the necessary requirements for the underlying coefficient domain. For convenience we assume at the outset that R is an integral domain (commutative ring with 1, and no zero divisors), and that K is its quotient field. Some results we will state require merely that R be a commutative ring with 1, and others require a stronger hypothesis, as we shall remark.

Let

$$f(x)=\sum_{i=0}^{m}a_ix^i,\quad g(x)=\sum_{j=0}^{n}b_jx^j$$

be polynomials of positive degrees m and n, respectively, in $R[x]$. If f and g have a common factor $d(x)$ of positive degree, then they have a common root in the algebraic closure $\overline{K}$ of K; so the system of equations

$$f(x)=g(x)=0 \tag{1}$$

has a solution in $\overline{K}$.

On the other hand, if $\alpha\in\overline{K}$ is a common root of f and g, then $\mathrm{norm}_{K(\alpha):K}(x-\alpha)$ is a common divisor of f and g in $K[x]$. So, by Gauss' Lemma (for which we need R to be a unique factorization domain) on primitive polynomials there is a similar (only differing by a factor in K) common factor of f and g in $R[x]$. We summarize these observations as follows:

Proposition 2.1. *Let R be a unique factorization domain (UFD) with quotient field K. For polynomials $f(x),g(x)\in R[x]$ the following are equivalent:*

(i) *f and g have a common solution in $\overline{K}$, the algebraic closure of K,*

(ii) *f and g have a common factor of positive degree in $R[x]$.*

So now we want to determine a necessary condition for f and g to have a common divisor of positive degree in $R[x]$. Suppose that f and g indeed have a common divisor $d(x)$ of positive degree in $R[x]$; i.e.,

$$f(x)=d(x)\overline{f}(x),\quad g(x)=d(x)\overline{g}(x). \tag{2}$$

Then for $p(x):=\overline{g}(x)$, $q(x):=-\overline{f}(x)$ we have

$$p(x)f(x)+q(x)g(x)=0. \tag{3}$$

So there are non-zero polynomials p and q with $\deg p<\deg g,\deg q<\deg f$, satisfying equation (3). This means that the linear system

$$\begin{pmatrix}p_{n-1} & \cdots & p_0 & q_{m-1} & \cdots & q_0\end{pmatrix}\cdot\begin{pmatrix}A\\ \cdots\\ B\end{pmatrix}\ =\ 0\ , \tag{4}$$

where

$$A = \begin{pmatrix} a_m & a_{m-1} & \cdots & a_0 & & & \\ & a_m & a_{m-1} & \cdots & a_0 & & \\ & & \ddots & & & \ddots & \\ & & & a_m & a_{m-1} & \cdots & a_0 \end{pmatrix} \in R^n_{m+n} \quad ,$$

$$B = \begin{pmatrix} b_n & b_{n-1} & \cdots & b_0 & & & \\ & b_n & b_{n-1} & \cdots & b_0 & & \\ & & \ddots & & & \ddots & \\ & & & b_n & b_{n-1} & \cdots & b_0 \end{pmatrix} \in R^m_{m+n} \; ,$$

has a non-trivial solution. The matrix of this system (4) is called the *Sylvester matrix* of f and g. Thus, the determinant of the Sylvester matrix of f and g is 0. The *resultant* of f and g, $\mathrm{res}(f, g)$, is this determinant, and it is clear that the resultant is a polynomial expression of the coefficients of f and g, and therefore an element of the integral domain R. This does not require R to be a UFD. Summarizing:

Proposition 2.2. *Let $f, g \in R[x]$, for R an integral domain.*
$\mathrm{res}(f, g) = 0$ *is a* **necessary condition** *for f and g to have a common factor of positive degree.*

If we identify a polynomial of degree d with the vector of its coefficients of length $d + 1$, we may also express this in terms of the linear map

$$\begin{array}{rlcl} S: & K^{m+n} & \longrightarrow & K^{m+n} \\ & (p_{n-1}, \ldots, p_0, q_{m-1}, \ldots, q_0) & \mapsto & \text{coefficients of } pf + qg \end{array}$$

Obviously the existence of a non-trivial linear combination (3) is equivalent to S having a non-trivial kernel, and therefore to S having determinant 0.

But is the vanishing of the resultant also a sufficient condition for f and g to have a common factor of positive degree? Suppose that $\mathrm{res}(f, g) = 0$. This means that (3) has a non-trivial solution $u(x), v(x)$ of bounded degree; so

$$u(x)f(x) = -v(x)g(x) \; .$$

These co-factors will be in $K[x]$, but of course we can clear denominators and have a similar relation with co-factors in $R[x]$. If we now require the coefficient domain R to be a unique factorization domain (UFD), we see that every irreducible factor of f must appear on the right hand side with at least the same multiplicity. Not all of these factors can be subsumed in v, because v is of lower degree than f. So at least one of the irreducible factors of f must divide g. Thus we have:

Proposition 2.3. *Let $f, g \in R[x]$, for R a UFD.*
$\mathrm{res}(f, g) = 0$ *is also a* **sufficient condition** *for f and g to have a common factor of positive degree; and therefore a common solution in $\overline{K}$.*

A further property is of interest and importance.

Proposition 2.4. *The resultant is a constant in the ideal generated by f and g in $R[x]$; i.e. we can write*

$$\mathrm{res}(f, g) \;=\; u(x)f(x) + v(x)g(x), \tag{5}$$

with $u, v \in R[x]$. Moreover, these cofactors satisfy the degree bounds $\deg(u) < \deg(g)$, $\deg(v) < \deg(f)$.

Proof: We follow an argument given in [6]. In fact, Collins proves this fact for R being a general commutative ring with 1.

Consider the Sylvester matrix $S = (A \vdots B)^T$; i.e. the $(m+n) \times (m+n)$ matrix, whose first n rows consist of the coefficients of

$$x^{n-1} \cdot f(x),\ \ldots\ , x \cdot f(x),\ f(x)\ ,$$

and whose last m rows consist of the coefficients of

$$x^{m-1} \cdot g(x),\ \ldots\ , x \cdot g(x),\ g(x)\ .$$

Now, for $1 \leq i < m+n$, multiply the ith column of S by x^{m+n-i} and add to the last column. This will result in a new matrix T, having the same determinant as S. The columns of T are the same as the corresponding columns of S, except for the last column, which consists of the polynomials

$$x^{n-1} \cdot f(x),\ \ldots\ , x \cdot f(x),\ f(x),\ x^{m-1} \cdot g(x),\ \ldots\ , x \cdot g(x),\ g(x)\ .$$

Expanding the determinant of T w.r.t. its last column, we obtain polynomials $u(x)$ and $v(x)$ satisfying the relation (5), and also the degree bounds. □

A proof of the above proposition could also be obtained by slightly modifying the proof of Proposition 9 in Section 5, Chapter 3 of [7] (proved for polynomials over a field).

An alternative approach (similar to that above but with a slightly different emphasis) to defining the Sylvester resultant of $f(x)$ and $g(x)$ is to regard all the coefficients a_i and b_j of f and g as distinct and unrelated indeterminates. The indeterminates a_m and b_n are then referred to as the *formal* leading coefficients of f and g, respectively. In effect we take R to be the domain $\mathbb{Z}[a_m, \ldots, a_0, b_n, \ldots, b_0]$. This approach allows us to study the resultant $\mathrm{res}(f,g)$ as a polynomial in the $m+n+2$ indeterminates a_i and b_j. Indeed it is not hard to see that $\mathrm{res}(f,g)$ is homogeneous in the a_i of degree n, homogeneous in the b_j of degree m, and has the "principal term" $a_m^n b_0^m$ (from the principal diagonal). With this approach, adjustment of some of the basic facts is needed. For example, the analogue of Proposition 2.3 would state that, for D a UFD, after replacement of all the coefficients a_i and b_j by elements of D, $\mathrm{res}(f,g) = 0$ is a sufficient condition for either $f(x)$ and $g(x)$ to have a common factor of positive degree, or $a_m = b_n = 0$.

Another variation on defining the Sylvester resultant of two polynomials is to start instead with two *homogeneous* polynomials $F(x,y) = \sum_{i=0}^{m} a_i x^i y^{m-i}$ and $G(x,y) = \sum_{j=0} b_j x^j y^{n-j}$. Let us similarly regard the coefficients a_i and b_j as indeterminates. Then the resultant of F and G is defined as $\mathrm{res}(F,G) = \mathrm{res}(f,g)$, where $f(x) = F(x,1)$ and $g(x) = G(x,1)$. Our analogue of Proposition 2.3 then becomes simpler. Combining it with homogeneous analogues of Propositions 2.1 and 2.2 we have:

Proposition 2.5. *After assigning values to the coefficients from a UFD D, $\mathrm{res}(F,G) = 0$ is a necessary and sufficient condition for $F(x,y)$ and $G(x,y)$ to have a common factor of positive degree over D, hence for a common zero to exist over an extension of the quotient field of D.*

2.2 Basic differential algebra

Let R be a commutative ring with 1. A *derivation* on R is a mapping $\partial : R \to R$ such that $\partial(a+b) = \partial(a) + \partial(b)$ and $\partial(ab) = \partial(a)b + a\partial(b)$ for all $a, b \in R$. That $\partial(0) = 0$ and $\partial(1) = 0$ follow readily from these axioms. We sometimes denote the derivative of a $\partial(a)$ by a'. Such a ring (or integral domain or field) R together with a derivation on R is called a *differential ring (or integral domain or field, respectively)*. In such a ring R elements r such that $r' = 0$ are known as *constants* and the set C of constants comprises a subring of R. If R is a field, C is a subfield of R. An ideal I of such a ring R is known as a *differential ideal* if $r \in I$ implies $r' \in I$. If $r_1, \ldots, r_n \in R$ we denote by $[r_1, \ldots, r_n]$ the differential ideal generated by $r_1, \ldots, r_n$, that is, the ideal generated by the r_i and all their derivatives.

Example 2.6. *The familiar rings such as $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ are differential rings if we set $\partial(a) = 0$ for all elements a. (In fact, any ring/field can be similarly made into a differential ring/field.)*

Example 2.7. *Let K be a field and t an indeterminate over K. Then $K[t]$, equipped with the derivation $\partial = d/dt$, is a differential integral domain and its quotient field $K(t)$ is a differential field, again with standard differentiation as its derivation. K is the ring (field) of constants of $K[t]$ ($K(t)$).*

Example 2.8. *Let (R, ∂) be a differential ring. Let $x = x^{(0)}, x^{(1)}, x^{(2)}, \ldots$ be distinct indeterminates over R. Put $\partial(x^{(i)}) = x^{(i+1)}$ for all $i \geq 0$. Then ∂ can be extended to a derivation on the polynomial ring $R\{x\} := R[x^{(0)}, x^{(1)}, \ldots]$ in a natural way, and we denote this extension also by ∂. The ring $R\{x\}$ together with this extended ∂ is a differential ring, called the ring of* differential polynomials *in the differential indeterminate x over R. An element $f(x) = \sum_{i=0}^{m} a_i x^{(i)}$ of $R\{x\}$ with $a_m \neq 0$ has* order m *and* leading coefficient a_m. *(***Remark***. It may be helpful to think of elements of R and of $x, x^{(1)}, \ldots$ as functions of an indeterminate t, and to regard ∂ as differentiation with respect to t.) If (K, ∂) is a differential field then $K\{x\}$ is a differential integral domain, and its derivation extends uniquely to the quotient field. We write $K\langle x\rangle$ for this quotient field; its elements are* differential rational functions *of x over K.*

Example 2.9. *Let $K \subset L$ be fields, with ∂ a derivation on K which extends to L. Then L is a* (differential) extension *of K. If $\eta \in L$, then the smallest differential field containing $K, \eta, \eta', \eta'', \ldots$ is denoted by $K\langle\eta\rangle$.*

Let (R, ∂) be a differential integral domain. An ultimate aim of this paper is to define and study a certain resultant of two elements of the differential ring (indeed domain) $R\{x\}$ introduced above. In the next section, we shall consider a simple and important R-submodule of $R\{x\}$ (considered as left R-module), namely that which comprises those elements of $R\{x\}$ which are linear and homogeneous. For two such elements we will introduce an analogue of the classical Sylvester resultant reviewed in Section 2.

3 Differential Sylvester resultant

Let (R, ∂) be a differential integral domain. Recall from Section 2 that the ring (indeed domain) of differential polynomials in the differential indeterminate x is denoted by $R\{x\}$. Then $R\{x\}$ is also a (left) R-module, and we denote by $R_{LH}\{x\}$ the R-submodule comprising those elements of $R\{x\}$ which are linear and homogeneous. We aim in this section to define a certain resultant, known as a differential Sylvester resultant, of two elements of $R_{LH}\{x\}$. We shall begin by studying a closely related noncommutative ring: namely, we consider the ring $R[\partial]$ of linear differential operators on R. As we shall see, there is an important relationship between $R[\partial]$, considered as left R-module, and $R_{LH}\{x\}$: these are isomorphic as left R-modules. Thus the differential theory of $R[\partial]$ and $R_{LH}\{x\}$ can to an extent be developed in parallel. The details are provided in the next two subsections.

3.1 Resultant of two linear differential operators

This subsection follows the presentation of [5], and elaborates on a number of points from that source. Let (R, ∂) be a differential integral domain. Recall that we sometimes denote $\partial(a)$ by a'. Then K, the quotient field of R, is naturally equipped with an extension of this derivation, which we will also denote by ∂ (and sometimes by $'$).

We consider the ring of linear differential operators $R[\partial]$, where the application of $A = \sum_{i=0}^{m} a_i \partial^i$ to $r \in R$ is defined as

$$A(r) = \sum_{i=0}^{m} a_i r^{(i)} \ .$$

Here $r^{(i)}$ denotes the i-fold application of ∂ (that is, $'$) to r. If $a_m \neq 0$, the *order* of A is m and a_m is the *leading coefficient* of A. Now the application of A can naturally be extended to K, and to any extension of K. If $A(\eta) = 0$, with η in R, K or any extension of K, we call η a *root* of the linear differential operator A.

The ring $R[\partial]$ is non-commutative; and the corresponding rule for the multiplication of ∂ by an element of $r \in R$ is

$$\partial r = r\partial + r' \ .$$

Note that ∂r, which denotes the operator product of ∂ and r, is distinct from $\partial(r)$ (that is, from r'), the application of map ∂ to r.

Proposition 3.1. *For $n \in N$: $\partial^n r \ = \ \sum_{i=0}^{n} \binom{n}{i} r^{(n-i)} \partial^i$.*

Proof: For $n = 0$ this obviously holds.
Assume the fact holds for some $n \in \mathbb{N}$. Then

$$\begin{aligned}
\partial^{n+1} r &= \partial(\partial^n r) \ = \ \partial\big(\textstyle\sum_{i=0}^{n} \binom{n}{i} r^{(n-i)} \partial^i\big) \\
&= \textstyle\sum_{i=0}^{n} \binom{n}{i} \partial r^{(n-i)} \partial^i \ = \ \sum_{i=0}^{n} \binom{n}{i} [r^{(n-i)}\partial + r^{(n-i+1)}] \partial^i \\
&= \textstyle\sum_{i=0}^{n} \binom{n}{i} r^{(n-i)} \partial^{i+1} \ + \ \sum_{i=0}^{n} \binom{n}{i} r^{(n-i+1)} \partial^i \\
&= \textstyle\sum_{i=1}^{n+1} \binom{n}{i-1} r^{(n+1-i)} \partial^i \ + \ \sum_{i=0}^{n} \binom{n}{i} r^{(n-i+1)} \partial^i \\
&= \textstyle\binom{n}{n} r^{(0)} \partial^{n+1} \ + \ \sum_{i=1}^{n} [\binom{n}{i-1} + \binom{n}{i}] r^{(n+1-i)} \partial^i \ + \ \binom{n}{0} r^{(n+1)} \partial^0 \\
&= \textstyle\sum_{i=0}^{n+1} \binom{n+1}{i} r^{(n+1-i)} \partial^i.
\end{aligned}$$

□

From a linear homogeneous ODE $f(x) = 0$, with $f(x) \in R\{x\}$, we can extract a linear differential operator $A = \mathcal{L}(f)$ such that the given ODE can be written as

$$A(x) = 0,$$

in which x is regarded as an unknown element of R, K or some extension of K. Such a linear homogeneous ODE always has the trivial solution $x = 0$; so a linear differential operator always has the trivial root 0.

In [5] it is stated that $K[\partial]$ is left-Euclidean, and a few brief remarks are provided by way of proof. Since the concept of a left-Euclidean ring is not as widely known as that of Euclidean ring, it may be helpful to recall its definition here. A ring R is *left-Euclidean* if there exists a function $d : R - \{0\} \rightarrow \mathbb{N}$ such that for all A, B in R, with $B \neq 0$, there exist Q and R in R such that $A = QB + R$, with $d(R) < d(B)$ or $R = 0$. If one wishes to provide a complete proof of the claim that $K[\partial]$ is left-Euclidean (in which we take $d(A)$ to be the order of A), Proposition 3.1 above is useful. For example, by way of proof hint, Chardin claims that the operator $A - (a/b)\partial^{m-n}B$ is of order less than m, where a and b are the leading coefficients of A and B, respectively, and m and n are their orders, with $m \geq n$ assumed. To show this claim, it suffices to show that the term $(a/b)\partial^{m-n}B$ consists of $a\partial^m$ plus terms of order less than m. This follows by applications of Proposition 3.1, putting $n = m - n$ and r equal to each coeficient of operator B in turn.

It follows from the left-Euclidean property that every left-ideal $_K I$ of the form $_K I = (A, B)$ is principle, and is generated by the right-gcd of A and B. As remarked in [5] with reference to [11], under suitable assumptions on K, any linear differential operator of positive order has a root in some extension of K. We state this result precisely.

Theorem 3.2. (Ritt-Kolchin). *Assume that the differential field K has characteristic 0 and that its field C of constants is algebraically closed. Then, for any linear differential operator A over K of positive order n, there exist n roots $\eta_1, \ldots, \eta_n$ in a suitable extension of K, such that the η_i are linearly independent over C. Moreover, the field $K\langle\eta_1, \ldots, \eta_n\rangle (= K\langle\eta_1\rangle \ldots \langle\eta_n\rangle)$ contains no constant not in C.*

This result is stated and proved in [13] using results from [12] and [17]. The field $K\langle\eta_1, \ldots, \eta_n\rangle$ associated with A is known as a *Picard-Vessiot extension* of K (for A). Henceforth assume the hypotheses of Theorem 3.2.

It follows from Theorem 3.2 that if the operators $A, B \in K[\partial]$ have a common factor F of positive order on the right, i.e.,

$$A = \overline{A} \cdot F, \quad \text{and } B = \overline{B} \cdot F, \tag{6}$$

then they have a non-trivial common root in a suitable extension of K. For by Theorem 3.2, F has a root $\eta \neq 0$ in an extension of K. We have $A(\eta) = \bar{A}(F(\eta)) = \bar{A}(0) = 0$ and similarly $B(\eta) = 0$.

On the other hand, if A and B have a non-trivial common root η in a suitable extension of K, we show that they have a common right factor of positive order in $K[\partial]$. Let F be a nonzero differential operator of lowest order s.t. $F(\eta) = 0$. Then F has positive order. Because the ring of operators is left-Euclidean, F is unique up to multiplication of non-zero elements of K. This F is a right divisior of both A and B. To see this, apply division in the left-Euclidean ring $K[\partial]$:

$$A = Q \cdot F + R,$$

with the order of R less than the order of F, or $R = 0$. Apply both sides of this equation to η:

$$A(\eta) = (Q \cdot F)(\eta) + R(\eta).$$

Since $A(\eta) = 0$ and $F(\eta) = 0$, $R(\eta) = 0$. Therefore, by minimality of F, $R = 0$. Hence F is a right divisor of A. We see that F is a right divisor of B similarly. We summarize our result in the following theorem, which is the closest analogue of Proposition 2.1 we can state.

Theorem 3.3. *Assume that K has characteristic 0 and that its field of constants is algebraically closed. Let A, B be differential operators of positive orders in $K[\partial]$. Then the following are equivalent:*

(i) *A and B have a common non-trivial root in an extension of K,*

(ii) *A and B have a common factor of positive order on the right in $K[\partial]$.*

Now let us see that the existence of a non-trivial factor (6) is equivalent to the existence of a non-trivial order-bounded linear combination

$$CA + DB = 0\ , \tag{7}$$

with order$(C) <$ order(B) and order$(D) <$ order(A), and $(C, D) \neq (0, 0)$.
For given $A, B \in K[\partial]$, with $m = \text{order}(A), n = \text{order}(B)$, consider the linear map

$$\begin{array}{llcl} S: & K^{m+n} & \longrightarrow & K^{m+n} \\ & (c_{n-1}, \ldots, c_0, d_{m-1}, \ldots, d_0) & \mapsto & \text{coefficients of } CA + DB \end{array} \tag{8}$$

Obviously the existence of a non-trivial linear combination (7) is equivalent to S having a non-trivial kernel, and therefore to S having determinant 0. Indeed we have the following result.

Theorem 3.4. $\det(S) = 0$ *if and only if A and B have a common factor (on the right) in $K[\partial]$ of positive order.*

Proof: Suppose $\det(S) = 0$. This means that S cannot be surjective. Now the right-gcd G of A and B can be written as an order-bounded linear combination of A and B, so it is in the image of the map S. This means that G cannot be trivial (that is, G cannot be an element of K), because otherwise S would be surjective.

On the other hand, suppose that $\det(S) \neq 0$. Then the linear map is invertible; in particular, it is surjective. Therefore there exist $C, D \in K[\partial]$ with appropriate degree bounds, s.t. $1 = CA+DB$. So every common divisor (on the right) of A and B is a common divisor of 1. Therefore no common divisor of A and B could have positive order. □

So let us see which linear conditions on the coefficients of A and B we get by requiring that (7) has a non-trivial solution of bounded order, i.e.,

$$\text{order}(C) < \text{order}(B) \quad \text{and} \quad \text{order}(D) < \text{order}(A).$$

Example 3.5. $\text{order}(A) = 2 = \text{order}(B)$

$$(c_1\partial + c_0)(a_2\partial^2 + a_1\partial + a_0) \;+\; (d_1\partial + d_0)(b_2\partial^2 + b_1\partial + b_0)$$

order 3:

$$c_1\partial a_2\partial^2 = c_1(a_2\partial + a_2')\partial^2 = c_1a_2\partial^3 + c_1a_2'\partial^2$$
$$d_1\partial b_2\partial^2 = d_1(b_2\partial + b_2')\partial^2 = d_1b_2\partial^3 + d_1b_2'\partial^2$$

order 2:

$$c_1a_2'\partial^2 \text{ (from above)} + c_1\partial a_1\partial + c_0a_2\partial^2 = c_1a_2'\partial^2 + c_1a_1\partial^2 + c_0a_2\partial^2 + c_1a_1'\partial$$

analogous for b and d

order 3:

$$c_1a_1'\partial \text{ (from above)} + c_1\partial a_0 + c_0a_1\partial = c_1a_1'\partial + c_1(a_0\partial + a_0') + c_0a_1\partial = c_1a_1'\partial + c_1a_0\partial + c_0a_1\partial + c_1a_0'$$

analogous for b and d

order 0:

$$c_1a_0' \text{ (from above)} + c_0a_0$$

analogous for b and d

So, finally,

$$\begin{pmatrix} c_1 & c_0 & d_1 & d_0 \end{pmatrix} \cdot \begin{pmatrix} a_2 & a_1 + a_2' & a_0 + a_1' & a_0' \\ 0 & a_2 & a_1 & a_0 \\ b_2 & b_1 + b_2' & b_0 + b_1' & b_0' \\ 0 & b_2 & b_1 & b_0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix} .$$

Observe, that the rows of this matrix consist of the coefficients of

$$\partial A,\ A,\ \partial B,\ B\ .$$

Comparing this to the example in [5], p.3, we see that after interchanging of rows this is the same matrix. □

Example 3.6. $\text{order}(A) = 2, \text{order}(B) = 3$

$$(c_2\partial + c_1\partial + c_0)(a_2\partial^2 + a_1\partial + a_0) \;+\; (d_1\partial + d_0)(b_3\partial^3 + b_2\partial^2 + b_1\partial + b_0)$$

order 4:

$$c_2\partial^2 a_2\partial^2 + d_1\partial b_3\partial^3 = 0$$
$$\underline{a_2c_2\partial^4} + 2a_2'c_2\partial^3 + a_2''c_2\partial^2 + \underline{b_3d_1\partial^4} + b_3'd_1\partial^3 = 0$$

order 3:

$$(2a_2'c_2\partial^3 + a_2''c_2\partial^2 \text{ from above}) + c_2\partial^2 a_1\partial + c_1\partial a_2\partial^2 + (b_3'd_1\partial^3 \text{ from above}) + d_1\partial b_2\partial^2 + d_0b_3\partial^3 = 0$$

$$\underline{2a_2'c_2\partial^3} + \underline{a_1c_2\partial^3} + \underline{a_2c_1\partial^3} + a_2''c_2\partial^2 + 2a_1'c_2\partial^2 + a_1''c_2\partial + \underline{b_3'd_1\partial^3} + \underline{b_2d_1\partial^3} + \underline{b_3d_0\partial^3} + b_2'd_1\partial^2 = 0$$

order 2:

$$\begin{aligned}(a_2''c_2\partial^2 + 2a_1'c_2\partial^2 + a_1''c_2\partial + a_2'c_1\partial^2 \text{ from above}) + c_2\partial^2 a_0 + c_1\partial a_1\partial + c_0a_2\partial^2 \\ +(b_2'd_1\partial^2 \text{ from above}) + d_1\partial b_1\partial + d_0b_2\partial^2 \quad = 0\end{aligned}$$

$$\begin{aligned}\underline{a_2''c_2\partial^2} + \underline{2a_1'c_2\partial^2} + a_1''c_2\partial + \underline{a_2'c_1\partial^2} + \underline{a_0c_2\partial^2} + 2a_0'c_2\partial + a_0''c_2 + \underline{a_1c_1\partial^2} + a_1'c_1\partial + \underline{a_2c_0\partial^2} \\ +\underline{b_2'd_1\partial^2} + \underline{b_1d_1\partial^2} + b_1'd_1\partial + \underline{b_2d_0\partial^2} \quad = 0\end{aligned}$$

order 1:

$$(a_1''c_2\partial+2a_0'c_2\partial+a_0''c_2+a_1'c_1\partial \text{ from above})+c_1\partial a_0+c_0a_1\partial+(b_1'd_1\partial \text{ from above})+d_1\partial b_0+d_0b_1\partial \ = 0$$

$$\underline{a_1''c_2\partial}+\underline{2a_0'c_2\partial}+\underline{a_1'c_1\partial}+\underline{a_0c_1\partial}+a_0'c_1+\underline{a_1c_0\partial}+a_0''c_2+\underline{b_1'd_1\partial}+\underline{b_0d_1\partial}+b_0'd_1+\underline{b_1d_0\partial} \ = 0$$

order 0:

$$(\underline{a_0'c_1}+\underline{a_0''c_2} \text{ from above})+\underline{a_0c_0}+(\underline{b_0'd_1} \text{ from above})+\underline{b_0d_0} \ = 0.$$

So, finally

$$\begin{pmatrix} c_2 & c_1 & c_0 & d_1 & d_0 \end{pmatrix}\cdot\begin{pmatrix} a_2 & a_1+2a_2' & a_0+2a_1'+a_2'' & 2a_0'+a_1'' & a_0'' \\ 0 & a_2 & a_1+a_2' & a_0+a_1' & a_0' \\ 0 & 0 & a_2 & a_1 & a_0 \\ b_3 & b_2+b_3' & b_1+b_2' & b_0+b_1' & b_0' \\ 0 & b_3 & b_2 & b_1 & b_0 \end{pmatrix} = \begin{pmatrix} 0 & \cdots & 0 \end{pmatrix} .$$

Observe, that the rows of this matrix consist of the coefficients of

$$\partial^2 A,\ \partial A,\ A, \partial B,\ B\ .$$

□

Theorem 3.7. *The linear map S in (8) corresponding to (7) is given by the matrix whose rows are $\partial^{n-1}A,\ldots,\partial A,\ A,\partial^{m-1}B,\ldots,\partial B,\ B$.*

Proof: Let $v=(c_{n-1},\ldots,c_0,d_{m-1},\ldots,d_0)$.
Consider an index i between 1 and n. If $c_{n-i}=1$, and all the other components of v are 0, then v is mapped by S to $\partial^{n-i}\cdot A+0\cdot B=\partial^{n-i}A$. So the i-th row of S has to consist of the coefficients of $\partial^{n-i}A$.
Consider an index j between 1 and m. If $d_{m-j}=1$, and all the other components of v are 0, then v is mapped by S to $0\cdot A+\partial^{m-j}\cdot B=\partial^{m-j}B$. So the $(n+j)$-th row of S has to consist of the coefficients of $\partial^{m-j}B$. □

Definition 3.8. *Let A,B be linear differential operators in $R[\partial]$ of $\operatorname{order}(A)=m, \operatorname{order}(B)=n$, with $m,n>0$.*
By $\partial\mathrm{syl}(A,B)$ we denote the (differential) Sylvester matrix*; i.e., the $(m+n)\times(m+n)$-matrix whose rows contain the coefficients of*

$$\partial^{n-1}A,\ldots,\partial A,\ A,\partial^{m-1}B,\ldots,\partial B,\ B\ .$$

The (differential Sylvester) resultant *of A and B, $\partial\mathrm{res}(A,B)$, is the determinant of $\partial\mathrm{syl}(A,B)$.* □

From Theorems 3.3 and 3.4 the following analogue of Propositions 2.2 and 2.3 is immediate.

Theorem 3.9. *Assume that K has characteristic 0 and that its field of constants is algebraically closed. Let A,B be linear differential operators over R of positive orders. Then the condition $\partial\mathrm{res}(A,B)=0$ is both necessary and sufficient for there to exist a common non-trivial root of A and B in an extension of K.*

We close this subsection by stating an analogue of Proposition 2.4.

Theorem 3.10. *Let $A,B\in R[\partial]$. The resultant of A and B is contained in (A,B), the ideal generated by A and B in $R[\partial]$. Moreover, $\partial\mathrm{res}(A,B)$ can be written as a linear combination $\partial\mathrm{res}(A,B)=CA+DB$, with $\operatorname{order}(C)<\operatorname{order}(B)$, and $\operatorname{order}(D)<\operatorname{order}(A)$.*

Proof: Let $S:=\partial\mathrm{syl}(A,B)$. Now proceed as in the proof of Proposition 2.4; only instead of multiplying the i-th column of S by x^{m+n-i}, multiply it by ∂^{m+n-i} from the right and add to the last column. This will result in a new matrix T, having the same determinant as S. The columns of T are the same as the corresponding columns of S, except for the last column, which consists of the operators

$$\partial^{n-1}A, \ldots, \partial A,\ A, \partial^{m-1}B, \ldots, \partial B,\ B\ .$$

Expanding the determinant of T w.r.t. its last column, we obtain operators C and D s.t.

$$\partial\mathrm{res}(A, B) = CA + DB,$$

and $\mathrm{order}(C) < \mathrm{order}(B)$, $\mathrm{order}(D) < \mathrm{order}(A)$. □

From Theorem 3.10 we readily obtain an alternative proof that $\partial\mathrm{res}(A, B) = 0$ is a necessary condition for the existence of a non-trivial common root of A and B in an extension of K. The details are left as an exercise for the reader.

3.2 Resultant of two linear homogeneous differential polynomials

The results for differential resultants which we have derived for linear differential operators can also be stated in terms of linear homogeneous differential polynomials. Such a treatment facilitates the generalization to the non-linear algebraic differential case.

Let (R, ∂) be a differential domain with quotient field K. Then elements of $R\{x\}$ can be interpreted as algebraic ordinary differential equations (AODEs). For instance, the differential polynomial

$$3\,x\,x^{(1)} + 2t\,x^{(2)} \ \in\ \mathbb{C}(t)\{x\}$$

corresponds to the AODE

$$3x(t)x'(t) + 2tx''(t) = 0\ .$$

The next proposition says that linear differential operators correspond to linear homogeneous differential polynomials in a natural way. Recall that $R_{LH}\{x\}$ denotes the left R-submodule of $R\{x\}$ comprising those elements of $R\{x\}$ which are linear and homogeneous.

Proposition 3.11. *$R[\partial]$ and $R_{LH}\{x\}$ are isomorphic as left R-modules. $K[\partial]$ and $K_{LH}\{x\}$ are isomorphic as left vector spaces over K.*

Proof: Define $\mathcal{P} : R[\partial] \to R_{LH}\{x\}$ as follows. Given $A = \sum_{i=0}^{m} a_i\partial^i$, let $\mathcal{P}(A) = f(x)$, where $f(x) = \sum_{i=0}^{m} a_i x^{(i)}$. ($\mathcal{P}$ stands for (linear homogeneous differential) polynomial.) Then we can easily verify that $\mathcal{P}$ is an isomorphism of left R-modules. The inverse of $\mathcal{P}$ is the mapping $\mathcal{L} : R_{LH\{x\}} \to R[\partial]$, with $\mathcal{L}(f(x)) = A$. ($\mathcal{L}$ stands for linear differential operator.) Note that $\mathcal{P}$ has a natural extension, also denoted by $\mathcal{P} : K[\partial] \to K_{LH}\{x\}$; and likewise for $\mathcal{L}$. The extended $\mathcal{P}$ is an isomorphism of vector spaces over K. □

Definition 3.12. *Let $f(x)$ and $g(x)$ be elements of $R_{LH}\{x\}$ of positive orders m and n, respectively. Then the* (differential) Sylvester matrix *of $f(x)$ and $g(x)$, denoted by $\partial\mathrm{syl}(f, g)$, is $\partial\mathrm{syl}(A, B)$, where $A = \mathcal{O}(f)$ and $B = \mathcal{O}(g)$. The* (differential Sylvester) resultant *of $f(x)$ and $g(x)$, denoted by $\partial\mathrm{res}(f, g)$, is $\partial\mathrm{res}(A, B)$.*

We may observe that the $m + n$ rows of $\partial\mathrm{syl}(f, g)$ contain the coefficients of

$$f^{(n-1)}(x), \ldots, f^{(1)}(x), f(x), g^{(m-1)}(x), \ldots, g^{(1)}(x), g(x).$$

The following analogue and slight reformulation of Theorem 3.9 is immediate.

Theorem 3.13. *Assume that K has characteristic 0 and that its field of constants is algebraically closed. Let $f(x), g(x)$ be linear homogeneous differential polynomials of positive orders over R. Then the condition $\partial\mathrm{res}(f, g) = 0$ is both necessary and sufficient for there to exist a common non-trivial solution of $f(x) = 0$ and $g(x) = 0$ in an extension of K.*

We have also an analogue and slight reformulation of Theorem 3.10:

Theorem 3.14. *Let $f(x), g(x) \in R_{LH}\{x\}$. Then $x\partial\text{res}(f,g)$ is contained in the differential ideal $[f,g]$.*

From the above theorem we readily obtain an alternative proof that $\partial\text{res}(f,g) = 0$ is a necessary condition for the existence of a non-trivial common solution of $f(x) = 0$ and $g(x) = 0$. The details are left to the reader.

4 The multipolynomial resultant

This section presents a relatively concise précis of classical material on elimination theory. The basic goal is to introduce the *multipolynomial resultant*, also known as *Macaulay's resultant*. For n generic homogeneous polynomials $F_1, \dots F_n$ in the n variables $x_1, \dots, x_n$, of positive (total) degrees d_i, we will see that there exists a multipolynomial resultant R, which is a polynomial in the indeterminate coefficients of the F_i, with the following property. If the coefficients of the F_i are assigned values from a field K, then the vanishing of R is necessary and sufficient for a nontrivial common zero of the F_i to exist in some extension of K. The theory of the multipolynomial resultant is much more involved than that of the Sylvester resultant. For this reason we shall provide only some of the highlights of the development, referring the reader to more comprehensive sources such as [8, 10, 15, 23, 26] for the full story. We will primarily follow the treatment of this topic in [26].

4.1 Resultant system of several univariate polynomials

We first need to describe how to construct a resultant system for several polynomials in a single variable. Let $f_1(x), \dots, f_r(x)$ be polynomials in a single variable x with indeterminate coefficients. Then there exists a system $\{r_1, \dots, r_h\}$ of integral polynomials in these coefficients with the property that if these coefficients are assigned values from a field K, the conditions $r_1 = 0, \dots, r_h = 0$ are necessary and sufficient in order that either the equations $f_1(x) = 0, \dots, f_r(x) = 0$ have a common solution in a suitable extension field or the formal leading coefficients of all the polynomials f_i vanish. Moreover each $r_j \in (f_1, \dots, f_r)$ (the classical algebraic ideal generated by the f_i).

Remarks.

1. The set $\{r_1, \dots, r_h\}$ is known as the *resultant system* of $f_1, \dots, f_r$.
2. In case $r = 1$, we may take the resultant system to be the set $\{0\}$.
3. In case $r = 2$, we may take the resultant system to be the set $\{\text{res}(f_1, f_2)\}$, where $\text{res}(f_1, f_2)$ denotes the Sylvester resultant, as in Section 2.
4. In case $r > 2$, if it is known in advance that the formal leading coefficient of f_1 does not vanish, put $\mathcal{R} = \text{res}(f_1, v_2 f_2 + \cdots + v_r f_r)$, where the v_i are new indeterminates. Then $\mathcal{R}$ may be expressed as $\mathcal{R} = \sum r_\alpha v_2^{\alpha_2} \cdots v_r^{\alpha_r}$, and we may take the resultant system to be the set $\{r_\alpha\}$.
5. Otherwise, the construction of the resultant system - still based upon the Sylvester resultant - is more involved, and we refer the reader to [26] for the details.

4.2 Solvability criteria for a system of homogeneous equations

In this subsection we investigate criteria for the solvability of a system of homogeneous polynomial equations in several variables. Let $F_1(x_1, \dots, x_n), \dots, F_r(x_1, \dots, x_n)$ be r homogeneous forms of positive (total) degrees d_i with indeterminate coefficients. Such polynomials always have the "trivial" zero $(0, \dots, 0)$. We shall see that there exists a resultant system $\{T_1, \dots, T_k\}$ for the F_i, with each T_j an integral polynomial in the coefficients of the F_i, such that for special values of the coefficients in K, the vanishing of all resultants T_j is necessary and sufficient for there to exist a nontrivial solution to the system $F_1 = 0, \ldots, F_r = 0$ in some extension of K.

We use the method of successive elimination of the variables, due to Kronecker and adapted by Kapferer. We begin this process by considering the forms $F_1, \ldots, F_r$ to be polynomials in x_n with coefficients depending on $x_1, \ldots, x_{n-1}$. Using the technique introduced in Subsection 4.1 (the resultant system for several univariate polynomials), we construct the resultant system $\{R_1, \ldots, R_h\}$ of $F_1, \ldots, F_r$. We show now that the existence of a nontrivial zero of the system $\{R_1, \ldots, R_h\}$ is necessary and sufficient for a nontrivial zero of $F_1, \ldots, F_r$ to exist.

There are two cases to consider. First, suppose that the coefficients of the terms in $F_1, \ldots, F_r$ consisting only of powers of x_n do not all vanish. In this case, by applying the properties of the resultant system, we see that every nontrivial zero $(\xi_1, \ldots, \xi_{n-1})$ of the R_j gives rise to at least one zero $(\xi_1, \ldots, \xi_{n-1}, \xi_n)$ of the F_i, which clearly cannot be trivial. Conversely, every nontrivial zero $(\xi_1, \ldots, \xi_{n-1}, \xi_n)$ of the F_i gives rise to a zero $(\xi_1, \ldots, \xi_{n-1})$ of the R_j, which also cannot be trivial since the vanishing of $\xi_1, \ldots, \xi_{n-1}$ would lead immediately to the vanishing of ξ_n. Second, suppose that the coefficients of the terms in $F_1, \ldots, F_r$ consisting only of powers of x_n all vanish. By Subsection 4.1 $R_1, \ldots, R_h$ vanish identically in this case. Hence the system $\{R_j = 0\}$ has a nontrivial zero, say $(1, 1, \ldots, 1)$. Moreover, in this case, the polynomials $F_1, \ldots, F_r$ have a nontrivial zero, namely, $(0, \ldots, 0, 1)$, since the terms with the highest power of x_n are all omitted. This proves our claim.

Since the F_i are homogeneous, and the construction of the R_j is based on the Sylvester resultant, $R_1, \ldots, R_h$ are homogeneous in $x_1, \ldots, x_{n-1}$. Hence the elimination process can be continued. After eliminating x_{n-1}, etc., and using simplified notation, we end up with $\{T_1 x_1^{\tau_1}, T_2 x_1^{\tau_2}, \ldots, T_k x_1^{\tau_k}\}$, where each T_j is a polynomial in the indeterminate coefficients of the F_i. The system $T_1 x_1^{\tau_1}, \ldots, T_k x_1^{\tau_k}$ has a nontrivial zero if and only if all the coefficients $T_1, \ldots, T_k$ vanish. The system $\{T_1, \ldots, T_k\}$ is termed a *resultant system* of the homogeneous forms $F_1, \ldots, F_r$ (meaning that the vanishing of all members of the system $\{T_j\}$ is necessary and sufficient for a common nontrivial zero of the F_i to exist).

By the properties of the resultants, each $T_j x_1^{\tau_j} \in (F_1, \ldots, F_r)$. Moreover, it is not difficult to see that the T_j are homogeneous in the coefficients of every individual form F_i. We summarize the result of this subsection as follows.

Theorem 4.1. *Let $F_1(x_1, \ldots, x_n), \ldots, F_r(x_1, \ldots, x_n)$ be r homogeneous forms of positive degrees with indeterminate coefficients. Then there exists a resultant system $\{T_1, \ldots, T_k\}$ for the F_i, with each T_j an integral polynomial in the coefficients of the F_i, such that for special values of the coefficients in a field K, the vanishing of all resultants T_j is necessary and sufficient for there to exist a nontrivial solution to the system $F_1 = 0, \ldots, F_r = 0$ in some extension of K. The T_j are homogeneous in the coefficients of every individual form F_i and satisfy $T_j x_1^{\tau_j} \in (F_1, \ldots, F_r)$, for suitable τ_j.*

4.3 Properties of inertia forms

The polynomials T_j which we obtained in the previous subsection are known as "inertia forms" for the given $F_1, \ldots, F_r$. This subsection lists some key properties of such polynomials. These properties will allow us, in the next subsection, to reach our goal for this section: namely, to show that for n generic homogeneous forms in n variables, there is a single resultant whose vanishing is necessary and sufficient for there to exist a nontrivial zero of the n given forms.

Definition 4.2. *Given r homogeneous polynomials $F_1, \ldots, F_r$ in $x_1, \ldots, x_n$, with indeterminate coefficients comprising a set A, an integral polynomial T in these indeterminates (that is, $T \in \mathbb{Z}[A]$) is called an* inertia form *for $F_1, \ldots, F_r$ if $x_i^\tau T \in (F_1, \ldots, F_r)$, for suitable i and τ.*

Remark 4.3. *In [26] this nomenclature is attributed to Hurwitz. (The reader may notice a slight misnomer. The definition above includes no requirement for an inertia form to be homogeneous, though in practice certain important inertia forms are typically homogeneous.)*

Every member of the resultant system for $F_1, \ldots, F_r$ derived in the previous subsection is an inertia form. It is shown in [26] that inertia forms may be defined equivalently by using any fixed variable (x_1, say) instead of x_i. The author proceeds to observe that the inertia forms comprise an ideal $\mathcal{I}$ of $\mathbb{Z}[A]$, and he shows further that $\mathcal{I}$ is a *prime* ideal of this ring.

Remark 4.4. *The concept of the ideal of inertia forms for $F_1, \ldots, F_r$ is closely related to the notion of the* projective elimination ideal *of $(F_1, \ldots, F_r)$ defined in Chapter 8 of [7]. (The latter notion seems slightly more general than the former.)*

It follows from these observations that we may take the ideal $\mathcal{I}$ of inertia forms to be a resultant system for the given $F_1, \ldots, F_r$. The reasoning is as follows. Suppose that (after assigning values to the coefficients) the system $F_1, \ldots, F_r$ has a nontrivial common zero $\xi = (\xi_1, \ldots, \xi_n)$. Then $\xi_j \neq 0$, for some j. Take an arbitrary inertia form $T \in \mathcal{I}$. Then $x_j^\tau T \in (F_1, \ldots, F_r)$ (since T may be defined equivalently with respect to the fixed variable x_j, as mentioned above.) Therefore, $x_j^\tau = A_1F_1 + \cdots + A_rF_r$, for suitable polynomials $A_1, \ldots, A_r$. Substituting ξ into this equation, we have $\xi_j^\tau T = 0$, which implies $T = 0$ since $\xi_j \neq 0$. Conversely, suppose that (after assigning values to the coefficients), all inertia forms vanish. Then in particular, all members of the resultant system derived in the previous subsection vanish. Consequently, the F_i have a common nontrivial zero. This establishes the assertion. Hence, any basis for the ideal $\mathcal{I}$ may also be used as a resultant system for the F_i.

In [26] the following result is proved.

Theorem 4.5. *If the number r of homogeneous forms F_i (of positive total degrees d_i) is less than the number of variables n, then there is no inertia form distinct from 0. If $r = n$, then every nonzero inertia form must include (that is, have positive degree in) the coefficient a of the term $x_n^{d_n}$ in F_n.*

The result just stated for $r = n$ can be usefully complemented.

Theorem 4.6. *If $r = n$, then there is a non-vanishing inertia form D_n. It is homogeneous in the coefficients of F_1, in those of F_2, etc., and has degree $d_1 d_2 \cdots d_{n-1}$ in the coefficients of F_n.*

Proof: Put $d = \sum_{i=1}^{n}(d_i - 1) + 1$. Arrange all power products in $x_1, \ldots, x_n$ of degree d as follows:

- *first*, all power products divisible by $x_1^{d_1}$;
- *second*, all power products divisible by $x_2^{d_2}$ but not $x_1^{d_1}$;
- *etc.*;
- *lastly*, all power products divisible by $x_n^{d_n}$ but not $x_1^{d_1}$, not $x_2^{d_2}$, etc..

Within each group use the lexicographic ordering of power products. This lists all power products of degree d. Designate the power products arranged in this way by

$$\mathcal{H}_1 x_1^{d_1}, \mathcal{H}_2 x_2^{d_2}, \ldots, \mathcal{H}_n x_n^{d_n}$$

where $\mathcal{H}_i$ denotes the appropriate list of power products of degree $d - d_i$. Note that the last group contains exactly $d_1 d_2 \cdots d_{n-1}$ power products.

Now consider the following system of polynomial equations:

$$\mathcal{H}_1 F_1 = 0,$$

$$\mathcal{H}_2 F_2 = 0,$$

$$\ldots$$

$$\mathcal{H}_n F_n = 0,$$

where the symbolic equation $\mathcal{H}_i F_i = 0$ represents the series of particular equations $x_1^{\nu_1} x_2^{\nu_2} \ldots x_n^{\nu_n} F_i = 0$, one for each power product $x_1^{\nu_1} x_2^{\nu_2} \ldots x_n^{\nu_n}$ contained in $\mathcal{H}_i$. Each particular equation should occupy a line. Since the total number of particular equations so listed equals the number of power products of degree d, the matrix M_n of coefficients of this system of particular equations is square, say $N \times N$. Its determinant D_n cannot vanish identically because by specializing $F_i = x_i^{d_i}$ we have $M_n = I$ and $D_n = 1$. Furthermore D_n is an inertia form. (To see this, use a variation of the proof of Proposition 2.4. Denote the power product of degree d at position i in the above listing by p_i. First mutliply the last column of M_n by $p_N = x_n^d$. Then, for $i = N-1$ downto 1, add the product of p_i and column i to the last column. Denote the matrix obtained in this way by M'_n. Then the determinant D'_n of M'_n satisfies $D'_n = x_n^d \times D_n$. Moreover the last column of M'_n contains the N polynomials $\mathcal{H}_1 F_1, \ldots, \mathcal{H}_n F_n$. Expanding D'_n down the last column we therefore see that $D'_n \in (F_1, \ldots, F_n)$. Hence $x_n^d D_n \in (F_1, \ldots, F_n)$.) By its construction D_n is clearly homogeneous in the coefficients of every individual form F_i, and has degree $d_1 d_2 \cdots d_{n-1}$ in the coeficients of F_n. This proves our result. □

Example 4.7. *Let $r = n = 3$ and use (x, y, z) for (x_1, x_2, x_3). Let*

$$F_1(x, y, z) = a_1 x + a_2 y + a_3 z,$$

$$F_2(x, y, z) = b_1 x + b_2 y + b_3 z,$$

$$F_3(x, y, z) = c_1 x^2 + c_2 xy + c_3 xz + c_4 y^2 + c_5 yz + c_6 z^2.$$

Then we have $d_1 = 1$, $d_2 = 1$, $d_3 = 2$. So $d = 2$. Arrange the power products of degree 2 as described in the proof of the theorem. We then have three groups designated as $\mathcal{H}_1 x, \mathcal{H}_2 y, \mathcal{H}_3 z^2$, where $\mathcal{H}_1 = \{x, y, z\}$, $\mathcal{H}_2 = \{y, z\}$ and $\mathcal{H}_3 = \{1\}$. The matrix M_3 is as follows:

$$\begin{pmatrix} a_1 & a_2 & a_3 & 0 & 0 & 0 \\ 0 & a_1 & 0 & a_2 & a_3 & 0 \\ 0 & 0 & a_1 & 0 & a_2 & a_3 \\ 0 & b_1 & 0 & b_2 & b_3 & 0 \\ 0 & 0 & b_1 & 0 & b_2 & b_3 \\ c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \end{pmatrix}$$

Observe that when $F_1 = x$, $F_2 = y$ and $F_3 = z^2$, $M_3 = I$ and $D_3 = 1$. So D_3 does not vanish identically. D_3 is an inertia form because $z^2 D_3 \in (F_1, F_2, F_3)$. D_3 contains the "principal term" $a_1^3 b_2^2 c_6$ (from the principal diagonal).

4.4 Resultant of n homogeneous forms in n variables

Let $F_1, \ldots, F_n$ be n generic homogeneous forms in $x_1, \ldots, x_n$ of positive total degrees $d_1, \ldots, d_n$. That is, every possible coefficient of each F_i is a distinct indeterminate, and the set of all such indeterminate coefficients is denoted by A. Let $\mathcal{I}$ denote the ideal of inertia forms for $F_1, \ldots, F_n$.

Theorem 4.8. *$\mathcal{I}$ is a nonzero principal ideal of $\mathbb{Z}[A]$: $\mathcal{I} = (R)$, for some $R \neq 0$.*

Proof: That $\mathcal{I}$ has a nonzero element follows from Theorem 4.6. Denote by a the coefficient of the term $x_n^{d_n}$ in F_n. Let P be a nonzero element of $\mathcal{I}$ of lowest degree in a. Then the degree in a of P is positive, by Theorem 4.5. Factorize P into irreducible factors in $\mathbb{Z}[A]$. Then at least one irreducible factor must belong to $\mathcal{I}$, since $\mathcal{I}$ is prime (mentioned previously). Any such irreducible factor must have the same degree in a as P by minimality of the degree in a of P. Since this degree is positive, there is exactly one such irreducible factor, and we denote this factor by R.

It remains to show that $\mathcal{I} = (R)$. Regard R as a polynomial in a, and let S and λ denote its leading coefficient and degree, respectively. Take any nonzero element $T \in \mathcal{I}$. Since the degree in a of T is at least λ, we can lower its degree in a by multiplying T by S and subtracting an appropriate multiple of R. We repeat this process (known as polynomial pseudodivision) until a polynomial is obtained whose degree is less than λ:

$$S^j T - QR = T'.$$

Clearly T' also belongs to $\mathcal{I}$ and its degree in a is less than λ. Therefore $T' = 0$, and so $S^j T$ is divisible by R. However R is irreducible and S is not divisible by R (since S is independent of a). Hence T is divisible by R. □

It follows from the theorem that R is uniquely dtermined up to sign. We call R the *(generic multipolynomial) resultant* of $F_1, \ldots, F_n$. The following fundamental property of R follows by remarks made in the previous subsection about $\mathcal{I}$.

Theorem 4.9. *The vanishing of R for particular $F_1, \ldots, F_n$ with coefficients in a field K is necessary and sufficient for the existence of a nontrivial zero of the system $F_1 = 0, \ldots, F_n = 0$ in some extension of K.*

In [26] some further properties of R are proved, which we summarize as follows.

Theorem 4.10. *R is homogeneous in the coefficients of F_1 of degree $\delta_1 = d_2 \cdots d_n$, in the coefficients of F_2 of degree $\delta_2 = d_1 d_3 \cdots d_n$, etc. R may be normalised to contain the "principal term" $+\alpha_1^{\delta_1} \cdots \alpha_n^{\delta_n}$, where α_i denotes the coefficient of the term $x_i^{d_i}$ in F_i. R is the greatest common divisor of the determinants $D_1, D_2, \ldots, D_n$, where for $i < n$, D_i is obtained by analogy with D_n (in the last subsection), by arranging $F_1, \ldots, F_n$ so that F_i occupies the last place.*

Practical aspects of computing the multipolynomial resultant are discussed in Chapter 3 (especially Section 4) of [8].

Example 4.11. *In the case of a system of n linear homogeneous forms F_i, the multipolynomial resultant of the F_i is the $n \times n$ determinant of the corresponding linear system $F_1 = 0, \ldots, F_n = 0$.*

Example 4.12. *If $n = 2$, the multipolynomial resultant of the homogeneous forms F_1 and F_2 is the Sylvester resultant of these two homogeneous polynomials (see remarks at the end of Subsection 2.1 concerning the Sylvester resultant of two homogeneous forms).*

Example 4.13. *Consider again Example 4.7 of the previous subsection. By expanding D_3, factorizing the resulting polynomial, and using the known degrees of the resultant (Theorem 4.10) we deduce that $D_3 = a_1 R$. (See [8], page 84, for an explicit expanded expression for R (with slightly different notation).*

Finally, suppose that $F_1, \ldots, F_n$ are particular homogeneous forms in $x_1, \ldots, x_n$ over some field K such that F_i has positive degree d_i. Then, using the generic resultant R, we may define the resultant $\mathrm{res}(F_1, \ldots, F_n)$ of these particular forms. Clearly $\mathrm{res}(F_1, \ldots, F_n) = 0$ if and only if the forms F_i have a common nontrivial zero over an extension of K.

4.5 Resultant of n non-homogeneous polynomials in $n-1$ variables

For a given non-homogeneous $f(x_1, \ldots, x_{n-1})$ over K of total degree d, we may write $f = H_d + H_{d-1} + \cdots + H_0$, where the H_j are homogeneous of degree j. Then H_d is known as the *leading form* of f. Recall that the *homogenization* $F(x_1, \ldots, x_n)$ of f is defined by $F = H_d + H_{d-1} x_n + \cdots + H_0 x_n^d$.

Let $f_1, \ldots, f_n$ be particular non-homogeneous polynomials in $x_1, \ldots, x_{n-1}$ over K of positive total degrees d_i, and with leading forms H_{i,d_i}. We put $\mathrm{res}(f_1, \ldots, f_n) = \mathrm{res}(F_1, \ldots, F_n)$, where F_i is the homogenization of f_i. We have:

Theorem 4.14. *The vanishing of $\mathrm{res}(f_1, \ldots, f_n)$ is necessary and sufficient for either the forms H_{i,d_i} to have a common nontrivial zero over an extension of K, or the polynomials f_i to have a common zero over an extension of K.*

A proof is found in [16] (see Theorem 2.4).

5 Resultant of two arbitrary differential polynomials

In this section we will review Carra'-Ferro's adaptation of the multipolynomial resultant to a pair of algebraic ordinary differential equations (AODEs) [2]. Such AODEs can be described by differential polynomials. We will deal with both homogeneous and non-homogeneous AODEs.

Suppose first we are given 2 homogeneous AODEs in the form of 2 homogeneous differential polynomial equations over a differential field K. Observe that a homogeneous AODE of positive order always has the solution 0. So we are interested in determining whether such a pair of homogeneous AODEs has a non-trivial common solution. Denote the given homogeneous AODEs by:

$$\begin{array}{ll} F(x) = 0, & \text{of order } m \ , \\ G(x) = 0, & \text{of order } n \ . \end{array} \tag{9}$$

So the differential polynomial $F(x) \in K\{x\}$ is of the form $F(x, x^{(1)}, \ldots, x^{(m)})$; and $G(x) \in K\{x\}$ is of analogous form.

The system (9) has the same solution set as the system

$$\begin{array}{ll} F^{(n-1)}(x) = \cdots = F^{(1)} = F(x) = 0, & n \text{ equations}, \\ G^{(m-1)}(x) = \cdots = G^{(1)} = G(x) = 0, & m \text{ equations}. \end{array} \tag{10}$$

This system (10) contains the variables $x, x^{(1)}, \ldots, x^{(m+n-1)}$. So it is a system of $m+n$ homogeneous equations in $m+n$ variables. Considered as a system of homogeneous algebraic equations (with the $x^{(i)}$ considered as unrelated indeterminates), it has a multipolynomial resultant $\mathrm{res}(F^{(n-1)}, \ldots, F, G^{(m-1)}, \ldots, G)$ (defined in Subsection 4.4) whose vanishing gives a necessary and sufficient condition for the existence of a non-trivial solution over an extension of K.

Definition 5.1. *For such homogeneous differential polynomials $F(x), G(x)$, we define the* (differential) resultant *$\partial\mathrm{res}(F, G)$ to be the multipolynomial resultant* $\mathrm{res}(F^{(n-1)}, \ldots, F, G^{(m-1)}, \ldots, G)$.

But, whereas a solution to the differential problem is also a solution to the algebraic problem, the converse is not true. So we do not expect the vanishing of this resultant to be a sufficient condition for the existence of a nontrivial solution to (9).

Example 5.2. *Condider Example 4 in [2], p.554.*

$$\begin{array}{rcll} F(x) & = & xx^{(1)} - x^2 & = 0 \ , \\ G(x) & = & xx^{(1)} & = 0 \ . \end{array}$$

The corresponding system (10) *would be the same in this case. Whereas the differential problem only has the trivial solution $x = x^{(1)} = 0$, the corresponding algebraic problem has also the non-trivial solutions $(0, a)$, for a in K.*

Indeed, in this case the differential resultant coincides with the Sylvester resultant: $\partial\mathrm{res}(F, G) = \mathrm{res}(F, G) = 0$. This reflects the fact that there are non-trivial algebraic solutions. But $x = 0$ does not lead to a non-trivial differential solution. □

The following theorem follows from Theorem 4.9.

Theorem 5.3. *For such homogeneous differential polynomials $F(x), G(x)$, the vanishing of $\partial\mathrm{res}(F, G)$ is a necessary condition for the existence of a non-trivial common solution of the system $F(x) = 0, G(x) = 0$ in an extension of K.*

Next we consider the more general case of a pair of non-homogeneous AODEs $f(x) = 0, g(x) = 0$ over K, of orders m and n, respectively. This system has the same solution set as the system

$$\begin{array}{ll} f^{(n)}(x) = \cdots = f^{(1)} = f(x) = 0, & n+1 \text{ equations}, \\ g^{(m)}(x) = \cdots = g^{(1)} = g(x) = 0, & m+1 \text{ equations}. \end{array} \tag{11}$$

This system contains the variables $x, x^{(1)}, \ldots, x^{(m+n)}$. So it is a system of $m+n+2$ non-homogeneous equations in $m+n+1$ variables. Considered as a system of non-homogeneous algebraic equations (with the $x^{(i)}$ considered as unrelated indeterminates), it has a multipolynomial resultant $\mathrm{res}(f^{(n)}, \ldots, f, g^{(m)}, \ldots, g)$ (defined in Subsection 4.5) whose vanishing gives a necessary condition for the existence of a common solution to the system in an extension of K.

Definition 5.4. *For such differential polynomials $f(x), g(x)$, we define the* (differential) resultant $\partial\mathrm{res}(f,g)$ *to be the multipolynomial resultant* $\mathrm{res}(f^{(n)},\dots,f,g^{(m)},\dots,g)$.

The following theorem follows from Theorem 4.14.

Theorem 5.5. *For such differential polynomials $f(x), g(x)$, the vanishing of $\partial\mathrm{res}(f,g)$ is a necessary condition for the existence of a common solution of the system $f(x)=0, g(x)=0$ in an extension of K.*

Example 5.6. *In the comprehensive resource [30] on differential equations there appears a short section entitled "Differential Resultants". The author briefly introduces the concept of differential resultants, claimed to be "... the analogue of [algebraic] resultants applied to differential systems". However no precise definition of the concept is given. Instead, the author simply considers an example of a system of two coupled algebraic ordinary differential equations (AODEs) for $x(t)$ and $z(t)$:*

$$\begin{cases} f(x,z) = 3xz + z - x' = 0, \\ g(x,z) = -z' + z^2 + x^2 + x = 0. \end{cases} \tag{12}$$

A single AODE involving only $z(t)$ is sought. (Our notation is slightly different from the author's. Note that x and z are regarded as differential indeterminates in the above system.) The author describes for this specific example how to derive a certain second order AODE for $z(t)$ only, but he does not give a general method in any sense. He suggests that the steps he follows are analogous in some sense to the steps done in constructing a Sylvester resultant in the case of two polynomial equations. It would appear, though, that the steps he follows for the given differential system (12) are more closely related to construction of a multipolynomial resultant. In other words, it would seem that the author has in mind a special case of the concept of differential resultant which we defined in this section (Definition 5.4).

Indeed, where $\mathbb{D}$ denotes the differential integral domain $\mathbb{C}(t)\{z\}$, the differential polynomials f and g occurring in (12) could be regarded as elements of the differential integral domain $\mathbb{D}\{x\}$. The first step of Zwillinger's process is to add the AODE

$$g'(x,z) = -z'' + 2zz' + 2xx' + x' = 0$$

to the given differential system (12), thereby obtaining a system of three AODEs. Next this expanded system is regarded as a system of three algebraic *polynomial equations, with each polynomial in the system belonging to $\mathbb{C}(t)[z,z',z''][x,x']$. Here z,z',z'',x,x' are considered to be unrelated algebraic (that is, nondifferential) indeterminates. That is, each polynomial is regarded as a polynomial in the variables x and x' whose coefficients lie in $\mathbb{C}(t)[z,z',z'']$.*

Zwillinger then constructs a certain 7×7 matrix M each of whose rows contains the coefficients of either f, g or g' so regarded: hence each entry of M is an element of $\mathbb{C}(t)[z,z',z'']$. The matrix M somewhat resembles a Macaulay matrix for f, g and g' with respect to x and x' (see below), but certain differences are apparent too. Finally the author computes the determinant of M, obtaining

$$(z'')^2 + (-16z' + 12z^2 - 3)zz'' + 64z^2(z')^2 + (23 - 96z^2)z^2z' + (36z^4 - 17z^2 + 2)z^2. \tag{13}$$

This is presumably what the author regards as the differential resultant of f and g with respect to the differential indeterminate x, though he does not explicitly name it as such. He seems to imply that the vanishing of this differential resultant for $z = \bar{z}(t)$ is a necessary condition for $(\bar{x}(t), \bar{z}(t)) \in F^2$ to be a solution of the given differential system (12), where F is a suitable differential extension field of $\mathbb{C}(t)$.

In [2] the author presents a clear and detailed treatment of the differential resultant of a system comprising two AODEs $f(x) = 0$ and $g(x) = 0$, where $f(x)$ and $g(x)$ are differential polynomials in the differential indeterminate x over some differential integral domain $\mathbb{D}$ of orders m and n, respectively. The author defines the differential resultant of f and g to be the multiploynomial resultant of the polynomials

$$f^{(n)}(x), f^{(n-1)}(x), \dots, f^{(0)}(x), g^{(m)}(x), g^{(m-1)}(x), \dots, g^{(0)}(x)$$

with respect to $x^{(m+n)}, x^{(m+n-1)}, \ldots, x^{(0)}$*, considered as unrelated algebraic indeterminates. This is consistent with our Definition 5.4. The author proves that the vanishing of the differential resultant of* f *and* g *is a necessary condition for* $x = \bar{x} \in F$ *to be a solution of the system* $f = g = 0$*, where* F *is a suitable differential field extension of (the quotient field of)* $\mathbb{D}$*. This confirms the intuition conveyed in [30], and is consistent with our Theorem 5.5.*

Regarding the computation of a multipolynomial resultant of such a system of $m+n+2$ *polynomials in* $m+n+1$ *variables, [2] offers two methods. The first expresses the resultant as the greatest common divisor of the determinants of certain matrices (which we like to call Macaulay matrices) involving the coefficients of the polynomials. This is the most classical computational definition of the resultant, and is also found in such references as [15], [26] and [8]. The second method, applicable in the generic case, expresses the resultant as the quotient of two such determinants. The second method is also found in [15] and [8].*

The system (12) is also treated in [2] (Examples 3 and 7). Like Zwillinger [30], Carra'-Ferro first adds the AODE

$$g'(x,z) = -z'' + 2zz' + 2xx' + x' = 0$$

to the given system, obtaining a system of three AODEs. Next (again like Zwillinger) this expanded system is regarded as a system of three algebraic polynomial equations, with each polynomial in the system belonging to $\mathbb{C}(t)[z, z', z''][x, x']$*. At this point, to compute the differential resultant, Carra'-Ferro uses the second definition of the multipolynomial resultant of the expanded system. It is observed that the value of the denominator determinant is 1. The numerator is the determinant of a* 10×10 *matrix as expected, but the matrix is not immediately recognizable as a Macaulay matrix. Nevertheless, Carra'-Ferro's answer (Example 7) agrees with that of Zwillinger.*

6 Further developments

A further interesting development concerning Example 5.6 was provided by T. Sturm [21]. Let us denote the differential resultant (13) of the system (12) by $D(z)$. The *separant* of $D(z)$, $\mathrm{sep}(D)$, is the partial derivative of $D(z)$, considered as a polynomial in z, z', z'', with respect to z''. Thus

$$\mathrm{sep}(D) = 2z'' - 16z'z + 12z^3 - 3z.$$

Using his software for differential elimination, Sturm found that

$$(z = 0) \vee (D(z) = 0 \wedge \mathrm{sep}(D) \neq 0)$$

is a necessary *and sufficient* condition on $\bar{z}(t)$ for the existence of a solution $(\bar{x}(t), \bar{z}(t)) \in F^2$ to the given differential system (12), where F is a suitable differential extension field of $\mathbb{C}(t)$. The question is thus raised as to the extent to which the vanishing of a differential resultant and the nonvanishing of its separant comes close to providing a sufficient condition for the existence of a common nontrivial solution to an arbitrary given coupled system of first order AODEs in a suitable differential extension. This question, amongst some others, is posed in the next section.

We now briefly summarize a further development on differential resultants which followed [2]. Carra'-Ferro herself [3] was the first person to try to extend the work of [2] by presenting Macaulay style formulas for a system $\mathcal{P}$ of n arbitrary ordinary differential polynomials in $n-1$ differential variables. The differential resultant of $\mathcal{P}$ defined by her is the multipolynomial resultant of a certain set of derivatives of the elements of $\mathcal{P}$. However Carra'-Ferro's construction does not take into consideration the sparsity of the differential polynomials in $\mathcal{P}$, and consequently her differential resultant vanishes in many cases, yielding no useful information. An important and contemporaneous advance in algebraic resultant theory was the definition of the *sparse* algebraic resultant [10, 22]. This concept stimulated the development of the theory of the *sparse* differential resultant. S. Rueda [18] provided sparse differential resultant formulas for the linear case. Such formulas for the nonlinear case appear in [27, 14, 19].

7 Research questions and directions

Let K be a differential field with derivation ∂ and x, y be differential indeterminates over K. Given algebraic polynomials P, Q in the polynomial ring $K[x, y]$, we consider the coupled, first order system

$$\begin{cases} x' = P(x, y), \\ y' = Q(x, y). \end{cases} \tag{14}$$

The system (14) is a system of two differential polynomials in the differential ring $K\{x, y\}$, which can be seen as differential polynomials in $D\{x\}$, with differential domain $D = K\{y\}$. Namely, put

$$\begin{cases} f_1(x) = x' - P(x, y), \\ f_2(x) = y' - Q(x, y) \end{cases} \tag{15}$$

and $\mathcal{P} = \{f_1, f_2\}$. Example 5.6 is a system of this kind.

Let $[f_1, f_2]$ be the differential ideal generated by f_1 and f_2 in $D\{x\} = K\{x, y\}$. Observe that the differential resultant $\partial\mathrm{res}(f_1, f_2)$ is a polynomial in $[f_1, f_2] \cap D$ and therefore in $D = K\{y\}$.

In this situation many questions arise and we list below some of them.

1. Under which conditions can we guarantee that $\partial\mathrm{res}(f_1, f_2) \neq 0$? Even for generic differential polynomials with the same monomial support, the differential resultant may vanish.

2. If $\partial\mathrm{res}(f_1, f_2) \neq 0$, is $\partial\mathrm{res}(f_1, f_2)$ the same as the differential resultant defined by Gao et al. in [9]? That is, do we have

$$[f_1, f_2] \cap D = \mathrm{sat}(\partial\mathrm{res}(f_1, f_2))?$$

   (In the above equation “sat” denotes saturation.)

3. If the system has a solution then $D = \partial\mathrm{res}(f_1, f_2) = 0$ (by Theorem 5.5). Can we find a direct, elementary and classical style proof that if $D = 0$ and the separant of D (defined in Section 6) is nonzero then the system has a non-trivial solution?

4. We can define the *sparse differential resultant* $\partial\mathrm{sres}(f_1, f_2)$ to be the sparse algebraic resultant of the set $\{f_1, f_2, \partial f_2\}$ (mentioned in Section 6). Under what conditions could we guarantee that $\partial\mathrm{sres}(f_1, f_2) \neq 0$? All the previous questions could be also applied to $\partial\mathrm{sres}(f_1, f_2)$.

Some potential research directions are listed as follows.

1. Differential resultant formulas for differential operators. A differential resultant formula for ordinary differential operators was studied by Chardin in [5] (cf. Section 3.1). A differential resultant formula for partial differential operators was defined by Carra'-Ferro in [4]. Differential elimination methods exist via noncommutative Gröbner bases. No formal definition of a differential resultant exists in the case of partial differential operators. One potential application of research on resultants of ordinary and partial differential operators is to integrability questions [20].

2. Differential resultant formulas for partial differential polynomials. The case of linear differential polynomials in one differential variable falls into the case of partial differential operators but the general situation is much more broad. No formal definition of a differential resultant exists in this case.

3. Differential resultant formulas for differential-difference operators or for Ore polynomials. No formal definition of a differential resultant exists in such cases.

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