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Summer Research Institute on q-Series

Symbolic Summation and (q-)Applications

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Some of the available summation tools:

- Abramov, S.A.: On the summation of rational functions. *Zh. vychisl. mat. Fiz.* **11**, 1071–1074 (1971)
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- Abramov, S.A.: Rational solutions of linear differential and difference equations with polynomial coefficients. *U.S.S.R. Comput. Math. Phys.* **29**(6), 7–12 (1989)
- Abramov, S.A., Petkovšek, M.: D'Alembertian solutions of linear differential and difference equations. In: J. von zur Gathen (ed.) *Proc. ISSAC'94*, pp. 169–174. ACM Press (1994)
- Abramov, S.A., Petkovšek, M.: Rational normal forms and minimal decompositions of hypergeometric terms. *J. Symbolic Comput.* **33**(5), 521–543 (2002)
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- Chen, S., Kauers, M.: Order-Degree Curves for Hypergeometric Creative Telescoping. In: J. van der Hoeven, M. van Hoeij (eds.) *Proceedings of ISSAC 2012*, pp. 122–129 (2012)
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- M. Kauers and P. Paule. *The concrete tetrahedron*. Texts and Monographs in Symbolic Computation. SpringerWienNewYork, Vienna, 2011. Symbolic sums, recurrence equations, generating functions, asymptotic estimates.



Some of the available summation tools:

⋮

- Koornwinder, T.H.: On Zeilberger's algorithm and its q -analogue. *J. Comp. Appl. Math.* **48**, 91–111 (1993)
- Koutschan, C.: Creative telescoping for holonomic functions. In: C. Schneider, J. Blümlein (eds.) *Computer Algebra in Quantum Field Theory: Integration, Summation and Special Functions*, Texts and Monographs in Symbolic Computation, pp. 171–194. Springer (2013). ArXiv:1307.4554 [cs.SC]
- Paule, P.: Greatest factorial factorization and symbolic summation. *J. Symbolic Comput.* **20**(3), 235–268 (1995)
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- Petkovšek, M.: Hypergeometric solutions of linear recurrences with polynomial coefficients. *J. Symbolic Comput.* **14**(2-3), 243–264 (1992)
- Petkovšek, M., Wilf, H.S., Zeilberger, D.: *$A = B$* . A. K. Peters, Wellesley, MA (1996)
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- Pirastu, R., Strehl, V.: Rational summation and Gosper-Petkovšek representation. *J. Symbolic Comput.* **20**(5-6), 617–635 (1995)
- Wegschaider, K., May 1997. Computer generated proofs of binomial multi-sum identities. Master's thesis, RISC, Johannes Kepler University.
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- Zeilberger, D.: The method of creative telescoping. *J. Symbolic Comput.* **11**, 195–204 (1991)

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- Paule, P., Riese, A.: A Mathematica q -analogue of Zeilberger's algorithm based on an algebraically motivated approach to q -hypergeometric telescoping. In: M. Ismail, M. Rahman (eds.) *Special Functions, q -Series and Related Topics*, vol. 14, pp. 179–210. AMS (1997)
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Here I will restrict to the setting of difference rings/fields.

A warm-up example: simplify

$$\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \left(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} \right. \\ \left. + \underbrace{\frac{j!k!(j+k+n)!(-S_1(j) + S_1(j+k) + S_1(j+n) - S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(j)} \right)$$

where

$$S_1(n) = \sum_{i=1}^n \frac{1}{i} \quad (= H_n)$$

Arose in the context of

I. Bierenbaum, J. Blümlein, and S. Klein, *Evaluating two-loop massive operator matrix elements with Mellin-Barnes integrals*. 2006

A warm-up example: simplify

$$\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \left(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} \right. \\ \left. + \underbrace{\frac{j!k!(j+k+n)!(-S_1(j) + S_1(j+k) + S_1(j+n) - S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(j)} \right)$$

FIND $g(j)$:

$$\boxed{f(j) = g(j+1) - g(j)}$$

A warm-up example: simplify

$$\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \left(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} \right. \\ \left. + \underbrace{\frac{j!k!(j+k+n)!(-S_1(j) + S_1(j+k) + S_1(j+n) - S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(j)} \right)$$

FIND $g(j)$:

$$\boxed{f(j) = g(j+1) - g(j)}$$

↑ summation package Sigma

$$g(j) = \frac{(j+k+1)(j+n+1)j!k!(j+k+n)!(S_1(j) - S_1(j+k) - S_1(j+n) + S_1(j+k+n))}{kn(j+k+1)!(j+n+1)!(k+n+1)!}$$

A warm-up example: simplify

$$\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \left(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} \right. \\ \left. + \underbrace{\frac{j!k!(j+k+n)!(-S_1(j) + S_1(j+k) + S_1(j+n) - S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(j)} \right)$$

FIND $g(j)$:

$$\boxed{f(j) = g(j+1) - g(j)}$$

Summing the telescoping equation over j from 0 to a gives

$$\sum_{j=0}^a f(j) = g(a+1) - g(0)$$

A warm-up example: simplify

$$\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \left(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} \right. \\ \left. + \underbrace{\frac{j!k!(j+k+n)!(-S_1(j) + S_1(j+k) + S_1(j+n) - S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(j)} \right)$$

FIND $g(j)$:

$$\boxed{f(j) = g(j+1) - g(j)}$$

Summing the telescoping equation over j from 0 to a gives

$$\sum_{j=0}^a f(j) = g(a+1) - g(0) \\ = \frac{(a+1)!(k-1)!(a+k+n+1)!(S_1(a) - S_1(a+k) - S_1(a+n) + S_1(a+k+n))}{n(a+k+1)!(a+n+1)!(k+n+1)!} \\ + \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)n!} + \frac{(2a+k+n+2)a!k!(a+k+n)!}{(a+k+1)(a+n+1)(a+k+1)!(a+n+1)!(k+n+1)!}}_{a \rightarrow \infty}$$

 $a \rightarrow \infty$

A warm-up example: simplify

$$\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \left(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} \right. \\ \left. + \underbrace{\frac{j!k!(j+k+n)!(-S_1(j) + S_1(j+k) + S_1(j+n) - S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(j)} \right)$$

$$\sum_{j=0}^{\infty} f(j) = \frac{1}{n!} \frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}$$

A warm-up example: simplify

$$\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \left(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} \right. \\ \left. + \frac{j!k!(j+k+n)!(-S_1(j) + S_1(j+k) + S_1(j+n) - S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!} \right) \\ \underbrace{\hspace{15em}}_{f(j)}$$

$$\sum_{k=1}^{\infty} \sum_{j=0}^{\infty} f(j) = \frac{1}{n!} \sum_{k=1}^{\infty} \frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}$$

Telescoping

GIVEN

$$A(n) := \sum_{k=1}^a \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=: f(k)}.$$

FIND $g(k)$:

$$\boxed{g(k+1) - g(k)} = \boxed{f(k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

no solution 😞

Zeilberger's creative telescoping paradigm

GIVEN

$$A(n) := \sum_{k=1}^n \frac{S_1(k) + S_1(n) - S_1(k+n)}{\underbrace{kn(k+n+1)}_{=: f(n,k)}}.$$

FIND $g(n, k)$

$$\boxed{g(n, k+1) - g(n, k)} = \boxed{f(n, k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.no solution 

Zeilberger's creative telescoping paradigm

GIVEN

$$A(n) := \sum_{k=1}^a \frac{S_1(k) + S_1(n) - S_1(k+n)}{\underbrace{kn(k+n+1)}_{=: f(n,k)}}.$$

FIND $g(n, k)$ and $c_0(n), c_1(n)$:

$$\boxed{g(n, k+1) - g(n, k)} = \boxed{c_0(n)f(n, k) + c_1(n)f(n+1, k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

Zeilberger's creative telescoping paradigm

GIVEN

$$A(n) := \sum_{k=1}^a \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=: f(n, k)}.$$

FIND $g(n, k)$ and $c_0(n), c_1(n)$:

$$\boxed{g(n, k+1) - g(n, k)} = \boxed{c_0(n)f(n, k) + c_1(n)f(n+1, k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

Sigma computes: $c_0(n) = -n, c_1(n) = (n+2)$ and

$$g(n, k) = \frac{kS_1(k) + (-n-1)S_1(n) - kS_1(k+n) - 2}{(k+n+1)(n+1)^2}$$

Zeilberger's creative telescoping paradigm

GIVEN

$$A(n) := \sum_{k=1}^a \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=: f(n, k)}.$$

FIND $g(n, k)$ and $c_0(n), c_1(n)$:

$$\boxed{g(n, k+1) - g(n, k)} = \boxed{c_0(n)f(n, k) + c_1(n)f(n+1, k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

Summing this equation over k from 1 to a gives:

$$\boxed{g(n, a+1) - g(n, 1)} = \boxed{\sum_{k=1}^a [c_0(n)f(n, k) + c_1(n)f(n+1, k)]}$$

Zeilberger's creative telescoping paradigm

GIVEN

$$A(n) := \sum_{k=1}^a \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=: f(n, k)}.$$

FIND $g(n, k)$ and $c_0(n), c_1(n)$:

$$\boxed{g(n, k+1) - g(n, k)} = \boxed{c_0(n)f(n, k) + c_1(n)f(n+1, k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

Summing this equation over k from 1 to a gives:

$$\boxed{g(n, a+1) - g(n, 1)} = \boxed{\sum_{k=1}^a c_0(n) f(n, k) + \sum_{k=1}^a c_1(n) f(n+1, k)}$$

Zeilberger's creative telescoping paradigm

GIVEN

$$A(n) := \sum_{k=1}^a \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=: f(n, k)}.$$

FIND $g(n, k)$ and $c_0(n), c_1(n)$:

$$\boxed{g(n, k+1) - g(n, k)} = \boxed{c_0(n)f(n, k) + c_1(n)f(n+1, k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

Summing this equation over k from 1 to a gives:

$$\boxed{g(n, a+1) - g(n, 1)} = \boxed{c_0(n) \sum_{k=1}^a f(n, k) + c_1(n) \sum_{k=1}^a f(n+1, k)}$$

Zeilberger's creative telescoping paradigm

GIVEN

$$A(n) := \sum_{k=1}^a \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=: f(n, k)}.$$

FIND $g(n, k)$ and $c_0(n), c_1(n)$:

$$\boxed{g(n, k+1) - g(n, k)} = \boxed{c_0(n)f(n, k) + c_1(n)f(n+1, k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

Summing this equation over k from 1 to a gives:

$$\boxed{g(n, a+1) - g(n, 1)} = \boxed{c_0(n)A(n) + c_1(n)A(n+1)}$$

Zeilberger's creative telescoping paradigm

GIVEN

$$A(n) := \sum_{k=1}^a \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=: f(n, k)}.$$

FIND $g(n, k)$ and $c_0(n), c_1(n)$:

$$\boxed{g(n, k+1) - g(n, k)} = \boxed{c_0(n)f(n, k) + c_1(n)f(n+1, k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

Summing this equation over k from 1 to a gives:

$$\begin{aligned} \boxed{g(n, a+1) - g(n, 1)} &= \boxed{c_0(n)A(n) + c_1(n)A(n+1)} \\ \parallel & \qquad \qquad \qquad \parallel \\ \frac{(a+1)(S_1(a)+S_1(n)-S_1(a+n))}{(n+1)^2(a+n+2)} & \qquad \qquad \qquad - nA(n) + (2+n)A(n+1) \\ + \frac{a(a+1)}{(n+1)^3(a+n+1)(a+n+2)} & \qquad \qquad \qquad \end{aligned}$$

$$(n+2)\mathbf{A}(n+1) - n\mathbf{A}(n) = \frac{(n+1)S_1(n) + 1}{(n+1)^3}$$

recurrence finder

$$A(n) = \sum_{k=1}^{\infty} \frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}$$

$$(n+2)\mathbf{A}(n+1) - n\mathbf{A}(n) = \frac{(n+1)S_1(n) + 1}{(n+1)^3}$$

recurrence solver

$$A(n) = \sum_{k=1}^{\infty} \frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}$$

\in

$$\left\{ c \times \frac{1}{n(n+1)} + \frac{S_1(n)^2 + S_2(n)}{2n(n+1)} \mid c \in \mathbb{R} \right\}$$

where

$$S_1(n) = \sum_{i=1}^n \frac{1}{i}$$

$$S_2(n) = \sum_{i=1}^n \frac{1}{i^2}$$

$$(n+2)\mathbf{A}(n+1) - n\mathbf{A}(n) = \frac{(n+1)S_1(n) + 1}{(n+1)^3}$$

Summation package Sigma

(based on difference field/ring algorithms/theory

see, e.g., Karr 1981, Bronstein 2000, Schneider 2001/2004/2005a-c/2007/2008/2010a-c)

$$A(n) = \sum_{k=1}^{\infty} \frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}$$

$$= 0 \times \frac{1}{n(n+1)} + \frac{S_1(n)^2 + S_2(n)}{2n(n+1)}$$

where

$$S_1(n) = \sum_{i=1}^n \frac{1}{i}$$

$$S_2(n) = \sum_{i=1}^n \frac{1}{i^2}$$

A warm-up example: simplify

$$\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \left(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} \right. \\ \left. + \underbrace{\frac{j!k!(j+k+n)!(-S_1(j) + S_1(j+k) + S_1(j+n) - S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(j)} \right)$$

$$\sum_{k=1}^{\infty} \sum_{j=0}^{\infty} f(j) = \frac{1}{n!} \sum_{k=1}^{\infty} \frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)} \\ = \frac{1}{n!} \frac{S_1(n)^2 + S_2(n)}{2n(n+1)}$$

where

$$S_1(n) = \sum_{i=1}^n \frac{1}{i} \qquad S_2(n) = \sum_{i=1}^n \frac{1}{i^2}$$

A warm-up example: simplify

$$\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \left(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} \right. \\ \left. + \underbrace{\frac{j!k!(j+k+n)!(-S_1(j) + S_1(j+k) + S_1(j+n) - S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(n,k,j)} \right)$$

$$\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} f(n,k,j) = \frac{S_1(n)^2 + 3S_2(n)}{2n(n+1)!}$$

where

$$S_1(n) = \sum_{i=1}^n \frac{1}{i} \qquad S_2(n) = \sum_{i=1}^n \frac{1}{i^2}$$

Toolbox 1: Indefinite summation

Toolbox 2: Definite summation

Toolbox 3: Special function algorithms

Toolbox 1: Indefinite summation

Telescoping

GIVEN $f(k) = S_1(k)$.

FIND $g(k)$:

$$f(k) = g(k + 1) - g(k)$$

for all $1 \leq k \leq n$ and $n \geq 0$.

Telescoping

GIVEN $f(k) = S_1(k)$.

FIND $g(k)$:

$$f(k) = g(k+1) - g(k)$$

for all $1 \leq k \leq n$ and $n \geq 0$.

Sigma compute

$$g(k) = (S_1(k) - 1)k.$$

Telescoping

GIVEN $f(k) = S_1(k)$.

FIND $g(k)$:

$$f(k) = g(k+1) - g(k)$$

for all $1 \leq k \leq n$ and $n \geq 0$.

Summing this equation over k from 1 to n gives

$$\sum_{k=1}^n S_1(k) = g(n+1) - g(1)$$

$$= (S_1(n+1) - 1)(n+1).$$

Telescoping in the given difference ring

FIND a closed form for

$$\sum_{k=1}^n S_1(k).$$

A difference ring for the [summand](#)

Consider a ring

$$\mathbb{A}$$

with the automorphism $\sigma : \mathbb{A} \rightarrow \mathbb{A}$ defined by

Telescoping in the given difference ring

FIND a closed form for

$$\sum_{k=1}^n S_1(k).$$

A difference ring for the **summand**

Consider a ring

$$\mathbb{A} := \mathbb{Q}$$

with the automorphism $\sigma : \mathbb{A} \rightarrow \mathbb{A}$ defined by

$$\sigma(c) = c \quad \forall c \in \mathbb{Q},$$

Telescoping in the given difference ring

FIND a closed form for

$$\sum_{k=1}^n S_1(k).$$

A difference ring for the **summand**

Consider a ring

$$\mathbb{A} := \mathbb{Q}(k)$$

with the automorphism $\sigma : \mathbb{A} \rightarrow \mathbb{A}$ defined by

$$\sigma(c) = c \quad \forall c \in \mathbb{Q},$$

$$\sigma(k) = k + 1,$$

$$S k = k + 1,$$

Telescoping in the given difference ring

FIND a closed form for

$$\sum_{k=1}^n S_1(k).$$

A difference ring for the **summand**

Consider a ring

$$\mathbb{A} := \mathbb{Q}(k)[h]$$

with the automorphism $\sigma : \mathbb{A} \rightarrow \mathbb{A}$ defined by

$$\sigma(c) = c \quad \forall c \in \mathbb{Q},$$

$$\sigma(k) = k + 1,$$

$$\sigma(h) = h + \frac{1}{k+1},$$

$$\mathcal{S}k = k + 1,$$

$$\mathcal{S}S_1(k) = S_1(k) + \frac{1}{k+1}.$$

Telescoping in the given difference ring

FIND $g \in \mathbb{A}$:

$$\sigma(g) - g = h.$$

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$$g(k + 1) - g(k) = S_1(k)$$

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Hence,

$$(S_1(n + 1) - 1)(n + 1) = \sum_{k=1}^n S_1(k).$$

Toolbox 1: Indefinite summation – the basic tactic

(inspired by Karr's algorithm, 1981)

CONSTRUCT a difference ring (\mathbb{A}, σ) :

- ▶ a ring (containing \mathbb{Q})

$$\mathbb{A} := \mathbb{K}$$

- ▶ with an automorphism where $\sigma(c) = c$ for all $c \in \mathbb{K}$ and where

CONSTRUCT a difference ring (\mathbb{A}, σ) :

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CONSTRUCT a difference ring (\mathbb{A}, σ) :

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$$\mathbb{A} := \mathbb{K}(k)(p_1)$$

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$$S_k! = (k+1)k! \quad \leftrightarrow \quad \sigma(p_1) = (k+1)p_1$$

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$$\text{(q-)hyper-geometric} \quad \leftrightarrow \quad \sigma(p_1) = a_1 p_1 \quad a_1 \in \mathbb{K}(k)^*$$

CONSTRUCT a difference ring (\mathbb{A}, σ) :

- ▶ a ring (containing \mathbb{Q})

$$\mathbb{A} := \mathbb{K}(k)(p_1)(p_2)$$

- ▶ with an automorphism where $\sigma(c) = c$ for all $c \in \mathbb{K}$ and where

$$\sigma(k) = k + 1$$

$$\begin{array}{l} \text{(q-)hyper-} \\ \text{geometric} \end{array} \quad \leftrightarrow \quad \begin{array}{l} \sigma(p_1) = a_1 p_1 \\ \sigma(p_2) = a_2 p_2 \end{array} \quad \begin{array}{l} a_1 \in \mathbb{K}(k)^* \\ a_2 \in \mathbb{K}(k)(p_1)^* \end{array}$$

CONSTRUCT a difference ring (\mathbb{A}, σ) :

- ▶ a ring (containing \mathbb{Q})

$$\mathbb{A} := \mathbb{K}(k)(p_1)(p_2) \dots (p_e)$$

- ▶ with an automorphism where $\sigma(c) = c$ for all $c \in \mathbb{K}$ and where

$$\sigma(k) = k + 1$$

$$\begin{array}{l}
 \text{(q-)hyper-} \\
 \text{geometric}
 \end{array}
 \leftrightarrow
 \begin{array}{l}
 \sigma(p_1) = a_1 p_1 \\
 \sigma(p_2) = a_2 p_2 \\
 \vdots \\
 \sigma(p_e) = a_e p_e
 \end{array}
 \begin{array}{l}
 a_1 \in \mathbb{K}(k)^* \\
 a_2 \in \mathbb{K}(k)(p_1)^* \\
 \vdots \\
 a_e \in \mathbb{K}(k)(p_1, \dots, p_{e-1})^*
 \end{array}$$

CONSTRUCT a difference ring (\mathbb{A}, σ) :

- ▶ a ring (containing \mathbb{Q})

$$\mathbb{A} := \mathbb{K}(k)(p_1)(p_2) \dots (p_e)[x]$$

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$$(-1)^k \quad \leftrightarrow \quad \begin{array}{l} \sigma(\mathbf{x}) = -\mathbf{x} \\ \mathbf{x}^2 = \mathbf{1} \end{array} \quad \begin{array}{l} a_e \in \mathbb{K}(k)(p_1, \dots, p_{e-1})^* \\ \mathbf{x}^2 = \mathbf{1} \end{array}$$

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$$\begin{array}{lll}
 \text{(q-)hyper-} & \leftrightarrow & \sigma(p_1) = a_1 p_1 \quad a_1 \in \mathbb{K}(k)^* \\
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 & & \vdots \\
 & & \sigma(p_e) = a_e p_e \quad a_e \in \mathbb{K}(k)(p_1, \dots, p_{e-1})^* \\
 (-1)^k & \leftrightarrow & \sigma(\mathbf{x}) = -\mathbf{x} \quad \mathbf{x}^2 = \mathbf{1}
 \end{array}$$

$$SS_1(k) = S_1(k) + \frac{1}{k+1} \quad \leftrightarrow \quad \sigma(s_1) = s_1 + \frac{1}{k+1}$$

CONSTRUCT a difference ring (\mathbb{A}, σ) :

- ▶ a ring (containing \mathbb{Q})

$$\mathbb{A} := \mathbb{K}(k)(p_1)(p_2) \dots (p_e)[x][s_1]$$

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$$\mathbb{A} := \mathbb{K}(k)(p_1)(p_2) \dots (p_e)[x][s_1][s_2]$$

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CONSTRUCT a $R\Pi\Sigma^*$ -ring (\mathbb{A}, σ) : (Karr81, ..., CS16, CS17)

- ▶ a ring (containing \mathbb{Q})

$$\mathbb{A} := \mathbb{K}(k)(p_1)(p_2) \dots (p_e)[x][s_1][s_2][s_3] \dots$$

- ▶ with an automorphism where $\sigma(c) = c$ for all $c \in \mathbb{K}$ and where

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such that $\text{const}_\sigma \mathbb{A} = \{c \in \mathbb{A} \mid \sigma(c) = c\} = \mathbb{K}$. ■

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$$\sigma(p_e) = a_e p_e \quad a_e \in \mathbb{K}(k)(p_1, \dots, p_{e-1})^*$$

$(-1)^k$

sum

GIVEN $f \in \mathbb{A}$;

FIND, in case of existence, a $g \in \mathbb{A}$ such that

$$\sigma(g) - g = f.$$

$$\sigma(s_3) = s_3 + f_3 \quad f_3 \in \mathbb{K}(k)(p_1, \dots, p_e)[x][s_1][s_2]$$

such that $\text{const}_\sigma \mathbb{A} = \{c \in \mathbb{A} \mid \sigma(c) = c\} = \mathbb{K}$. ■

Telescoping in the given difference ring

FIND a closed form for

$$\sum_{k=1}^n S_1(k).$$

A $R\Pi\Sigma^*$ -ring for the **summand**

$$\text{const}_\sigma \mathbb{A} = \mathbb{Q}$$

Consider a ring

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with the automorphism $\sigma : \mathbb{A} \rightarrow \mathbb{A}$ defined by

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We compute

$$g = (h - 1)k \in \mathbb{A}.$$

This gives

$$g(k+1) - g(k) = S_1(k)$$

with

$$g(k) = (S_1(k) - 1)k.$$

Hence,

$$(S_1(n+1) - 1)(n+1) = \sum_{k=1}^n S_1(k).$$

Toolbox 1: Improved indefinite summation

– symbolic simplification

For algorithmic details see:

- ▶ CS. Symbolic summation with single-nested sum extensions. In J. Gutierrez, editor, *Proc. ISSAC'04*, pages 282–289. ACM Press, 2004.
- ▶ CS. Product representations in $\Pi\Sigma$ -fields. *Ann. Comb.*, 9(1):75–99, 2005.
- ▶ CS. Simplifying Sums in $\Pi\Sigma$ -Extensions. *J. Algebra Appl.*, 6(3):415–441, 2007.
- ▶ CS. A refined difference field theory for symbolic summation. *J. Symbolic Comput.*, 43(9):611–644, 2008. [arXiv:0808.2543v1].
- ▶ S.A. Abramov, M. Petkovšek. Polynomial ring automorphisms, rational (w, σ) -canonical forms, and the assignment problem. *J. Symbolic Comput.*, 45(6): 684–708, 2010.
- ▶ CS, A Symbolic Summation Approach to Find Optimal Nested Sum Representations. In: A. Carey, D. Ellwood, S. Paycha, S. Rosenberg (eds.) *Motives, Quantum Field Theory, and Pseudodifferential Operators*, Clay Mathematics Proceedings, vol. 12, pp. 285–308. Amer. Math. Soc (2010). ArXiv:0808.2543
- ▶ CS, Parameterized Telescoping Proves Algebraic Independence of Sums. *Ann. Comb.* 14(4), 533–552 (2010). [arXiv:0808.2596]
- ▶ CS. Structural Theorems for Symbolic Summation. *Appl. Algebra Engrg. Comm. Comput.*, 21(1):1–32, 2010.
- ▶ CS. Fast Algorithms for Refined Parameterized Telescoping in Difference Fields. To appear in *Computer Algebra and Polynomials*, Lecture Notes in Computer Science (LNCS), Springer, 2014. arXiv:1307.7887 [cs.SC].

For special cases see:

- ▶ S.A. Abramov. On the summation of rational functions. *Zh. vychisl. mat. Fiz.*, 11: 1071-1074, 1971.
- ▶ P. Paule. Greatest factorial factorization and symbolic summation, *J. Symbolic Comput.*, 20(3): 235-268, 1995.

The basic difference ring approach

GIVEN a $R\Pi\Sigma^*$ -ring/ $\Pi\Sigma^*$ -field (\mathbb{A}, σ) with $f \in \mathbb{A}$.

FIND $g \in \mathbb{A}$:

$$\sigma(g) - g = f.$$

A symbolic summation approach

1. FIND an appropriate $R\Pi\Sigma^*$ -ring/ $\Pi\Sigma^*$ -field (\mathbb{A}, σ) with $f \in \mathbb{A}$.

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A symbolic summation approach

1. FIND an appropriate $R\Pi\Sigma^*$ -ring/ $\Pi\Sigma^*$ -field (\mathbb{A}, σ) with $f \in \mathbb{A}$.

2. FIND an appropriate extension $\mathbb{E} > \mathbb{A}$ with $g \in \mathbb{E}$:

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A symbolic summation approach

1. FIND an **appropriate** $R\Pi\Sigma^*$ -ring/ $\Pi\Sigma^*$ -field (\mathbb{A}, σ) with $f \in \mathbb{A}$.

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$$\sigma(g) - g = f.$$

appropriate = degrees in denominators minimal

Example:

$$\sum_{k=1}^a \left(\frac{-2+k}{10(1+k^2)} + \frac{(1-4k-2k^2)S_1(k)}{10(1+k^2)(2+2k+k^2)} + \frac{(1-4k-2k^2)S_3(k)}{5(1+k^2)(2+2k+k^2)} \right)$$

=?

A symbolic summation approach

1. FIND an appropriate $R\Pi\Sigma^*$ -ring/ $\Pi\Sigma^*$ -field (\mathbb{A}, σ) with $f \in \mathbb{A}$.

2. FIND an appropriate extension $\mathbb{E} > \mathbb{A}$ with $g \in \mathbb{E}$:

$$\sigma(g) - g = f.$$

appropriate = degrees in denominators minimal

Example:

$$\begin{aligned} \sum_{k=1}^a \left(\frac{-2+k}{10(1+k^2)} + \frac{(1-4k-2k^2)S_1(k)}{10(1+k^2)(2+2k+k^2)} + \frac{(1-4k-2k^2)S_3(k)}{5(1+k^2)(2+2k+k^2)} \right) \\ = \frac{a^2+4a+5}{10(a^2+2a+2)}S_1(a) - \frac{(a-1)(a+1)}{5(a^2+2a+2)}S_3(a) - \frac{2}{5} \sum_{k=1}^a \frac{1}{k^2} \end{aligned}$$

A symbolic summation approach

1. FIND an appropriate $R\Pi\Sigma^*$ -ring/ $\Pi\Sigma^*$ -field (\mathbb{A}, σ) with $f \in \mathbb{A}$.

2. FIND an appropriate extension $\mathbb{E} > \mathbb{A}$ with $g \in \mathbb{E}$:

$$\sigma(g) - g = f.$$

appropriate = sum representations with optimal nesting depth

Example:

$$\sum_{k=1}^n \frac{\sum_{j=1}^k \frac{\sum_{i=1}^j 1}{i}}{j} = ?$$

A symbolic summation approach

1. FIND an appropriate $R\Pi\Sigma^*$ -ring/ $\Pi\Sigma^*$ -field (\mathbb{A}, σ) with $f \in \mathbb{A}$.

2. FIND an appropriate extension $\mathbb{E} > \mathbb{A}$ with $g \in \mathbb{E}$:

$$\sigma(g) - g = f.$$

appropriate = sum representations with optimal nesting depth

Example:

$$\sum_{k=1}^n \frac{\sum_{j=1}^k \frac{\sum_{i=1}^j \frac{1}{i}}{j}}{k} = \frac{1}{6} \left(\sum_{i=1}^n \frac{1}{i} \right)^3 + \frac{1}{2} \left(\sum_{i=1}^n \frac{1}{i^2} \right) \left(\sum_{i=1}^n \frac{1}{i} \right) + \frac{1}{3} \sum_{i=1}^n \frac{1}{i^3}$$

depth 3

depth 1

Simplification of nested product-sum expressions

$A(k)$: nested product-sum expression (sums/products not in the denominator)

↓ `SigmaReduce[A,k]`

$B(k)$: nested product-sum expression (sums/products not in the denominator)

► such that

$$A(\lambda) = B(\lambda) \quad \text{for almost all } \lambda \in \mathbb{N}$$

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- ▶ such that

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- ▶ such that all the sums in $B(k)$ are **simplified** as above
- ▶ and such that the arising sums in $B(k)$ are **algebraically independent** (i.e., they do not satisfy any polynomial relation)

Toolbox 2: Definite summation

$$(n+2)\mathbf{A}(n+1) - n\mathbf{A}(n) = \frac{(n+1)S_1(n) + 1}{(n+1)^3}$$

Summation package Sigma

(based on difference field/ring algorithms/theory

see, e.g., Karr 1981, Bronstein 2000, Schneider 2001/2004/2005a-c/2007/2008/2010a-c)

$$A(n) = \sum_{k=1}^{\infty} \frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}$$

$$= 0 \times \frac{1}{n(n+1)} + \frac{S_1(n)^2 + S_2(n)}{2n(n+1)}$$

where

$$S_1(n) = \sum_{i=1}^n \frac{1}{i}$$

$$S_2(n) = \sum_{i=1}^n \frac{1}{i^2}$$

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$A(n) = \sum_{k=1}^n f(n, k);$$

$f(n, k)$: indefinite nested product-sum in k ;
 n : extra parameter

FIND a recurrence for $A(n)$

Zeilberger's creative telescoping paradigm

GIVEN

$$A(n) := \sum_{k=1}^n \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=: f(n,k)}.$$

FIND $g(n, k)$ and $c_0(n), c_1(n)$:

$$\boxed{g(n, k+1) - g(n, k)} = \boxed{c_0(n)f(n, k) + c_1(n)f(n+1, k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

Sigma computes: $c_0(n) = -n, c_1(n) = (n+2)$ and

$$g(n, k) = \frac{kS_1(k) + (-n-1)S_1(n) - kS_1(k+n) - 2}{(k+n+1)(n+1)^2}$$

Zeilberger's creative telescoping paradigm

GIVEN

$$A(n) := \sum_{k=1}^a \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=: f(n, k)}.$$

FIND $g(n, k)$ and $c_0(n), c_1(n)$:

$$\boxed{g(n, k+1) - g(n, k)} = \boxed{c_0(n)f(n, k) + c_1(n)f(n+1, k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.Summing this equation over k from 1 to a gives:

$$\begin{aligned} \boxed{g(n, a+1) - g(n, 1)} &= \boxed{c_0(n)A(n) + c_1(n)A(n+1)} \\ \parallel & \qquad \qquad \qquad \parallel \\ \frac{(a+1)(S_1(a) + S_1(n) - S_1(a+n))}{(n+1)^2(a+n+2)} & \qquad \qquad \qquad - nA(n) + (2+n)A(n+1) \\ + \frac{a(a+1)}{(n+1)^3(a+n+1)(a+n+2)} & \qquad \qquad \qquad \end{aligned}$$

Zeilberger's creative telescoping paradigm

GIVEN

$$A(n) := \sum_{k=1}^a \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=: f(n, k)}.$$

FIND $g(n, k)$ and $c_0(n), c_1(n)$:

$$\boxed{g(n, k+1) - g(n, k)} = \boxed{c_0(n)f(n, k) + c_1(n)f(n+1, k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.Summing this equation over k from 1 to a gives:

$$\boxed{g(n, a+1) - g(n, 1)} = \boxed{c_0(n)A(n) + c_1(n)A(n+1)}$$

$$\lim_{a \rightarrow \infty} \left\| \frac{(n+1)S_1(n) + 1}{(n+1)^3} \right\| \quad \left\| -nA(n) + (2+n)A(n+1) \right\|$$

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$A(n) = \sum_{k=1}^n f(n, k);$$

$f(n, k)$: indefinite nested product-sum in k ;
 n : extra parameter

FIND a recurrence for $A(n)$

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2. Recurrence solving

GIVEN a recurrence

$a_0(n), \dots, a_d(n), h(n)$:
 indefinite nested product-sum expressions in n .

$$a_0(n)A(n) + \dots + a_d(n)A(n+d) = h(n);$$

FIND all solutions expressible by indefinite nested products/sums in n .
 (d'Alembertian solutions)

(Abramov/Bronstein/Petkovšek/CS, in preparation)

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Note: the sum solutions are highly nested
 (possibly with denominators of high degrees)

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GIVEN a definite sum

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FIND all solutions expressible by indefinite nested products/sums in n .
 (d'Alembertian solutions)

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3. Simplify the solutions (using difference ring/field theory) s.t.

- ▶ the sums are algebraically independent;
- ▶ the sums are flattened;
- ▶ the sums can be given in terms of special functions.

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

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FIND all solutions expressible by indefinite nested products/sums in n .
 (d'Alembertian solutions)

(Abramov/Bronstein/Petkovšek/CS, in preparation)

4. Find a "closed form"

$A(n)$ = combined solutions in terms of indefinite nested sums in n .

In[1]:= << **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= **mySum** = $\sum_{k=1}^A \frac{S_1[k] + S_1[n] - S_1[k + n]}{kn(k + n + 1)}$;

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$$\sum_{k=1}^A \frac{S_1[k] + S_1[n] - S_1[k+n]}{kn(k+n+1)};$$

Compute a recurrence

In[3]:= **rec** = **GenerateRecurrence**[**mySum**, **n**][[1]]Out[3]=
$$-n\text{SUM}[n] + (1+n)(2+n)\text{SUM}[n+1] == \frac{(a+1)(S_1[a]+S_1[n]-S_1[a+n])}{(n+1)^2(a+n+2)n!} + \frac{a(a+1)}{(n+1)^3(a+n+1)(a+n+2)n!}$$
In[4]:= **rec** = **LimitRec**[**rec**, **SUM**[**n**], {**n**}, **A**]Out[4]=
$$-n\text{SUM}[n] + (1+n)(2+n)\text{SUM}[n+1] == \frac{(n+1)S_1[n] + 1}{(n+1)^3}$$

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In[3]:= rec = GenerateRecurrence[mySum, n][[1]]

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In[4]:= rec = LimitRec[rec, SUM[n], {n}, A]

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Solve a recurrence

In[5]:= recSol = SolveRecurrence[rec, SUM[n], IndefiniteSummation → False]

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Solve a recurrence

In[5]:= recSol = SolveRecurrence[rec, SUM[n], IndefiniteSummation → False]

Out[5]=
$$\left\{ \left\{ 0, \frac{1}{n(n+1)} \right\}, \left\{ 1, \frac{\sum_{i=1}^n \frac{S_1[i]}{i}}{n(n+1)} \right\} \right\}$$

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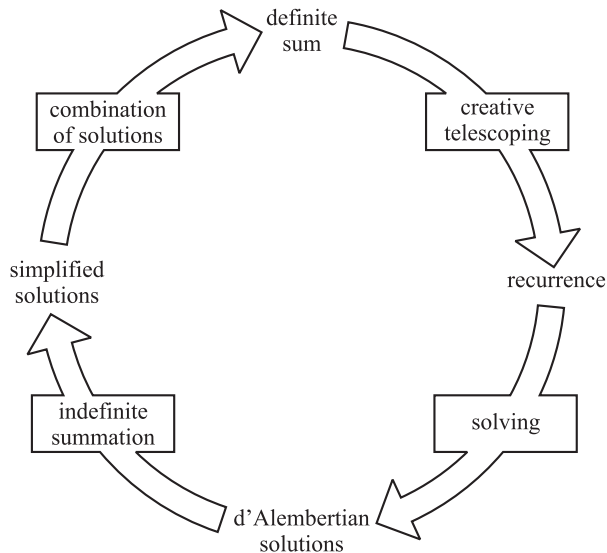
Out[5]=
$$\left\{ \left\{ 0, \frac{1}{n(n+1)} \right\}, \left\{ 1, \frac{S_1[n]^2 + \sum_{i=1}^n \frac{1}{i^2}}{2n(n+1)} \right\} \right\}$$

Combine the solutions

In[6]:= FindLinearCombination[recSol, {1, {1/2}}, n, 2]

Out[6]=
$$\frac{S_1[n]^2 + \sum_{i=1}^n \frac{1}{i^2}}{2n(n+1)}$$

Sigma's summation spiral



Toolbox 3: Special function algorithms

Computer algebra and special functions:

Harmonic sums (Borwein, Hoffman, Broadhurst, Vermaseren, Remiddi, Blümlein, . . .)

$$\sum_{i=1}^n \frac{1}{i^2} \sum_{j=1}^i \frac{1}{j}$$

Computer algebra and special functions:

Harmonic sums (Borwein, Hoffman, Broadhurst, Vermaseren, Remiddi, Blümlein, . . .)

$$\boxed{\sum_{i=1}^n \frac{1}{i^2} \sum_{j=1}^i \frac{1}{j}}$$

Integral representation:

$$= \int_0^1 \frac{x^n - 1}{1 - x} \left(\int_0^x \frac{\int_0^y \frac{1}{1-z} dz}{y} dy - \zeta_2 \right) dx, \quad \zeta_z := \sum_{i=1}^{\infty} 1/i^z$$

Computer algebra and special functions:

Harmonic sums (Borwein, Hoffman, Broadhurst, Vermaseren, Remm, Blümlein, ...)

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Asymptotic expansion:

$$= \left(\frac{1}{30n^5} - \frac{1}{6n^3} + \frac{1}{2n^2} - \frac{1}{n} \right) \ln(n) \\ - \frac{1}{100n^5} - \frac{1}{6n^4} + \frac{13}{36n^3} - \frac{1}{4n^2} - \frac{1}{n} + 2\zeta_3 + O\left(\frac{\ln(n)}{n^6}\right).$$

limit computations

numerical evaluation

► Generalized algorithms for generalized harmonic sums

$$\begin{aligned}
 \sum_{k=1}^N \frac{2^k \sum_{i=1}^k \frac{2^{-i} \sum_{j=1}^i \frac{S_1(j)}{j}}{i}}{k} &= -\frac{21\zeta_2^2}{20} \frac{1}{N} + \frac{1}{8N^2} + \frac{295}{216N^3} - \frac{1115}{96N^4} + O(N^{-5}) \\
 &+ \left(\frac{1}{2N} - \frac{3}{4N^2} + \frac{19}{12N^3} - \frac{5}{N^4} + O(N^{-5}) \right) \zeta_2 \\
 &+ 2^N \left(\frac{3}{2N} + \frac{3}{2N^2} + \frac{9}{2N^3} + \frac{39}{2N^4} + O(N^{-5}) \right) \zeta_3 \\
 &+ \left(\frac{1}{N} + \frac{3}{4N^2} - \frac{157}{36N^3} + \frac{19}{N^4} + O(N^{-5}) \right) (\log(N) + \gamma) \\
 &+ \left(\frac{1}{2N} - \frac{3}{4N^2} + \frac{19}{12N^3} - \frac{5}{N^4} + O(N^{-5}) \right) (\log(N) + \gamma)^2
 \end{aligned}$$

[Ablinger, Blümlein, CS, J. Math. Phys. 54, 2013, arXiv:1302.0378 [math-ph]]

► Generalized algorithms for cyclotomic harmonic sums

$$\begin{aligned}
 \sum_{k=1}^N \frac{\sum_{j=1}^k \frac{1}{1+2i}}{j^2} &= \left(-3 + \frac{35\zeta_3}{16}\right)\zeta_2 - \frac{31\zeta_5}{8} \\
 &+ \frac{1}{N} - \frac{33}{32N^2} + \frac{17}{16N^3} - \frac{4795}{4608N^4} + O(N^{-5}) \\
 &+ \log(2)\left(6\zeta_2 - \frac{1}{N} + \frac{9}{8N^2} - \frac{7}{6N^3} + \frac{209}{192N^4} + O(N^{-5})\right) \\
 &+ \left(-\frac{7}{4} - \frac{7}{16N} + \frac{7}{16N^2} - \frac{77}{192N^3} + \frac{21}{64N^4} + O(N^{-5})\right)\zeta_3 \\
 &+ \left(\frac{1}{16N^2} - \frac{1}{8N^3} + \frac{65}{384N^4} + O(N^{-5})\right)(\log(N) + \gamma)
 \end{aligned}$$

[Ablinger, Blümlein, CS, J. Math. Phys. 52, 2011, arXiv:1302.0378 [math-ph]]

► Generalized algorithms for nested binomial sums

$$\sum_{j=1}^N \frac{4^j S_1(j-1)}{\binom{2j}{j} j^2} = 7\zeta_3 + \sqrt{\pi}\sqrt{N} \left\{ \left[-\frac{2}{N} + \frac{5}{12N^2} - \frac{21}{320N^3} - \frac{223}{10752N^4} + \frac{671}{49152N^5} \right. \right. \\ + \frac{11635}{1441792N^6} - \frac{1196757}{136314880N^7} - \frac{376193}{50331648N^8} + \frac{201980317}{18253611008N^9} \\ \left. \left. + O(N^{-10}) \right] \ln(\bar{N}) - \frac{4}{N} + \frac{5}{18N^2} - \frac{263}{2400N^3} + \frac{579}{12544N^4} + \frac{10123}{1105920N^5} \right. \\ \left. - \frac{1705445}{71368704N^6} - \frac{27135463}{11164188672N^7} + \frac{197432563}{7927234560N^8} + \frac{405757489}{775778467840N^9} \right. \\ \left. + O(N^{-10}) \right\}$$

Ablinger, Blümlein, CS, ACAT 2013, arXiv:1310.5645 [math-ph]

Ablinger, Blümlein, Raab, CS, J. Math. Phys. 55, 2014. arXiv:1407.1822 [hep-th]

The full machinery:

Toolbox 1 + Toolbox 2 + Toolbox 3

The full machinery:

In[1]:= << **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= << **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[3]:= << **EvaluateMultiSums.m**

EvaluateMultiSums by Carsten Schneider © RISC-Linz

In[4]:= **EvaluateMultiSum**[

$$\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \left(\frac{j!k!(j+k+n)! (-S_1(j) + S_1(j+k) + S_1(j+n) - S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!} \right. \\ \left. + \frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} \right), \{n\}, \{1\}$$

The full machinery:

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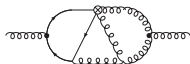
$$\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \left(\frac{j!k!(j+k+n)! (-S_1(j) + S_1(j+k) + S_1(j+n) - S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!} \right. \\ \left. + \frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} \right), \{n\}, \{1\}$$

Out[4]= $\frac{S_1(n)^2 + 3S_2(n)}{2n(n+1)!}$

Example 1: Feynman integrals

joint work with J. Ablinger, A. Behring, J. Blümlein, A. Hasselhuhn,
A. de Freitas, C. Raab, M. Round, Kay Schönwald, F. Wissbrock
(RISC-DESY)

Evaluation of Feynman Integrals

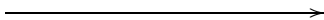


Behavior of particles

Evaluation of Feynman Integrals



Behavior of particles



$$\int \Phi(n, \epsilon, x) dx$$

Feynman integrals

Evaluation of Feynman Integrals



Behavior of particles



$$\int \Phi(n, \epsilon, x) dx$$

Feynman integrals

DESY
(J. Blümlein)

$$\sum f(n, \epsilon, k)$$

complicated
multi-sums

Evaluation of Feynman Integrals



Behavior of particles



$$\int \Phi(n, \epsilon, x) dx$$

Feynman integrals

DESY
(J. Blümlein)



$$\sum f(n, \epsilon, k)$$

complicated
multi-sums

expression in
special functions



RISC
(Sigma-package)

Evaluation of Feynman Integrals



Behavior of particles



$$\int \Phi(n, \epsilon, x) dx$$

Feynman integrals



LHC at CERN

DESY
(J. Blümlein)



$$\sum f(n, \epsilon, k)$$

complicated
multi-sums

applicable

expression in
special functions



RISC

(Sigma-package)

A warm-up example: simplify

$$\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \left(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} \right. \\ \left. + \frac{j!k!(j+k+n)!(-S_1(j) + S_1(j+k) + S_1(j+n) - S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!} \right)$$

$$\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} f(n, k, j) = \frac{S_1(n)^2 + 3S_2(n)}{2n(n+1)!}$$

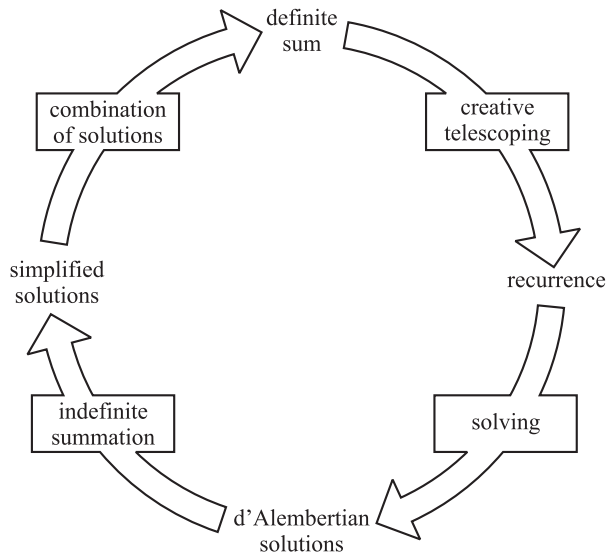
where

$$S_1(n) = \sum_{i=1}^n \frac{1}{i} \quad S_2(n) = \sum_{i=1}^n \frac{1}{i^2}$$

Arose in the context of

I. Bierenbaum, J. Blümlein, and S. Klein, *Evaluating two-loop massive operator matrix elements with Mellin-Barnes integrals*. 2006

Recall: Sigma's summation spiral



$$\sum_{j=0}^{n-2} \sum_{r=0}^{j+1} \sum_{s=0}^{n-j+r-2} \frac{(-1)^{r+s} \binom{j+1}{r} \binom{-j+n+r-2}{s} (-j+n-2)! r!}{(n-s)(s+1)(-j+n+r)!}$$

Simple sum

$$\sum_{j=0}^{n-2} \sum_{r=0}^{j+1} \sum_{s=0}^{n-j+r-2} \frac{(-1)^{r+s} \binom{j+1}{r} \binom{-j+n+r-2}{s} (-j+n-2)! r!}{(n-s)(s+1)(-j+n+r)!}$$

||

$$\sum_{j=0}^{n-2} \sum_{r=0}^{j+1} \boxed{\sum_{s=0}^{n-j+r-2} \frac{(-1)^{r+s} \binom{j+1}{r} \binom{-j+n+r-2}{s} (-j+n-2)! r!}{(n-s)(s+1)(-j+n+r)!}}$$

$$\sum_{j=0}^{n-2} \sum_{r=0}^{j+1} \sum_{s=0}^{n-j+r-2} \frac{(-1)^{r+s} \binom{j+1}{r} \binom{-j+n+r-2}{s} (-j+n-2)! r!}{(n-s)(s+1)(-j+n+r)!}$$

||

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||

$$\left(\binom{j+1}{r} \left(\frac{(-1)^r (-j+n-2)! r!}{(n+1)(-j+n+r-1)(-j+n+r)!} + \frac{(-1)^{n+r} (j+1)! (-j+n-2)! (-j+n-1)_r r!}{(n-1)n(n+1)(-j+n+r)! (-j-1)_r (2-n)_j} \right) \right)$$

$$\sum_{j=0}^{n-2} \sum_{r=0}^{j+1} \sum_{s=0}^{n-j+r-2} \frac{(-1)^{r+s} \binom{j+1}{r} \binom{-j+n+r-2}{s} (-j+n-2)! r!}{(n-s)(s+1)(-j+n+r)!}$$

$$\sum_{j=0}^{n-2} \left(\sum_{r=0}^{j+1} \binom{j+1}{r} \left(\frac{(-1)^r (-j+n-2)! r!}{(n+1)(-j+n+r-1)(-j+n+r)!} + \frac{(-1)^{n+r} (j+1)! (-j+n-2)! (-j+n-1)_r r!}{(n-1)n(n+1)(-j+n+r)! (-j-1)_r (2-n)_j} \right) \right)$$

$$\sum_{j=0}^{n-2} \sum_{r=0}^{j+1} \sum_{s=0}^{n-j+r-2} \frac{(-1)^{r+s} \binom{j+1}{r} \binom{-j+n+r-2}{s} (-j+n-2)! r!}{(n-s)(s+1)(-j+n+r)!}$$

$$\sum_{j=0}^{n-2} \left(\sum_{r=0}^{j+1} \binom{j+1}{r} \left(\frac{(-1)^r (-j+n-2)! r!}{(n+1)(-j+n+r-1)(-j+n+r)!} + \frac{(-1)^{n+r} (j+1)! (-j+n-2)! (-j+n-1)_r r!}{(n-1)n(n+1)(-j+n+r)! (-j-1)_r (2-n)_j} \right) \right)$$

$$\left(\frac{n^2 - n + 1}{(n-1)^2 n^2 (n+1)(2-n)_j} + \frac{\sum_{i=1}^j \frac{(2-n)_i}{(-i+n-1)^2 (i+1)!}}{(n+1)(2-n)_j} + \frac{(-1)^{j+n} (-j-2)(-j+n-2)!}{(j-n+1)(n+1)^2 n!} \right) (j+1)! - \frac{1}{(n+1)^2 (-j+n-1)}$$

$$\sum_{j=0}^{n-2} \sum_{r=0}^{j+1} \sum_{s=0}^{n-j+r-2} \frac{(-1)^{r+s} \binom{j+1}{r} \binom{-j+n+r-2}{s} (-j+n-2)! r!}{(n-s)(s+1)(-j+n+r)!}$$

||

$$\sum_{j=0}^{n-2} \left(\left(\frac{n^2 - n + 1}{(n-1)^2 n^2 (n+1)(2-n)_j} + \frac{\sum_{i=1}^j \frac{(2-n)_i}{(-i+n-1)^2 (i+1)!}}{(n+1)(2-n)_j} + \frac{(-1)^{j+n} (-j-2)(-j+n-2)!}{(j-n+1)(n+1)^2 n!} \right) (j+1)! - \frac{1}{(n+1)^2 (-j+n-1)} \right)$$

$$\sum_{j=0}^{n-2} \sum_{r=0}^{j+1} \sum_{s=0}^{n-j+r-2} \frac{(-1)^{r+s} \binom{j+1}{r} \binom{-j+n+r-2}{s} (-j+n-2)! r!}{(n-s)(s+1)(-j+n+r)!}$$

||

$$\sum_{j=0}^{n-2} \left(\left(\frac{n^2 - n + 1}{(n-1)^2 n^2 (n+1)(2-n)_j} + \frac{\sum_{i=1}^j \frac{(2-n)_i}{(-i+n-1)^2 (i+1)!}}{(n+1)(2-n)_j} + \frac{(-1)^{j+n} (-j-2)(-j+n-2)!}{(j-n+1)(n+1)^2 n!} \right) (j+1)! - \frac{1}{(n+1)^2 (-j+n-1)} \right)$$

||

$$\frac{-n^2 - n - 1}{n^2 (n+1)^3} + \frac{(-1)^n (n^2 + n + 1)}{n^2 (n+1)^3} - \frac{2S_{-2}(n)}{n+1} + \frac{S_1(n)}{(n+1)^2} + \frac{S_2(n)}{-n-1}$$

Note: $S_a(n) = \sum_{i=1}^n \frac{\text{sign}(a)^i}{i^{|a|}}$, $a \in \mathbb{Z} \setminus \{0\}$.

In[9]:= << **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[10]:= << **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[11]:= << **EvaluateMultiSums.m**

EvaluateMultiSums by Carsten Schneider © RISC-Linz

In[9]:= << **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[10]:= << **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[11]:= << **EvaluateMultiSums.m**

EvaluateMultiSums by Carsten Schneider © RISC-Linz

$$\text{In[12]:= mySum} = \sum_{j=0}^{n-2} \sum_{r=0}^{j+1} \sum_{s=0}^{n-j+r-2} \frac{(-1)^{r+s} \binom{j+1}{r} \binom{-j+n+r-2}{s} (-j+n-2)! r!}{(n-s)(s+1)(-j+n+r)!};$$

In[13]:= **EvaluateMultiSum**[mySum, {}, {n}, {1}]

In[9]:= << **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[10]:= << **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[11]:= << **EvaluateMultiSums.m**

EvaluateMultiSums by Carsten Schneider © RISC-Linz

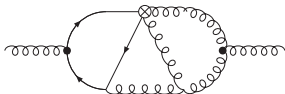
$$\text{In[12]:= mySum} = \sum_{j=0}^{n-2} \sum_{r=0}^{j+1} \sum_{s=0}^{n-j+r-2} \frac{(-1)^{r+s} \binom{j+1}{r} \binom{-j+n+r-2}{s} (-j+n-2)! r!}{(n-s)(s+1)(-j+n+r)!};$$

In[13]:= **EvaluateMultiSum**[mySum, {}, {n}, {1}]

$$\text{Out[13]=} \frac{-n^2 - n - 1}{n^2(n+1)^3} + \frac{(-1)^n (n^2 + n + 1)}{n^2(n+1)^3} - \frac{2S[-2, n]}{n+1} + \frac{S[1, n]}{(n+1)^2} + \frac{S[2, n]}{-n-1}$$



$$= F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + F_{-1}(n)\varepsilon^{-1} + \boxed{F_0(n)} + \dots$$



$$= F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + F_{-1}(n)\varepsilon^{-1} + \boxed{F_0(n)} + \dots$$

Simplify

||

$$\sum_{j=0}^{n-3} \sum_{k=0}^j \sum_{l=0}^k \sum_{q=0}^{-j+n-3-l+n-q-3} \sum_{s=1}^{-l+n-q-3} \sum_{r=0}^{-l+n-q-s-3} (-1)^{-j+k-l+n-q-3} \times$$

$$\times \frac{\binom{j+1}{k+1} \binom{k}{l} \binom{n-1}{j+2} \binom{-j+n-3}{q} \binom{-l+n-q-3}{s} \binom{-l+n-q-s-3}{r} r! (-l+n-q-r-s-3)! (s-1)!}{(-l+n-q-2)! (-j+n-1) (n-q-r-s-2) (q+s+1)}$$

$$\left[4S_1(-j+n-1) - 4S_1(-j+n-2) - 2S_1(k) \right.$$

$$\left. - (S_1(-l+n-q-2) + S_1(-l+n-q-r-s-3) - 2S_1(r+s)) \right.$$

$$\left. + 2S_1(s-1) - 2S_1(r+s) \right] + \mathbf{3 \text{ further 6-fold sums}}$$

$$\boxed{F_0(N)} =$$

$$\begin{aligned} & \frac{7}{12}S_1(N)^4 + \frac{(17N+5)S_1(N)^3}{3N(N+1)} + \left(\frac{35N^2-2N-5}{2N^2(N+1)^2} + \frac{13S_2(N)}{2} + \frac{5(-1)^N}{2N^2} \right) S_1(N)^2 \\ & + \left(-\frac{4(13N+5)}{N^2(N+1)^2} + \left(\frac{4(-1)^N(2N+1)}{N(N+1)} - \frac{13}{N} \right) S_2(N) + \left(\frac{29}{3} - (-1)^N \right) S_3(N) \right. \\ & + \left(2 + 2(-1)^N \right) S_{2,1}(N) - 28S_{-2,1}(N) + \frac{20(-1)^N}{N^2(N+1)} S_1(N) + \left(\frac{3}{4} + (-1)^N \right) S_2(N)^2 \\ & - 2(-1)^N S_{-2}(N)^2 + S_{-3}(N) \left(\frac{2(3N-5)}{N(N+1)} + (26 + 4(-1)^N) S_1(N) + \frac{4(-1)^N}{N+1} \right) \\ & + \left(\frac{(-1)^N(5-3N)}{2N^2(N+1)} - \frac{5}{2N^2} \right) S_2(N) + S_{-2}(N) (10S_1(N)^2 + \left(\frac{8(-1)^N(2N+1)}{N(N+1)} \right. \\ & + \left. \frac{4(3N-1)}{N(N+1)} \right) S_1(N) + \frac{8(-1)^N(3N+1)}{N(N+1)^2} + (-22 + 6(-1)^N) S_2(N) - \frac{16}{N(N+1)} \\ & + \left(\frac{(-1)^N(9N+5)}{N(N+1)} - \frac{29}{3N} \right) S_3(N) + \left(\frac{19}{2} - 2(-1)^N \right) S_4(N) + (-6 + 5(-1)^N) S_{-4}(N) \\ & + \left(-\frac{2(-1)^N(9N+5)}{N(N+1)} - \frac{2}{N} \right) S_{2,1}(N) + (20 + 2(-1)^N) S_{2,-2}(N) + (-17 + 13(-1)^N) S_{3,1}(N) \\ & - \frac{8(-1)^N(2N+1) + 4(9N+1)}{N(N+1)} S_{-2,1}(N) - (24 + 4(-1)^N) S_{-3,1}(N) + (3 - 5(-1)^N) S_{2,1,1}(N) \\ & + 32S_{-2,1,1}(N) + \left(\frac{3}{2} S_1(N)^2 - \frac{3S_1(N)}{N} + \frac{3}{2} (-1)^N S_{-2}(N) \right) \zeta(2) \end{aligned}$$

$$F_0(N) =$$

$$\begin{aligned} & \frac{7}{12} S_1(N)^4 + \frac{(17N+5)S_1(N)^3}{3N(N+1)} + \left(\frac{35N^2-2N-5}{2N^2(N+1)^2} + \frac{13S_2(N)}{2} + \frac{5(-1)^N}{2N^2} \right) S_1(N)^2 \\ & + \left(-\frac{(-1)^N(2N+1)}{N(N+1)} - \frac{13}{N} \right) S_2(N) + \left(\frac{29}{3} - (-1)^N \right) S_3(N) \\ & + \left(2 + \frac{28S_{-2,1}(N) + \frac{20(-1)^N}{N^2(N+1)}}{S_1(N)} + \left(\frac{3}{4} + (-1)^N \right) S_2(N)^2 \right. \\ & - 2(-1)^N S_{-2}(N)^2 + S_{-3}(N) \left(\frac{2(3N-5)}{N(N+1)} + (26 + 4(-1)^N) S_1(N) + \frac{4(-1)^N}{N+1} \right) \\ & + \left(\frac{(-1)^N(5-3N)}{2N^2(N+1)} - \frac{5}{2N^2} \right) S_2(N) + S_{-2}(N) (10S_1(N)^2 + \left(\frac{8(-1)^N(2N+1)}{N(N+1)} \right. \\ & + \left. \frac{4(3N-1)}{N(N+1)} \right) S_1(N) + \frac{8(-1)^N(3N+1)}{N(N+1)^2} + (-22 + 6(-1)^N) S_2(N) - \frac{16}{N(N+1)} \\ & + \left(\frac{(-1)^N(9N+5)}{N(N+1)} - \frac{29}{3N} \right) S_3(N) + \left(\frac{19}{2} - 2(-1)^N \right) S_4(N) + (-6 + 5(-1)^N) S_{-4}(N) \\ & + \left(-\frac{2(-1)^N(9N+5)}{N(N+1)} - \frac{2}{N} \right) S_{2,1}(N) + (20 + 2(-1)^N) S_{2,-2}(N) + (-17 + 13(-1)^N) S_{3,1}(N) \\ & - \frac{8(-1)^N(2N+1) + 4(9N+1)}{N(N+1)} S_{-2,1}(N) - (24 + 4(-1)^N) S_{-3,1}(N) + (3 - 5(-1)^N) S_{2,1,1}(N) \\ & + 32S_{-2,1,1}(N) + \left(\frac{3}{2} S_1(N)^2 - \frac{3S_1(N)}{N} + \frac{3}{2} (-1)^N S_{-2}(N) \right) \zeta(2) \end{aligned}$$

$$F_0(N) =$$

$$\begin{aligned} & \frac{7}{12} S_1(N)^4 + \frac{(17N+5)S_1(N)^3}{3N(N+1)} + \left(\frac{35N^2-2N-5}{2N^2(N+1)^2} + \frac{13S_2(N)}{2} + \frac{5(-1)^N}{2N^2} \right) S_1(N)^2 \\ & + \left(-\frac{(-1)^N(2N+1)}{N(N+1)} - \frac{13}{N} \right) S_2(N) + \left(\frac{29}{3} - (-1)^N \right) S_3(N) \\ & + (2 + \frac{20(-1)^N}{N^2(N+1)}) S_2(N)^2 + 28S_{-2,1}(N) + \frac{20(-1)^N}{N^2(N+1)} S_2(N)^2 \\ & - 2(-1)^N S_{-2}(N)^2 + S_{-3}(N) \left(\frac{2(3N-5)}{N(N+1)} + (26+4) \right) S_2(N) \\ & + \left(\frac{(-1)^N(5-3N)}{2N^2(N+1)} - \frac{5}{2N^2} \right) S_2(N) + S_{-2}(N) (10S_1(N)^2 + \frac{5(-1)^N(2N+1)}{N(N+1)}) \\ & + \frac{4(3N-1)}{N(N+1)} S_1(N) + \frac{8(-1)^N(3N+1)}{N(N+1)^2} + (-22+6(-1)^N) S_2(N) - \frac{16}{N(N+1)} \\ & + \left(\frac{(-1)^N(9N+5)}{N(N+1)} - \frac{29}{3N} \right) S_3(N) + \left(\frac{19}{2} - 2(-1)^N \right) S_4(N) + (-6+5(-1)^N) S_{-4}(N) \\ & + \left(-\frac{2(-1)^N(9N+5)}{N(N+1)} - \frac{2}{N} \right) S_{2,1}(N) + (20+2(-1)^N) S_{2,-2}(N) + (-17+13(-1)^N) S_{3,1}(N) \\ & - \frac{8(-1)^N(2N+1)+4(9N+1)}{N(N+1)} S_{-2,1}(N) - (24+4(-1)^N) S_{-3,1}(N) + (3-5(-1)^N) S_{2,1,1}(N) \\ & + 32S_{-2,1,1}(N) + \left(\frac{3}{2} S_1(N)^2 - \frac{3S_1(N)}{N} + \frac{3}{2} (-1)^N S_{-2}(N) \right) \zeta(2) \end{aligned}$$

$$F_0(N) =$$

$$\frac{7}{12}S_1(N)^4 + \frac{(17N+5)S_1(N)^3}{3N(N+1)} + \left(\frac{35N^2-2N-5}{2N^2(N+1)^2} + \frac{13S_2(N)}{2} + \frac{5(-1)^N}{2N^2}\right)S_1(N)^2$$

$$+ \left(\frac{(-1)^N(2N+1)}{N(N+1)} - \frac{13}{N}\right)S_2(N) + \left(\frac{29}{3} - (-1)^N\right)S_3(N)$$

$$+ (2 + \frac{20(-1)^N}{N^2(N+1)})S_2(N)^2$$

$$- 2(-1)^N S_{-2}(N)^2 + S_{-3}(N)\left(\frac{2(3N-5)}{N(N+1)} + (26+4\frac{(-1)^N}{N})\right)$$

$$+ \left(\frac{(-1)^N(5-3N)}{2N^2(N+1)} - \frac{5}{N(N+1)}\right)S_2(N) + S_{-2}(N)(10S_1(N)^2 + \frac{(-1)^N(2N+1)}{N(N+1)})$$

$$+ \frac{4(3N-5)}{N(N+1)}S_2(N) - \frac{16}{N(N+1)}(-1)^N S_2(N)$$

$$+ \left(\frac{(-1)^N}{N(N+1)}\right)S_2(N) + (-6 + 5(-1)^N)S_{-4}(N)$$

$$+ \left(\frac{2(-1)^N}{N(N+1)}\right)S_{-2}(N) + (-17 + 13(-1)^N)S_{3,1}(N)$$

$$- \frac{8(-1)^N}{N(N+1)}S_{-2,1}(N) - (24 + 4(-1)^N)S_{-3,1}(N) + (3 - 5(-1)^N)S_{2,1,1}(N)$$

$$+ 32S_{-2,1,1}(N) + \left(\frac{3}{2}S_1(N)^2 - \frac{3S_1(N)}{N} + \frac{3}{2}(-1)^N S_{-2}(N)\right)\zeta(2)$$

$$S_1(N) = \sum_{i=1}^N \frac{1}{i}$$

$$S_2(N) = \sum_{i=1}^N \frac{1}{i^2}$$

$$S_{-2,1,1}(N) = \sum_{i=1}^N \frac{(-1)^i \sum_{j=1}^i \frac{1}{k}}{i^2}$$

So far derived results (see Nucl. Phys. B and Phys. Review D)

1. I. Bierenbaum, J. Blümlein, S. Klein, and C. Schneider. Two-Loop Massive Operator Matrix Elements for Unpolarized Heavy Flavor Production to $\mathcal{O}(\epsilon)$. *Nucl.Phys. B* 803(1-2):1–41, 2008.
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18. J. Ablinger, J. Blümlein, A. De Freitas, A. Goedicke, C. Schneider, K. Schönwald The Two-mass Contribution to the Three-Loop Gluonic Operator Matrix Element $A_{gg, Q}^{(3)}$. *Nucl. Phys. B* 932, pp. 129-240. 2018. arXiv:1804.02226 [hep-ph].
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So far derived results (see Nucl. Phys. B and Phys. Review D)

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Determine the coupling constant of the strong force (5 % error \rightarrow 1% error)

One (of many) motivations: the central value hints if and how the fundamental forces unite to one elementary force

Example 2: Super-congruences

(S. Ahlgren, E. Mortenson, R. Osburn, Sigma)

Sigma's contribution to harmonic number congruences

- ▶ S. Ahlgren (2001):

$$\sum_{j=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{j}^2 \binom{j + \frac{p-1}{2}}{j} (H_{j + \frac{p-1}{2}} - H_{\frac{p-1}{2}}) \equiv 0 \pmod{p}$$

Sigma's contribution to harmonic number congruences

- ▶ S. Ahlgren (2001):

$$\sum_{j=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{j}^2 \binom{j + \frac{p-1}{2}}{j} (H_{j+\frac{p-1}{2}} - H_{\frac{p-1}{2}}) \equiv 0 \pmod{p}$$

- ▶ E. Mortenson (2003):

$$\sum_{j=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{j}^2 \binom{j + \frac{p-1}{2}}{j} (1 + 3jH_{j+\frac{p-1}{2}} - 3jH_j) \equiv 0 \pmod{p}$$

$$\sum_{j=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{j} \binom{j + \frac{p-1}{2}}{j} (1 + 2jH_{j+\frac{p-1}{2}} - 2jH_j) \equiv 0 \pmod{p}$$

Sigma's contribution to harmonic number congruences

- ▶ S. Ahlgren (2001):

$$\sum_{j=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{j}^2 \binom{j + \frac{p-1}{2}}{j} (H_{j+\frac{p-1}{2}} - H_{\frac{p-1}{2}}) \equiv 0 \pmod{p}$$

- ▶ E. Mortenson (2003):

$$\sum_{j=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{j}^2 \binom{j + \frac{p-1}{2}}{j} (1 + 3jH_{j+\frac{p-1}{2}} - 3jH_j) \equiv 0 \pmod{p}$$

$$\sum_{j=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{j} \binom{j + \frac{p-1}{2}}{j} (1 + 2jH_{j+\frac{p-1}{2}} - 2jH_j) \equiv 0 \pmod{p}$$

- ▶ R. Osburn:

$$p^2 E_2(p) + p E_1(p) + p^0 E_0(p) \equiv (-1)^{\frac{p-1}{2}} \pmod{p^3}$$

For a prime $p > 2$,

$$p^2 E_2(p)$$

$$+pE_1(p)$$

$$+p^0 E_0(p) \equiv (-1)^{\frac{p-1}{2}} \pmod{p^3}$$

For a prime $p > 2$,

$$p^2 E_2(p)$$

$$+ p E_1(p)$$

$$+ p^0 \sum_{j=0}^{\frac{p-1}{2}} \binom{2j}{j}^2 16^{-j} \equiv (-1)^{\frac{p-1}{2}} \pmod{p^3}$$

For a prime $p > 2$,

$$p^2 E_2(p)$$

$$\begin{aligned}
 &+ p \left[\sum_{j=0}^{\frac{p-1}{2}} (-1)^j \binom{\frac{p-1}{2}}{j} \binom{j + \frac{p-1}{2}}{j} (1 + j(+ H_{j+\frac{p-1}{2}} + H_{-j+\frac{p-1}{2}} - 2H_j)) \right] \\
 &+ p^0 \sum_{j=0}^{\frac{p-1}{2}} \binom{2j}{j}^2 16^{-j} \equiv (-1)^{\frac{p-1}{2}} \pmod{p^3}
 \end{aligned}$$

For a prime $p > 2$,

$$\begin{aligned}
 & p^2 \left[\sum_{j=1}^{\frac{p-3}{2}} \left(\frac{(-1)^j}{\binom{\frac{p-1}{2}}{j} \binom{j+\frac{p-1}{2}}{j}} \right) \right. \\
 & \quad + \sum_{j=0}^{\frac{p-1}{2}} (-1)^j \binom{\frac{p-1}{2}}{j} \binom{j+\frac{p-1}{2}}{j} (1 + 4j(H_{j+\frac{p-1}{2}} - H_j) \\
 & \quad \quad \left. + j^2(2(H_{j+\frac{p-1}{2}} - H_j)^2 + H_j^{(2)} - H_{j+\frac{p-1}{2}}^{(2)})) \right] \\
 & + p \left[\sum_{j=0}^{\frac{p-1}{2}} (-1)^j \binom{\frac{p-1}{2}}{j} \binom{j+\frac{p-1}{2}}{j} (1 + j(+H_{j+\frac{p-1}{2}} + H_{-j+\frac{p-1}{2}} - 2H_j)) \right] \\
 & + p^0 \sum_{j=0}^{\frac{p-1}{2}} \binom{2j}{j}^2 16^{-j} \equiv (-1)^{\frac{p-1}{2}} \pmod{p^3}
 \end{aligned}$$

For a prime $p > 2$,

$$\begin{aligned}
 & p^2 \left[\sum_{j=1}^{\frac{p-3}{2}} \left(\frac{(-1)^j}{\binom{\frac{p-1}{2}}{j} \binom{j+\frac{p-1}{2}}{j}} \right) \right. \\
 & + \sum_{j=0}^{\frac{p-1}{2}} (-1)^j \binom{\frac{p-1}{2}}{j} \binom{j+\frac{p-1}{2}}{j} (1 + 4j(H_{j+\frac{p-1}{2}} - H_j) \\
 & \left. + j^2(2(H_{j+\frac{p-1}{2}} - H_j)^2 + H_j^{(2)} - H_{j+\frac{p-1}{2}}^{(2)})) \right] \\
 & + p \left[\sum_{j=0}^{\frac{p-1}{2}} (-1)^j \binom{\frac{p-1}{2}}{j} \binom{j+\frac{p-1}{2}}{j} (1 + j(+H_{j+\frac{p-1}{2}} + H_{-j+\frac{p-1}{2}} - 2H_j)) \right] \\
 & + p^0 \sum_{j=0}^{\frac{p-1}{2}} \binom{2j}{j}^2 16^{-j} \equiv (-1)^{\frac{p-1}{2}} \pmod{p^3}
 \end{aligned}$$

$$\begin{aligned}
& \sum_{j=1}^{\frac{p-3}{2}} \left(\frac{(-1)^j}{\binom{\frac{p-1}{2}}{j} \binom{j+\frac{p-1}{2}}{j}} \right) \\
& + \sum_{j=0}^{\frac{p-1}{2}} (-1)^j \binom{\frac{p-1}{2}}{j} \binom{j+\frac{p-1}{2}}{j} (1 + 4j(H_{j+\frac{p-1}{2}} - H_j) \\
& \quad + j^2(2(H_{j+\frac{p-1}{2}} - H_j)^2 + H_j^{(2)} - H_{j+\frac{p-1}{2}}^{(2)}))
\end{aligned}$$

$$\begin{aligned}
& \sum_{j=1}^{n-1} \left(\frac{(-1)^j}{\binom{n}{j} \binom{j+n}{j}} \right) \\
& + \sum_{j=0}^n (-1)^j \binom{n}{j} \binom{j+n}{j} (1 + 4j(H_{j+n} - H_j) \\
& \quad + j^2(2(H_{j+n} - H_j)^2 + H_j^{(2)} - H_{j+n}^{(2)}))
\end{aligned}$$

$$\begin{aligned}
& \sum_{j=1}^{n-1} \left(\frac{(-1)^j}{\binom{n}{j} \binom{j+n}{j}} \right) \\
& + \sum_{j=0}^n (-1)^j \binom{n}{j} \binom{j+n}{j} (1 + 4j(H_{j+n} - H_j) \\
& \quad + j^2(2(H_{j+n} - H_j)^2 + H_j^{(2)} - H_{j+n}^{(2)}))
\end{aligned}$$

||

summation spiral

$$(-1)^n ((n+1)(2n+1) - \binom{2n}{n})$$

$$\sum_{j=1}^{\frac{p-3}{2}} \left(\frac{(-1)^j}{\binom{\frac{p-1}{2}}{j} \binom{j+\frac{p-1}{2}}{j}} \right) \\
+ \sum_{j=0}^{\frac{p-1}{2}} (-1)^j \binom{\frac{p-1}{2}}{j} \binom{j+\frac{p-1}{2}}{j} (1 + 4j(H_{j+\frac{p-1}{2}} - H_j) \\
+ j^2(2(H_{j+\frac{p-1}{2}} - H_j)^2 + H_j^{(2)} - H_{j+\frac{p-1}{2}}^{(2)}))$$

||

$$(-1)^{\frac{p-1}{2}} \left(\left(\frac{p-1}{2} + 1 \right) p - \binom{p-1}{\frac{p-1}{2}} \right)$$

For a prime $p > 2$,

$$\begin{aligned}
 & p^2 \left[\sum_{j=1}^{\frac{p-3}{2}} \left(\frac{(-1)^j}{\binom{\frac{p-1}{2}}{j} \binom{j+\frac{p-1}{2}}{j}} \right) \right. \\
 & + \sum_{j=0}^{\frac{p-1}{2}} (-1)^j \binom{\frac{p-1}{2}}{j} \binom{j+\frac{p-1}{2}}{j} (1 + 4j(H_{j+\frac{p-1}{2}} - H_j) \\
 & \left. + j^2(2(H_{j+\frac{p-1}{2}} - H_j)^2 + H_j^{(2)} - H_{j+\frac{p-1}{2}}^{(2)})) \right] \\
 & + p \left[\sum_{j=0}^{\frac{p-1}{2}} (-1)^j \binom{\frac{p-1}{2}}{j} \binom{j+\frac{p-1}{2}}{j} (1 + j(+H_{j+\frac{p-1}{2}} + H_{-j+\frac{p-1}{2}} - 2H_j)) \right] \\
 & + p^0 \sum_{j=0}^{\frac{p-1}{2}} \binom{2j}{j}^2 16^{-j} \equiv (-1)^{\frac{p-1}{2}} \pmod{p^3}
 \end{aligned}$$

For a prime $p > 2$,

$$\begin{aligned}
 & p^2 \left[\right. \\
 & \quad \left. (-1)^{\frac{p-1}{2}} \left(\left(\frac{p-1}{2} + 1 \right) p - \binom{p-1}{\frac{p-1}{2}} \right) \right. \\
 & \quad \left. \left. \right. \right] \\
 & + p \left[\sum_{j=0}^{\frac{p-1}{2}} (-1)^j \binom{\frac{p-1}{2}}{j} \binom{j + \frac{p-1}{2}}{j} (1 + j(+ H_{j+\frac{p-1}{2}} + H_{-j+\frac{p-1}{2}} - 2H_j)) \right] \\
 & + p^0 \sum_{j=0}^{\frac{p-1}{2}} \binom{2j}{j}^2 16^{-j} \equiv (-1)^{\frac{p-1}{2}} \pmod{p^3}
 \end{aligned}$$

For a prime $p > 2$,

$$\begin{aligned}
 & p^2 \left[\right. \\
 & \quad \left. - (-1)^{\frac{p-1}{2}} \binom{p-1}{\frac{p-1}{2}} \right. \\
 & \quad \left. \left. + p \left[\sum_{j=0}^{\frac{p-1}{2}} (-1)^j \binom{\frac{p-1}{2}}{j} \binom{j + \frac{p-1}{2}}{j} (1 + j(+ H_{j+\frac{p-1}{2}} + H_{-j+\frac{p-1}{2}} - 2H_j)) \right] \right. \right. \\
 & \quad \left. \left. + p^0 \sum_{j=0}^{\frac{p-1}{2}} \binom{2j}{j}^2 16^{-j} \equiv (-1)^{\frac{p-1}{2}} \pmod{p^3} \right. \right. \\
 & \quad \left. \left. \right. \right]
 \end{aligned}$$

For a prime $p > 2$,

$$\begin{aligned}
 & p^2 \left[\right. \\
 & \quad - (-1)^{\frac{p-1}{2}} \binom{p-1}{\frac{p-1}{2}} \\
 & \quad \left. \right] \\
 & + p \left[\sum_{j=0}^{\frac{p-1}{2}} (-1)^j \binom{\frac{p-1}{2}}{j} \binom{j + \frac{p-1}{2}}{j} (1 + j(+H_{j+\frac{p-1}{2}} + H_{-j+\frac{p-1}{2}} - 2H_j)) \right] \\
 & + p^0 \sum_{j=0}^{\frac{p-1}{2}} \binom{2j}{j}^2 16^{-j} \equiv (-1)^{\frac{p-1}{2}} \pmod{p^3}
 \end{aligned}$$

$$\sum_{j=0}^{\frac{p-1}{2}} (-1)^j \binom{\frac{p-1}{2}}{j} \binom{j + \frac{p-1}{2}}{j} \left(1 + j \left(-2H_j + H_{j+\frac{p-1}{2}} + H_{-j+\frac{p-1}{2}} \right) \right)$$

$$\sum_{j=0}^n (-1)^j \binom{n}{j} \binom{j+n}{j} (1 + j(-2H_j + H_{j+n} + H_{-j+n}))$$

$$\sum_{j=0}^n (-1)^j \binom{n}{j} \binom{j+n}{j} (1 + j(-2H_j + H_{j+n} + H_{-j+n}))$$

||

summation spiral

$$-\frac{3}{2}(-1)^n n(n+1) \sum_{j=1}^n \frac{\binom{2j}{j}}{j} + (-1)^n (2n+1) \binom{2n}{n}$$

$$\sum_{j=0}^{\frac{p-1}{2}} (-1)^j \binom{\frac{p-1}{2}}{j} \binom{j + \frac{p-1}{2}}{j} \left(1 + j \left(-2H_j + H_{j+\frac{p-1}{2}} + H_{-j+\frac{p-1}{2}}\right)\right)$$

||

$$-\frac{3}{2} (-1)^{\frac{p-1}{2}} \left(\frac{p^2}{4} - \frac{1}{4}\right) \sum_{j=1}^{\frac{p-1}{2}} \frac{\binom{2j}{j}}{j} + (-1)^{\frac{p-1}{2}} p \binom{p-1}{\frac{p-1}{2}}$$

For a prime $p > 2$,

$$p^2 \left[- (-1)^{\frac{p-1}{2}} \binom{p-1}{\frac{p-1}{2}} \right]$$

$$+ p \left[\sum_{j=0}^{\frac{p-1}{2}} (-1)^j \binom{\frac{p-1}{2}}{j} \binom{j + \frac{p-1}{2}}{j} (1 + j(H_{j+\frac{p-1}{2}} + H_{-j+\frac{p-1}{2}} - 2H_j)) \right]$$

$$+ p^0 \sum_{j=0}^{\frac{p-1}{2}} \binom{2j}{j}^2 16^{-j} \equiv (-1)^{\frac{p-1}{2}} \pmod{p^3}$$

For a prime $p > 2$,

$$\begin{aligned}
 & p^2 \left[\right. \\
 & \quad \left. - (-1)^{\frac{p-1}{2}} \binom{p-1}{\frac{p-1}{2}} \right] \\
 & + p \left[-\frac{3}{2} (-1)^{\frac{p-1}{2}} \left(\frac{p^2}{4} - \frac{1}{4} \right) \sum_{j=1}^{\frac{p-1}{2}} \frac{\binom{2j}{j}}{j} + (-1)^{\frac{p-1}{2}} p \binom{p-1}{\frac{p-1}{2}} \right] \\
 & + p^0 \sum_{j=0}^{\frac{p-1}{2}} \binom{2j}{j}^2 16^{-j} \equiv (-1)^{\frac{p-1}{2}} \pmod{p^3}
 \end{aligned}$$

For a prime $p > 2$,

$$p^2 \left[\begin{aligned} & \\ & 0 \end{aligned} \right]$$

$$+p \left[-\frac{3}{2}(-1)^{\frac{p-1}{2}} \left(\frac{p^2}{4} - \frac{1}{4} \right) \sum_{j=1}^{\frac{p-1}{2}} \frac{\binom{2j}{j}}{j} \right]$$

$$+p^0 \sum_{j=0}^{\frac{p-1}{2}} \binom{2j}{j}^2 16^{-j} \equiv (-1)^{\frac{p-1}{2}} \pmod{p^3}$$

For a prime $p > 2$,

$$\begin{aligned}
 & p^2 \left[\sum_{j=1}^{\frac{p-3}{2}} \left(\frac{(-1)^j}{\binom{\frac{p-1}{2}}{j} \binom{j+\frac{p-1}{2}}{j}} \right) \right. \\
 & \quad + \sum_{j=0}^{\frac{p-1}{2}} (-1)^j \binom{\frac{p-1}{2}}{j} \binom{j+\frac{p-1}{2}}{j} \left(1 + 4j(H_{j+\frac{p-1}{2}} - H_j) \right. \\
 & \quad \left. \left. + j^2(2(H_{j+\frac{p-1}{2}} - H_j)^2 + H_j^{(2)} - H_{j+\frac{p-1}{2}}^{(2)}) \right) \right] \\
 & + p \left[\sum_{j=0}^{\frac{p-1}{2}} (-1)^j \binom{\frac{p-1}{2}}{j} \binom{j+\frac{p-1}{2}}{j} \left(1 + j(+H_{j+\frac{p-1}{2}} + H_{-j+\frac{p-1}{2}} - 2H_j) \right) \right] \\
 & + p^0 \sum_{j=0}^{\frac{p-1}{2}} \binom{2j}{j}^2 16^{-j} \equiv (-1)^{\frac{p-1}{2}} \pmod{p^3}
 \end{aligned}$$

Sigma's contribution to harmonic number congruences

- ▶ S. Ahlgren (2001):

$$\sum_{j=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{j}^2 \binom{j + \frac{p-1}{2}}{j} (H_{j+\frac{p-1}{2}} - H_{\frac{p-1}{2}}) \equiv 0 \pmod{p}$$

- ▶ E. Mortenson (2003):

$$\sum_{j=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{j}^2 \binom{j + \frac{p-1}{2}}{j} (1 + 3jH_{j+\frac{p-1}{2}} - 3jH_j) \equiv 0 \pmod{p}$$

$$\sum_{j=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{j} \binom{j + \frac{p-1}{2}}{j} (1 + 2jH_{j+\frac{p-1}{2}} - 2jH_j) \equiv 0 \pmod{p}$$

- ▶ R. Osburn/CS (2008):

$$p \frac{3}{8} (-1)^{\frac{p-1}{2}} \sum_{j=1}^{\frac{p-1}{2}} \frac{\binom{2j}{j}}{j} + \sum_{j=0}^{\frac{p-1}{2}} \binom{2j}{j}^2 16^{-j} \equiv (-1)^{\frac{p-1}{2}} \pmod{p^3}$$

Example 3: Remarks on Calkin's identity

G.E. Andrews, P. Paule. MacMahon's Partition Analysis IV: Hypergeometric Multisums, From the Andrews Festschrift, Springer, Berlin (2001), pp. 189-208.

N.J. Calkin. A curious binomial identities Discrete Math., 131 (1994), pp. 335-337.

M. Hirschhorn Calkin's binomial identity Discrete Math., 159 (1996), pp. 273-278.

C. Schneider. C. Schneider Symbolic Summation Assists Combinatorics. Sem. Lothar. Combin. 56, pp. 1-36. 2007.

J. Wang, Z.Z. Zhang. On extensions of Calkin's binomial identities Discrete Math., 274 (2004), pp. 331-342.

Z.Z. Zhang. A kind of curious binomial identity Discrete Math., 306 (2006), pp. 2740-2754.

▶ Case 1:

$$\sum_{k=0}^n \left(\sum_{i=0}^k \binom{n}{i} \right)^1$$

▶ Case 2:

$$\sum_{k=0}^n \left(\sum_{i=0}^k \binom{n}{i} \right)^2$$

▶ Case 3:

$$\sum_{k=0}^n \left(\sum_{i=0}^k \binom{n}{i} \right)^3$$

▶ Case 1:

$$\sum_{k=0}^n \left(\sum_{i=0}^k \binom{n}{i} \right)^1 = (2+n)2^{-1+n}$$

▶ Case 2:

$$\sum_{k=0}^n \left(\sum_{i=0}^k \binom{n}{i} \right)^2$$

▶ Case 3:

$$\sum_{k=0}^n \left(\sum_{i=0}^k \binom{n}{i} \right)^3$$

▶ Case 1:

$$\sum_{k=0}^n \left(\sum_{i=0}^k \binom{n}{i} \right)^1 = (2+n)2^{-1+n}$$

▶ Case 2:

$$\sum_{k=0}^n \left(\sum_{i=0}^k \binom{n}{i} \right)^2 = (2+n)2^{-1+2n} - \frac{1}{2}n \binom{2n}{n}$$

▶ Case 3:

$$\sum_{k=0}^n \left(\sum_{i=0}^k \binom{n}{i} \right)^3$$

▶ Case 1:

$$\sum_{k=0}^n \left(\sum_{i=0}^k \binom{n}{i} \right)^1 = (2+n)2^{-1+n}$$

▶ Case 2:

$$\sum_{k=0}^n \left(\sum_{i=0}^k \binom{n}{i} \right)^2 = (2+n)2^{-1+2n} - \frac{1}{2}n \binom{2n}{n}$$

▶ Case 3:

$$\sum_{k=0}^n \left(\sum_{i=0}^k \binom{n}{i} \right)^3 = (2+n)2^{-1+3n} - 3n2^{-2+n} \binom{2n}{n}$$

► Case 1:

$$\sum_{k=0}^n (-1)^k \left(\sum_{i=0}^k \binom{n}{i} \right)^1$$

► Case 2:

$$\sum_{k=0}^{2n} (-1)^k \left(\sum_{i=0}^k \binom{2n}{i} \right)^2$$

$$\sum_{k=0}^{2n+1} (-1)^k \left(\sum_{i=0}^k \binom{2n+1}{i} \right)^2$$

► Case 3:

$$\sum_{k=0}^{2n+1} (-1)^k \left(\sum_{i=0}^k \binom{2n+1}{i} \right)^3$$

$$\sum_{k=0}^{2n} (-1)^k \left(\sum_{i=0}^k \binom{2n}{i} \right)^3$$

► Case 1:

$$\sum_{k=0}^n (-1)^k \left(\sum_{i=0}^k \binom{n}{i} \right)^1 = (-1)^n 2^{-1+n}$$

► Case 2:

$$\sum_{k=0}^{2n} (-1)^k \left(\sum_{i=0}^k \binom{2n}{i} \right)^2$$

$$\sum_{k=0}^{2n+1} (-1)^k \left(\sum_{i=0}^k \binom{2n+1}{i} \right)^2$$

► Case 3:

$$\sum_{k=0}^{2n+1} (-1)^k \left(\sum_{i=0}^k \binom{2n+1}{i} \right)^3$$

$$\sum_{k=0}^{2n} (-1)^k \left(\sum_{i=0}^k \binom{2n}{i} \right)^3$$

► Case 1:

$$\sum_{k=0}^n (-1)^k \left(\sum_{i=0}^k \binom{n}{i} \right)^1 = (-1)^n 2^{-1+n}$$

► Case 2:

$$\sum_{k=0}^{2n} (-1)^k \left(\sum_{i=0}^k \binom{2n}{i} \right)^2 = 2^{-1+4n}$$

$$\sum_{k=0}^{2n+1} (-1)^k \left(\sum_{i=0}^k \binom{2n+1}{i} \right)^2 = -2^{1+4n} + (-1)^{1+n} \binom{2n}{n}$$

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► Case 3:

$$\sum_{k=0}^{2n+1} (-1)^k \left(\sum_{i=0}^k \binom{2n+1}{i} \right)^3 = -2^{2+6n} - 3(-1)^n 2^{2n} \binom{2n}{n}$$

$$\sum_{k=0}^{2n} (-1)^k \left(\sum_{i=0}^k \binom{2n}{i} \right)^3 = 2^{-1+6n} + \frac{(-1)^{1+n} 4^{-2+3n} \sum_{i=0}^{-1+n} 64^{-i} (3+11i) \binom{2i}{i}^2 \binom{3i}{i}}{n \binom{2n}{n}}$$

Case 1:

$$\sum_{k=0}^a x^k \sum_{i=0}^k y^i \binom{n}{i} = ?$$

Case 1:

$$\text{▶ } x \neq 1$$
$$\sum_{k=0}^a x^k \sum_{i=0}^k y^i \binom{n}{i} = \frac{x^{1+a} \sum_{i=0}^a y^i \binom{n}{i}}{-1+x} - \frac{\sum_{i=0}^a x^i y^i \binom{n}{i}}{-1+x}$$

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$\downarrow \quad a = n$

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q -Case 1:

$$\sum_{k=0}^a x^k \sum_{i=0}^k q^{\frac{1}{2}(-1+i)i} y^i \begin{bmatrix} n \\ i \end{bmatrix}$$

q -Case 1:▶ $x \neq 1$:

$$\sum_{k=0}^a x^k \sum_{i=0}^k q^{\frac{1}{2}(-1+i)i} y^i \begin{bmatrix} n \\ i \end{bmatrix} = \frac{x^{1+a} \sum_{i=0}^a q^{\frac{1}{2}(-1+i)i} y^i \begin{bmatrix} n \\ i \end{bmatrix} - \sum_{i=0}^a q^{\frac{1}{2}(-1+i)i} x^i y^i \begin{bmatrix} n \\ i \end{bmatrix}}{-1+x}$$

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Case 2:

$$\sum_{k=0}^a x^k \left(\sum_{i=0}^k y^i \binom{n}{i} \right)^2 = ?$$

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► $x \mapsto \bar{x}^2, y \mapsto \frac{1}{\bar{x}}, a = n$:

$$\begin{aligned} \sum_{k=0}^n (\bar{x}^2)^k \left(\sum_{i=0}^k \frac{1}{\bar{x}^i} \binom{n}{i} \right)^2 &= \frac{-2 + \bar{x}^{1+n} + \bar{x}^{2+n}}{-1 + \bar{x}} \bar{x}^{1-n} (1 + \bar{x})^{-2+2n} \\ &+ \frac{\binom{2n}{n}}{(1 + \bar{x})^2} - \bar{x}^{-n} (1 + \bar{x})^{-2+2n} \sum_{i=1}^n \bar{x}^i (1 + \bar{x})^{-2i} \binom{2i}{i} \end{aligned}$$

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► $x \mapsto \bar{x}^2, y \mapsto -\frac{1}{\bar{x}}, a = n$:

$$\begin{aligned} \sum_{k=0}^n (\bar{x}^2)^k \left(\sum_{i=0}^k \frac{(-1)^i}{\bar{x}^i} \binom{n}{i} \right)^2 &= \frac{\bar{x}^2 (-1 + \bar{x})^{-1+2n}}{1 + \bar{x}} - \frac{2(-1)^n (-1 + \bar{x})^{-2+2n} \bar{x}^{1-n}}{1 + \bar{x}} \\ &+ \frac{\binom{2n}{n}}{(-1 + \bar{x})^2} + \frac{(-1)^{1+n} (-1 + \bar{x})^{-2+2n}}{\bar{x}^n} \sum_{i=1}^n \frac{(-1)^i \bar{x}^i \binom{2i}{i}}{(-1 + \bar{x})^{2i}} \end{aligned}$$

► $x = 1$

► $y \neq -1$

$$\sum_{k=0}^a \left(\sum_{i=0}^k y^i \binom{n}{i} \right)^2 = - \frac{2(a-n)y^{1+a} \binom{n}{a} \sum_{i=0}^a y^i \binom{n}{i}}{1+y} - \frac{ny \sum_{i=0}^a y^{2i} \binom{n}{i}^2}{1+y} \\ + \frac{1+a+y+ay-ny}{1+y} \left(\sum_{i=0}^a y^i \binom{n}{i} \right)^2 + \frac{(-1+y) \sum_{i=0}^a iy^{2i} \binom{n}{i}^2}{1+y}$$

► $x = 1$

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$$\downarrow \quad a = n, y = 1$$

$$\sum_{k=0}^n \left(\sum_{i=0}^k \binom{n}{i} \right)^2 = (2+n)2^{-1+2n} - \frac{1}{2}n \binom{2n}{n}$$

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$$\sum_{k=0}^a \left(\sum_{i=0}^k (-1)^i \binom{n}{i} \right)^2 = \frac{-2a^2 - 2a^3 + 4an + 7a^2n - 2n^2 - 8an^2 + 3n^3}{2n^2(-1+2n)} \binom{n}{a}^2 + \frac{n \sum_{i=0}^a \binom{n}{i}^2}{2(-1+2n)}$$

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$$\sum_{k=0}^n \left(\sum_{i=0}^k (-1)^i \binom{n}{i} \right)^2 = \frac{n \binom{2n}{n}}{2(-1+2n)}$$

$$\blacktriangleright x = -1$$

$$\blacktriangleright y = -1$$

$$\sum_{k=0}^a (-1)^k \left(\sum_{i=0}^k (-1)^i \binom{n}{i} \right)^2 = \frac{(-a+n)^2 (-1)^a \binom{n}{a}^2}{2n^2} + \frac{1}{2} \sum_{i=0}^a (-1)^i \binom{n}{i}^2 - \frac{\sum_{i=0}^a (-1)^i i \binom{n}{i}^2}{n}$$

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$$\downarrow \quad a = n, y = 1$$

$$\sum_{k=0}^n (-1)^k \left(\sum_{i=0}^k (-1)^i \binom{n}{i} \right)^2 = \begin{cases} 0 & n \text{ even} \\ (-1)^{\frac{n-1}{2}} \binom{n-1}{\frac{n-1}{2}} & n \text{ odd} \end{cases}$$

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$$\sum_{k=0}^a (-1)^k \left(\sum_{i=0}^k \binom{n}{i} \right)^2 = \left(\frac{(-a+n) \binom{n}{a} \sum_{i=0}^a \binom{n}{i}}{n} + \frac{1}{2} \left(\sum_{i=0}^a \binom{n}{i} \right)^2 \right) (-1)^a - \frac{1}{2} \sum_{i=0}^a (-1)^i \binom{n}{i}^2 + \frac{1}{n} \sum_{i=0}^a (-1)^i i \binom{n}{i}^2$$

$$\blacktriangleright x = -1$$

$$\blacktriangleright y = -1$$

$$\sum_{k=0}^a (-1)^k \left(\sum_{i=0}^k (-1)^i \binom{n}{i} \right)^2 = \frac{(-a+n)^2 (-1)^a \binom{n}{a}^2}{2n^2} + \frac{1}{2} \sum_{i=0}^a (-1)^i \binom{n}{i}^2 - \frac{\sum_{i=0}^a (-1)^i i \binom{n}{i}^2}{n}$$

$$\downarrow a = n, y = 1$$

$$\sum_{k=0}^n (-1)^k \left(\sum_{i=0}^k (-1)^i \binom{n}{i} \right)^2 = \begin{cases} 0 & n \text{ even} \\ (-1)^{\frac{n-1}{2}} \binom{n-1}{\frac{n-1}{2}} & n \text{ odd} \end{cases}$$

$$\blacktriangleright y = 1$$

$$\sum_{k=0}^a (-1)^k \left(\sum_{i=0}^k \binom{n}{i} \right)^2 = \left(\frac{(-a+n) \binom{n}{a} \sum_{i=0}^a \binom{n}{i}}{n} + \frac{1}{2} \left(\sum_{i=0}^a \binom{n}{i} \right)^2 \right) (-1)^a - \frac{1}{2} \sum_{i=0}^a (-1)^i \binom{n}{i}^2 + \frac{1}{n} \sum_{i=0}^a (-1)^i i \binom{n}{i}^2$$

$$\downarrow a = n, y = 1$$

$$\sum_{k=0}^n (-1)^k \left(\sum_{i=0}^k \binom{n}{i} \right)^2 = \begin{cases} 2^{-1+2n} & n \text{ even} \\ -2^{2n-1} + (-1)^{\frac{n+1}{2}} \binom{n-1}{\frac{n-1}{2}} & n \text{ odd} \end{cases}$$

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q -Case 2:

$$\sum_{k=0}^a x^k \left(\sum_{i=0}^k y^i q^{\frac{1}{2}i(i-1)} \begin{bmatrix} n \\ i \end{bmatrix} \right)^2 = ?$$

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► $x \mapsto \bar{x}^2, y \mapsto \frac{1}{\bar{x}}$:

$$\begin{aligned} \sum_{k=0}^n (\bar{x}^2)^k \left(\sum_{i=0}^k \frac{1}{\bar{x}^i} q^{\frac{1}{2}i(i-1)} \begin{bmatrix} n \\ i \end{bmatrix} \right)^2 &= \frac{q^2 \bar{x}^3 (1+q^2 - 2q\bar{x}^2) (-\frac{1}{q\bar{x}}; q)_{1+n} (-\frac{\bar{x}}{q}; q)_{1+n}}{(-1+\bar{x})(1+\bar{x})(q+\bar{x})(1+q\bar{x})(-q+\bar{x}^2)(-1+q\bar{x}^2)} \\ &+ \frac{(-\frac{1}{q}; q)_{1+n} (\frac{1}{q}; q^2)_{1+n}}{(-1+q)(1+q)(q-\bar{x}^2)(-1+q\bar{x}^2)(q; q)_{1+n}} (-q^3 - q^3\bar{x} - q^4\bar{x} - q^3\bar{x}^2) \\ &+ \frac{(-\frac{1}{q}; q)_{1+n} (\frac{1}{q}; q^2)_{1+n}}{(-1+q)(1+q)(q-\bar{x}^2)(-1+q\bar{x}^2)(q; q)_{1+n}} (q^4 + q^3\bar{x} + q^4\bar{x} + q^4\bar{x}^2) q^n \\ &+ \frac{q^2 ((-\frac{1}{q\bar{x}}; q)_{1+n})^2}{(-1+\bar{x})(1+\bar{x})(1+q\bar{x})^2} \bar{x}^{4+2n} \\ &+ \frac{q^2 (1+q^2) \bar{x} (-\frac{1}{q\bar{x}}; q)_{1+n} (-\frac{\bar{x}}{q}; q)_{1+n}}{(q-\bar{x}^2)(-1+q\bar{x}^2)} \sum_{i=1}^n \frac{(-\frac{1}{q}; q)_{1+i} (\frac{1}{q}; q^2)_{1+i}}{(q; q)_{1+i} (-\frac{1}{q\bar{x}}; q)_{1+i} (-\frac{\bar{x}}{q}; q)_{1+i}} \\ &- \frac{q^3 (1+q^2) \bar{x} (-\frac{1}{q\bar{x}}; q)_{1+n} (-\frac{\bar{x}}{q}; q)_{1+n}}{(q-\bar{x}^2)(-1+q\bar{x}^2)} \sum_{i=1}^n \frac{(-\frac{1}{q}; q)_{1+i} (\frac{1}{q}; q^2)_{1+i}}{(q+q^i)(q; q)_{1+i} (-\frac{1}{q\bar{x}}; q)_{1+i} (-\frac{\bar{x}}{q}; q)_{1+i}} \end{aligned}$$

q -Case 2:

$$\sum_{k=0}^a x^k \left(\sum_{i=0}^k y^i q^{\frac{1}{2}i(i-1)} \begin{bmatrix} n \\ i \end{bmatrix} \right)^2 = ?$$

► $x \mapsto \bar{x}^2, y \mapsto \frac{1}{\bar{x}}$:

$$\begin{aligned} \sum_{k=0}^n (\bar{x}^2)^k \left(\sum_{i=0}^k \frac{1}{\bar{x}^i} q^{\frac{1}{2}i(i-1)} \begin{bmatrix} n \\ i \end{bmatrix} \right)^2 &= \frac{q^2 \bar{x}^3 (1+q^2-2q\bar{x}^2) (-\frac{1}{q\bar{x}}; q)_{1+n} (-\frac{\bar{x}}{q}; q)_{1+n}}{(-1+\bar{x})(1+\bar{x})(q+\bar{x})(1+q\bar{x})(-q+\bar{x}^2)(-1+q\bar{x}^2)} \\ &+ \frac{(-\frac{1}{q}; q)_{1+n} (\frac{1}{q}; q^2)_{1+n}}{(-1+q)(1+q)(q-\bar{x}^2)(-1+q\bar{x}^2)(q; q)_{1+n}} (-q^3 - q^3\bar{x} - q^4\bar{x} - q^3\bar{x}^2) \\ &+ \frac{(-\frac{1}{q}; q)_{1+n} (\frac{1}{q}; q^2)_{1+n}}{(-1+q)(1+q)(q-\bar{x}^2)(-1+q\bar{x}^2)(q; q)_{1+n}} (q^4 + q^3\bar{x} + q^4\bar{x} + q^4\bar{x}^2) q^n \\ &+ \frac{q^2 ((-\frac{1}{q\bar{x}}; q)_{1+n})^2}{(-1+\bar{x})(1+\bar{x})(1+q\bar{x})^2} \bar{x}^{4+2n} \\ &+ \frac{q^2(1+q^2)\bar{x} (-\frac{1}{q\bar{x}}; q)_{1+n} (-\frac{\bar{x}}{q}; q)_{1+n}}{(q-\bar{x}^2)(-1+q\bar{x}^2)} \sum_{i=1}^n \frac{(-\frac{1}{q}; q)_{1+i} (\frac{1}{q}; q^2)_{1+i}}{(q; q)_{1+i} (-\frac{1}{q\bar{x}}; q)_{1+i} (-\frac{\bar{x}}{q}; q)_{1+i}} \\ &- \frac{q^3(1+q^2)\bar{x} (-\frac{1}{q\bar{x}}; q)_{1+n} (-\frac{\bar{x}}{q}; q)_{1+n}}{(q-\bar{x}^2)(-1+q\bar{x}^2)} \sum_{i=1}^n \frac{(-\frac{1}{q}; q)_{1+i} (\frac{1}{q}; q^2)_{1+i}}{(q+q^i)(q; q)_{1+i} (-\frac{1}{q\bar{x}}; q)_{1+i} (-\frac{\bar{x}}{q}; q)_{1+i}} \end{aligned}$$

► $x \mapsto \bar{x}^2, y \mapsto -\frac{1}{\bar{x}}$: similar

$$q\text{-Case 2: } \sum_{k=0}^a x^k \left(\sum_{i=0}^k y^i q^{\frac{1}{2}i(i-1)} \begin{bmatrix} n \\ i \end{bmatrix} \right)^2 = ?$$

Observation: (Indefinite) summable for

$(x = q^r \ [r \neq 0] \text{ and } y = q^s)$ or $(x = q^r \text{ and } y = -1)$

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$(x = q^r \ [r \neq 0] \text{ and } y = q^s) \text{ or } (x = q^r \text{ and } y = -1)$

E.g., $x = q, y = 1$

$$\begin{aligned} \sum_{k=0}^a q^k \left(\sum_{i=0}^k q^{\frac{1}{2}i(i-1)} \begin{bmatrix} n \\ i \end{bmatrix} \right)^2 &= \frac{-1 + 2q^{1+a} - q^n}{2(-1 + q)} \left(\left(\sum_{i=0}^a q^{\frac{1}{2}(-1+i)i} \begin{bmatrix} n \\ i \end{bmatrix} \right)^2 \right. \\ &\quad \left. - (-1 + q^n) \sum_{i=0}^a q^{(-1+i)i} \begin{bmatrix} n \\ i \end{bmatrix}^2 \right. \\ &\quad \left. + 2(-q^a + q^n) q^{\frac{1}{2}(-1+a)a} \sum_{i=0}^a q^{\frac{1}{2}(-1+i)i} \begin{bmatrix} n \\ i \end{bmatrix} \begin{bmatrix} n \\ a \end{bmatrix} \right) \end{aligned}$$

$$q\text{-Case 2: } \sum_{k=0}^a x^k \left(\sum_{i=0}^k y^i q^{\frac{1}{2}i(i-1)} \begin{bmatrix} n \\ i \end{bmatrix} \right)^2 = ?$$

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$$\downarrow \quad a = n$$

$$\begin{aligned} &\left(q^2 \left(-\frac{1}{q}; q \right)_{1+n} ((1+q)(-1+q^n)(-1+q^{1+n})) \left(\frac{1}{q}; q^2 \right)_{1+n} \right. \\ &\left. + (-1+q)(-1+q^n(-1+2q)) \left(-\frac{1}{q}; q \right)_{1+n} (q; q)_{1+n} \right) / (2(-1+q)^2(1+q)^2(q; q)_{1+n}) \end{aligned}$$

$$q\text{-Case 2: } \sum_{k=0}^a x^k \left(\sum_{i=0}^k y^i q^{\frac{1}{2}i(i-1)} \begin{bmatrix} n \\ i \end{bmatrix} \right)^2 = ?$$

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E.g., $x = q, y = q$

$$\begin{aligned} \sum_{k=0}^a q^k \left(\sum_{i=0}^k q^i q^{\frac{1}{2}i(i-1)} \begin{bmatrix} n \\ i \end{bmatrix} \right)^2 &= \left((-1 + (1+q)q^{1+a} - q^{1+n}) \left(\sum_{i=0}^a q^{i+\frac{1}{2}(-1+i)i} \begin{bmatrix} n \\ i \end{bmatrix} \right)^2 \right. \\ &+ (1 - q^{1+n}) \sum_{i=0}^a q^{2i+(-1+i)i} \begin{bmatrix} n \\ i \end{bmatrix}^2 + (-1 + q) \sum_{i=0}^a q^{3i+(-1+i)i} \begin{bmatrix} n \\ i \end{bmatrix}^2 \\ &\left. + 2(-q^a + q^n) q^{1+a+\frac{1}{2}(-1+a)a} \begin{bmatrix} n \\ a \end{bmatrix} \sum_{i=0}^a q^{i+\frac{1}{2}(-1+i)i} \begin{bmatrix} n \\ i \end{bmatrix} \right) / (-1+q)(1+q) \end{aligned}$$

$$q\text{-Case 2: } \sum_{k=0}^a x^k \left(\sum_{i=0}^k y^i q^{\frac{1}{2}i(i-1)} \begin{bmatrix} n \\ i \end{bmatrix} \right)^2 = ?$$

Observation: (Indefinite) summable for

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E.g., $x = q, y = q$

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$$\downarrow \quad a = n$$

$$\begin{aligned} &\left((-1+q)^2(-1+q^{2+n})((-1; q)_{1+n})^2 - \frac{2(-1+q)^2(1+q^{2+2n})(-1; q)_{1+n}(q; q^2)_{1+n}}{(-1+q^{1+n})(q; q)_{1+n}} \right. \\ &\left. + \frac{4(1+q^2) + ((q; q)_{1+n})^2}{-1+q^{1+n}} + \frac{4(-1+q)^2 q ((q; q)_{1+n})^2}{-1+q^{1+n}} \sum_{i=1}^n \frac{q^i (-1; q)_{1+i} (q; q^2)_{1+i}}{((q; q)_{1+i})^3} \right) / (4(-1+q)^3(1+q)) \end{aligned}$$

▶ Case 1:

$$\sum_{k=0}^n \left(\sum_{i=0}^k \binom{n}{i} \right)^1 = (2+n)2^{-1+n}$$

▶ Case 2:

$$\sum_{k=0}^n \left(\sum_{i=0}^k \binom{n}{i} \right)^2 = (2+n)2^{-1+2n} - \frac{1}{2}n \binom{2n}{n}$$

▶ Case 3:

$$\sum_{k=0}^n \left(\sum_{i=0}^k \binom{n}{i} \right)^3 = (2+n)2^{-1+3n} - 3n2^{-2+n} \binom{2n}{n}$$

Sigma