The Generators of all Polynomial Relations among Jacobi Theta Functions

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Abstract

In this article, we consider the classical Jacobi theta functions $\theta_i(z)$, $i = 1, 2, 3, 4$ and show that the ideal of all polynomial relations among them with coefficients in $K := \mathbb{Q}(\theta_2(0|\tau), \theta_3(0|\tau), \theta_4(0|\tau))$ is generated by just two polynomials, that correspond to well known identities among Jacobi theta functions.

1 Introduction

Let $\theta_j(z|\tau)$ ($j = 1, \ldots, 4$, $z \in \mathbb{C}$, $\tau \in \mathbb{H}$) denote the four classical Jacobi theta functions where $\mathbb{H}$ denotes the upper complex half plane. In this article we show that if $p \in K[T_1, T_2, T_3, T_4]$ is a polynomial with coefficients in $K := \mathbb{Q}(\theta_2(0|\tau), \theta_3(0|\tau), \theta_4(0|\tau))$ such that for every $z \in \mathbb{C}$ and every $\tau \in \mathbb{H}$

$$p(\theta_1(z|\tau), \theta_2(z|\tau), \theta_3(z|\tau), \theta_4(z|\tau)) = 0, \tag{1}$$

then $p = p_1 b_1 + p_2 b_2$ for some $p_1, p_2 \in K[T_1, T_2, T_3, T_4]$ where

$$b_1 := \theta_2(0|\tau)^2\, T_2^2 - \theta_3(0|\tau)^2\, T_3^2 + \theta_4(0|\tau)^2\, T_4^2, \tag{2}$$

$$b_2 := \theta_2(0|\tau)^2\, T_1^2 + \theta_4(0|\tau)^2\, T_3^2 - \theta_3(0|\tau)^2\, T_4^2. \tag{3}$$

Note that b_1 and b_2 correspond to [4, Eq. 20.7.1] and [4, Eq. 20.7.2], respectively.

The polynomials b_1 and b_2 form a Gröbner basis of the ideal of all such relations. Thus, one can check whether relation of the form (1) holds by simply reducing p by b_1 and b_2. The result of the reduction is zero if and only if the identity holds.

After introducing some notation, we give the precise formulation of our problem in Section 2. In Section 3, we reduce the problem of finding relations among theta functions to finding relations among quotients of theta functions that, additionally are elliptic. In Section 4, we then show that the ideal of relations among elliptic theta quotients is generated by two elements. These two elements are

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then used to setup the generators for the ideal of polynomial relations among Jacobi theta functions in Section 5. To actually, compute the Gröbner basis of this ideal, we show computability of K in Section 6. Eventually, we show the steps to compute the polynomials b_1 and b_2 in the computer algebra system FriCAS.

2 Notation and Problem Formulation

The classical Jacobi theta functions $\theta_j(z|\tau)$ $(j = 1, \ldots, 4)$ are defined as follows.

Definition 1. *(cf. [4, Eq. 20.2(i)]) Let* $\tau \in \mathbb{H} := \{z \in \mathbb{C} \mid \Im(z) > 0\}$ *and* $q := e^{\pi i \tau}$*, then*

$$\begin{aligned}
\theta_1(z, q) &:= \theta_1(z|\tau) := 2\sum_{n=0}^{\infty}(-1)^n q^{(n+\frac{1}{2})^2} \sin((2n+1)z),\\
\theta_2(z, q) &:= \theta_2(z|\tau) := 2\sum_{n=0}^{\infty} q^{(n+\frac{1}{2})^2} \cos((2n+1)z),\\
\theta_3(z, q) &:= \theta_3(z|\tau) := 1 + 2\sum_{n=1}^{\infty} q^{n^2} \cos(2nz),\\
\theta_4(z, q) &:= \theta_4(z|\tau) := 1 + 2\sum_{n=1}^{\infty}(-1)^n q^{n^2} \cos(2nz).
\end{aligned}$$

For simplicity, we write $\theta_j(z) := \theta_j(z|\tau)$.

Throughout the paper, we use multi-index notation, i.e., for $n \in \mathbb{N}$, $\alpha \in \mathbb{Z}^n$ and objects $x_1, \ldots, x_n$ we simply write x^α instead of $x_1^{\alpha_1} \ldots x_n^{\alpha_n}$. We mostly use $n = 3$ or $n = 4$. In particular, if $\alpha \in \mathbb{Z}^4$,

$$\theta(z)^\alpha := \theta_1(z)^{\alpha_1}\theta_2(z)^{\alpha_2}\theta_3(z)^{\alpha_3}\theta_4(z)^{\alpha_4}. \tag{4}$$

If L is a ring and S is a subset of an L-module, we denote by $\langle S\rangle_L$ the set of L-linear combinations of elements of S. If L is a field, then $\langle S\rangle_L$ is a vector space. If $S \subset L$, then $\langle S\rangle_L$ is an ideal of L.

We define the field $K := \mathbb{Q}(\theta_2(0), \theta_3(0), \theta_4(0))$ and set

$$\begin{aligned}
&\theta := \{\theta_i(z) \mid i = 1, 2, 3, 4\},\\
&T := \{T_1, T_2, T_3, T_4\},\\
&\phi : K[T] \to K[\theta], \quad T_i \mapsto \theta_i(z), \quad i = 1, 2, 3, 4.
\end{aligned}$$

The problem we are dealing with in this article is to determine (algorithmically) the set $\ker\phi \subset K[T]$. Note that $\ker\phi$ is an ideal of $K[T]$ and, thus, by Hilbert's basis theorem, finitely generated.

In order to describe $\ker\phi$, we first consider the map

$$\Phi : K[T, T^{-1}] \to K[\theta, \theta^{-1}], \quad T_i \mapsto \theta_i(z), \quad i = 1, 2, 3, 4.$$

Note that $\phi = \Phi|_{K[T]}$ and $\ker\phi = \ker\Phi \cap K[T]$. Define $L := K[T, T^{-1}]$. For $p \in L$, we sometimes write $p(\theta)$ instead of $\Phi(p)$.

3 Reduction to elliptic theta quotients

Definition 2. *A meromorphic function f on $\mathbb{C}$ is called* elliptic*, if there are two non-complex numbers ω_1 and ω_2 with $\frac{\omega_1}{\omega_2} \notin \mathbb{R}$ such that $f(z+\omega_1) = f(z)$ and $f(z+\omega_2) = f(z)$ for all $z \in \mathbb{C}$.*

In [9], an algorithm was given to decide whether $f = 0$ for $f \in K[\theta]$ by reducing the problem to such f that are additionally, "quasi-elliptic" functions. More precisely, for our problem it is enough to find all relations among quotients of theta functions that are elliptic.

In view of the following lemma, we can connect theta functions with elliptic functions. Note that whenever we say elliptic function, we mean elliptic function with respect to the argument z.

Lemma 1. *(cf. [8, p. 465]) Let $N := e^{-\pi i\tau - 2iz}$. For $j \in \{1,2,3,4\}$ we have $\theta_j(z+\pi\tau|\tau) = \varepsilon_1(j)\theta_j(z|\tau)$ and $\theta_j(z+\pi|\tau) = \varepsilon_2(j)\theta_j(z|\tau)$ where $\varepsilon_1(j)$ and $\varepsilon_2(j)$ are defined in the following table.*

j	1	2	3	4
$\varepsilon_1(j)$	$-N$	N	N	$-N$
$\varepsilon_2(j)$	-1	-1	1	1

Definition 3. *(cf. [9, Def. 2.2]) Given $\alpha, \beta \in \mathbb{Z}^4$, we say that α and β are similar, denoted by $\alpha \sim \beta$, if $\alpha_1+\alpha_2+\alpha_3+\alpha_4 = \beta_1+\beta_2+\beta_3+\beta_4$, $\alpha_1+\alpha_2 \equiv \beta_1+\beta_2 \pmod 2$, and $\alpha_1+\alpha_4 \equiv \beta_1+\beta_4 \pmod 2$.*

It is easy to prove that $\sim$ is a congruence relation on the $\mathbb{Z}$-module $\mathbb{Z}^4$.

The conditions in Definition 3 have been chosen according to the table in Lemma 1, so that $\theta(z)^\alpha$ is elliptic if $\alpha \sim 0$, cf. Lemma 3.1 in [9]. Similar to Definition 4.1 in [9] we define $R^* := \left\{\alpha \in \mathbb{Z}^4 \,\middle|\, \alpha \sim 0\right\}$.

Theorem 2.7 from [9] can be formulated as follows.

Theorem 1. *Let M be a finite subset of $\mathbb{Z}^4$, $M/{\sim} = \{M_1, \dots, M_n\}$. For $i \in \{1,\dots,n\}$ let $p_i = \sum_{\alpha\in M_i} c_\alpha T^\alpha$ with $c_\alpha \in K$ and let $p = \sum_{i=1}^n p_i$. Then $p(\theta) = 0$ if and only if $p_i(\theta) = 0$ for all $i \in \{1,\dots,n\}$.*

With the same notation as in Theorem 1 we can write

$$p = \sum_{i=1}^n p_i = \sum_{i=1}^n T^{\beta_i} \frac{p_i}{T^{\beta_i}} = \sum_{i=1}^n T^{\beta_i} \sum_{\alpha\in M_i} c_\alpha T^{\alpha-\beta_i}$$

for some $\beta_i \in M_i$. Note that if $\alpha \in M_i$, then $\alpha - \beta_i \in R^*$.

Let L^* be the set of K-linear combinations of monomials $T^\alpha \in L$ with $\alpha \in R^*$. Theorem 1 says that $\ker\Phi = \langle L^* \cap \ker\Phi\rangle_L$.

Lemma 2. *(cf. [9, Lemma 4.2]) The set R^* forms an (additive) $\mathbb{Z}$-module that is generated by the vectors $\iota_1 = (-2,2,0,0)$, $\iota_2 = (-2,0,2,0)$, $\iota_3 = (-3,1,1,1)$, i.e., $R^* = \langle\iota_1,\iota_2,\iota_3\rangle_{\mathbb{Z}}$.*

Proof. Clearly, $\langle\iota_1,\iota_2,\iota_3\rangle_{\mathbb{Z}} \subseteq R^*$. For $R^* \subseteq \langle\iota_1,\iota_2,\iota_3\rangle_{\mathbb{Z}}$ note that if $\alpha \in R^*$, then

$$\alpha = \iota_1\frac{\alpha_2-\alpha_4}{2} + \iota_2\frac{\alpha_3-\alpha_4}{2} + \iota_3\alpha_4.$$

□

4 The ideal of relations among elliptic theta quotients

From Lemma 2 follows $L^* = K[T^{\iota_1}, T^{\iota_2}, T^{\iota_3}]$, i.e.,

$$\ker \Phi = \left\langle K[T^{\iota_1}, T^{\iota_2}, T^{\iota_3}] \cap \ker \Phi \right\rangle_L .$$

In other words, any relation among theta functions can be expressed as a L-linear combination of polynomials in $T^{\iota_1}, T^{\iota_2}, T^{\iota_3}$ whose coefficients are in K. We would like to find polynomials p in $T^{\iota_1}, T^{\iota_2}, T^{\iota_3}$ such that $\Phi(p) = 0$.

Let us define the elliptic functions corresponding to the above generators.

$$\begin{aligned} j_1(z) &:= \Phi(T^{\iota_1}) = \theta(z)^{\iota_1} = \frac{\theta_2(z)^2}{\theta_1(z)^2}, \\ j_2(z) &:= \Phi(T^{\iota_1}) = \theta(z)^{\iota_2} = \frac{\theta_3(z)^2}{\theta_1(z)^2}, \\ j_3(z) &:= \Phi(T^{\iota_1}) = \theta(z)^{\iota_2} = \frac{\theta_2(z)\theta_3(z)\theta_4(z)}{\theta_1(z)^3}. \end{aligned}$$

Let $J = \{J_1, J_2, J_3\}$ be a new set of indeterminates. As an intermediate step to solve our original problem, we consider the map $\Psi : K[J, J^{-1}] \to K[\theta, \theta^{-1}]$, which is defined by $\Psi = \Phi \circ \sigma$ for the ring homomorphism $\sigma : K[J, J^{-1}] \to L^*$, $J_i \mapsto T^{\iota_i}$, $i = 1, 2, 3$. Note that because $L^* = K[T^{\iota_1}, T^{\iota_2}, T^{\iota_3}]$, σ is surjective, i.e., $p \in K[T^{\iota_1}, T^{\iota_2}, T^{\iota_3}] \cap \ker \Phi$, there exists $f \in K[J, J^{-1}]$ such that $\sigma(f) = p$. Therefore, $\sigma(\ker \Psi) = K[T^{\iota_1}, T^{\iota_2}, T^{\iota_3}] \cap \ker \Phi$.

Clearly, Ψ maps J_i to $j_i(z)$, $i = 1, 2, 3$. In the following we are going to show that $\ker \Psi$ is an ideal in $K[J, J^{-1}]$ that is generated by the two polynomial

$$h_1 := J_2 - c_3 J_1 - c_4, \tag{5}$$

$$h_2 := J_3^2 - J_1 J_2 (c_4 J_1 + c_3) \tag{6}$$

where $c_3 = \frac{\theta_3(0)^2}{\theta_2(0)^2}$, $c_4 = \frac{\theta_4(0)^2}{\theta_2(0)^2}$.

Let $I_\Psi := \langle h_1, h_2 \rangle_{K[J, J^{-1}}$. One can verify by Algorithm 6.6 from [9] that $\Psi(h_1) = 0$, and $\Psi(h_2) = 0$. Hence $I_\Psi \subseteq \ker \Psi$. In order to prove $\ker \Psi \subseteq I_\Psi$, assume that $f \in \ker \Psi$. Because $h_1 \in I_\Psi$, we have

$$f(J_1, J_2, J_3) + I_\Psi = f(J_1, c_3 J_1 + c_4, J_3) + I_\Psi = J_1^{\alpha_1} J_3^{\alpha_3} \tilde{f}(J_1, J_3) + I_\Psi$$

for some $\alpha_1, \alpha_3 \in \mathbb{Z}$ and $\tilde{f} \in K[J_1, J_3]$.

Clearly, we can split $\tilde{f}$ with respect to even and odd powers of J_3 in such a way that for some polynomials $\tilde{f}_1$ and $\tilde{f}_2$ we have the representation

$$\tilde{f}(J_1, J_3) = \tilde{f}_1(J_1, J_3^2) + J_3 \tilde{f}_2(J_1, J_3^2).$$

Since $h_1, h_2 \in I_\Psi$, we can replace J_3^2 by $J_1(c_3 J_1 + c_4)(c_4 J_1 + c_3) \in K[J_1]$ and obtain

$$\tilde{f}(J_1, J_3) + I_\Psi = f_1(J_1) + J_3 f_2(J_1) + I_\Psi$$

for some $f_1, f_2 \in K[J_1]$. Hence,

$$f(J_1, J_2, J_3) + I_\Psi = J_1^{\alpha_1} J_3^{\alpha_3} \left(f_1(J_1) + J_3 f_2(J_1) \right) + I_\Psi$$

From $f \in \ker \Psi$ and $I_\Psi \subseteq \ker \Psi$, we conclude

$$j_1^{\alpha_1} j_3^{\alpha_3} \left(f_1(j_1) + j_3 f_2(j_1)\right) = 0.$$

Since $j_1^{\alpha_1} j_3^{\alpha_3}$ is a nonzero meromorphic function, it follows that

$$f_1(j_1) + j_3 f_2(j_1) = 0. \tag{7}$$

Note that expanding $j_1(z)$ and $j_3(z)$ as Laurent series in z with coefficients in K, we observe that

$$j_1(z) = z^{-2} + \text{ higher order terms}$$

and

$$j_3(z) = z^{-3} + \text{ higher order terms.}$$

If we assume that $f_1, f_2 \neq 0$ and $\deg(f_1) = d_1$ and $\deg(f_2) = d_2$ for $d_1, d_2 \in \mathbb{N}$ then

$$f_1(j_1(z)) = c_1 z^{-2d_1} + \text{ higher order terms}$$

and

$$j_3(z) f_2(j_1(z)) = c_2 z^{-2d_2-3} + \text{ higher order terms}$$

for some $c_1, c_2 \in K \setminus \{0\}$.

Since $-2d_1$ is even and $-2d_2 - 3$ is odd, the leading terms cannot cancel and, therefore, $f_1(j_1(z)) + j_3(z) f_2(j_1(z)) \neq 0$. Thus, either $f_1 = 0$ or $f_2 = 0$. However, if one of these polynomials is zero, the other must also be zero, since otherwise the respective leading term of the Laurent series expansion cannot be made to vanish as required by (7).

In summary, for $f \in \ker \Psi$ we have shown

$$\begin{aligned} f(J_1, J_2, J_3) + I_\Psi &= J_1^{\alpha_1} J_3^{\alpha_3} \tilde{f}(J_1, J_3) + I_\Psi \\ &= J_1^{\alpha_1} J_3^{\alpha_3} (f_1(J_1) + J_3 f_2(J_1)) + I_\Psi \\ &= J_1^{\alpha_1} J_3^{\alpha_3} (0 + J_3 \cdot 0) + I_\Psi \\ &= 0 + I_\Psi. \end{aligned}$$

Therefore $f \in I_\Psi$ and $\ker \Psi = I_\Psi = \langle h_1, h_2 \rangle_{K[J, J^{-1}]}$.

5 The ideal of relations among theta functions

From the previous section we have $\sigma(\ker \Psi) = K[T^{\iota_1}, T^{\iota_2}, T^{\iota_3}] \cap \ker \Phi$ and, therefore, $\ker \Phi = \langle \sigma(I_\Psi) \rangle_L$. Let $H^L := \left\{h_1^L, h_2^L\right\}$ for $h_1^L := \sigma(h_1)$, $h_2^L := \sigma(h_2)$.

We are left with the problem of computing $\left\langle H^L \right\rangle_L \cap K[T] = \ker \phi$.

A solution of this problem is well-known in the computer algebra community. Let us denote by $P = K[S, T]$ the polynomial ring in the indeterminates $S = \{S_1, S_2, S_3, S_4\}$ and $T = \{T_1, T_2, T_3, T_4\}$. Let $U = \{1 - S_i T_i \mid i \in \{1, 2, 3, 4\}\}$ and $I = \langle U \rangle_P$ be the ideal generated by the elements of U. By [7, Proposition 7.1], $\ker \chi = I$ for the surjective homomorphism $\chi : P \to L$ with $\chi(S_i) = T_i^{-1}$ and $\chi(T_i) = T_i$ for $i \in \{1, 2, 3, 4\}$, i.e., $P/I \cong L$.

Let $\chi' : L \to P$ be such that $\chi'(T_i) = T_i$, $\chi'(T_i^{-1}) = S_i$, i.e., $\chi(\chi'(p)) = p$ for every $p \in L$. Then $\ker\phi = \ker\Phi \cap K[T] = \langle \chi'(H^L) \cup U \rangle_P \cap K[T]$. Note that

$$\chi'(h_1^L) := S_1^2 T_3^2 - c_3 S_1^2 T_2^2 - c_4,$$
$$\chi'(h_2^L) := (S_1^3 T_1 T_2 T_3)^2 - (S_1^2 T_2^2)(S_1^2 T_3^2)(c_4 S_1^2 T_2^2 + c_3).$$

A generating set for the latter intersection can be computed by Buchberger's algorithm (cf. [3] or [1]) applied to $\chi'(H^L) \cup U$ with respect to a term ordering such that monomials with indeterminates exclusively from the set T are smaller than any monomial involving indeterminates from S. Then by [1, Cor. 5.51] the polynomials $g_1, \dots, g_t$ in this Gröbner basis that only involve indeterminates from the set T form a Gröbner basis G of all the relations among the theta functions $\theta_1, \theta_2, \theta_3, \theta_3$ with coefficients in K.

6 Computability of K

Up to now the field of coefficients has not played an essential role in the derivation. However, in order to actually compute the Gröbner basis from the previous section, we must find a good representation of the elements of K. Note that $\theta_2(0)$, $\theta_3(0)$, and $\theta_4(0)$, and therefore, also c_3 and c_4 are actually Puiseux series in q.

In the following, we employ results from [6] in order to show that the well known Jacobi identity

$$\theta_2(0|\tau)^4 - \theta_3(0|\tau)^4 + \theta_4(0|\tau)^4 = 0$$

is a "factor" of any other identity among $\theta_2(0)$, $\theta_3(0)$, and $\theta_4(0)$ and then use it to model K in a finitary way.

Let

$$\eta : \mathbb{H} \to \mathbb{C}, \quad \tau \mapsto \exp\left(\frac{\pi i \tau}{12}\right) \prod_{n=1}^{\infty} (1 - e^{2\pi i n})$$

be the Dedekind eta function and denote for $\delta = 1, 2, 4$ by $\eta_\delta : \mathbb{H} \to \mathbb{C}$ the function $\eta_\delta(\tau) := \eta(\delta\tau)$.

By simple rewriting of formulas for theta functions in Section 21.42 of [8] or rewriting of q-series expansions from Entry 22 together with formulas (0.12) and (0.13) of Chapter 20 of [2], we can express the Jacobi theta functions in terms of in Dedekind η functions:

$$\theta_2(0|\tau) = \frac{2\eta(2\tau)^2}{\eta(\tau)}, \qquad \theta_3(0|\tau) = \frac{\eta(\tau)^5}{\eta(\frac{1}{2}\tau)^2 \eta(2\tau)^2}, \qquad \theta_4(0|\tau) = \frac{\eta(\frac{1}{2}\tau)^2}{\eta(\tau)}. \quad (8)$$

The relations among the theta functions are given by the kernel of the following map.

$$\xi : \mathbb{Q}[t_2, t_3, t_4] \to \mathbb{Q}[\theta_2(0), \theta_3(0), \theta_4(0)],$$
$$t_j \mapsto \theta_j(0), \quad j = 2, 3, 4,$$

where t_2, t_3, t_4 are indeterminates. In order to find $\ker \xi$, we extend this map to

$$\Xi : \mathbb{Q}[Y, E, t] \to \mathbb{Q}[\eta^{-1}, \eta, \theta],$$
$$Y_\delta \mapsto \eta_\delta(\tau/2)^{-1}, \quad E_\delta \mapsto \eta_\delta(\tau/2), \quad \delta = 1, 2, 4, \quad t_j \mapsto \theta_j(0|\tau), \quad j = 2, 3, 4.$$

Define $r := E_2^{24} - E_1^{16} E_4^8 - 16 E_1^8 E4^{16}$ and the ideal $I = \langle W_1 \cup W_2 \cup W_3 \rangle_{\mathbb{Q}[Y,E,t]}$ in $\mathbb{Q}[Y, E, t]$ where

$$\begin{aligned} W_1 &:= \left\{ t_2 - 2Y_2E_4^2, t_3 - Y_1^2Y_4^2E_2^5, t_4 - Y_2E_1^2 \right\}, \\ W_2 &:= \left\{ Y_\delta E_\delta - 1 \mid \delta = 1, 2, 4 \right\}, \\ W_2 &:= \{ r \}. \end{aligned}$$

W_1 encodes the relations (8) and W_2 just says that Y_δ models the inverse of E_δ.

Computing the relations among eta functions of level 4 as described in [6] leads to an ideal that is generated by only one polynomial, namely r, i.e.,

$$\ker(\Xi|_{\mathbb{Q}[E]}) = \langle r \rangle_{\mathbb{Q}[E]} \tag{9}$$

where $\Xi|_{\mathbb{Q}[E]}$ denotes the restriction of the map Ξ to $\mathbb{Q}[E]$.

Clearly, $I \subseteq \ker \Xi$. To prove $\ker \Xi \subseteq I$, consider $f \in \ker \Xi$. By W_1 we can find a polynomial $f_1 \in \mathbb{Q}[Y, E]$ with $f + I = f_1 + I$. Note that by W_2 we have $Y_\delta E_\delta + I = 1 + I$. Thus, similar to "clearing a common denominator", by multiplication of each term of f_1 with an appropriate power of $Y_\delta E_\delta$, we can find a polynomial $f_2 \in \mathbb{Q}[E]$ and a vector $\alpha \in \mathbb{N}^3$ such that $f + I = Y^\alpha f_2 + I$. Since $\Xi(Y^\alpha) \neq 0$, it follows $\Xi(f_2) = 0$ and, thus, $f_2 \in \ker(\Xi|_{\mathbb{Q}[E]})$. From (9) we conclude that there is $\tilde{p} \in \mathbb{Q}[E]$ such that $f_2 = \tilde{p} \cdot r$. Therefore, $f \in I = \ker \Xi$.

Since we are actually interested in $\ker \xi = \ker \Xi \cap \mathbb{Q}[t]$, we can simply compute a Gröbner basis of I and intersect with $\mathbb{Q}[t]$. We find $I \cap \mathbb{Q}[t] = \left\langle t_2^4 - t_3^4 + t_4^4 \right\rangle_{\mathbb{Q}[t]}$. This polynomial corresponds to [4, Eq. 20.7.5]. In particular, that result says that there is no polynomial $p \in \mathbb{Q}[t_2, t_4]$ such that $p(\theta_2(0), \theta_4(0)) = 0$. Hence, $F := \mathbb{Q}(t_2, t_4)$ is isomorphic to $\mathbb{Q}(\theta_2(0), \theta_4(0))$. Since $t_2^4 - t_3^4 + t_4^4$ is irreducible over $F[t_3]$, it follows from the First Isomorphism Theorem that

$$K \cong F(\theta_3(0|\tau)) \cong F[t_3] / \left\langle t_2^4 - t_3^4 + t_4^4 \right\rangle. \tag{10}$$

7 Computation of the ideal of relations in FriCAS

Having a finite (and computable) representation for the coefficient field K, we now demonstrate the steps to compute $\ker \phi$ in the computer algebra system FriCAS[1]. Due to its type system, FriCAS allows to almost naturally enter the respective data structures in order to compute the Gröbner basis of $\ker \phi$.

We try to use almost the same identifiers in the following FriCAS session as we use in the mathematical notation above.

Let us start with setting up the field K and the two coefficients c_3 and c_4 that are used in the definition of h_1 and h_2 in (5) and (6).

```
N ==> NonNegativeInteger; Q ==> Fraction Integer
D ==> HomogeneousDistributedMultivariatePolynomial([t2,t4], Q)
F ==> Fraction D; R ==> UnivariatePolynomial('t3, F)
```

[1]FriCAS 1.3.4 [5]

```
4 r: R := t2^4 -t3^4 + t4^4;
5 K := SimpleAlgebraicExtension(F, R, r)
6 t2: K := 't2; t3: K := 't3; t4: K := 't4::K
7 c3 := (t3/t2)^2; c4 := (t4/t2)^2;
```

Next, we create the data structure for $P = K[S, T]$.

```
8 vars := [S1, S2, S3, S4, T1, T2, T3, T4];
9 E ==> SplitHomogeneousDirectProduct(8, 4, N)
10 P ==> GeneralDistributedMultivariatePolynomial(vars, K, E)
```

Now, we setup the generators of $\ker \Phi$ and compute a Gröbner basis.

```
11 U: List(P) := [S1*T1-1, S2*T2-1, S3*T3-1, S4*T4-1]
12 h1: P := (S1*T3)^2 - c3*(S1*T2)^2 - c4
13 h2: P := (S1^3*T2*T3*T4)^2 - (c4*S1^2*T2^2+c3)*S1^4*T2^2*T3^2
14 B := groebner(concat [U, [h1, h2]])
```

Eventually, we compute a Gröbner basis of the intersection $\ker \phi = \ker \Phi \cap K[T]$ and take advantage of the fact that, if B is a Gröbner basis with respect to a termorder where any term that involves only variables from the set T is smaller than any term that involves at least one variable from the set S, then $B \cap K[T]$ is a Gröbner basis. We have defined the terms E in line 9 in exactly such a way, i.e., we can simply extract all the polynomials from G that have a vanishing total degree in the indeterminates S.

```
15 G := [x for x in B | zero? reduce(_+, degree(x, vars(1..4)))]
16 [(t2::K)^2*x for x in gbT]  -- make it denominator-free
```

The computation returns the polynomials

$$
\begin{aligned}
g_1 &:= t_2^2 T_1^2 + t_4^2 T_3^2 - t_3^2 T_4^2, \\
g_2 &:= t_2^2 T_2^2 - t_3^2 T_3^2 + t_4^2 T_4^2.
\end{aligned}
$$

as generators of $\ker \phi$, i.e., $G := \langle g_1, g_2 \rangle_{K[T]}$. In view of the isomorphism given in 10, these are exactly the polynomials b_1 and b_2 as given by (2) and (3).

Having a Gröbner basis of the ideal of all polynomial relations among the classical Jacobi theta functions with coefficients involving $\theta_2(0)$, $\theta_3(0)$, and $\theta_4(0)$, allows for a simple decision procedure to check whether a given polynomial expression p in $\theta_2(0)$, $\theta_3(0)$, $\theta_4(0)$, $\theta_1(z)$, $\theta_2(z)$, $\theta_3(z)$, $\theta_4(z)$ is zero or not. One would simply have to translate this expression into a polynomial p in $t_2, t_3, t_4, T_1, T_2, T_3, T_4$ and then apply the function `normalForm` in FriCAS.

As an example, take the identity [4, Eq. 20.7.3]. We can enter it into FriCAS like

```
17 p: P := t3^2*T1^2 + t4^2*T2^2 - t2^2*T4^2
18 normalForm(p, G)
```

FriCAS returns 0 if an only if identity $p(\theta) = 0$ holds. In this case 0 is indeed computed.

One can easily program an extended normalform computation that collects the cofactors during the normalform computation and that leads to a representation of the form $p = p_2 g_1 + p_2 g_2$. In the above, we get $p_1 = c_3$ and $p_2 = c_4$.

8 Conclusion

We have shown that any polynomial identity in Jacobi theta functions can be expressed as a $K[T]$-linear combination of just two polynomials. Moreover such a linear combination can be computed algorithmically by a simple reduction process.

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