

# The functional equation of Dedekind's $\eta$ -function

Tobias Magnusson\*

June 15, 2018

## Abstract

We give an account of a proof of the functional equation of Dedekind's  $\eta$ -function due to Siegel [Sie54] and Gordon. The exposition is based on Apostol's book "Modular Functions and Dirichlet Series in Number Theory".

## Introduction

In this report we give a complete account of a proof of a well-known property of Dedekind's  $\eta$ -function, namely the functional equation that it satisfies with respect to the action of  $\mathrm{PSL}_2(\mathbb{Z})$  on the upper half-plane  $\mathbb{H}$ . In the first part we prove the functional equation in a special case, and then this is used in the second part to prove the general functional equation.

## Part 1

Let  $\mathrm{PSL}_2(\mathbb{Z})$  act on  $\mathbb{H}$  by Möbius transformations and write

$$[A].\tau = \frac{A_{1,1}\tau + A_{1,2}}{A_{2,1}\tau + A_{2,2}},$$

where  $A \in \mathrm{SL}_2(\mathbb{Z})$ ,  $[A]$  denotes the coset  $A\{I, -I\}$ , and  $\tau \in \mathbb{H}$ . Recall that  $\mathrm{SL}_2(\mathbb{Z})$  is freely generated by the matrices

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \text{ and } T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

We also recalled that Dedekind's  $\eta$ -function, is given by

$$\eta(\tau) = q^{1/24} \prod_{n \geq 1} (1 - q^n),$$

where  $q = q(\tau) = e^{2\pi i \tau}$ . We recall the following proposition.

**Proposition 1.** The function  $\eta$  is holomorphic on  $\mathbb{H}$ .

*Proof.* We need only prove that  $\prod_{n=1}^N (1 - q^n)$  converges uniformly on compact subsets of  $\mathbb{H}$  as  $N \rightarrow \infty$ . That the limit is holomorphic then follows by Morera's theorem.<sup>1</sup> Let therefore  $C$  be any compact subset of  $\mathbb{H}$ . We

\*This research was funded by grant SFBF50-06 of the Austrian Science Fund (FWF).

<sup>1</sup>Let  $\{f_n\}_{n \geq 1}$  be a sequence of holomorphic functions on a domain  $D$  that converges uniformly to  $f : D \rightarrow \mathbb{C}$  on all compact subsets of  $D$ . Let  $\Delta \subset D$  be any closed triangle in  $D$ . Since each  $f_n$  is holomorphic, we've by Cauchy's theorem that  $\int_{\partial\Delta} f_n(z) dz = 0$  for every  $n$ . Hence

$$0 \leq \left| \int_{\partial\Delta} f(z) dz \right| = \left| \int_{\partial\Delta} (f(z) - f_n(z)) dz \right| \leq \text{length}(\partial\Delta) \sup_{z \in \partial\Delta} |f(z) - f_n(z)|.$$

By hypothesis, we have that  $f_n$  converges uniformly to  $f$  on  $\Delta$ . Hence the right hand side goes to zero as  $n$  goes to infinity. In conclusion  $\int_{\partial\Delta} f(z) dz \rightarrow 0$  and so by Morera's theorem  $f(z)$  is holomorphic on  $D$ .

have by the extreme value theorem that  $\sup_{z \in C} |q^n| = e^{-2\pi n \Im(\tau_0)}$  for some  $\tau_0 \in C$  and  $\sum_{n \geq 1} e^{-2\pi n \Im(\tau_0)} = e^{-2\pi \Im(\tau_0)} / (1 - e^{-2\pi \Im(\tau_0)})$  whence it follows by the  $M$ -test that  $\prod_{n \geq 1} (1 - q^n)$  converges uniformly on  $C$ .  $\square$

In this part of the report, we concern ourselves with proving the following theorem.

**Theorem 1.** Let  $\tau \in \mathbb{H}$ . Then

$$\eta([S].\tau) = (-i\tau)^{\frac{1}{2}} \eta(\tau), \quad (\text{FUNCEQ})$$

where  $x^{\frac{1}{2}} > 0$  for  $x > 0$ .

*Proof.* We have that zeroes of non-zero holomorphic functions are isolated and that both the left hand side and right hand side of the identity are holomorphic. Therefore their difference is also holomorphic. We shall prove that the difference is zero on a path-connected subset  $L$  of  $\mathbb{H}$ . This then implies that the difference is identically zero, whence we have the theorem.

Let  $L = \{iy : 0 < y \in \mathbb{R}\}$  and let  $\tau \in L$ . Then (2) is equivalent to

$$\eta(i/y) = \sqrt{y} \eta(iy),$$

which can be rewritten as

$$e^{-\frac{\pi}{12y}} \prod_{n \geq 1} (1 - e^{-2\pi n/y}) = \sqrt{y} e^{-\frac{\pi y}{12}} \prod_{n \geq 1} (1 - e^{-2\pi n y}). \quad (1)$$

Let now  $\alpha > 0$  be arbitrary and notice that

$$\begin{aligned} \log\left(\prod_{n \geq 1} (1 - e^{-2\pi n \alpha})\right) &= \sum_{n \geq 1} \log(1 - e^{-2\pi n \alpha}) \\ &= - \sum_{n \geq 1} \sum_{m \geq 1} \frac{e^{-2\pi n m \alpha}}{m} \\ &= - \sum_{m \geq 1} \frac{1}{m} \sum_{n \geq 1} (e^{-2\pi m \alpha})^n \\ &= - \sum_{m \geq 1} \frac{1}{m} \frac{e^{-2\pi m \alpha}}{1 - e^{-2\pi m \alpha}} \\ &= \sum_{m \geq 1} \frac{1}{m} \frac{1}{1 - e^{2\pi m \alpha}}. \end{aligned}$$

Noticing that  $y, 1/y > 0$  and taking logarithms on both side of (1) we thus see that (2) is equivalent to

$$-\frac{\pi}{12y} + \sum_{m \geq 1} \frac{1}{m} \frac{1}{1 - e^{2\pi m/y}} = \frac{1}{2} \log(y) - \frac{\pi y}{12} + \sum_{m \geq 1} \frac{1}{m} \frac{1}{1 - e^{2\pi m y}},$$

which we can rearrange to

$$\sum_{m \geq 1} \frac{1}{m} \frac{1}{1 - e^{2\pi m y}} - \sum_{m \geq 1} \frac{1}{m} \frac{1}{1 - e^{2\pi m/y}} + \frac{\pi}{12y} - \frac{\pi y}{12} = -\frac{1}{2} \log(y), \quad (\text{FUNCEQ2})$$

and this is what we shall prove. The genius of Siegel is to notice that we can prove (FUNCEQ2) by evaluating a complex integral in two different ways.

Let  $y > 0$  be fixed, let  $n \geq 1$  be an integer, put  $N = n + \frac{1}{2}$ , and define the integrand we shall use as follows

$$F_n(z) = -\frac{1}{8z} \cot(\pi i N z) \cot\left(\frac{\pi N z}{y}\right).$$

Let  $C$  be the parallelogram connecting  $y, i, -y, -i$ , and  $y$ , and let  $\partial C$  be the boundary (positively oriented). By the residue theorem we have that

$$\int_{\partial C} F_n(z) dz = 2\pi i \sum_{\rho \in C} \text{Res}_{z=\rho}(F_n(z)),$$

where  $\rho$  are the poles of  $F_n(z)$  inside  $C$ . To find the poles we write

$$F_n(z) = -\frac{1}{8} \frac{\cos(\pi i N z) \cos(\frac{\pi N z}{y})}{z \sin(\pi i N z) \sin(\frac{\pi N z}{y})},$$

and since neither the numerator nor denominator has poles, we need only find the zeros of the numerator and denominator. Clearly we have that

$$\begin{aligned} \cos(\pi i N z) &= 0 \text{ if and only if } z = \frac{i(2k+1)}{2N} \\ \cos(\frac{\pi N z}{y}) &= 0 \text{ if and only if } z = \frac{y(2k+1)}{2N} \\ \sin(\pi i N z) &= 0 \text{ if and only if } z = \frac{ik}{N} \\ \sin(\frac{\pi N z}{y}) &= 0 \text{ if and only if } z = \frac{ky}{N} \end{aligned}$$

where the  $k$  are independent from one another. Trivially we also have the zero  $z = 0$  of the denominator. We see that the zeros of the numerator and denominator do not coincide and hence the poles of  $F_n(z)$  are precisely the zeros of the denominator.

Every zero of  $\sin$  is simple, and clearly the zero of  $z$  is simple. Therefore, concerning the denominator, we have simple zeros in  $ik/N$  and  $ky/N$  for  $k \neq 0$ , and a triple zero in  $z = 0$ . Consequently the poles of  $F_n(z)$  inside of  $C$  are given by the table below.

Zero	Range	Order
$ik/N$	$-n \leq k \leq n$ and $k \neq 0$	1
$ky/N$	$-n \leq k \leq n$ and $k \neq 0$	1
0	N/A	3

We now compute the integral in the first way – by residue calculus. To find the residue of  $z = 0$  we compute the Laurent series about  $z = 0$ . We have that

$$\begin{aligned} \cos(\pi i N z) &= 1 + \frac{(\pi N z)^2}{2} + O(z^4), \text{ and} \\ \cos(\frac{\pi N z}{y}) &= 1 - \frac{(\pi N z)^2}{2y^2} + O(z^4). \end{aligned}$$

Hence we have that

$$\cos(\pi i N z) \cos(\frac{\pi N z}{y}) = 1 + \frac{(\pi N z)^2}{2} (1 - \frac{1}{y^2}) + O(z^4).$$

We furthermore have that

$$\begin{aligned} \sin(\pi i N z) &= \pi i N z (1 + \frac{(\pi N z)^2}{3!} + O(z^4)), \text{ and} \\ \sin(\frac{\pi N z}{y}) &= \frac{\pi N z}{y} (1 - \frac{(\pi N z)^2}{3! y^2} + O(z^4)). \end{aligned}$$

Hence we have that

$$\sin(\pi i N z) \sin\left(\frac{\pi N z}{y}\right) = \frac{i(\pi N z)^2}{y} \left(1 + \frac{(\pi N z)^2}{3!} \left(1 - \frac{1}{y^2}\right) + O(z^4)\right).$$

Taking geometric series we find that

$$\begin{aligned} \frac{1}{1 + \frac{(\pi N z)^2}{3!} \left(1 - \frac{1}{y^2}\right) + O(z^4)} &= 1 - \left(\frac{(\pi N z)^2}{3!} \left(1 - \frac{1}{y^2}\right) + O(z^4)\right) + \left(\frac{(\pi N z)^2}{3!} \left(1 - \frac{1}{y^2}\right) + O(z^4)\right)^2 + \dots \\ &= 1 - \frac{(\pi N z)^2}{3!} \left(1 - \frac{1}{y^2}\right) + O(z^4). \end{aligned}$$

We now have that

$$\begin{aligned} F_n(z) &= \frac{i y}{8(\pi N)^2 z^3} \left(1 + \frac{(\pi N z)^2}{2} \left(1 - \frac{1}{y^2}\right) + O(z^4)\right) \left(1 - \frac{(\pi N z)^2}{3!} \left(1 - \frac{1}{y^2}\right) + O(z^4)\right) \\ &= \frac{i y}{8(\pi N)^2 z^3} \left(1 + \frac{(\pi N z)^2}{3} \left(1 - \frac{1}{y^2}\right) + O(z^4)\right) \\ &= \frac{i y}{8(\pi N)^2} \frac{1}{z^3} + \frac{i(y - 1/y)}{24} \frac{1}{z} + O(z). \end{aligned}$$

Hence we conclude that

$$\text{Res}_{z=0}(F_n(z)) = \frac{i}{24} \left(y - \frac{1}{y}\right).$$

As for the simple poles, we have that<sup>2</sup>

$$\begin{aligned} \text{Res}_{z=ik/N}(F_n(z)) &= \frac{-\cos(-\pi k) \cos(i\pi k/y)}{\frac{d}{dz} (8z \sin(\pi i N z) \sin(\pi N z/y))|_{z=ik/N}} \\ &= \frac{-\cos(-\pi k) \cos(i\pi k/y)}{8(ik/N)(\pi i N) \cos(-\pi k) \sin(i\pi k/y)} \\ &= \frac{\cot(i\pi k/y)}{8\pi k}, \end{aligned}$$

and

$$\begin{aligned} \text{Res}_{z=ky/N}(F_n(z)) &= \frac{-\cos(i\pi ky) \cos(\pi k)}{8(ky/N) \sin(i\pi ky) (\pi N/y) \cos(\pi k)} \\ &= -\frac{\cot(i\pi ky)}{8\pi k}. \end{aligned}$$

Notice now that

$$\text{Res}_{z=ik/N}(F_n(z)) = \text{Res}_{z=-ik/N}(F_n(z)) \text{ and } \text{Res}_{z=ky/N}(F_n(z)) = \text{Res}_{z=-ky/N}(F_n(z)),$$

and also that

$$\cot(i\theta) = i \left( \frac{e^{-\theta} + e^{\theta}}{e^{-\theta} - e^{\theta}} \right) = \frac{1}{i} \left( 1 - \frac{2}{1 - e^{2\theta}} \right).$$

---

<sup>2</sup>Recall that if  $f(z) = \frac{g(z)}{h(z)}$  with  $g(c) \neq 0$  and  $h(c) = 0$  simple, then  $\text{Res}_{z=c}(f(z)) = \frac{g(c)}{h'(c)}$  as can be easily seen through Taylor expansion.

By the residue theorem and the above two facts we now have that

$$\begin{aligned}
\int_{\partial C} F_n(z) dz &= 2\pi i \left( \frac{i}{24} \left( y - \frac{1}{y} \right) + \sum_{\substack{-n \leq k \leq n \\ k \neq 0}} \operatorname{Res}_{z=ik/N} (F_n(z)) + \sum_{\substack{-n \leq k \leq n \\ k \neq 0}} \operatorname{Res}_{z=ky/N} (F_n(z)) \right) \\
&= -\frac{\pi}{12} \left( y - \frac{1}{y} \right) + 4\pi i \left( \sum_{k=1}^n \frac{1}{8\pi i k} \left( 1 - \frac{2}{1 - e^{2\pi k/y}} \right) - \sum_{k=1}^n \frac{1}{8\pi i k} \left( 1 - \frac{2}{1 - e^{2\pi ky}} \right) \right) \\
&= -\frac{\pi}{12} \left( y - \frac{1}{y} \right) + \frac{1}{2} \left( 2 \sum_{k=1}^n \frac{1}{k} \frac{1}{1 - e^{2\pi ky}} - 2 \sum_{k=1}^n \frac{1}{k} \frac{1}{1 - e^{2\pi k/y}} \right) \\
&= \sum_{k=1}^n \frac{1}{k} \frac{1}{1 - e^{2\pi ky}} - \sum_{k=1}^n \frac{1}{k} \frac{1}{1 - e^{2\pi k/y}} + \frac{\pi}{12y} - \frac{\pi y}{12}.
\end{aligned}$$

From this it is evident that (FUNCEQ2) is true if and only if

$$\lim_{n \rightarrow \infty} \int_{\partial C} F_n(z) dz = -\frac{1}{2} \log(y).$$

We now evaluate the integral in the second way. As is shown in the appendix, we have that  $F_n(z)$  is uniformly bounded on  $\partial C$  and therefore, by Arzelà's bounded convergence theorem<sup>3</sup>, we have that

$$\lim_{n \rightarrow \infty} \int_{\partial C} F_n(z) dz = \int_{\partial C} \lim_{n \rightarrow \infty} F_n(z) dz.$$

To compute the limit, we parametrize  $\partial C$  into line segments  $S_1$ ,  $S_2$ ,  $S_3$  and  $S_4$ , in the first, second, third and fourth quadrants, respectively. Clearly we have that

$$\begin{aligned}
S_1(t) &= (1-t)y + it \\
S_2(t) &= (1-t)i - yt \\
S_3(t) &= -S_1(t) \\
S_4(t) &= -S_2(t).
\end{aligned}$$

We also have that

$$\lim_{b \rightarrow \infty} \cot(a + bi) = \lim_{b \rightarrow \infty} i \left( \frac{e^{i2a} e^{-2b} + 1}{e^{i2a} e^{-2b} - 1} \right) = -i,$$

and that

$$\begin{aligned}
\pi i N S_1(t) &= -\pi N t + i\pi N (1-t)y \\
\frac{\pi N S_1(t)}{y} &= \pi N (1-t) + i \frac{\pi N t}{y} \\
\pi i N S_2(t) &= -\pi N (1-t) - i\pi N y t \\
\frac{\pi N S_2(t)}{y} &= -\pi N t + i \frac{\pi N (1-t)}{y}.
\end{aligned}$$

---

<sup>3</sup>Let  $f_n(z)$  be a series of functions from some domain  $D \subset \mathbb{C}$  to  $\mathbb{C}$  that converges pointwise to some integrable function  $f : D \rightarrow \mathbb{C}$  and let  $\gamma$  be a curve in  $D$ . Then if  $f_n$  is uniformly bounded on  $\gamma$  we have that  $\lim_{n \rightarrow \infty} \int_{\gamma} f_n(z) dz = \int_{\gamma} f(z) dz$ .

We thus conclude that

$$\begin{aligned}
\lim_{n \rightarrow \infty} \cot(\pi i N S_1(t)) &= -i \\
\lim_{n \rightarrow \infty} \cot\left(\frac{\pi N S_1(t)}{y}\right) &= -i \\
\lim_{n \rightarrow \infty} \cot(\pi i N S_2(t)) &= i \\
\lim_{n \rightarrow \infty} \cot\left(\frac{\pi N S_2(t)}{y}\right) &= -i \\
\lim_{n \rightarrow \infty} \cot(\pi i N S_3(t)) &= i \\
\lim_{n \rightarrow \infty} \cot\left(\frac{\pi N S_3(t)}{y}\right) &= i \\
\lim_{n \rightarrow \infty} \cot(\pi i N S_4(t)) &= -i \\
\lim_{n \rightarrow \infty} \cot\left(\frac{\pi N S_4(t)}{y}\right) &= i,
\end{aligned}$$

where  $0 < t < 1$  and where we used that  $\cot$  is continuous and odd in equalities 3 and 5 to 8. From this we can conclude that

$$\begin{aligned}
\lim_{n \rightarrow \infty} F_n(S_1(t)) &= -\frac{1}{8S_1(t)}(-i)^2 = \frac{1}{8S_1(t)} \\
\lim_{n \rightarrow \infty} F_n(S_2(t)) &= -\frac{1}{8S_2(t)}(-i^2) = -\frac{1}{8S_2(t)} \\
\lim_{n \rightarrow \infty} F_n(S_3(t)) &= -\frac{1}{8S_3(t)}i^2 = \frac{1}{8S_3(t)} \\
\lim_{n \rightarrow \infty} F_n(S_4(t)) &= -\frac{1}{8S_4(t)}(-i^2) = -\frac{1}{8S_4(t)},
\end{aligned}$$

where  $0 < t < 1$ .

Hence we have that

$$\begin{aligned}
\int_{\partial C} \lim_{n \rightarrow \infty} F_n(z) dz &= \frac{1}{8} \left( \int_{S_1} \frac{dz}{z} - \int_{S_2} \frac{dz}{z} + \int_{S_3} \frac{dz}{z} - \int_{S_4} \frac{dz}{z} \right) \\
&= \frac{1}{8} \left( -\log(y) + \frac{i\pi}{2} - (\log(y) + \frac{i\pi}{2}) - \log(y) + \frac{i\pi}{2} - (\log(y) + \frac{i\pi}{2}) \right) \\
&= -\frac{1}{2} \log(y),
\end{aligned}$$

and we are done.<sup>4</sup> □

---

<sup>4</sup>It is important to notice that we cannot use the principal branch of the logarithm as a primitive on the segments  $S_2$  and  $S_3$ , as it has a jump discontinuity at  $-y$ . In the computation above, we used the  $[0, 2\pi)$  branch.

## Part 2

Before we can state the general functional equation, we need to have a basic understanding of so-called Dedekind sums.

**Definition 1.** Let  $h, k$  be integers with  $\gcd(h, k) = 1$  and  $k > 0$ . Then the Dedekind sum  $s(h, k)$  is defined as

$$s(h, k) = \sum_{r=1}^{k-1} \frac{r}{k} \left( \frac{hr}{k} - \left\lfloor \frac{hr}{k} \right\rfloor - \frac{1}{2} \right).$$

**Proposition 2.** Let  $h, k$  be as in definition 1, and let  $m$  be an integer. Then

$$s(-h, k) = -s(h, k) \tag{2}$$

$$s(h + km, k) = s(h, k). \tag{3}$$

*Proof.* As for (2), let  $x \in \mathbb{R}$  and notice that

$$\lfloor -x \rfloor = -\lfloor x \rfloor - 1,$$

so that

$$-x - \lfloor -x \rfloor - \frac{1}{2} = -(x + \lfloor -x \rfloor + \frac{1}{2}) = -(x - \lfloor x \rfloor - \frac{1}{2}),$$

and thus (2) holds. As for (3), let  $l$  be an integer and notice that

$$\lfloor x + l \rfloor = \lfloor x \rfloor + l,$$

so that

$$\frac{(h + km)r}{k} - \left\lfloor \frac{(h + km)r}{l} \right\rfloor - \frac{1}{2} = \frac{hr}{k} + mr - \left( \left\lfloor \frac{hr}{k} \right\rfloor + mr \right) - \frac{1}{2} = \frac{hr}{k} - \left\lfloor \frac{hr}{k} \right\rfloor - \frac{1}{2},$$

where  $1 \leq r \leq k - 1$ . Therefore (3) holds. □

**Proposition 3** (Reciprocity law). Let  $h, k$  be integers with  $\gcd(h, k) = 1$  and  $h, k > 0$ . Then

$$s(h, k) + s(k, h) = \frac{1}{12hk} (h^2 + k^2 - 3hk + 1).$$

*Proof.* See [Apo90, Theorem 3.7, pp. 62-64]. □

We are now ready to state the general functional equation.

**Theorem 2.** If  $A \in \mathrm{SL}_2(\mathbb{Z})$  and  $A_{2,1} \neq 0$ , then for every  $\tau \in \mathbb{H}$  we have

$$\eta([A].\tau) = \begin{cases} \epsilon(A) \{-i(A_{2,1}\tau + A_{2,2})\}^{1/2} \eta(\tau) & \text{if } A_{2,1} > 0 \\ \epsilon(-A) \{i(A_{2,1}\tau + A_{2,2})\}^{1/2} \eta(\tau) & \text{if } A_{2,1} < 0 \end{cases}$$

where

$$\epsilon(A) = \exp \left\{ \pi i \left( \frac{A_{1,1} + A_{2,2}}{12A_{2,1}} - s(A_{2,2}, A_{2,1}) \right) \right\},$$

and  $s(\cdot, \cdot)$  is a Dedekind sum.

If  $A \in \mathrm{SL}_2(\mathbb{Z})$  and  $A_{2,1} = 0$ , then there exists an integer  $k$  such that for every  $\tau \in \mathbb{H}$  we have

$$\eta([A].\tau) = e^{\frac{i\pi k}{12}} \eta(\tau).$$

The proof will rely on induction with  $\eta([S].\tau) = (-i\tau)^{1/2} \eta(\tau)$  as a base case.<sup>5</sup> The starting point is the following proposition.

---

<sup>5</sup>Also  $\eta([T^m].\tau) = e^{\frac{i\pi m}{12}} \eta(\tau)$  would do as a base case.

**Proposition 4.** The group  $\mathrm{SL}_2(\mathbb{Z})$  is freely generated by the matrices  $S$  and  $T$ .

For the induction step we thus need to prove that if theorem 2 is true for a matrix  $A \in \mathrm{SL}_2(\mathbb{Z})$ , then it also true for the matrices  $AT^m$  and  $AS$ , where  $m$  is an integer.

**Lemma 1.** If  $A \in \mathrm{SL}_2(\mathbb{Z})$  with  $A_{2,1} > 0$ , then for every integer  $m$  we have

$$\epsilon(AT^m) = e^{\frac{i\pi m}{12}} \epsilon(A).$$

*Proof.* For the sake of convenience, let us write

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then we have that

$$AT^m = \begin{pmatrix} a & b + am \\ c & d + cm \end{pmatrix},$$

whence

$$\begin{aligned} \epsilon(AT^m) &= \exp \left\{ \pi i \left( \frac{a + d + cm}{12c} - s(d + cm, c) \right) \right\} \\ &= \exp \left\{ \frac{i\pi m}{12} + \pi i \left( \frac{a + d}{12c} - s(d + cm, c) \right) \right\} \\ &= e^{\frac{i\pi m}{12}} \epsilon(A), \end{aligned}$$

where the last step follows from equation (3) of proposition 2. □

**Lemma 2.** If  $A \in \mathrm{SL}_2(\mathbb{Z})$  with  $A_{2,1} > 0$  and  $d = (AS)_{2,1} \neq 0$ , then we have

$$\epsilon(\mathrm{sgn}(d)AS) = e^{-i\mathrm{sgn}(d)\pi/4} \epsilon(A).$$

*Proof.* With the same notation as before, we have that

$$AS = \begin{pmatrix} b & -a \\ d & -c \end{pmatrix}.$$

So that

$$\epsilon(AS) = \exp \left\{ \pi i \left( \frac{b - c}{12d} - s(-c, d) \right) \right\}.$$

By proposition 2 we have that  $s(-c, d) = -s(c, d)$  and by proposition 3 we have that

$$\begin{aligned} s(c, d) + s(d, c) &= \frac{d}{12c} + \frac{c}{12d} - \frac{1}{4} + \frac{1}{12cd} \\ &= \frac{d}{12c} + \frac{c}{12d} - \frac{1}{4} + \frac{ad - bc}{12cd} \\ &= \frac{a + d}{12c} + \frac{c - d}{12d} - \frac{1}{4}. \end{aligned}$$

Rearranging we get

$$\frac{b - c}{12d} + s(c, d) = \frac{a + d}{12c} - s(d, c) - \frac{1}{4},$$

whence

$$\begin{aligned} \exp \left\{ \pi i \left( \frac{b - c}{12d} - s(-c, d) \right) \right\} &= \exp \left\{ \pi \left( \frac{b - c}{12d} + s(c, d) \right) \right\} \\ &= \exp \left\{ \pi i \left( \frac{a + d}{12c} - s(d, c) \right) - \frac{i\pi}{4} \right\} \\ &= \exp \left\{ \frac{-i\pi}{4} \right\} \epsilon(A), \end{aligned}$$

and we are done. For the other case, simply substitute  $A$  with  $-A$  in the above. □



We are now ready to prove theorem 2. Evidently the theorem is true for the matrix  $S$ , and so we are done with the base case. Suppose that the theorem holds for the matrix  $A \in \text{SL}_2(\mathbb{Z})$ . We either have that  $A_{2,1} < 0$ ,  $A_{2,1} > 0$  or  $A_{2,1} = 0$  and split into cases accordingly.

### The case $A_{2,1} > 0$

Let  $m \in \mathbb{Z}$  be arbitrary. Then

$$\begin{aligned}
\eta([AT^m].\tau) &= \eta([A].([T^m].\tau)) \\
&= \epsilon(A)\{-i(A_{2,1}([T^m].\tau) + A_{2,2})\}^{1/2}\eta([T^m].\tau) \\
&= \epsilon(A)\{-i(A_{2,1}(\tau + m) + A_{2,2})\}^{1/2}e^{\frac{i\pi m}{12}}\eta(\tau) \\
&= \epsilon(AT^m)\{-i(A_{2,1}\tau + A_{2,2} + mA_{2,1})\}^{1/2}\eta(\tau) \\
&= \epsilon(AT^m)\{-i((AT^m)_{2,1}\tau + (AT^m)_{2,2})\}^{1/2}\eta(\tau).
\end{aligned}$$

If  $A_{2,2} > 0$  we have  $(AS)_{2,1} > 0$  and so

$$\begin{aligned}
\eta([AS].\tau) &= \eta([A].([S].\tau)) \\
&= \epsilon(A)\{-i(A_{2,1}([S].\tau) + A_{2,2})\}^{1/2}\eta([S].\tau) \\
&= \epsilon(A)\{-i(A_{2,1}(-\frac{1}{\tau}) + A_{2,2})\}^{1/2}\{-i\tau\}^{1/2}\eta(\tau) \\
&= \epsilon(A)\{-i(\frac{A_{2,2}\tau - A_{2,1}}{\tau})\}^{1/2}\{-i\tau\}^{1/2}\eta(\tau) \\
&= \epsilon(A)\{e^{-i\pi/2}(\frac{-i(A_{2,2}\tau - A_{2,1})}{-i\tau})\}^{1/2}\{-i\tau\}^{1/2}\eta(\tau) \\
&= \epsilon(A)e^{-i\pi/4}\{-i(A_{2,2}\tau - A_{2,1})\}^{1/2}\eta(\tau) \\
&= \epsilon(AS)\{-i((AS)_{2,1}\tau + (AS)_{2,2})\}^{1/2}\eta(\tau).
\end{aligned}$$

If  $A_{2,2} < 0$  we have  $(AS)_{2,1} < 0$  and so

$$\begin{aligned}
\eta([AS].\tau) &= \eta([A].([S].\tau)) \\
&= \epsilon(A)\{-i(A_{2,1}([S].\tau) + A_{2,2})\}^{1/2}\eta([S].\tau) \\
&= \epsilon(A)\{-i(\frac{A_{2,2}\tau - A_{2,1}}{\tau})\}^{1/2}\{-i\tau\}^{1/2}\eta(\tau) \\
&= \epsilon(A)\{i(\frac{i(A_{2,2}\tau - A_{2,1})}{-i\tau})\}^{1/2}\{-i\tau\}^{1/2}\eta(\tau) \\
&= e^{i\pi/4}\epsilon(A)\{i((AS)_{2,1}\tau + (AS)_{2,2})\}^{1/2}\eta(\tau) \\
&= \epsilon(-AS)\{i((AS)_{2,1}\tau + (AS)_{2,2})\}^{1/2}\eta(\tau).
\end{aligned}$$

If  $A_{2,2} = 0$  we have  $(AS)_{2,1} = 0$  and so

$$A = \begin{pmatrix} k & -1 \\ 1 & 0 \end{pmatrix} \text{ or } A = \begin{pmatrix} k & 1 \\ -1 & 0 \end{pmatrix},$$

where the  $k$  are independent integers. Hence we have that

$$AS = -T^k \text{ or } AS = T^{-k}.$$

Therefore we get  $[AS].\tau = \tau + k$  for some independent integer  $k$ . Clearly now

$$\eta([AS].\tau) = e^{\frac{i\pi k}{12}}\eta(\tau).$$

### The case $A_{2,1} < 0$

Let  $m \in \mathbb{Z}$  be arbitrary. Then

$$\begin{aligned}
\eta([AT^m].\tau) &= \eta([-A].([T^m].\tau)) \\
&= \epsilon(-A)\{i(A_{2,1}([T^m].\tau) + A_{2,2})\}^{1/2} e^{\frac{i\pi m}{12}} \eta(\tau) \\
&= \epsilon(-A)\{i(A_{2,1}\tau + A_{2,2} + mA_{2,1})\}^{1/2} e^{\frac{i\pi m}{12}} \eta(\tau) \\
&= \epsilon(-AT^m)\{i((AT^m)_{2,1}\tau + (AT^m)_{2,2})\}^{1/2} \eta(\tau).
\end{aligned}$$

If  $A_{2,2} > 0$  we have  $(AS)_{2,1} > 0$  and so

$$\begin{aligned}
\eta([AS].\tau) &= \eta([-A].([S].\tau)) \\
&= \epsilon(-A)\{i(\frac{A_{2,2}\tau - A_{2,1}}{\tau})\}^{1/2} \{-i\tau\}^{1/2} \eta(\tau) \\
&= \epsilon(-A)\{e^{i\pi/2}(\frac{-i(A_{2,2}\tau - A_{2,1})}{-i\tau})\}^{1/2} \{-i\tau\}^{1/2} \eta(\tau) \\
&= e^{i\pi/4} \epsilon(-A)\{-i((AS)_{2,1}\tau + (AS)_{2,2})\}^{1/2} \eta(\tau) \\
&= \epsilon(-(-AS))\{-i((AS)_{2,1}\tau + (AS)_{2,2})\}^{1/2} \eta(\tau) \\
&= \epsilon(AS)\{-i((AS)_{2,1}\tau + (AS)_{2,2})\}^{1/2} \eta(\tau).
\end{aligned}$$

If  $A_{2,2} < 0$  we have that  $(AS)_{2,1} < 0$  and so

$$\begin{aligned}
\eta([AS].\tau) &= \eta([-A].([S].\tau)) \\
&= \epsilon(-A)\{i(\frac{A_{2,2}\tau - A_{2,1}}{\tau})\}^{1/2} \{-i\tau\}^{1/2} \eta(\tau) \\
&= \epsilon(-A)\{-i(\frac{i(A_{2,2}\tau - A_{2,1})}{-i\tau})\}^{1/2} \{-i\tau\}^{1/2} \eta(\tau) \\
&= e^{-i\pi/4} \epsilon(-A)\{i(A_{2,2}\tau - A_{2,1})\}^{1/2} \eta(\tau) \\
&= \epsilon(-AS)\{i((AS)_{2,1}\tau + (AS)_{2,2})\}^{1/2} \eta(\tau).
\end{aligned}$$

If  $A_{2,2} = 0$  we can use the same technique as in the  $A_{2,1} > 0$  case and we are done.

### The case $A_{2,1} = 0$

Now we have that  $A = \pm T^k$  for some integer  $k$ . Clearly then

$$\eta([AT^m].\tau) = e^{\frac{i\pi(k+m)}{12}} \eta(\tau).$$

If  $A = T^k$  then  $(AS)_{2,1} > 0$  and

$$\begin{aligned}
\eta([AS].\tau) &= e^{\frac{i\pi k}{12}} \eta([S].\tau) \\
&= e^{\frac{i\pi k}{12}} \{-i\tau\}^{1/2} \eta(\tau) \\
&= \epsilon(AS)\{-i((AS)_{2,1}\tau + (AS)_{2,2})\}^{1/2} \eta(\tau),
\end{aligned}$$

where the last step follows from the fact that  $s(h, 1) = 0$  for any integer  $h$ . Similarly if  $A = -T^k$ , then  $(AS)_{2,1} < 0$  and

$$\begin{aligned}
\eta([AS].\tau) &= e^{\frac{i\pi k}{12}} \eta([S].\tau) \\
&= \epsilon(-AS)\{i((AS)_{2,1}\tau + (AS)_{2,2})\}^{1/2} \eta(\tau).
\end{aligned}$$

Theorem 2 now follows by induction.

## Appendix

**Proposition 5.** The function  $F_n(z)$  is uniformly bounded on  $\partial C$ .

*Proof.* Notice that we have

$$|F_n(z)| = \frac{|\cos(\pi i N z)| |\cos(\frac{\pi N z}{y})|}{8|z| |\sin(\pi i N z)| |\sin(\frac{\pi N z}{y})|} \leq \frac{1}{8} \frac{1}{|z| |\sin(\pi i N z)| |\sin(\frac{\pi N z}{y})|}.$$

So we only have to bound  $|z|$ ,  $|\sin(\pi i N z)|$  and  $|\sin(\frac{\pi N z}{y})|$  from below. As we have seen before, we have that  $S_3(t) = -S_1(t)$  and  $S_4(t) = -S_2(t)$ , and since  $|-z| = |z|$  and  $|\sin(-z)| = |\sin(z)|$  we therefore only need to find bounds on  $S_1$  and  $S_2$ .

Let us start off by noticing that

$$|\sin(a + bi)|^2 = \frac{1}{4}(e^{2b} + e^{-2b} - 2\cos(2a)).$$

This will come in handy later.

### Bounds on $S_1$

We have that

$$\pi i N S_1(t) = -\pi N t + i\pi N(1-t)y,$$

and

$$\frac{\pi N S_1(t)}{y} = \pi N(1-t) + \frac{i\pi N t}{y}.$$

Consequently

$$|\sin(\pi i N S_1(t))|^2 = \frac{1}{4}(e^{2\pi N(1-t)y} + e^{-2\pi N(1-t)y} - 2\cos(2\pi N t)),$$

and

$$|\sin(\frac{\pi N S_1(t)}{y})|^2 = \frac{1}{4}(e^{2\pi N t/y} + e^{-2\pi N t/y} - 2\cos(2\pi N(1-t))).$$

Let now  $0 < \epsilon_1, \epsilon_2 < 1$  be parameters and say

$$e^{2\pi N(1-t)y} + e^{-2\pi N(1-t)y} > 2 + \epsilon_1, \tag{INEQ1}$$

and

$$e^{2\pi N t/y} + e^{-2\pi N t/y} > 2 + \epsilon_2. \tag{INEQ1}$$

Completing the square and using that  $2\pi N(1-t)y, 2\pi N t/y \geq 0$ , we find that (INEQ1) is equivalent to

$$t < 1 - \frac{1}{2\pi N y} \log\left(1 + \frac{\epsilon_1}{2} + \sqrt{\epsilon_1 + \frac{\epsilon_1^2}{4}}\right),$$

and that (INEQ1) is equivalent to

$$t > \frac{y}{2\pi N} \log\left(1 + \frac{\epsilon_2}{2} + \sqrt{\epsilon_2 + \frac{\epsilon_2^2}{4}}\right).$$

Clearly the negation of (INEQ1) is equivalent to

$$2\pi N - \frac{1}{y} \log\left(1 + \frac{\epsilon_1}{2} + \sqrt{\epsilon_1 + \frac{\epsilon_1^2}{4}}\right) \leq 2\pi N t \leq 2\pi N,$$

and the negation of (INEQ1) is equivalent to

$$2\pi N - y \log\left(1 + \frac{\epsilon_2}{2} + \sqrt{\epsilon_2 + \frac{\epsilon_2^2}{4}}\right) \leq 2\pi N(1-t) \leq 2\pi N.$$

We have that  $\cos(x)$  is decreasing when  $2\pi n \leq x \leq (2n+1)\pi$  so by fine-tuning  $\epsilon_1$  and  $\epsilon_2$  we can control the cos term. We want that

$$\frac{1}{y} \log\left(1 + \frac{\epsilon_1}{2} + \sqrt{\epsilon_1 + \frac{\epsilon_1^2}{4}}\right) < \pi, \quad (\text{FINETUNE1})$$

and

$$y \log\left(1 + \frac{\epsilon_2}{2} + \sqrt{\epsilon_2 + \frac{\epsilon_2^2}{4}}\right) < \pi. \quad (\text{FINETUNE2})$$

Since  $0 < \epsilon_i < 1$  we have that

$$1 + \frac{\epsilon_i}{2} + \sqrt{\epsilon_i + \frac{\epsilon_i^2}{4}} \leq 1 + \alpha\sqrt{\epsilon_i},$$

where  $\alpha = (1 + \sqrt{5})/2$ . Therefore we can force (FINETUNE1) and (FINETUNE2) to be true by letting

$$\log(1 + \alpha\sqrt{\epsilon_1}) < \pi y,$$

and

$$\log(1 + \alpha\sqrt{\epsilon_2}) < \frac{\pi}{y}.$$

Equivalently

$$\epsilon_1 < \left(\frac{e^{\pi y} - 1}{\alpha}\right)^2,$$

and

$$\epsilon_2 < \left(\frac{e^{\frac{\pi}{y}} - 1}{\alpha}\right)^2.$$

We therefore put

$$\epsilon_1 = \beta_1 \min\left\{1, \left(\frac{e^{\pi y} - 1}{\alpha}\right)^2\right\}, \quad (\text{EP1DEF})$$

and

$$\epsilon_2 = \beta_2 \min\left\{1, \left(\frac{e^{\frac{\pi}{y}} - 1}{\alpha}\right)^2\right\}, \quad (\text{EP2DEF})$$

for fixed (arbitrary)  $0 < \beta_1, \beta_2 < 1$ . If the negation of (INEQ1) is true, we now have that

$$-1 \leq \cos(2\pi Nt) \leq -\cos\left(\frac{1}{y} \log\left(1 + \frac{\epsilon_1}{2} + \sqrt{\epsilon_1 + \frac{\epsilon_1^2}{4}}\right)\right),$$

and if the negation of (INEQ1) is true, we now have that

$$-1 \leq \cos(2\pi N(1-t)) \leq -\cos\left(y \log\left(1 + \frac{\epsilon_2}{2} + \sqrt{\epsilon_2 + \frac{\epsilon_2^2}{4}}\right)\right).$$

This yields then

$$|\sin(\pi i N S_1(t))|^2 \geq \frac{1}{2} \left(1 + \cos\left(\frac{1}{y} \log\left(1 + \frac{\epsilon_1}{2} + \sqrt{\epsilon_1 + \frac{\epsilon_1^2}{4}}\right)\right)\right),$$

and

$$|\sin(\frac{\pi N S_1(t)}{y})|^2 \geq \frac{1}{2}(1 + \cos(y \log(1 + \frac{\epsilon_2}{2} + \sqrt{\epsilon_2 + \frac{\epsilon_2^2}{4}}))),$$

respectively.

In conclusion, we have that

$$|\sin(\pi i N S_1(t))|^2 \geq \min\{\frac{\epsilon_1}{4}, \frac{1}{2}(1 + \cos(\frac{1}{y} \log(1 + \frac{\epsilon_1}{2} + \sqrt{\epsilon_1 + \frac{\epsilon_1^2}{4}}))\},$$

and

$$|\sin(\frac{\pi N S_1(t)}{y})|^2 \geq \min\{\frac{\epsilon_2}{4}, \frac{1}{2}(1 + \cos(y \log(1 + \frac{\epsilon_2}{2} + \sqrt{\epsilon_2 + \frac{\epsilon_2^2}{4}}))\},$$

where  $\epsilon_1$  and  $\epsilon_2$  are given as in (EP1DEF) and (EP2DEF). For convenience we introduce the notation

$$A(\beta_1, y) = \min\{\frac{\epsilon_1}{4}, \frac{1}{2}(1 + \cos(\frac{1}{y} \log(1 + \frac{\epsilon_1}{2} + \sqrt{\epsilon_1 + \frac{\epsilon_1^2}{4}}))\},$$

and

$$B(\beta_2, y) = \min\{\frac{\epsilon_2}{4}, \frac{1}{2}(1 + \cos(y \log(1 + \frac{\epsilon_2}{2} + \sqrt{\epsilon_2 + \frac{\epsilon_2^2}{4}}))\}.$$

By completing the square, it is easy to see that  $|S_1(t)|^2 \geq y^2/(y^2 + 1)$  and hence we have that

$$|F_n(S_1(t))| \leq \frac{\sqrt{y^2 + 1}}{8y\sqrt{A(\beta_1, y)B(\beta_2, y)}}.$$

## Bounds on $S_2$

We apply the same technique as above. We have that

$$\pi i N S_2(t) = -\pi N(1-t) - i\pi N y t,$$

and

$$\frac{\pi N S_2(t)}{y} = -\pi N t + \frac{i\pi N(1-t)}{y},$$

and therefore

$$|\sin(\pi i N S_2(t))|^2 = \frac{1}{4}(e^{2\pi N y t} + e^{-2\pi N y t} - 2 \cos(2\pi N(1-t))),$$

and

$$|\sin(\frac{\pi N S_2(t)}{y})|^2 = \frac{1}{4}(e^{2\pi N(1-t)/y} + e^{-2\pi N(1-t)/y} - 2 \cos(2\pi N t)).$$

Putting as before

$$\epsilon_3 = \beta_3 \min\{1, (\frac{e^{\pi y} - 1}{\alpha})^2\},$$

and

$$\epsilon_4 = \beta_4 \min\{1, (\frac{e^{\frac{\pi}{y}} - 1}{\alpha})^2\},$$

for fixed (arbitrary)  $0 < \beta_3, \beta_4 < 1$ , we get

$$|\sin(\pi i N S_2(t))|^2 \geq C(\beta_3, y),$$

and

$$|\sin(\frac{\pi N S_2(t)}{y})|^2 \geq D(\beta_4, y),$$

where  $C(\beta_3, y)$  and  $D(\beta_4, y)$  are defined as  $A(\beta_1, y)$  and  $B(\beta_2, y)$ . It's again clear that  $|S_2(t)|^2 \geq y^2/(y^2 + 1)$  and hence we conclude that

$$|F_n(S_2(t))| \leq \frac{\sqrt{y^2 + 1}}{8y\sqrt{C(\beta_3, y)D(\beta_4, y)}}.$$

We conclude that  $F_n(S_2(t))$  is uniformly bounded on  $\partial C$ , as desired.  $\square$

## References

- [Apo90] Tom M. Apostol. *Modular functions and Dirichlet series in number theory*, volume 41 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1990.
- [Sie54] Carl Ludwig Siegel. A simple proof of  $\eta(-1/\tau) = \eta(\tau)\sqrt{\tau}/i$ . *Mathematika*, 1:4, 1954.