Recurrence and Bound for A166105

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Abstract

In this work we consider a sequence closely related to those discussed in “Some Doubly Exponential Sequences”, that is A166105. Both the Non-linear Recurrence and bound we present for the growth of A166105 are known, however, not from the provided combinatorial interpretation. We provide an elementary proof of these results based on a set construction.

1 Set Construction

Let $P(S) = \{(s,t) \mid s,t \in S \ \&\ s \neq t\}$. We define the sets $S(n)$ as follows:

$$S(1) = \{\alpha_1, \cdots, \alpha_n\} \qquad\qquad S(n+1) = S(n) \cup P(S(n))$$

where $\alpha_1, \cdots, \alpha_m$ are arbitrary individuals, $m > 1$ and (s,t) is a pairing of the individuals such that $(s,t) \neq (t,s)$, unless $s = t$. In particular we are interested in $F(n) = |S(n)|$ for $n > 0$ and $F(0) = 1$.

2 Enumeration and Bound

Now consider Figure 1 which provides proof that $F(n)$ enumerates the number of objects in $S(n)$. Adding the area of the two rectangles and the large square plus the small diagonal results in the following recurrence:

$$F(n+1) = F(n)^2 - F(n-1)^2 + F(n-1) \tag{1}$$

Furthermore the ratio between the area of the smaller and larger square of Figure 1 is:

$$\frac{(F(n-1)^2 - F(n-1))^2}{F(n-1)^2} = F(n-1)^2 - 2 \cdot F(n-1) + 1 \tag{2}$$

The ratio is quadratic implying that the size of $|S(n)|$ can be closely bounded by the recurrence $F(n-1)^2$ which is bounded by $O(2^{2^n})$.

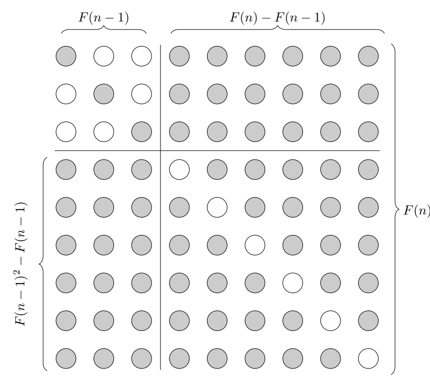

Figure 1: Construction of $S(2)$ when S(1)=$\{\alpha_1, \alpha_2, \alpha_3\}$

References

[1] V. Aho, A and Sloane, N. Some doubly exponential sequences *The Fibonacci Quarterly* **11** (1973) 429–437.