REPRESENTATION AND MANAGEMENT OF MATHEMATICS IN THEOREMA 2.0

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Research Institute for Symbolic Computation (RISC)
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Minisymposium From Information to Knowledge Management
Salzburg, September 12, 2017

JOHANNES KEPLER UNIVERSITÄT LINZ

RESEARCH INSTITUTE FOR SYMBOLIC COMPUTATION

THE THEOREMA SYSTEM: OVERVIEW

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- User Interface: prepare mathematical document + support activities whose results get incorporated into the document.

EXAMPLE (USER INTERFACE)

Alice prepares lecture notes for Linear Algebra.

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ALL IN ONE DOCUMENT

THEORY DEVELOPMENT

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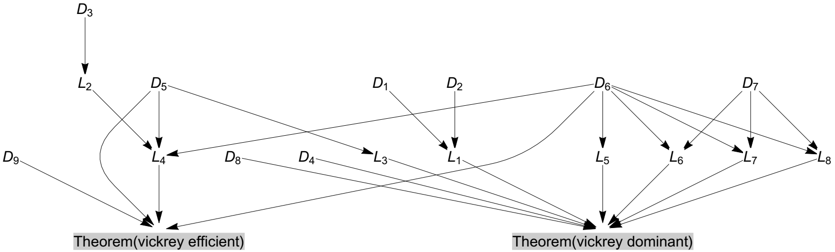
Theorema archive:

- file containing formulas only
- in a format that can be imported into Theorema again
- in other formats to be read by other systems, eg. MMT (Kohlhase/Rabe)

SUPPORT FOR THEORY DEVELOPMENT

Work in progress: Automatically maintain and display a formula graph such as



where

- vertices are formula labels and
- $X \to Y$ means: formula X is needed in the proof of formula Y.

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- Apply graph algorithms in order to investigate theory structure.

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Solution:

PROOF SIMPLIFICATION.

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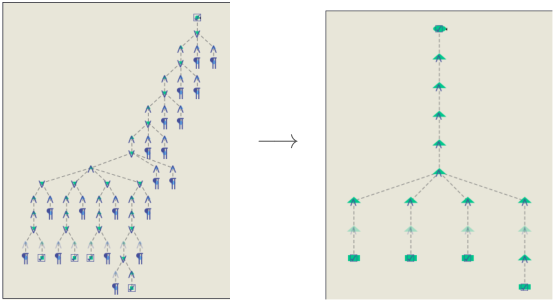
1. Eliminate unnecessary paths in the proof tree



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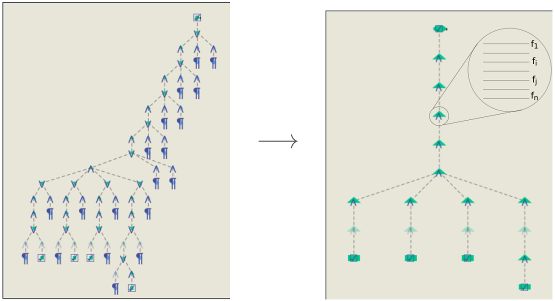

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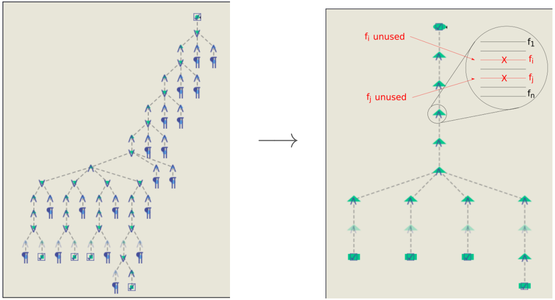

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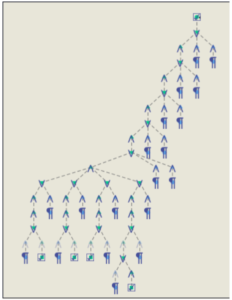
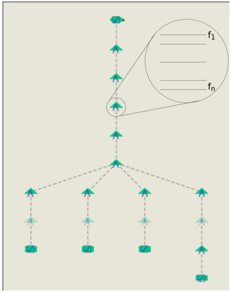


$\longrightarrow$



DETECTION OF UNNECESSARY FORMULAS

Proof tree node: proof situation (proof goal + knowledge base + additional prover info)

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Proof tree node (detail): proof info + proof situation

Proof info: contains information necessary for displaying proof in human-readable form, eg. formulas needed to perform the proof step.

Proof info cannot be extracted automatically, it has to be specified when implementing the prover.

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Proof Step: Reduce proof goal g by knowledge k_i to new goal g'.

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$\downarrow$	$\downarrow$	$\ldots\ldots\ldots\ldots$
child of N	$g', (k_1, \ldots, k_i, \ldots, k_n)$	

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Proof info allows to generate text:

"In order to prove g, because of k_i, it is sufficient to prove g'."

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$$\longrightarrow \qquad ((u_1, \ldots, u_m), (g_1, \ldots, g_m))$$

where

u_i are sets of formulas used and

g_i are sets of formulas generated in a node, st.

all formulas in u_i are needed to generate the formulas in g_i.

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u_i are sets of formulas used and

g_i are sets of formulas generated in a node, st.

all formulas in u_i are needed to generate the formulas in g_i.

Example above: $((\{g, k_i\}), (\{g'\}))$.

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where

$$g'_i = g_i \cap F \qquad u'_i = \begin{cases} u_i & \text{if } g'_i \neq \emptyset \\ \emptyset & \text{if } g'_i = \emptyset \end{cases}$$

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N	$((u_1', \ldots, u_m'), (g_1', \ldots, g_m'))$	$F = F \cup u_1' \cup \cdots \cup u_m'$
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Terminal Node	$((u_1), (\{\}))$	$F = u_1$

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$$g_i^* = g_i \cap F \qquad u_i^* = \begin{cases} u_i & \text{if } g_i^* \neq \emptyset \\ \emptyset & \text{if } g_i^* = \emptyset \end{cases}$$

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Terminal Node	$((u_1), (\{\}))$	$F = u_1$

Upon termination: F contains all necessary formulas!

Root node proof info: $((\{\}), (K^*))$ where K^* contains just those formulas of the original knowledge base that were actually needed.

DEMO: PROOF SIMPLIFICATION

Lemma (Winner's Payment)

$$\forall_{b,x,p,i}$$

If b_i wins and also $b_{i\leftarrow a}$ wins, then i pays the same amount because the maximum of the others did not change:

$$\texttt{secondPriceAuctionWinner}[b, x, p, i] \Rightarrow \forall_{y,q,a} \left(\texttt{secondPriceAuctionWinner}[b_{i\leftarrow a}, y, q, i] \Rightarrow p_i == q_i \wedge x_i == 1 \wedge y_i == 1\right)$$

(win – win)

If b_i wins and $b_{i\leftarrow a}$ loses, then a is not the maximum bid, hence the maximum of the rest must be greater or equal to a:

$$\texttt{secondPriceAuctionWinner}[b, x, p, i] \Rightarrow \forall_{y,q,a} \left(\texttt{secondPriceAuctionLoser}[b_{i\leftarrow a}, y, q, i] \Rightarrow q_i == 0 \wedge x_i == 1 \wedge y_i == 0 \wedge 0 \geq a - p_i\right)$$

(win – lose)

If b_i loses and $b_{i\leftarrow a}$ wins, then a is the maximum bid and, hence, greater or equal the payment, because this is the maximum of the rest:

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(lose – win)

If b_i loses and also $b_{i\leftarrow a}$ loses, then i does not get anything and has to pay nothing:

$$\texttt{secondPriceAuctionLoser}[b, x, p, i] \Rightarrow \forall_{y,q,a} \left(\texttt{secondPriceAuctionLoser}[b_{i\leftarrow a}, y, q, i] \Rightarrow p_i == 0 \wedge x_i == 0 \wedge y_i == 0 \wedge q_i == 0\right)$$

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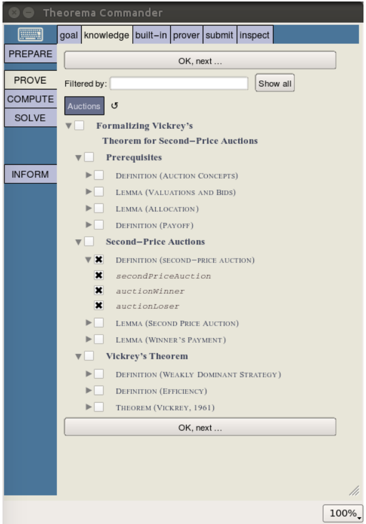

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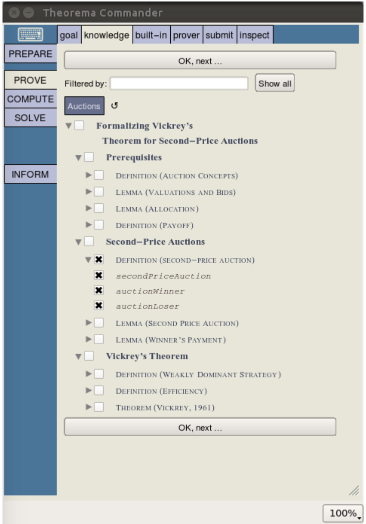

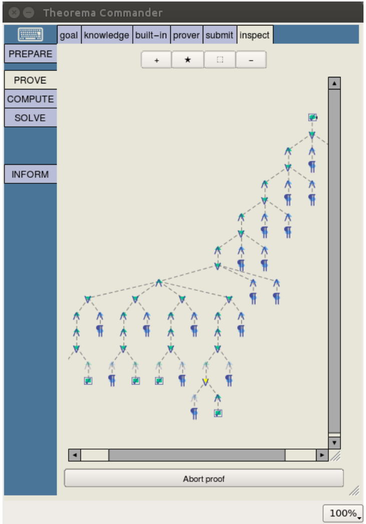

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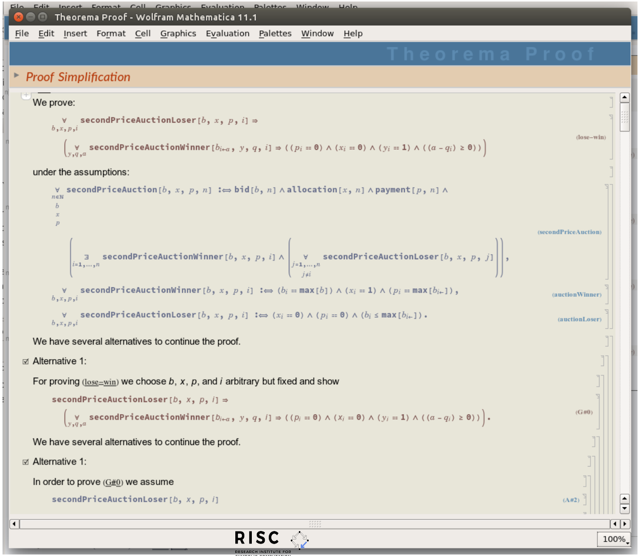

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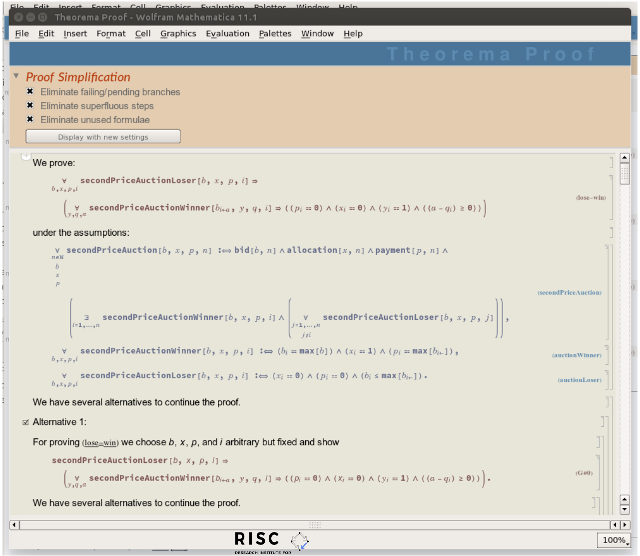

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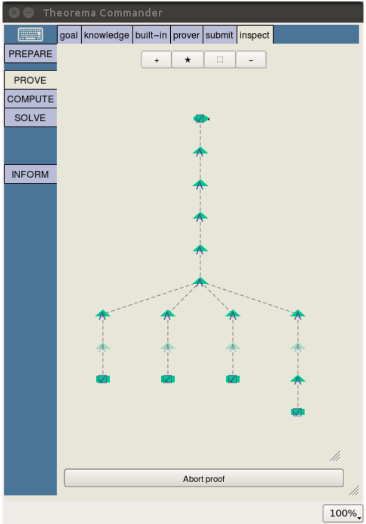

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- Formula dependency graph is a nice/valuable tool for theory exploration/development.
- Proof simplification is a necessary requirement, if we want to automatically maintain a formula dependency graph.