Festkolloquium for Prof. Cigler, Vienna, October 25, 2017

Symbolic Summation for Combinatorics and Particle Physics

Carsten Schneider

Research Institute for Symbolic Computation (RISC)
Johannes Kepler University Linz

From: Prof. Cigler
To: Carsten Schneider

Lieber Herr Schneider,

Ich suche einen Beweis, dass die Summe

$$r[n,t] := \sum_{k=0}^{n} (-t)^{n-k} \binom{n}{k} \binom{n+k}{k} \sum_{j=1}^{k} \frac{t^j}{j}$$

die Rekursion

$$nr[n,t] + (t-2)(2n-1)r[n-1,t] + t^2(n-1)r[n-2,t] = 0$$

erfüllt. Ich nehme an, das lässt sich mit Ihrem Softwarepaket Sigma beweisen. Leider gelingt es mir nicht, Sigma auf meinen Computer zu laden, so dass es sich öffnen lässt.
Könnten Sie mir bitte ein File senden, das sich öffnen lässt.

Mit bestem Dank im Voraus,
Johann Cigler

GIVEN

$$\mathsf{A}(n) := \sum_{k=0}^{n} \overbrace{(-t)^{n-k} \binom{n}{k} \binom{n+k}{k} \sum_{j=1}^{k} \frac{t^j}{j}}^{=f(n,k)}$$

Telescoping

GIVEN

$$\mathsf{A}(n) := \sum_{k=0}^{n} \overbrace{(-t)^{n-k} \binom{n}{k} \binom{n+k}{k} \sum_{j=1}^{k} \frac{t^j}{j}}^{=f(n,k)}$$

FIND $g(n,k)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{f(n,k)}$$

for all $n \geq 0$ and $n \geq k \geq 0$.

Telescoping

GIVEN

$$\mathsf{A}(n) := \sum_{k=0}^{n} \overbrace{(-t)^{n-k}\binom{n}{k}\binom{n+k}{k}\sum_{j=1}^{k}\frac{t^j}{j}}^{=f(n,k)}$$

FIND $g(n,k)$:

$$\boxed{g(n,k+1)-g(n,k)} = \boxed{f(n,k)}$$

for all $n \geq 0$ and $n \geq k \geq 0$.

$$\boxed{g(n,n+1)-g(n,0)} = \boxed{\sum_{k=0}^{n} f(n,k)}$$

Telescoping

GIVEN

$$\mathsf{A}(n) := \sum_{k=0}^{n} \overbrace{(-t)^{n-k} \binom{n}{k} \binom{n+k}{k} \sum_{j=1}^{k} \frac{t^j}{j}}^{=f(n,k)}$$

FIND $g(n,k)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{f(n,k)}$$

for all $n \geq 0$ and $n \geq k \geq 0$.

no solution ☹

Zeilberger's creative telescoping paradigm

GIVEN

$$\mathsf{A}(n) := \sum_{k=0}^{n} \overbrace{(-t)^{n-k}\binom{n}{k}\binom{n+k}{k}\sum_{j=1}^{k}\frac{t^j}{j}}^{=f(n,k)}$$

FIND $g(n,k)$ and $c_0(n), c_1(n)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n) f(n,k) + c_1(n)\, f(n+1,k)}$$

for all $n \geq 0$ and $n \geq k \geq 0$.

no solution ☹

Zeilberger's creative telescoping paradigm

GIVEN

$$\mathsf{A}(n) := \sum_{k=0}^{n} \overbrace{(-t)^{n-k} \binom{n}{k} \binom{n+k}{k} \sum_{j=1}^{k} \frac{t^j}{j}}^{=f(n,k)}$$

FIND $g(n,k)$ and $c_0(n), c_1(n), c_2(n)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n) f(n,k) + c_1(n)\, f(n+1,k) + c_2(n)\, f(n+2,k)}$$

for all $n \geq 0$ and $n \geq k \geq 0$.

solution 

Zeilberger's creative telescoping paradigm

GIVEN

$$\mathsf{A}(n) := \sum_{k=0}^{n} \overbrace{(-t)^{n-k}\binom{n}{k}\binom{n+k}{k}\sum_{j=1}^{k}\frac{t^j}{j}}^{=f(n,k)}$$

FIND $g(n,k)$ and $c_0(n), c_1(n), c_2(n)$:

$$\boxed{g(n,k+1)-g(n,k)} = \boxed{c_0(n)f(n,k)+c_1(n)\,f(n+1,k)+c_2(n)\,f(n+2,k)}$$

for all $n \geq 0$ and $n \geq k \geq 0$.

Sigma *computes*: $c_0(n) = -(1+n)t^2, c_1(n) = (3+2n)(2-t), c_2(n) = -2-n$

and

$$g(n,k) = \frac{2k(3+2n)}{-1+k-n}(-t)^n\binom{n}{k}\binom{k+n}{k}\left(\frac{k(-t)^{-k}\sum_{j=1}^{k}\frac{t^j}{j}}{-2+k-n} + \frac{1+k+n}{(1+n)(2+n)}(-t)^{-k}t^k\right)$$

Zeilberger's creative telescoping paradigm

GIVEN

$$\mathsf{A}(n) := \sum_{k=0}^{n} \overbrace{(-t)^{n-k}\binom{n}{k}\binom{n+k}{k}\sum_{j=1}^{k}\frac{t^j}{j}}^{=f(n,k)}$$

FIND $g(n,k)$ and $c_0(n), c_1(n), c_2(n)$:

$$\boxed{g(n,k+1)-g(n,k)} = \boxed{c_0(n)f(n,k)+c_1(n)\,f(n+1,k)+c_2(n)\,f(n+2,k)}$$

for all $n \geq 0$ and $n \geq k \geq 0$.

Summing this equation over k from 0 to n gives:

$$\boxed{g(n,n+1)-g(n,0)} = \boxed{\sum_{k=0}^{n}\Big[c_0(n)f(n,k)+c_1(n)f(n+1,k)+c_2(n)f(n+2,k)\Big]}$$

Zeilberger's creative telescoping paradigm

GIVEN

$$\mathsf{A}(n) := \sum_{k=0}^{n} \overbrace{(-t)^{n-k} \binom{n}{k} \binom{n+k}{k} \sum_{j=1}^{k} \frac{t^j}{j}}^{=f(n,k)}$$

FIND $g(n,k)$ and $c_0(n), c_1(n), c_2(n)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n) f(n,k) + c_1(n)\, f(n+1,k) + c_2(n)\, f(n+2,k)}$$

for all $n \geq 0$ and $n \geq k \geq 0$.

Summing this equation over k from 0 to n gives:

$$\boxed{g(n,n+1) - g(n,0)} = \boxed{\sum_{k=0}^{n} c_0(n) f(n,k) + \sum_{k=0}^{n} c_1(n) f(n+1,k) + \sum_{k=0}^{n} c_2(n) f(n+2,k)}$$

Zeilberger's creative telescoping paradigm

GIVEN

$$\mathsf{A}(n) := \sum_{k=0}^{n} \overbrace{(-t)^{n-k}\binom{n}{k}\binom{n+k}{k}\sum_{j=1}^{k}\frac{t^j}{j}}^{=f(n,k)}$$

FIND $g(n,k)$ and $c_0(n), c_1(n), c_2(n)$:

$$\boxed{g(n,k+1)-g(n,k)} = \boxed{c_0(n)f(n,k)+c_1(n)\,f(n+1,k)+c_2(n)\,f(n+2,k)}$$

for all $n \geq 0$ and $n \geq k \geq 0$.

Summing this equation over k from 0 to n gives:

$$\boxed{g(n,n+1)-g(n,0)} = \boxed{c_0(n)\sum_{k=0}^{n} f(n,k)+c_1(n)\sum_{k=0}^{n} f(n+1,k)+c_2(n)\sum_{k=0}^{n} f(n+2,k)}$$

Zeilberger's creative telescoping paradigm

GIVEN

$$\mathsf{A}(n) := \sum_{k=0}^{n} \overbrace{(-t)^{n-k} \binom{n}{k} \binom{n+k}{k} \sum_{j=1}^{k} \frac{t^j}{j}}^{=f(n,k)}$$

FIND $g(n,k)$ and $c_0(n), c_1(n), c_2(n)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n) f(n,k) + c_1(n)\, f(n+1,k) + c_2(n)\, f(n+2,k)}$$

for all $n \geq 0$ and $n \geq k \geq 0$.

Summing this equation over k from 0 to n gives:

$$\boxed{\begin{array}{l} g(n,n+1) - g(n,0) \\ +c_1(n) f(n+1,n+1) \\ +c_2(n)\big(f(n+2,n+1) \\ \qquad +f(n+2,n+2)\big) \end{array}} = \boxed{c_0(n)\mathsf{A}(n) + c_1(n)\mathsf{A}(n+1) + c_2(n)\mathsf{A}(n+2)}$$

Zeilberger's creative telescoping paradigm

GIVEN

$$\mathsf{A}(n) := \sum_{k=0}^{n} \overbrace{(-t)^{n-k} \binom{n}{k} \binom{n+k}{k} \sum_{j=1}^{k} \frac{t^j}{j}}^{=f(n,k)}$$

FIND $g(n,k)$ and $c_0(n), c_1(n), c_2(n)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n) f(n,k) + c_1(n)\, f(n+1,k) + c_2(n)\, f(n+2,k)}$$

for all $n \geq 0$ and $n \geq k \geq 0$.

Summing this equation over k from 0 to n gives:

$$\boxed{\begin{array}{l} g(n,n+1) - g(n,0) \\ +c_1(n) f(n+1,n+1) \\ +c_2(n)\big(f(n+2,n+1) \\ \qquad +f(n+2,n+2)\big) \end{array}} = \boxed{c_0(n)\mathsf{A}(n) + c_1(n)\mathsf{A}(n+1) + c_2(n)\mathsf{A}(n+2)}$$

$$\| \qquad\qquad\qquad\qquad \|$$

$$0 \qquad -(1+n)t^2\mathsf{A}(n) + (3+2n)(2-t)\mathsf{A}(n+1) - (2+n)\mathsf{A}(n+2)$$

Hankel determinants of harmonic numbers and related topics, J. Cigler, 2017

Definition. The Hankel determinant of $(a(k))_{k\geq 0}$ is

$$\det(a(i+j))_{i,j=0}^{n} = \det \begin{pmatrix} a(0) & a(1) & a(2) & \dots & a(n) \\ a(1) & a(2) & a(3) & \dots & a(n+1) \\ a(2) & a(3) & a(4) & \dots & a(n+2) \\ \vdots & & & & \vdots \\ a(n) & a(n+1) & a(n+2) & a\dots & a(2n) \end{pmatrix}$$

Hankel determinants of harmonic numbers and related topics, J. Cigler, 2017

Definition. The Hankel determinant of $(a(k))_{k\geq 0}$ is

$$\det(a(i+j))_{i,j=0}^{n} = \det\begin{pmatrix} a(0) & a(1) & a(2) & \dots & a(n) \\ a(1) & a(2) & a(3) & \dots & a(n+1) \\ a(2) & a(3) & a(4) & \dots & a(n+2) \\ \vdots & & & & \vdots \\ a(n) & a(n+1) & a(n+2) & a\dots & a(2n) \end{pmatrix}$$

Theorem (Cigler,2017). For $a(k) = \sum_{j=1}^{k} \frac{t^j}{j}$ we have

$$\det(a(i+j))_{i,j=0}^{n} = \frac{(-1)^n t^{n^2}}{\binom{2n}{n}\prod_{j=1}^{n-1}(2j+1)\binom{2j}{j}^2} r(n,t)$$

Hankel determinants of harmonic numbers and related topics, J. Cigler, 2017

Definition. The Hankel determinant of $(a(k))_{k\geq 0}$ is

$$\det(a(i+j))_{i,j=0}^{n} = \det\begin{pmatrix} a(0) & a(1) & a(2) & \dots & a(n) \\ a(1) & a(2) & a(3) & \dots & a(n+1) \\ a(2) & a(3) & a(4) & \dots & a(n+2) \\ \vdots & & & & \vdots \\ a(n) & a(n+1) & a(n+2) & a\dots & a(2n) \end{pmatrix}$$

Theorem (Cigler,2017). For $a(k) = \sum_{j=1}^{k} \frac{t^j}{j}$ we have

$$\det(a(i+j))_{i,j=0}^{n} = \frac{(-1)^n t^{n^2}}{\binom{2n}{n}\prod_{j=1}^{n-1}(2j+1)\binom{2j}{j}^2} r(n,t)$$

with

$$r(n,t) = \sum_{k=0}^{n}(-t)^{n-k}\binom{n}{k}\binom{n+k}{k}\sum_{j=1}^{k}\frac{t^j}{j} \quad (= A(n)).$$

Hankel determinants of harmonic numbers and related topics, J. Cigler, 2017

Definition. The Hankel determinant of $(a(k))_{k\geq 0}$ is

$$\det(a(i+j))_{i,j=0}^{n} = \det\begin{pmatrix} a(0) & a(1) & a(2) & \dots & a(n) \\ a(1) & a(2) & a(3) & \dots & a(n+1) \\ a(2) & a(3) & a(4) & \dots & a(n+2) \\ \vdots & & & & \vdots \\ a(n) & a(n+1) & a(n+2) & a\dots & a(2n) \end{pmatrix}$$

Theorem (Cigler,2017). For $a(k) = \sum_{j=1}^{k} \frac{t^j}{j}$ we have

$$\det(a(i+j))_{i,j=0}^{n} = \frac{(-1)^n t^{n^2}}{\binom{2n}{n}\prod_{j=1}^{n-1}(2j+1)\binom{2j}{j}^2} r(n,t)$$

and using Sigma we get

$$nr(n,t) + (t-2)(2n-1)r(n-1,t) + t^2(n-1)r(n-2,t) = 0$$

$$nr(n,t) + (t-2)(2n-1)r(n-1,t) + t^2(n-1)r(n-2,t) = 0$$

Special cases (explored in Cigler's article)

- $t = 1$:

$$r(n,1) = 2H_n$$

$$nr(n,t) + (t-2)(2n-1)r(n-1,t) + t^2(n-1)r(n-2,t) = 0$$

Special cases (explored in Cigler's article)

- $t = 1$:

$$r(n,1) = 2H_n$$

Note: there are other proofs (communicated to Prof. Cigler) using

1. generating functions (F. Petrov)
2. Chu-Vandermonde plus derivation (C. Krattenthaler)
3. Zeilberger's algorithm plus a variant of derivation (T. Amdeberhan) similar to the Andrews–Newton–Zeilberger alg. [Paule/Schneider 2003]

$$nr(n,t) + (t-2)(2n-1)r(n-1,t) + t^2(n-1)r(n-2,t) = 0$$

Special cases (explored in Cigler's article)

- $t = 1$:

$$r(n,1) = 2H_n$$

Note: there are other proofs (communicated to Prof. Cigler) using

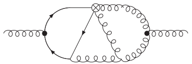
1. generating functions (F. Petrov)
2. Chu-Vandermonde plus derivation (C. Krattenthaler)
3. Zeilberger's algorithm plus a variant of derivation (T. Amdeberhan) similar to the Andrews–Newton–Zeilberger alg. [Paule/Schneider 2003]

- $t = 2$:

$$r(2n,t) = 0$$
$$r(2n+1,t) = (-1)^n \frac{(2n+1)!2^{2n+2}}{((2n+1)!!)^2}$$

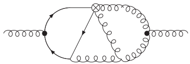
```
From: Prof. Cigler
To: Carsten Schneider

Lieber Herr Schneider,

besten Dank für Ihre prompte Reaktion. Speziell möchte ich mich
dafür bedanken, dass Sie mein Problem vorgerechnet haben.
Denn ohne diese Hilfe hätte ich wahrscheinlich einige Tage
benötigt, um damit vertraut zu werden.

...

Für t=1 und t=2 habe ich die Formel und für allgemeines t
die Rekursion erraten, konnte jedoch nichts davon beweisen.
Ich habe dann meine Vermutungen für t=1 und t=2 auf MathOverflow
gepostet und für diese Fälle auch trickreiche Beweise erhalten.
Durch Ihr Programm Sigma ist nun alles viel einfacher geworden.

Mit besten Grüßen,
Johann Cigler
```

Evaluation of Feynman Integrals



Behavior of particles

Evaluation of Feynman Integrals

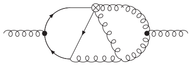


Behavior of particles

$$\longrightarrow \quad \int \Phi(N, \epsilon, x)dx$$

Feynman integrals

Evaluation of Feynman Integrals



Behavior of particles

$$\int \Phi(N,\epsilon,x)dx$$

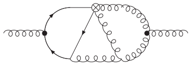
Feynman integrals

DESY

$$\sum f(N,\epsilon,k)$$

complicated multi-sums

Evaluation of Feynman Integrals



Behavior of particles → Feynman integrals

$$\int \Phi(N,\epsilon,x)dx$$

Feynman integrals

↓ **DESY**

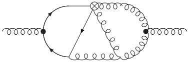
$$\sum f(N,\epsilon,k)$$

complicated multi-sums

← **RISC** (Sigma-package)

expression in special functions

Evaluation of Feynman Integrals



Behavior of particles

$$\int \Phi(N,\epsilon,x)dx$$

Feynman integrals

LHC at CERN

DESY

applicable

$$\sum f(N,\epsilon,k)$$

complicated multi-sums

RISC

(Sigma-package)

expression in
special functions

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Big(\underbrace{\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(j)}\Big)$$

where

$$S_1(n)=\sum_{i=1}^{n}\frac{1}{i}\quad(=H_n)$$

Arose in the context of

I. Bierenbaum, J. Blümlein, and S. Klein, *Evaluating two-loop massive operator matrix elements with Mellin-Barnes integrals*. 2006

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\underbrace{\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}\Big)}_{f(j)}$$

FIND $g(j)$:

$$\boxed{f(j) = g(j+1) - g(j)}$$

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\underbrace{\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}\Big)}_{f(j)}$$

FIND $g(j)$:

$$\boxed{f(j)=g(j+1)-g(j)}$$

$\uparrow$ summation package Sigma

$$g(j)=\frac{(j+k+1)(j+n+1)j!k!(j+k+n)!\Big(S_1(j)-S_1(j+k)-S_1(j+n)+S_1(j+k+n)\Big)}{kn(j+k+1)!(j+n+1)!(k+n+1)!}$$

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\underbrace{\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}\Big)}_{f(j)}$$

FIND $g(j)$:

$$\boxed{f(j) = g(j+1) - g(j)}$$

Summing the telescoping equation over j from 0 to a gives

$$\sum_{j=0}^{a} f(j) = g(a+1) - g(0)$$

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\underbrace{\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}\Big)}_{f(j)}$$

FIND $g(j)$:

$$\boxed{f(j) = g(j+1) - g(j)}$$

Summing the telescoping equation over j from 0 to a gives

$$\sum_{j=0}^{a} f(j) = g(a+1) - g(0)$$

$$= \underbrace{\frac{(a+1)!(k-1)!(a+k+n+1)!(S_1(a)-S_1(a+k)-S_1(a+n)+S_1(a+k+n))}{n(a+k+1)!(a+n+1)!(k+n+1)!} + \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)n!} + \frac{(2a+k+n+2)a!k!(a+k+n)!}{(a+k+1)(a+n+1)(a+k+1)!(a+n+1)!(k+n+1)!}}_{a\,\to\,\infty}$$

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\underbrace{\Bigg(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}\Bigg)}_{f(j)}$$

$$\sum_{j=0}^{\infty} f(j) = \qquad \frac{1}{n!}\frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}$$

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\underbrace{\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}\Big)}_{f(j)}$$

$$\sum_{k=1}^{\infty}\sum_{j=0}^{\infty}f(j)=\frac{1}{n!}\sum_{k=1}^{\infty}\frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}$$

Telescoping

GIVEN

$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=: f(k)}.$$

FIND $g(k)$:

$$\boxed{g(k+1) - g(k)} = \boxed{f(k)}$$

for all $0 \le k \le n$ and all $n \ge 0$.

no solution ☹

Zeilberger's creative telescoping paradigm

GIVEN

$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=: f(n,k)}.$$

FIND $g(n, k)$

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{f(n,k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

no solution ☹

Zeilberger's creative telescoping paradigm

GIVEN

$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=:\, f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n), c_1(n)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n) f(n,k) + c_1(n)\, f(n+1,k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

Zeilberger's creative telescoping paradigm

GIVEN

$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=:\, f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n), c_1(n)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n) f(n,k) + c_1(n)\, f(n+1,k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

Sigma *computes*: $c_0(n) = -n$, $c_1(n) = (n+2)$ and

$$g(n,k) = \frac{kS_1(k) + (-n-1)S_1(n) - kS_1(k+n) - 2}{(k+n+1)(n+1)^2}$$

Zeilberger's creative telescoping paradigm

GIVEN

$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=:\, f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n), c_1(n)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n) f(n,k) + c_1(n)\, f(n+1,k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

Summing this equation over k from 1 to a gives:

$$\boxed{g(n,a+1) - g(n,1)} = \boxed{\sum_{k=1}^{a} \Big[c_0(n)\, f(n,k) + c_1(n)\, f(n+1,k) \Big]}$$

Zeilberger's creative telescoping paradigm

GIVEN

$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=:\, f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n), c_1(n)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n) f(n,k) + c_1(n)\, f(n+1,k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

Summing this equation over k from 1 to a gives:

$$\boxed{g(n,a+1) - g(n,1)} = \boxed{\sum_{k=1}^{a} c_0(n)\, f(n,k) + \sum_{k=1}^{a} c_1(n)\, f(n+1,k)}$$

Zeilberger's creative telescoping paradigm

GIVEN

$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=: f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n), c_1(n)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n) f(n,k) + c_1(n)\, f(n+1,k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

Summing this equation over k from 1 to a gives:

$$\boxed{g(n,a+1) - g(n,1)} = \boxed{c_0(n) \sum_{k=1}^{a} f(n,k) + c_1(n) \sum_{k=1}^{a} f(n+1,k)}$$

Zeilberger's creative telescoping paradigm

GIVEN

$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=:\, f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n), c_1(n)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n) f(n,k) + c_1(n)\, f(n+1,k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

Summing this equation over k from 1 to a gives:

$$\boxed{g(n,a+1) - g(n,1)} = \boxed{c_0(n)\, \mathsf{A}(n) + c_1(n)\, \mathsf{A}(n+1)}$$

Zeilberger's creative telescoping paradigm

GIVEN
$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}}_{=:\, f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n), c_1(n)$:

$$\boxed{g(n,k+1)-g(n,k)} = \boxed{c_0(n) f(n,k) + c_1(n)\, f(n+1,k)}$$

for all $0 \le k \le n$ and all $n \ge 0$.

Summing this equation over k from 1 to a gives:

$$\boxed{g(n,a+1)-g(n,1)} = \boxed{c_0(n)\,\mathsf{A}(n) + c_1(n)\,\mathsf{A}(n+1)}$$

$$\frac{(a+1)(S_1(a)+S_1(n)-S_1(a+n))}{(n+1)^2(a+n+2)} + \frac{a(a+1)}{(n+1)^3(a+n+1)(a+n+2)} \quad = \quad -n\mathsf{A}(n) + (2+n)\mathsf{A}(n+1)$$

Zeilberger's creative telescoping paradigm

GIVEN
$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=:\, f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n), c_1(n)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n) f(n,k) + c_1(n)\, f(n+1,k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

Summing this equation over k from 1 to a gives:

$$\boxed{g(n,a+1) - g(n,1)} = \boxed{c_0(n)\,\mathsf{A}(n) + c_1(n)\,\mathsf{A}(n+1)}$$

$$\lim_{a\to\infty} \quad || \qquad\qquad\qquad ||$$

$$\frac{(n+1)S_1(n)+1}{(n+1)^3} \qquad\qquad -n\mathsf{A}(n) + (2+n)\mathsf{A}(n+1)$$

$$(n+2)\mathbf{A}(n+1) - n\mathbf{A}(n) = \frac{(n+1)S_1(n)+1}{(n+1)^3}$$

recurrence finder

$$A(n) = \sum_{k=1}^{\infty} \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}$$

$$(n+2)\mathbf{A}(n+1) - n\mathbf{A}(n) = \frac{(n+1)S_1(n)+1}{(n+1)^3}$$

recurrence solver

$$\boxed{A(n) = \sum_{k=1}^{\infty} \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}} \in \boxed{\left\{c \times \frac{1}{n(n+1)} + \frac{S_1(n)^2+S_2(n)}{2n(n+1)} \middle| c \in \mathbb{R}\right\}}$$

where

$$S_1(n) = \sum_{i=1}^{n} \frac{1}{i} \qquad S_2(n) = \sum_{i=1}^{n} \frac{1}{i^2}$$

$$(n+2)\mathbf{A}(n+1) - n\mathbf{A}(n) = \frac{(n+1)S_1(n)+1}{(n+1)^3}$$

Summation package Sigma

(based on difference field/ring algorithms/theory
see, e.g., Karr 1981, Bronstein 2000, Schneider 2001–)

$$\boxed{A(n) = \sum_{k=1}^{\infty} \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}} = \boxed{0 \times \frac{1}{n(n+1)} + \frac{S_1(n)^2+S_2(n)}{2n(n+1)}}$$

where

$$S_1(n) = \sum_{i=1}^{n} \frac{1}{i} \qquad S_2(n) = \sum_{i=1}^{n} \frac{1}{i^2}$$

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\underbrace{\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}\Big)}_{f(j)}$$

$$\sum_{k=1}^{\infty}\sum_{j=0}^{\infty} f(j) = \frac{1}{n!}\sum_{k=1}^{\infty}\frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)} = \frac{1}{n!}\frac{S_1(n)^2+S_2(n)}{2n(n+1)}$$

where

$$S_1(n)=\sum_{i=1}^{n}\frac{1}{i} \qquad S_2(n)=\sum_{i=1}^{n}\frac{1}{i^2}$$

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\underbrace{\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}\Big)}_{f(n,k,j)}$$

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} f(n,k,j) = \frac{S_1(n)^2+3S_2(n)}{2n(n+1)!}$$

where

$$S_1(n)=\sum_{i=1}^{n}\frac{1}{i} \qquad S_2(n)=\sum_{i=1}^{n}\frac{1}{i^2}$$

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$A(n) = \sum_{k=1}^{n} f(n,k);$$

$f(n,k)$: indefinite nested product-sum in k;
n: extra parameter

FIND a recurrence for $A(n)$

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$A(n) = \sum_{k=1}^{n} f(n,k);$$

$f(n,k)$: indefinite nested product-sum in k;
n: extra parameter

FIND a recurrence for $A(n)$

2. Recurrence solving

GIVEN a recurrence

$a_0(n), \ldots, a_d(n), h(n)$:
indefinite nested product-sum expressions in n.

$$a_0(n)A(n) + \cdots + a_d(n)A(n+d) = h(n);$$

FIND all solutions expressible by indefinite nested products/sums in n.
(d'Alembertian solutions)
(Abramov/Bronstein/Petkovšek/CS, in preparation)

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$A(n) = \sum_{k=1}^{n} f(n,k);$$

$f(n,k)$: indefinite nested product-sum in k;
n: extra parameter

FIND a recurrence for $A(n)$

2. Recurrence solving

GIVEN a recurrence

$a_0(n), \dots, a_d(n), h(n)$:
indefinite nested product-sum expressions in n.

$$a_0(n)A(n) + \cdots + a_d(n)A(n+d) = h(n);$$

FIND all solutions expressible by indefinite nested products/sums in n.
(d'Alembertian solutions)
(Abramov/Bronstein/Petkovšek/CS, in preparation)

Note: the sum solutions are highly nested
(possibly with denominators of high degrees)

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$A(n) = \sum_{k=1}^{n} f(n,k);$$

$f(n,k)$: indefinite nested product-sum in k;
n: extra parameter

FIND a recurrence for $A(n)$

2. Recurrence solving

GIVEN a recurrence

$a_0(n), \dots, a_d(n), h(n)$:
indefinite nested product-sum expressions in n.

$$a_0(n)A(n) + \dots + a_d(n)A(n+d) = h(n);$$

FIND all solutions expressible by indefinite nested products/sums in n.
(d'Alembertian solutions)
(Abramov/Bronstein/Petkovšek/CS, in preparation)

3. Simplify the solutions (using difference ring/field theory) s.t.

- the sums are algebraically independent;
- the sums are flattened;
- the sums can be given in terms of special functions.

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$A(n) = \sum_{k=1}^{n} f(n,k);$$

$f(n,k)$: indefinite nested product-sum in k;
n: extra parameter

FIND a recurrence for $A(n)$

2. Recurrence solving

GIVEN a recurrence

$a_0(n), \dots, a_d(n), h(n)$:
indefinite nested product-sum expressions in n.

$$a_0(n)A(n) + \cdots + a_d(n)A(n+d) = h(n);$$

FIND all solutions expressible by indefinite nested products/sums in n.
(d'Alembertian solutions)
(Abramov/Bronstein/Petkovšek/CS, in preparation)

4. Find a "closed form"

A(n)=combined solutions in terms of indefinite nested sums in n.

In[1]:= $<<$ **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= $$\mathbf{mySum} = \sum_{k=1}^{A} \frac{S[1,k] + S[1,n] - S[1,k+n]}{kn(k+n+1)};$$

In[1]:= $<<$ **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= $\mathbf{mySum} = \sum_{\mathbf{k}=1}^{\mathbf{A}} \frac{\mathbf{S[1,k]+S[1,n]-S[1,k+n]}}{\mathbf{kn(k+n+1)}};$

Compute a recurrence

In[3]:= $\mathbf{rec = GenerateRecurrence[mySum, n][[1]]}$

Out[3]= $n\mathsf{SUM}[n]+(1+n)(2+n)\mathsf{SUM}[n+1] == \frac{(a+1)(S[1,a]+S[1,n]-S[1,a+n])}{(n+1)^2(a+n+2)n!} + \frac{a(a+1)}{(n+1)^3(a+n+1)(a+n+2)n!}$

In[4]:= $\mathbf{rec = LimitRec[rec, SUM[n], \{n\}, A]}$

Out[4]= $-n\mathsf{SUM}[n]+(1+n)(2+n)\mathsf{SUM}[n+1] == \frac{(n+1)S[1,n]+1}{(n+1)^3}$

In[1]:= $<<$ **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= $\mathbf{mySum} = \sum_{\mathbf{k=1}}^{\mathbf{A}} \frac{\mathbf{S[1,k] + S[1,n] - S[1,k+n]}}{\mathbf{kn(k+n+1)}};$

Compute a recurrence

In[3]:= $\mathbf{rec = GenerateRecurrence[mySum, n][[1]]}$

Out[3]= $n\mathsf{SUM}[n]+(1+n)(2+n)\mathsf{SUM}[n+1] == \frac{(a+1)(S[1,a]+S[1,n]-S[1,a+n])}{(n+1)^2(a+n+2)n!} + \frac{a(a+1)}{(n+1)^3(a+n+1)(a+n+2)n!}$

In[4]:= $\mathbf{rec = LimitRec[rec, SUM[n], \{n\}, A]}$

Out[4]= $-n\mathsf{SUM}[n] + (1+n)(2+n)\mathsf{SUM}[n+1] == \frac{(n+1)S[1,n]+1}{(n+1)^3}$

Solve a recurrence

In[5]:= $\mathbf{recSol = SolveRecurrence[rec, SUM[n], IndefiniteSummation \to False]}$

In[1]:= $<<$ **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= $\mathbf{mySum} = \sum_{\mathbf{k=1}}^{\mathbf{A}} \frac{\mathbf{S[1,k] + S[1,n] - S[1,k+n]}}{\mathbf{kn(k+n+1)}};$

Compute a recurrence

In[3]:= $\mathbf{rec = GenerateRecurrence[mySum, n][[1]]}$

Out[3]= $n\mathsf{SUM}[n] + (1+n)(2+n)\mathsf{SUM}[n+1] == \frac{(a+1)(S[1,a]+S[1,n]-S[1,a+n])}{(n+1)^2(a+n+2)n!} + \frac{a(a+1)}{(n+1)^3(a+n+1)(a+n+2)n!}$

In[4]:= $\mathbf{rec = LimitRec[rec, SUM[n], \{n\}, A]}$

Out[4]= $-n\mathsf{SUM}[n] + (1+n)(2+n)\mathsf{SUM}[n+1] == \frac{(n+1)S[1,n]+1}{(n+1)^3}$

Solve a recurrence

In[5]:= $\mathbf{recSol = SolveRecurrence[rec, SUM[n], IndefiniteSummation \to False]}$

Out[5]= $\{\{0, \frac{1}{n(n+1)}\}, \{1, \frac{\sum_{i=1}^{n} \frac{S[1,i]}{i}}{n(n+1)}\}\}$

In[1]:= $<<$ **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= $\mathbf{mySum} = \sum_{k=1}^{A} \frac{S[1,k] + S[1,n] - S[1,k+n]}{kn(k+n+1)};$

Compute a recurrence

In[3]:= $\mathbf{rec = GenerateRecurrence[mySum, n][[1]]}$

Out[3]= $n\mathsf{SUM}[n] + (1+n)(2+n)\mathsf{SUM}[n+1] == \frac{(a+1)(S[1,a]+S[1,n]-S[1,a+n])}{(n+1)^2(a+n+2)n!} + \frac{a(a+1)}{(n+1)^3(a+n+1)(a+n+2)n!}$

In[4]:= $\mathbf{rec = LimitRec[rec, SUM[n], \{n\}, A]}$

Out[4]= $-n\mathsf{SUM}[n] + (1+n)(2+n)\mathsf{SUM}[n+1] == \frac{(n+1)S[1,n]+1}{(n+1)^3}$

Solve a recurrence

In[5]:= $\mathbf{recSol = SolveRecurrence[rec, SUM[n], IndefiniteSummation \to True]}$

In[1]:= $<<$ **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= $\mathbf{mySum} = \sum_{k=1}^{A} \frac{S[1,k]+S[1,n]-S[1,k+n]}{kn(k+n+1)};$

Compute a recurrence

In[3]:= $\mathbf{rec = GenerateRecurrence[mySum, n][[1]]}$

Out[3]= $n\mathrm{SUM}[n]+(1+n)(2+n)\mathrm{SUM}[n+1]==\frac{(a+1)(S[1,a]+S[1,n]-S[1,a+n])}{(n+1)^2(a+n+2)n!}+\frac{a(a+1)}{(n+1)^3(a+n+1)(a+n+2)n!}$

In[4]:= $\mathbf{rec = LimitRec[rec, SUM[n], \{n\}, A]}$

Out[4]= $-n\mathrm{SUM}[n]+(1+n)(2+n)\mathrm{SUM}[n+1]==\frac{(n+1)S[1,n]+1}{(n+1)^3}$

Solve a recurrence

In[5]:= $\mathbf{recSol = SolveRecurrence[rec, SUM[n], IndefiniteSummation \rightarrow True]}$

Out[5]= $\{\{0, \frac{1}{n(n+1)}\}, \{1, \frac{S[1,n]^2+\sum_{i=1}^{n}\frac{1}{i^2}}{2n(n+1)}\}\}$

In[1]:= $<<$ **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= $\mathbf{mySum} = \sum_{k=1}^{A} \frac{S[1,k]+S[1,n]-S[1,k+n]}{kn(k+n+1)};$

Compute a recurrence

In[3]:= $\mathbf{rec = GenerateRecurrence[mySum, n][[1]]}$

Out[3]= $n\mathsf{SUM}[n]+(1+n)(2+n)\mathsf{SUM}[n+1]==\frac{(a+1)(S[1,a]+S[1,n]-S[1,a+n])}{(n+1)^2(a+n+2)n!}+\frac{a(a+1)}{(n+1)^3(a+n+1)(a+n+2)n!}$

In[4]:= $\mathbf{rec = LimitRec[rec, SUM[n], \{n\}, A]}$

Out[4]= $-n\mathsf{SUM}[n]+(1+n)(2+n)\mathsf{SUM}[n+1]==\frac{(n+1)S[1,n]+1}{(n+1)^3}$

Solve a recurrence

In[5]:= $\mathbf{recSol = SolveRecurrence[rec, SUM[n], IndefiniteSummation \to True]}$

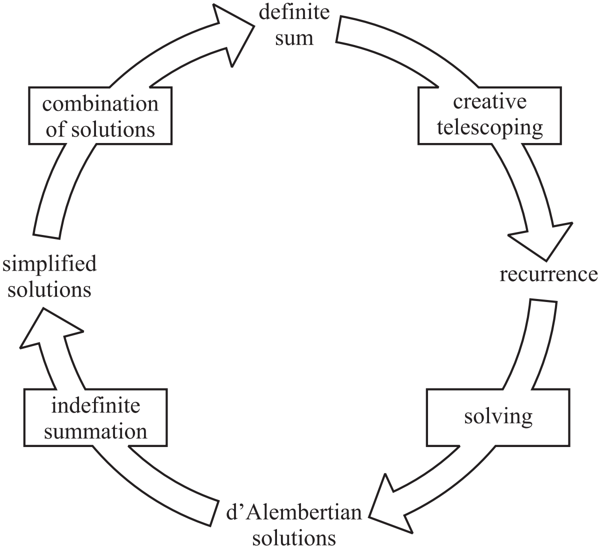
Out[5]= $\{\{0, \frac{1}{n(n+1)}\}, \{1, \frac{S[1,n]^2+\sum_{i=1}^{n}\frac{1}{i^2}}{2n(n+1)}\}\}$

Combine the solutions

In[6]:= $\mathbf{FindLinearCombination[recSol, \{1, \{1/2\}, n, 2]}$

Out[6]= $\frac{S[1,n]^2+\sum_{i=1}^{n}\frac{1}{i^2}}{2n(n+1)}$

Sigma's summation spiral



$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

Simple sum

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\boxed{\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}}$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\boxed{\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}}$$

$$||$$

$$\boxed{\binom{j+1}{r}\Big(\frac{(-1)^r(-j+n-2)!r!}{(n+1)(-j+n+r-1)(-j+n+r)!}+ \frac{(-1)^{n+r}(j+1)!(-j+n-2)!(-j+n-1)_r r!}{(n-1)n(n+1)(-j+n+r)!(-j-1)_r(2-n)_j}\Big)}$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\sum_{j=0}^{n-2}\boxed{\sum_{r=0}^{j+1}\binom{j+1}{r}\Big(\frac{(-1)^r(-j+n-2)!r!}{(n+1)(-j+n+r-1)(-j+n+r)!}+\frac{(-1)^{n+r}(j+1)!(-j+n-2)!(-j+n-1)_r r!}{(n-1)n(n+1)(-j+n+r)!(-j-1)_r(2-n)_j}\Big)}$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\sum_{j=0}^{n-2}\left[\sum_{r=0}^{j+1}\binom{j+1}{r}\Big(\frac{(-1)^r(-j+n-2)!r!}{(n+1)(-j+n+r-1)(-j+n+r)!}+\frac{(-1)^{n+r}(j+1)!(-j+n-2)!(-j+n-1)_r r!}{(n-1)n(n+1)(-j+n+r)!(-j-1)_r(2-n)_j}\Big)\right]$$

$$||$$

$$\Big(\frac{n^2-n+1}{(n-1)^2n^2(n+1)(2-n)_j}+\frac{\sum_{i=1}^{j}\frac{(2-n)_i}{(-i+n-1)^2(i+1)!}}{(n+1)(2-n)_j}+\frac{(-1)^{j+n}(-j-2)(-j+n-2)!}{(j-n+1)(n+1)^2n!}\Big)(j+1)!-\frac{1}{(n+1)^2(-j+n-1)}$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\sum_{j=0}^{n-2}\Bigg(\Big(\frac{n^2-n+1}{(n-1)^2n^2(n+1)(2-n)_j}+\frac{\sum_{i=1}^{j}\frac{(2-n)_i}{(-i+n-1)^2(i+1)!}}{(n+1)(2-n)_j}+$$
$$\frac{(-1)^{j+n}(-j-2)(-j+n-2)!}{(j-n+1)(n+1)^2n!}\Big)(j+1)!-\frac{1}{(n+1)^2(-j+n-1)}\Bigg)$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

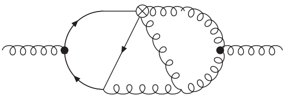
$$||$$

$$\sum_{j=0}^{n-2}\Bigg(\Big(\frac{n^2-n+1}{(n-1)^2n^2(n+1)(2-n)_j}+\frac{\sum_{i=1}^{j}\frac{(2-n)_i}{(-i+n-1)^2(i+1)!}}{(n+1)(2-n)_j}+$$
$$\frac{(-1)^{j+n}(-j-2)(-j+n-2)!}{(j-n+1)(n+1)^2n!}\Big)(j+1)!-\frac{1}{(n+1)^2(-j+n-1)}\Bigg)$$

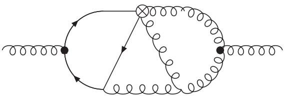
$$||$$

$$\frac{-n^2-n-1}{n^2(n+1)^3}+\frac{(-1)^n\left(n^2+n+1\right)}{n^2(n+1)^3}-\frac{2S_{-2}(n)}{n+1}+\frac{S_1(n)}{(n+1)^2}+\frac{S_2(n)}{-n-1}$$

Note: $S_a(n)=\sum_{i=1}^{N}\frac{\mathsf{sign}(a)^i}{i^{|a|}}$, $a\in\mathbb{Z}\setminus\{0\}$.



$$= F_{-3}(N)\varepsilon^{-3} + F_{-2}(N)\varepsilon^{-2} + F_{-1}(N)\varepsilon^{-1} + \boxed{F_0(N)}$$



$$= F_{-3}(N)\varepsilon^{-3} + F_{-2}(N)\varepsilon^{-2} + F_{-1}(N)\varepsilon^{-1} + \boxed{F_0(N)}$$

Simplify

$$||$$

$$\sum_{j=0}^{N-3}\sum_{k=0}^{j}\sum_{l=0}^{k}\sum_{q=0}^{-j+N-3}\sum_{s=1}^{-l+N-q-3}\sum_{r=0}^{-l+N-q-s-3}(-1)^{-j+k-l+N-q-3}\times$$
$$\times\frac{\binom{j+1}{k+1}\binom{k}{l}\binom{N-1}{j+2}\binom{-j+N-3}{q}\binom{-l+N-q-3}{s}\binom{-l+N-q-s-3}{r}r!(-l+N-q-r-s-3)!(s-1)!}{(-l+N-q-2)!(-j+N-1)(N-q-r-s-2)(q+s+1)}$$
$$\Big[4S_1(-j+N-1)-4S_1(-j+N-2)-2S_1(k)$$
$$-(S_1(-l+N-q-2)+S_1(-l+N-q-r-s-3)-2S_1(r+s))$$
$$+2S_1(s-1)-2S_1(r+s)\Big] + \textbf{3 further 6–fold sums}$$

$$\boxed{F_0(N)} =$$

$$\frac{7}{12}S_1(N)^4 + \frac{(17N+5)S_1(N)^3}{3N(N+1)} + \Big(\frac{35N^2-2N-5}{2N^2(N+1)^2} + \frac{13S_2(N)}{2} + \frac{5(-1)^N}{2N^2}\Big)S_1(N)^2$$
$$+ \Big(-\frac{4(13N+5)}{N^2(N+1)^2} + \Big(\frac{4(-1)^N(2N+1)}{N(N+1)} - \frac{13}{N}\Big)S_2(N) + \Big(\frac{29}{3} - (-1)^N\Big)S_3(N)$$
$$+ \big(2+2(-1)^N\big)S_{2,1}(N) - 28S_{-2,1}(N) + \frac{20(-1)^N}{N^2(N+1)}\Big)S_1(N) + \Big(\frac{3}{4} + (-1)^N\Big)S_2(N)^2$$
$$-2(-1)^N S_{-2}(N)^2 + S_{-3}(N)\Big(\frac{2(3N-5)}{N(N+1)} + \big(26+4(-1)^N\big)S_1(N) + \frac{4(-1)^N}{N+1}\Big)$$
$$+ \Big(\frac{(-1)^N(5-3N)}{2N^2(N+1)} - \frac{5}{2N^2}\Big)S_2(N) + S_{-2}(N)\Big(10S_1(N)^2 + \Big(\frac{8(-1)^N(2N+1)}{N(N+1)}$$
$$+ \frac{4(3N-1)}{N(N+1)}\Big)S_1(N) + \frac{8(-1)^N(3N+1)}{N(N+1)^2} + \big(-22+6(-1)^N\big)S_2(N) - \frac{16}{N(N+1)}\Big)$$
$$+ \Big(\frac{(-1)^N(9N+5)}{N(N+1)} - \frac{29}{3N}\Big)S_3(N) + \Big(\frac{19}{2} - 2(-1)^N\Big)S_4(N) + \big(-6+5(-1)^N\big)S_{-4}(N)$$
$$+ \Big(-\frac{2(-1)^N(9N+5)}{N(N+1)} - \frac{2}{N}\Big)S_{2,1}(N) + \big(20+2(-1)^N\big)S_{2,-2}(N) + \big(-17+13(-1)^N\big)S_{3,1}(N)$$
$$-\frac{8(-1)^N(2N+1)+4(9N+1)}{N(N+1)}S_{-2,1}(N) - \big(24+4(-1)^N\big)S_{-3,1}(N) + \big(3-5(-1)^N\big)S_{2,1,1}(N)$$
$$+ 32S_{-2,1,1}(N) + \left(\frac{3}{2}S_1(N)^2 - \frac{3S_1(N)}{N} + \frac{3}{2}(-1)^N S_{-2}(N)\right)\zeta(2)$$

$$\boxed{F_0(N)} =$$

$$\frac{7}{12} S_1(N)^4 + \frac{(17N+5)S_1(N)^3}{3N(N+1)} + \Big(\frac{35N^2-2N-5}{2N^2(N+1)^2} + \frac{13S_2(N)}{2} + \frac{5(-1)^N}{2N^2}\Big) S_1(N)^2$$

$$+ \Big(\ldots \frac{\ldots 1)^N(2N+1)}{\ldots N(N+1)} - \frac{13}{N}\Big) S_2(N) + \Big(\frac{29}{3} - (-1)^N\Big) S_3(N)$$

$$+ \big(2 + \ldots 28 S_{-2,1}(N) + \frac{20(-1)^N}{N^2(N+1)}\big) S_1(N) + \Big(\frac{3}{4} + (-1)^N\Big) S_2(N)^2$$

$$- 2(-1)^N S_{-2}(N)^2 + S_{-3}(N)\Big(\frac{2(3N-5)}{N(N+1)} + \big(26 + 4(-1)^N\big) S_1(N) + \frac{4(-1)^N}{N+1}\Big)$$

$$+ \Big(\frac{(-1)^N(5-3N)}{2N^2(N+1)} - \frac{5}{2N^2}\Big) S_2(N) + S_{-2}(N)\Big(10 S_1(N)^2 + \Big(\frac{8(-1)^N(2N+1)}{N(N+1)}$$

$$+ \frac{4(3N-1)}{N(N+1)}\Big) S_1(N) + \frac{8(-1)^N(3N+1)}{N(N+1)^2} + \big(-22 + 6(-1)^N\big) S_2(N) - \frac{16}{N(N+1)}\Big)$$

$$+ \Big(\frac{(-1)^N(9N+5)}{N(N+1)} - \frac{29}{3N}\Big) S_3(N) + \Big(\frac{19}{2} - 2(-1)^N\Big) S_4(N) + \big(-6 + 5(-1)^N\big) S_{-4}(N)$$

$$+ \Big(-\frac{2(-1)^N(9N+5)}{N(N+1)} - \frac{2}{N}\Big) S_{2,1}(N) + \big(20 + 2(-1)^N\big) S_{2,-2}(N) + \big(-17 + 13(-1)^N\big) S_{3,1}(N)$$

$$- \frac{8(-1)^N(2N+1) + 4(9N+1)}{N(N+1)} S_{-2,1}(N) - \big(24 + 4(-1)^N\big) S_{-3,1}(N) + \big(3 - 5(-1)^N\big) S_{2,1,1}(N)$$

$$+ 32 S_{-2,1,1}(N) + \left(\frac{3}{2} S_1(N)^2 - \frac{3S_1(N)}{N} + \frac{3}{2}(-1)^N S_{-2}(N)\right)\zeta(2)$$

$$S_1(N) = \sum_{i=1}^{N} \frac{1}{i}$$

$\boxed{F_0(N)} =$

$$\frac{7}{12}S_1(N)^4 + \frac{(17N+5)S_1(N)^3}{3N(N+1)} + \Big(\frac{35N^2-2N-5}{2N^2(N+1)^2} + \frac{13S_2(N)}{2} + \frac{5(-1)^N}{2N^2}\Big)S_1(N)^2$$

$$+\Big(- \text{[illegible]} \frac{\text{[illegible]}1)^N(2N+1)}{\text{[illegible]}N(N+1)} - \frac{13}{N}\Big)S_2(N) + \Big(\frac{29}{3} - (-1)^N\Big)S_3(N)$$

$$+\Big(2 + \text{[illegible]}\, 28S_{-2,1}(N) + \frac{20(-1)^N}{N^2(N+1\text{[illegible]}} \text{[illegible]})^N\Big)S_2(N)^2$$

$$-2(-1)^N S_{-2}(N)^2 + S_{-3}(N)\Big(\frac{2(3N-5)}{N(N+1)} + \big(26 + 4 \text{[illegible]})^N \text{[illegible]} 1\Big)$$

$$+\Big(\frac{(-1)^N(5-3N)}{2N^2(N+1)} - \frac{5}{2N^2}\Big)S_2(N) + S_{-2}(N)\Big(10S_1(N)^2 + \Big(\frac{8(-1)^N(2N+1)}{N(N+1)}$$

$$+\frac{4(3N-1)}{N(N+1)}\Big)S_1(N) + \frac{8(-1)^N(3N+1)}{N(N+1)^2} + \big(-22 + 6(-1)^N\big)S_2(N) - \frac{16}{N(N+1)}\Big)$$

$$+\Big(\frac{(-1)^N(9N+5)}{N(N+1)} - \frac{29}{3N}\Big)S_3(N) + \Big(\frac{19}{2} - 2(-1)^N\Big)S_4(N) + \big(-6 + 5(-1)^N\big)S_{-4}(N)$$

$$+\Big(-\frac{2(-1)^N(9N+5)}{N(N+1)} - \frac{2}{N}\Big)S_{2,1}(N) + \big(20 + 2(-1)^N\big)S_{2,-2}(N) + \big(-17 + 13(-1)^N\big)S_{3,1}(N)$$

$$-\frac{8(-1)^N(2N+1) + 4(9N+1)}{N(N+1)}S_{-2,1}(N) - \big(24 + 4(-1)^N\big)S_{-3,1}(N) + \big(3 - 5(-1)^N\big)S_{2,1,1}(N)$$

$$+32S_{-2,1,1}(N) + \left(\frac{3}{2}S_1(N)^2 - \frac{3S_1(N)}{N} + \frac{3}{2}(-1)^N S_{-2}(N)\right)\zeta(2)$$

$$S_1(N) = \sum_{i=1}^{N}\frac{1}{i}$$

$$S_2(N) = \sum_{i=1}^{N}\frac{1}{i^2}$$

$\boxed{F_0(N)} =$

$$\frac{7}{12}S_1(N)^4 + \frac{(17N+5)S_1(N)^3}{3N(N+1)} + \Big(\frac{35N^2-2N-5}{2N^2(N+1)^2} + \frac{13S_2(N)}{2} + \frac{5(-1)^N}{2N^2}\Big)S_1(N)^2$$

$$+ \Big(\ldots 1)^N(2N+1) \over N(N+1)} - \frac{13}{N}\Big)S_2(N) + \Big(\frac{29}{3} - (-1)^N\Big)S_3(N)$$

$$+ \big(2 + \ldots 28S_{-2,1}(N) + \frac{20(-1)^N}{N^2(N+1)} \ldots)^N\big)S_2(N)^2$$

$$- 2(-1)^N S_{-2}(N)^2 + S_{-3}(N)\Big(\frac{2(3N-5)}{N(N+1)} + (26 + 4 \ldots\Big)$$

$$+ \Big(\frac{(-1)^N(5-3N)}{2N^2(\ldots} - \frac{5}{\ldots}\Big)S_2(N) + S_{-2}(N)\Big(10S_1(N)^2 + \Big(\frac{\ldots(2N+1)}{N(N+1)}$$

$$+ \frac{4(3N - \ldots}{N(N + \ldots} \ldots -1)^N\big)S_2(N) - \frac{16}{N(N+1)}\Big)$$

$$+ \Big(\frac{(-1)^N \ldots}{N(\ldots} \ldots (N) + \big(-6 + 5(-1)^N\big)S_{-4}(N)$$

$$+ \Big(-\frac{2(-\ldots}{\ldots} \ldots\Big)S_{2,-2}(N) + \big(-17 + 13(-1)^N\big)S_{3,1}(N)$$

$$- \frac{8(-1)^N \ldots}{N(N+1)} S_{-2,1}(N) - \big(24 + 4(-1)^N\big)S_{-3,1}(N) + \big(3 - 5(-1)^N\big)S_{2,1,1}(N)$$

$$+ 32S_{-2,1,1}(N) + \left(\frac{3}{2}S_1(N)^2 - \frac{3S_1(N)}{N} + \frac{3}{2}(-1)^N S_{-2}(N)\right)\zeta(2)$$

$$S_1(N) = \sum_{i=1}^{N} \frac{1}{i}$$

$$S_2(N) = \sum_{i=1}^{N} \frac{1}{i^2}$$

$$S_{-2,1,1}(N) = \sum_{i=1}^{N} \frac{(-1)^i \sum_{j=1}^{i} \frac{\sum_{k=1}^{j} \frac{1}{k}}{j}}{i^2}$$

Johann Cigler's work reloaded

Hankel determinants of harmonic numbers and related topics, J. Cigler, 2017

Definition. The Hankel determinant of $(a(k))_{k\geq 0}$ is

$$\det(a(i+j))_{i,j=0}^{n} = \det\begin{pmatrix} a(0) & a(1) & a(2) & \dots & a(n) \\ a(1) & a(2) & a(3) & \dots & a(n+1) \\ a(2) & a(3) & a(4) & \dots & a(n+2) \\ \vdots & & & & \vdots \\ a(n) & a(n+1) & a(n+2) & a\dots & a(2n) \end{pmatrix}$$

Theorem (Cigler,2017). For $a(k)=\sum_{j=1}^{k}\frac{t^j}{j}$ we have

$$\det(a(i+j))_{i,j=0}^{n} = \frac{(-1)^n t^{n^2}}{\binom{2n}{n}\prod_{j=1}^{n-1}(2j+1)\binom{2j}{j}^2} r(n,t)$$

and using Sigma we get

$$nr(n,t)+(t-2)(2n-1)r(n-1,t)+t^2(n-1)r(n-2,t)=0$$

Hankel determinants of harmonic numbers and related topics, J. Cigler, 2017

Definition. The Hankel determinant of $(a(k))_{k\geq 0}$ is

$$\det(a(i+j))_{i,j=0}^{n} = \det\begin{pmatrix} a(0) & a(1) & a(2) & \dots & a(n) \\ a(1) & a(2) & a(3) & \dots & a(n+1) \\ a(2) & a(3) & a(4) & \dots & a(n+2) \\ \vdots & & & & \vdots \\ a(n) & a(n+1) & a(n+2) & a\dots & a(2n) \end{pmatrix}$$

Theorem (Cigler,2017). For $a(k) = \sum_{j=1}^{k} \frac{s\, t^j}{j+s-1}$ we have

$$\det(a(i+j))_{i,j=0}^{n} = \frac{(-1)^n t^{n^2}}{\binom{2n}{n}\prod_{j=1}^{n-1}(2j+1)\binom{2j}{j}^2} r(n,t)$$

and using Sigma we get

$$nr(n,t) + (t-2)(2n-1)r(n-1,t) + t^2(n-1)r(n-2,t) = 0$$

Theorem(Cigler,2017). For $a(k) = \sum_{j=1}^{k} \frac{s\,t^j}{j+s-1}$ we have

$$\det(a(i+j))_{i,j=0}^{n} = \frac{(-1)^n s^{n-1} t^{n^2}}{\binom{2n+s-1}{n} \prod_{j=1}^{n-1} (2j+s)\binom{2j+s-1}{j}^2} r(n,t,s)$$

Theorem(Cigler,2017). For $a(k) = \sum_{j=1}^{k} \frac{s\,t^j}{j+s-1}$ we have

$$\det(a(i+j))_{i,j=0}^{n} = \frac{(-1)^n s^{n-1} t^{n^2}}{\binom{2n+s-1}{n} \prod_{j=1}^{n-1} (2j+s) \binom{2j+s-1}{j}^2} r(n,t,s)$$

with

$$r(n,t,s) = \sum_{k=0}^{n} (-t)^{n-j} \binom{n}{j} \binom{n+k+s-1}{n} \sum_{j=1}^{k} \frac{st^j}{j+s-1}.$$

Theorem(Cigler,2017). For $a(k) = \sum_{j=1}^{k} \frac{s\,t^j}{j+s-1}$ we have

$$\det(a(i+j))_{i,j=0}^{n} = \frac{(-1)^n s^{n-1} t^{n^2}}{\binom{2n+s-1}{n} \prod_{j=1}^{n-1} (2j+s) \binom{2j+s-1}{j}^2} r(n,t,s)$$

with

$$r(n,t,s) = \sum_{k=0}^{n} (-t)^{n-j} \binom{n}{j} \binom{n+k+s-1}{n} \sum_{j=1}^{k} \frac{st^j}{j+s-1}.$$

$\downarrow$ J. Cigler's usage of `Sigma`

$$\begin{aligned} &(-1-n)t^3(n+s)(3+2n+s)r(n,t,s) \\ &-t(2+2n+s)\big(-3-8n-4n^2-4s-4ns-s^2+2t+4nt+2n^2t+st+2nst+s^2t\big)r(1+n,t,s) \\ &\qquad - (2+n)t(1+n+s)(1+2n+s)r(2+n,t,s) = 0 \end{aligned}$$

Theorem(Cigler,2017). For $a(k) = \displaystyle\sum_{j=1}^{k} \frac{s\,t^j}{j+s-1}$ we have

$$\det(a(i+j))_{i,j=0}^{n} = \frac{(-1)^n s^{n-1} t^{n^2}}{\binom{2n+s-1}{n} \prod_{j=1}^{n-1} (2j+s)\binom{2j+s-1}{j}^2} r(n,t,s)$$

with

$$r(n,t,s) = \sum_{k=0}^{n} (-t)^{n-j} \binom{n}{j} \binom{n+k+s-1}{n} \sum_{j=1}^{k} \frac{st^j}{j+s-1}.$$

$\downarrow$ J. Cigler's usage of `Sigma`

$$\begin{aligned}
&(-1-n)t^3(n+s)(3+2n+s)r(n,t,s)\\
&-t(2+2n+s)\big(-3-8n-4n^2-4s-4ns-s^2+2t+4nt+2n^2t+st+2nst+s^2t\big)r(1+n,t,s)\\
&\qquad - (2+n)t(1+n+s)(1+2n+s)r(2+n,t,s) = 0
\end{aligned}$$

$t=1 \quad \downarrow$ `Sigma`

$$r(n,1,s) = s\,H_n + \sum_{j=1}^{n} \frac{s}{j+s-1}$$

Theorem(Cigler,2017). For $a(k) = \sum_{j=1}^{k} \frac{s\,t^j}{j+s-1}$ we have

$$\det(a(i+j))_{i,j=0}^{n} = \frac{(-1)^n s^{n-1} t^{n^2}}{\binom{2n+s-1}{n} \prod_{j=1}^{n-1} (2j+s) \binom{2j+s-1}{j}^2} r(n,t,s)$$

with

$$r(n,t,s) = \sum_{k=0}^{n} (-t)^{n-j} \binom{n}{j} \binom{n+k+s-1}{n} \sum_{j=1}^{k} \frac{st^j}{j+s-1}.$$

$\downarrow$ Sigma

Theorem(Cigler,2017). For $a(k)=\sum_{j=1}^{k}\frac{s\,t^j}{j+s-1}$ we have

$$\det(a(i+j))_{i,j=0}^{n}=\frac{(-1)^n s^{n-1}t^{n^2}}{\binom{2n+s-1}{n}\prod_{j=1}^{n-1}(2j+s)\binom{2j+s-1}{j}^2}r(n,t,s)$$

with

$$r(n,t,s)=\sum_{k=0}^{n}(-t)^{n-j}\binom{n}{j}\binom{n+k+s-1}{n}\sum_{j=1}^{k}\frac{st^j}{j+s-1}.$$

$\downarrow$ Sigma

$$r(2n,-4n,\tfrac{1}{2})=(-1)^n 2^{-2-6n}\tbinom{2n}{n}n\Big(2\sum_{i=1}^{n}\tfrac{2^{4i}}{i}+4\sum_{i=1}^{n}\tfrac{2^{4i}}{-1+4i}+\sum_{i=1}^{n}\tfrac{2^{4i}}{-1+2i}+\sum_{i=1}^{n}\tfrac{2^{4i}}{-3+4i}\Big)$$

$$r(2n+1,-4n-2,\tfrac{1}{2})=\frac{(-1)^{1+n}2^{-1-2n}}{\binom{2n}{n}}$$