

Computer Algebra and Elementary Particle Theory at the Large Scale:

10 Years of JKU-DESY Collaboration, February 7, 2017

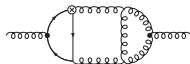
# All You Can Sum (part 2)

Carsten Schneider

Research Institute for Symbolic Computation (RISC)  
Johannes Kepler University Linz

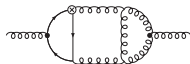


# Evaluation of Feynman Integrals



Behavior of particles

# Evaluation of Feynman Integrals



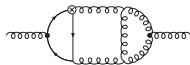
Behavior of particles



$$\int \Phi(N, \epsilon, x) dx$$

Feynman integrals

# Evaluation of Feynman Integrals



Behavior of particles



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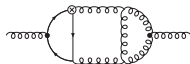
Feynman integrals

**DESY**

$$\sum f(N, \epsilon, k)$$

complicated  
multi-sums

# Evaluation of Feynman Integrals



Behavior of particles



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Feynman integrals

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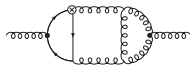
complicated  
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**RISC**

(Sigma-package)

expression in  
special functions

# Evaluation of Feynman Integrals



Behavior of particles



$$\int \Phi(N, \epsilon, x) dx$$

Feynman integrals



LHC at CERN

**DESY**



$$\sum f(N, \epsilon, k)$$

complicated  
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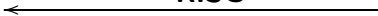
applicable



expression in  
special functions

**RISC**

(Sigma-package)



$$F(\varepsilon, N) = \int \frac{d^{4+\varepsilon}k_1}{(2\pi)^{4+\varepsilon}} \frac{d^{4+\varepsilon}k_2}{(2\pi)^{4+\varepsilon}} \frac{d^{4+\varepsilon}k_3}{(2\pi)^{4+\varepsilon}} \frac{(\Delta \cdot k_3)^N}{k_2^4 ((k_1 - k_3)^2 - m^2) (k_1 - k_2)^2 ((k_3 - p)^2 - m^2)}$$

||?

$$F_{-3}(N)\varepsilon^{-3} + F_{-2}(N)\varepsilon^{-2} + F_{-1}(N)\varepsilon^{-1} + F_0(N)\varepsilon^0 + \dots$$

$$F(\varepsilon, N) = \int \frac{d^{4+\varepsilon}k_1}{(2\pi)^{4+\varepsilon}} \frac{d^{4+\varepsilon}k_2}{(2\pi)^{4+\varepsilon}} \frac{d^{4+\varepsilon}k_3}{(2\pi)^{4+\varepsilon}} \frac{(\Delta.k_3)^N}{k_2^4((k_1-k_3)^2-m^2)(k_1-k_2)^2((k_3-p)^2-m^2)}$$

||

$$\sum_{k=1}^N (-1)^k e^{-\frac{3\varepsilon\gamma}{2}} \Gamma\left(-1 - \frac{3\varepsilon}{2}\right) \times \\ \times B\left(2+k, \frac{\varepsilon}{2}\right) B(-\varepsilon+k, -\varepsilon) B\left(1 - \frac{\varepsilon}{2} + k, 1 + \frac{\varepsilon}{2}\right) \binom{N}{k}$$

where

$$B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$



$$F(\varepsilon, N) = \int \frac{d^{4+\varepsilon}k_1}{(2\pi)^{4+\varepsilon}} \frac{d^{4+\varepsilon}k_2}{(2\pi)^{4+\varepsilon}} \frac{d^{4+\varepsilon}k_3}{(2\pi)^{4+\varepsilon}} \frac{(\Delta.k_3)^N}{k_2^4((k_1-k_3)^2-m^2)(k_1-k_2)^2((k_3-p)^2-m^2)}$$

||

$$\sum_{k=1}^N (-1)^k e^{-\frac{3\varepsilon\gamma}{2}} \Gamma\left(-1 - \frac{3\varepsilon}{2}\right) \times$$

$$\underbrace{\times B\left(2+k, \frac{\varepsilon}{2}\right) B(-\varepsilon+k, -\varepsilon) B\left(1 - \frac{\varepsilon}{2} + k, 1 + \frac{\varepsilon}{2}\right) \binom{N}{k}}_{= f_{-3}(N, k)\varepsilon^{-3} + f_{-2}(N, k)\varepsilon^{-2} + f_{-1}(N, k)\varepsilon^{-1} + \dots}$$

G.E. Andrews, R. Askey, R. Roy: Special Functions

$$F(\varepsilon, N) = \int \frac{d^{4+\varepsilon}k_1}{(2\pi)^{4+\varepsilon}} \frac{d^{4+\varepsilon}k_2}{(2\pi)^{4+\varepsilon}} \frac{d^{4+\varepsilon}k_3}{(2\pi)^{4+\varepsilon}} \frac{(\Delta.k_3)^N}{k_2^4((k_1-k_3)^2-m^2)(k_1-k_2)^2((k_3-p)^2-m^2)}$$

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||

$$\left(\sum_{k=1}^N f_{-3}(N, k)\right)\varepsilon^{-3} + \left(\sum_{k=1}^N f_{-2}(N, k)\right)\varepsilon^{-2} + \left(\sum_{k=1}^N f_{-1}(N, k)\right)\varepsilon^{-1} + \dots$$

$$F(\varepsilon, N) = \int \frac{d^{4+\varepsilon}k_1}{(2\pi)^{4+\varepsilon}} \frac{d^{4+\varepsilon}k_2}{(2\pi)^{4+\varepsilon}} \frac{d^{4+\varepsilon}k_3}{(2\pi)^{4+\varepsilon}} \frac{(\Delta \cdot k_3)^N}{k_2^4 ((k_1 - k_3)^2 - m^2) (k_1 - k_2)^2 ((k_3 - p)^2 - m^2)}$$

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$$\underbrace{\sum_{k=1}^N (-1)^k e^{-\frac{3\varepsilon\gamma}{2}} \Gamma\left(-1 - \frac{3\varepsilon}{2}\right) \times \\ \times B\left(2+k, \frac{\varepsilon}{2}\right) B(-\varepsilon+k, -\varepsilon) B\left(1 - \frac{\varepsilon}{2} + k, 1 + \frac{\varepsilon}{2}\right) \binom{N}{k}}_{= f_{-3}(N, k)\varepsilon^{-3} + f_{-2}(N, k)\varepsilon^{-2} + f_{-1}(N, k)\varepsilon^{-1} + \dots}$$

||

$$\left(\sum_{k=1}^N f_{-3}(N, k)\right)\varepsilon^{-3} + \left(\sum_{k=1}^N f_{-2}(N, k)\right)\varepsilon^{-2} + \left(\sum_{k=1}^N f_{-1}(N, k)\right)\varepsilon^{-1} + \dots$$

## Simplify

$$F_{-1}(N) = \sum_{k=1}^N (-1)^{k+1} \binom{N}{k} \left( \frac{(2+3k)(-2+3k+7k^2+3k^3)}{3k^2(1+k)^3} + \frac{2S_2(k)}{1+k} + \frac{\zeta_2}{2(1+k)} \right)$$

where

$$S_a(N) = \sum_{i=1}^N \frac{\text{sign}(a)^i}{i^a} \quad \text{and} \quad \zeta_a = \sum_{i=1}^{\infty} \frac{1}{i^a}$$

## Simplify

$$F_{-1}(N) = \sum_{k=1}^N (-1)^{k+1} \binom{N}{k} \left( \frac{(2+3k)(-2+3k+7k^2+3k^3)}{3k^2(1+k)^3} + \frac{2S_2(k)}{1+k} + \frac{\zeta_2}{2(1+k)} \right)$$

↓ (summation package Sigma.m)

$$\begin{aligned} & (16N^3 + 144N^2 + 413N + 384)(N+1)^2 F_{-1}(N) \\ & - (N+2)(2N+5)(16N^3 + 112N^2 + 221N + 113) F_{-1}(N+1) \\ & + (N+3)^2(16N^3 + 96N^2 + 173N + 99) F_{-1}(N+2) \\ & = \frac{1}{2}(4N^2 + 21N + 29)\zeta_2 + \frac{-64N^5 - 500N^4 - 1133N^3 + 203N^2 + 3516N + 3090}{3(N+2)(N+3)} \end{aligned}$$

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$$\begin{aligned} & (16N^3 + 144N^2 + 413N + 384)(N+1)^2 F_{-1}(N) \\ & - (N+2)(2N+5)(16N^3 + 112N^2 + 221N + 113) F_{-1}(N+1) \\ & + (N+3)^2(16N^3 + 96N^2 + 173N + 99) F_{-1}(N+2) \\ & = \frac{1}{2}(4N^2 + 21N + 29)\zeta_2 + \frac{-64N^5 - 500N^4 - 1133N^3 + 203N^2 + 3516N + 3090}{3(N+2)(N+3)} \end{aligned}$$

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$$\begin{aligned} & \left\{ c_1 \frac{1-4N}{N+1} + c_2 \frac{-14N-13}{(N+1)^2} \right. \\ & + \frac{(4N-1)S_1(N)}{N+1} + \frac{(1-4N)S_1(N)^2}{6(N+1)} + \frac{(14N+13)S_1(N)}{3(N+1)^2} \\ & \left. + \frac{175N^2 + 334N + 155}{12(N+1)^3} + \frac{(1-4N)S_2(N)}{6(N+1)} + \frac{\zeta_2}{8(N+1)} \mid c_1, c_2 \in \mathbb{Q} \right\} \end{aligned}$$

## Simplify

$$F_{-1}(N) = \sum_{k=1}^N (-1)^{k+1} \binom{N}{k} \left( \frac{(2+3k)(-2+3k+7k^2+3k^3)}{3k^2(1+k)^3} + \frac{2S_2(k)}{1+k} + \frac{\zeta_2}{2(1+k)} \right)$$



$$\left\{ c_1 \frac{1-4N}{N+1} + c_2 \frac{-14N-13}{(N+1)^2} + \frac{(4N-1)S_1(N)}{N+1} + \frac{(1-4N)S_1(N)^2}{6(N+1)} + \frac{(14N+13)S_1(N)}{3(N+1)^2} + \frac{175N^2+334N+155}{12(N+1)^3} + \frac{(1-4N)S_2(N)}{6(N+1)} + \frac{\zeta_2}{8(N+1)} \mid c_1, c_2 \in \mathbb{Q} \right\}$$

## Simplify

$$F_{-1}(N) = \sum_{k=1}^N (-1)^{k+1} \binom{N}{k} \left( \frac{(2+3k)(-2+3k+7k^2+3k^3)}{3k^2(1+k)^3} + \frac{2S_2(k)}{1+k} + \frac{\zeta_2}{2(1+k)} \right)$$

|| (recurrence finding and solving)

$$\begin{aligned} & \left( \frac{1}{12} - \frac{1}{8}\zeta_2 \right) \frac{1-4N}{N+1} + 1 \frac{-14N-13}{(N+1)^2} \\ & + \frac{(4N-1)S_1(N)}{N+1} + \frac{(1-4N)S_1(N)^2}{6(N+1)} + \frac{(14N+13)S_1(N)}{3(N+1)^2} \\ & + \frac{175N^2+334N+155}{12(N+1)^3} + \frac{(1-4N)S_2(N)}{6(N+1)} + \frac{\zeta_2}{8(N+1)} \end{aligned}$$



## 1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$F(N) = \sum_{k=0}^N f(N, k);$$

$f(N, k)$ : indefinite nested product-sum in  $k$ ;  
 $N$ : extra parameter

FIND a recurrence for  $F(N)$

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FIND a **recurrence** for  $F(N)$

## 2. Recurrence solving

GIVEN a recurrence

$a_0(N), \dots, a_d(N), h(N)$ :  
indefinite nested product-sum expressions.

$$a_0(N)F(N) + \dots + a_d(N)F(N + d) = h(N);$$

FIND **all solutions** expressible by indefinite nested products/sums

(Abramov/Bronstein/Petkovšek/CS, in preparation)

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## 3. Find a “closed form”

F(N)=combined solutions in terms of **indefinite nested** sums.

# Sigma.m is based on difference ring/field theory

1. M. Karr. Summation in finite terms. *J. ACM*, 28:305–350, 1981.
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In[1]:= << **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= << **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[3]:= << **EvaluateMultiSums.m**

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In[4]:= **mySum =**

$$\sum_{k=1}^N (-1)^k e^{-\frac{3\epsilon\gamma}{2}} \left(-2 - \frac{3\epsilon}{2}\right)! B\left(2+k, \frac{\epsilon}{2}\right) B(-\epsilon+k, -\epsilon) B\left(1 - \frac{\epsilon}{2} + k, 1 + \frac{\epsilon}{2}\right) \binom{N}{k};$$

In[5]:= **EvaluateMultiSum[mySum, {}, {N}, {1}, ExpandIn → {ε, -3, -3}]**

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In[5]:= **EvaluateMultiSum**[**mySum**, {}, {**N**}, {**1**}, **ExpandIn** → {**ε**, -**3**, -**3**}]

$$\text{Out[5]} = \left\{ \frac{59N^2 + 120N + 49}{9(N+1)^2} - \frac{2(N+3)S_1[N]}{3(N+1)} \right\}$$

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Sigma - A summation package by Carsten Schneider © RISC-Linz

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$$\text{Out[5]} = \left\{ \frac{59N^2 + 120N + 49}{9(N+1)^2} - \frac{2(N+3)S_1[N]}{3(N+1)}, \right. \\ \left. - \frac{2(20N^3 + 58N^2 + 57N + 22)}{3(N+1)^3} + \frac{2(N+2)(2N-1)S_1[N]}{3(N+1)^2} - \frac{S_1[N]^2}{N+1} - \frac{S_2[N]}{N+1} \right\}$$



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$$\begin{aligned} \text{Out[5]} = & \left\{ \frac{59N^2 + 120N + 49}{9(N+1)^2} - \frac{2(N+3)S_1[N]}{3(N+1)}, \right. \\ & - \frac{2(20N^3 + 58N^2 + 57N + 22)}{3(N+1)^3} + \frac{2(N+2)(2N-1)S_1[N]}{3(N+1)^2} - \frac{S_1[N]^2}{N+1} - \frac{S_2[N]}{N+1}, \\ & \left( \frac{1}{12} - \frac{1}{8}\zeta(2) \right) \frac{1-4N}{N+1} + \frac{-14N-13}{(N+1)^2} + \frac{(4N-1)S_1(N)}{N+1} + \frac{(1-4N)S_1(N)^2}{6(N+1)} + \\ & \left. \frac{(14N+13)S_1(N)}{3(N+1)^2} + \frac{175N^2 + 334N + 155}{12(N+1)^3} + \frac{(1-4N)S_2(N)}{6(N+1)} + \frac{\zeta(2)}{8(N+1)} \right\} \end{aligned}$$

$$\sum_{j=0}^{n-2} \sum_{r=0}^{j+1} \sum_{s=0}^{n-j+r-2} \frac{(-1)^{r+s} \binom{j+1}{r} \binom{-j+n+r-2}{s} (-j+n-2)! r!}{(n-s)(s+1)(-j+n+r)!}$$

Simple sum

$$\sum_{j=0}^{n-2} \sum_{r=0}^{j+1} \sum_{s=0}^{n-j+r-2} \frac{(-1)^{r+s} \binom{j+1}{r} \binom{-j+n+r-2}{s} (-j+n-2)! r!}{(n-s)(s+1)(-j+n+r)!}$$

||

$$\sum_{j=0}^{n-2} \sum_{r=0}^{j+1} \boxed{\sum_{s=0}^{n-j+r-2} \frac{(-1)^{r+s} \binom{j+1}{r} \binom{-j+n+r-2}{s} (-j+n-2)! r!}{(n-s)(s+1)(-j+n+r)!}}$$

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||

$$\boxed{\binom{j+1}{r} \left( \frac{(-1)^r (-j+n-2)! r!}{(n+1)(-j+n+r-1)(-j+n+r)!} + \frac{(-1)^{n+r} (j+1)! (-j+n-2)! (-j+n-1)_r r!}{(n-1)n(n+1)(-j+n+r)! (-j-1)_r (2-n)_j} \right)}$$

$$\sum_{j=0}^{n-2} \sum_{r=0}^{j+1} \sum_{s=0}^{n-j+r-2} \frac{(-1)^{r+s} \binom{j+1}{r} \binom{-j+n+r-2}{s} (-j+n-2)! r!}{(n-s)(s+1)(-j+n+r)!}$$

$$\sum_{j=0}^{n-2} \left( \sum_{r=0}^{j+1} \binom{j+1}{r} \left( \frac{(-1)^r (-j+n-2)! r!}{(n+1)(-j+n+r-1)(-j+n+r)!} + \frac{(-1)^{n+r} (j+1)! (-j+n-2)! (-j+n-1)_r r!}{(n-1)n(n+1)(-j+n+r)! (-j-1)_r (2-n)_j} \right) \right)$$

$$\sum_{j=0}^{n-2} \sum_{r=0}^{j+1} \sum_{s=0}^{n-j+r-2} \frac{(-1)^{r+s} \binom{j+1}{r} \binom{-j+n+r-2}{s} (-j+n-2)! r!}{(n-s)(s+1)(-j+n+r)!}$$

$$\sum_{j=0}^{n-2} \left( \sum_{r=0}^{j+1} \binom{j+1}{r} \left( \frac{(-1)^r (-j+n-2)! r!}{(n+1)(-j+n+r-1)(-j+n+r)!} + \frac{(-1)^{n+r} (j+1)! (-j+n-2)! (-j+n-1)_r r!}{(n-1)n(n+1)(-j+n+r)! (-j-1)_r (2-n)_j} \right) \right)$$

$$\left( \frac{n^2 - n + 1}{(n-1)^2 n^2 (n+1) (2-n)_j} + \frac{\sum_{i=1}^j \frac{(2-n)_i}{(-i+n-1)^2 (i+1)!}}{(n+1) (2-n)_j} + \frac{(-1)^{j+n} (-j-2) (-j+n-2)!}{(j-n+1) (n+1)^2 n!} \right) (j+1)! - \frac{1}{(n+1)^2 (-j+n-1)}$$

$$\sum_{j=0}^{n-2} \sum_{r=0}^{j+1} \sum_{s=0}^{n-j+r-2} \frac{(-1)^{r+s} \binom{j+1}{r} \binom{-j+n+r-2}{s} (-j+n-2)! r!}{(n-s)(s+1)(-j+n+r)!}$$

||

$$\sum_{j=0}^{n-2} \left( \left( \frac{n^2 - n + 1}{(n-1)^2 n^2 (n+1)(2-n)_j} + \frac{\sum_{i=1}^j \frac{(2-n)_i}{(-i+n-1)^2 (i+1)!}}{(n+1)(2-n)_j} + \frac{(-1)^{j+n} (-j-2)(-j+n-2)!}{(j-n+1)(n+1)^2 n!} \right) (j+1)! - \frac{1}{(n+1)^2 (-j+n-1)} \right)$$

$$\sum_{j=0}^{n-2} \sum_{r=0}^{j+1} \sum_{s=0}^{n-j+r-2} \frac{(-1)^{r+s} \binom{j+1}{r} \binom{-j+n+r-2}{s} (-j+n-2)! r!}{(n-s)(s+1)(-j+n+r)!}$$

$$\parallel$$

$$\sum_{j=0}^{n-2} \left( \left( \frac{n^2 - n + 1}{(n-1)^2 n^2 (n+1)(2-n)_j} + \frac{\sum_{i=1}^j \frac{(2-n)_i}{(-i+n-1)^2 (i+1)!}}{(n+1)(2-n)_j} + \right. \right.$$

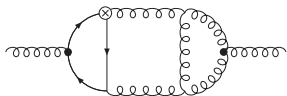
$$\left. \left. \frac{(-1)^{j+n} (-j-2)(-j+n-2)!}{(j-n+1)(n+1)^2 n!} \right) (j+1)! - \frac{1}{(n+1)^2 (-j+n-1)} \right)$$

$$\parallel$$

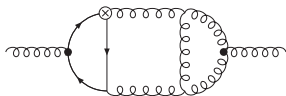
$$\frac{-n^2 - n - 1}{n^2 (n+1)^3} + \frac{(-1)^n (n^2 + n + 1)}{n^2 (n+1)^3} - \frac{2S_{-2}(n)}{n+1} + \frac{S_1(n)}{(n+1)^2} + \frac{S_2(n)}{-n-1}$$

Note:  $S_a(n) = \sum_{i=1}^n \frac{\text{sign}(a)^i}{i^{|a|}}$ ,  $a \in \mathbb{Z} \setminus \{0\}$ .





$$= F_{-3}(N)\varepsilon^{-3} + F_{-2}(N)\varepsilon^{-2} + F_{-1}(N)\varepsilon^{-1} + \boxed{F_0(N)}$$



$$= F_{-3}(N)\varepsilon^{-3} + F_{-2}(N)\varepsilon^{-2} + F_{-1}(N)\varepsilon^{-1} + \boxed{F_0(N)}$$

||

Simplify

$$\sum_{j=0}^{N-3} \sum_{k=0}^j \sum_{l=0}^k \sum_{q=0}^{-j+N-3} \sum_{s=1}^{-l+N-q-3} \sum_{r=0}^{-l+N-q-s-3} (-1)^{-j+k-l+N-q-3} \times$$

$$\times \frac{\binom{j+1}{k+1} \binom{k}{l} \binom{N-1}{j+2} \binom{-j+N-3}{q} \binom{-l+N-q-3}{s} \binom{-l+N-q-s-3}{r} r! (-l+N-q-r-s-3)! (s-1)!}{(-l+N-q-2)! (-j+N-1) (N-q-r-s-2) (q+s+1)}$$

$$\left[ 4S_1(-j+N-1) - 4S_1(-j+N-2) - 2S_1(k) \right.$$

$$\left. - (S_1(-l+N-q-2) + S_1(-l+N-q-r-s-3) - 2S_1(r+s)) \right.$$

$$\left. + 2S_1(s-1) - 2S_1(r+s) \right] + \mathbf{3 \text{ further 6-fold sums}}$$

$$\boxed{F_0(N)} =$$

$$\begin{aligned} & \frac{7}{12}S_1(N)^4 + \frac{(17N+5)S_1(N)^3}{3N(N+1)} + \left(\frac{35N^2-2N-5}{2N^2(N+1)^2} + \frac{13S_2(N)}{2} + \frac{5(-1)^N}{2N^2}\right)S_1(N)^2 \\ & + \left(-\frac{4(13N+5)}{N^2(N+1)^2} + \left(\frac{4(-1)^N(2N+1)}{N(N+1)} - \frac{13}{N}\right)S_2(N) + \left(\frac{29}{3} - (-1)^N\right)S_3(N)\right) \\ & + (2 + 2(-1)^N)S_{2,1}(N) - 28S_{-2,1}(N) + \frac{20(-1)^N}{N^2(N+1)}S_1(N) + \left(\frac{3}{4} + (-1)^N\right)S_2(N)^2 \\ & - 2(-1)^NS_{-2}(N)^2 + S_{-3}(N)\left(\frac{2(3N-5)}{N(N+1)} + (26 + 4(-1)^N)S_1(N) + \frac{4(-1)^N}{N+1}\right) \\ & + \left(\frac{(-1)^N(5-3N)}{2N^2(N+1)} - \frac{5}{2N^2}\right)S_2(N) + S_{-2}(N)(10S_1(N)^2 + \frac{8(-1)^N(2N+1)}{N(N+1)}) \\ & + \frac{4(3N-1)}{N(N+1)}S_1(N) + \frac{8(-1)^N(3N+1)}{N(N+1)^2} + (-22 + 6(-1)^N)S_2(N) - \frac{16}{N(N+1)} \\ & + \left(\frac{(-1)^N(9N+5)}{N(N+1)} - \frac{29}{3N}\right)S_3(N) + \left(\frac{19}{2} - 2(-1)^N\right)S_4(N) + (-6 + 5(-1)^N)S_{-4}(N) \\ & + \left(-\frac{2(-1)^N(9N+5)}{N(N+1)} - \frac{2}{N}\right)S_{2,1}(N) + (20 + 2(-1)^N)S_{2,-2}(N) + (-17 + 13(-1)^N)S_{3,1}(N) \\ & - \frac{8(-1)^N(2N+1) + 4(9N+1)}{N(N+1)}S_{-2,1}(N) - (24 + 4(-1)^N)S_{-3,1}(N) + (3 - 5(-1)^N)S_{2,1,1}(N) \\ & + 32S_{-2,1,1}(N) + \left(\frac{3}{2}S_1(N)^2 - \frac{3S_1(N)}{N} + \frac{3}{2}(-1)^NS_{-2}(N)\right)\zeta(2) \end{aligned}$$

$$F_0(N) =$$

$$\begin{aligned} & \frac{7}{12}S_1(N)^4 + \frac{(17N+5)S_1(N)^3}{3N(N+1)} + \left( \frac{35N^2-2N-5}{2N^2(N+1)^2} + \frac{13S_2(N)}{2} + \frac{5(-1)^N}{2N^2} \right) S_1(N)^2 \\ & + \left( - \frac{1^N(2N+1)}{N(N+1)} - \frac{13}{N} \right) S_2(N) + \left( \frac{29}{3} - (-1)^N \right) S_3(N) \\ & + \left( 2 + \frac{20(-1)^N}{N^2(N+1)} \right) S_1(N) + \left( \frac{3}{4} + (-1)^N \right) S_2(N)^2 \\ & - 2(-1)^N S_{-2}(N)^2 + S_{-3}(N) \left( \frac{2(3N-5)}{N(N+1)} + (26 + 4(-1)^N) S_1(N) + \frac{4(-1)^N}{N+1} \right) \\ & + \left( \frac{(-1)^N(5-3N)}{2N^2(N+1)} - \frac{5}{2N^2} \right) S_2(N) + S_{-2}(N) (10S_1(N)^2 + \frac{8(-1)^N(2N+1)}{N(N+1)}) \\ & + \frac{4(3N-1)}{N(N+1)} S_1(N) + \frac{8(-1)^N(3N+1)}{N(N+1)^2} + (-22 + 6(-1)^N) S_2(N) - \frac{16}{N(N+1)} \\ & + \left( \frac{(-1)^N(9N+5)}{N(N+1)} - \frac{29}{3N} \right) S_3(N) + \left( \frac{19}{2} - 2(-1)^N \right) S_4(N) + (-6 + 5(-1)^N) S_{-4}(N) \\ & + \left( - \frac{2(-1)^N(9N+5)}{N(N+1)} - \frac{2}{N} \right) S_{2,1}(N) + (20 + 2(-1)^N) S_{2,-2}(N) + (-17 + 13(-1)^N) S_{3,1}(N) \\ & - \frac{8(-1)^N(2N+1) + 4(9N+1)}{N(N+1)} S_{-2,1}(N) - (24 + 4(-1)^N) S_{-3,1}(N) + (3 - 5(-1)^N) S_{2,1,1}(N) \\ & + 32S_{-2,1,1}(N) + \left( \frac{3}{2}S_1(N)^2 - \frac{3S_1(N)}{N} + \frac{3}{2}(-1)^N S_{-2}(N) \right) \zeta(2) \end{aligned}$$

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$$F_0(N) =$$

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$$S_1(N) = \sum_{i=1}^N \frac{1}{i}$$

$$S_2(N) = \sum_{i=1}^N \frac{1}{i^2}$$

$$S_{-2,1,1}(N) = \sum_{i=1}^N \frac{(-1)^i \sum_{k=1}^j \frac{1}{k}}{i^2}$$

## The general tactic

Feynman integrals

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↓ non-trivial transformations (DESY)

multiple sums



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Tactic 1: Expand the summand and simplify

Ablinger, Blümlein, Klein, CS, LL2010, arXiv:1006.4797 [math-ph]

Blümlein, Hasselhuhn, CS, RADCOR'10, arXiv:1202.4303 [math-ph]

CS, ACAT 2013, arXiv:1310.0160 [cs.SC]

## The general tactic

Feynman integrals

↓ non-trivial transformations (DESY)

multiple sums

↓ symbolic summation

compact expression in terms  
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## Tactic 2: Expand a recurrence in $\varepsilon$

Blümlein, Klein, CS, Stan, J. Symbol. Comput. 2012; arXiv:1011.2656 [cs.SC]

Ablinger, Blümlein, Round, CS, LL2012, arXiv:1210.1685 [cs.SC]

$$F(N) = \sum_{k=1}^N (-1)^k e^{-\frac{3\varepsilon\gamma}{2}} \Gamma\left(-1 - \frac{3\varepsilon}{2}\right) B\left(2+k, \frac{\varepsilon}{2}\right) B(-\varepsilon+k, -\varepsilon) B\left(1 - \frac{\varepsilon}{2} + k, 1 + \frac{\varepsilon}{2}\right) \binom{N}{k}$$

↓ (summation package Sigma.m)

$$2(N+1)^2 F(N) + (3\varepsilon^2 + 3\varepsilon N + 9\varepsilon - 4N^2 - 12N - 8) F(N+1) - (2\varepsilon - N - 1)(\varepsilon + 2N + 6) F(N+2) = 0\varepsilon^{-3} - \frac{16}{3}\varepsilon^{-2} + \frac{40}{3}\varepsilon^{-1} - \left(2\zeta_2 - \frac{68}{3}\right)\varepsilon^0 + \dots$$

$$F(N) = \sum_{k=1}^N (-1)^k e^{-\frac{3\varepsilon\gamma}{2}} \Gamma\left(-1 - \frac{3\varepsilon}{2}\right) B\left(2+k, \frac{\varepsilon}{2}\right) B(-\varepsilon+k, -\varepsilon) B\left(1 - \frac{\varepsilon}{2} + k, 1 + \frac{\varepsilon}{2}\right) \binom{N}{k}$$

↓ (summation package Sigma.m)

$$2(N+1)^2 F(N) + (3\varepsilon^2 + 3\varepsilon N + 9\varepsilon - 4N^2 - 12N - 8) F(N+1) - (2\varepsilon - N - 1)(\varepsilon + 2N + 6) F(N+2) = 0\varepsilon^{-3} - \frac{16}{3}\varepsilon^{-2} + \frac{40}{3}\varepsilon^{-1} - (2\zeta_2 - \frac{68}{3})\varepsilon^0 + \dots$$

$$F(1) = \frac{2}{3}\varepsilon^{-3} - \frac{11}{6}\varepsilon^{-2} + \left(\frac{\zeta_2}{4} + \frac{79}{24}\right)\varepsilon^{-1} + \dots$$

$$F(2) = \frac{8}{9}\varepsilon^{-3} - \frac{73}{27}\varepsilon^{-2} + \left(\frac{\zeta_2}{3} + \frac{1415}{324}\right)\varepsilon^{-1} + \dots$$

↓

$$F(N) = F_{-3}(N) \varepsilon^{-3} + F_{-2}(N) \varepsilon^{-2} + F_{-1}(N) \varepsilon^{-1} + \dots$$

## Ansatz (for power series)

$$\begin{aligned} & a_0(\varepsilon, N) [F(N)] \\ & + a_1(\varepsilon, N) [F(N + 1)] \\ & + \\ & \vdots \\ & + a_d(\varepsilon, N) [F(N + d)] \end{aligned}$$

$= h_0(N) + h_1(N)\varepsilon + h_1(N)\varepsilon^2 + \dots$

given (in terms of indefinite nested sums and products)


## Ansatz (for power series)

$$\begin{aligned}
 & a_0(\varepsilon, N) \left[ F_0(N) + F_1(N)\varepsilon + F_2(N)\varepsilon^2 + \dots \right] \\
 & + a_1(\varepsilon, N) \left[ F(N + 1) \right] \\
 & + \\
 & \vdots \\
 & + a_d(\varepsilon, N) \left[ F(N + d) \right] \\
 & \qquad \qquad \qquad = h_0(N) + h_1(N)\varepsilon + h_1(N)\varepsilon^2 + \dots
 \end{aligned}$$

given (in terms of indefinite nested sums and products)

## Ansatz (for power series)

$$\begin{aligned}
 & a_0(\varepsilon, N) \left[ F_0(N) + F_1(N)\varepsilon + F_2(N)\varepsilon^2 + \dots \right] \\
 & + a_1(\varepsilon, N) \left[ F_0(N+1) + F_1(N+1)\varepsilon + F_2(N+1)\varepsilon^2 + \dots \right] \\
 & + \\
 & \vdots \\
 & + a_d(\varepsilon, N) \left[ F(N+d) \right] \\
 & \qquad \qquad \qquad = h_0(N) + h_1(N)\varepsilon + h_1(N)\varepsilon^2 + \dots
 \end{aligned}$$


**given** (in terms of indefinite nested sums and products)



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 & + \\
 & \vdots \\
 & + a_d(\varepsilon, N) \left[ F_0(N+d) + F_1(N+d)\varepsilon + F_2(N+d)\varepsilon^2 + \dots \right] \\
 & \qquad \qquad \qquad = h_0(N) + h_1(N)\varepsilon + h_1(N)\varepsilon^2 + \dots
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 & + \\
 & \vdots \\
 & + a_d(\varepsilon, N) \left[ F_0(N+d) + F_1(N+d)\varepsilon + F_2(N+d)\varepsilon^2 + \dots \right] \\
 & \qquad \qquad \qquad = h_0(N) + h_1(N)\varepsilon + h_1(N)\varepsilon^2 + \dots
 \end{aligned}$$

↓ constant terms must agree

$$a_0(0, N)F_0(N) + a_1(0, N)F_0(N+1) + \dots + a_d(0, N)F_0(N+d) = h_0(N)$$

## Ansatz (for power series)

$$\begin{aligned}
 & a_0(\varepsilon, N) \left[ F_0(N) + F_1(N)\varepsilon + F_2(N)\varepsilon^2 + \dots \right] \\
 & + a_1(\varepsilon, N) \left[ F_0(N+1) + F_1(N+1)\varepsilon + F_2(N+1)\varepsilon^2 + \dots \right] \\
 & + \\
 & \vdots \\
 & + a_d(\varepsilon, N) \left[ F_0(N+d) + F_1(N+d)\varepsilon + F_2(N+d)\varepsilon^2 + \dots \right] \\
 & \qquad \qquad \qquad = h_0(N) + h_1(N)\varepsilon + h_1(N)\varepsilon^2 + \dots
 \end{aligned}$$

⇓ constant terms must agree

$$a_0(0, N)F_0(N) + a_1(0, N)F_0(N+1) + \dots + a_d(0, N)F_0(N+d) = h_0(N)$$

REC solver: Given the initial values  $F_0(1), F_0(2), \dots, F_0(d-1)$ ,  
**decide** if  $F_0(N)$  can be written in terms of indefinite  
 nested sums and products.

## Ansatz (for power series)

$$\begin{aligned}
 & a_0(\varepsilon, N) \left[ F_0(N) + F_1(N)\varepsilon + F_2(N)\varepsilon^2 + \dots \right] \\
 & + a_1(\varepsilon, N) \left[ F_0(N+1) + F_1(N+1)\varepsilon + F_2(N+1)\varepsilon^2 + \dots \right] \\
 & + \\
 & \vdots \\
 & + a_d(\varepsilon, N) \left[ F_0(N+d) + F_1(N+d)\varepsilon + F_2(N+d)\varepsilon^2 + \dots \right] \\
 & \qquad \qquad \qquad = h_0(N) + h_1(N)\varepsilon + h_1(N)\varepsilon^2 + \dots
 \end{aligned}$$

↓ constant terms must agree

$$a_0(0, N)F_0(N) + a_1(0, N)F_0(N+1) + \dots + a_d(0, N)F_0(N+d) = h_0(N)$$

## Ansatz (for power series)

$$\begin{aligned}
 & a_0(\varepsilon, N) \left[ F_0(N) + F_1(N)\varepsilon + F_2(N)\varepsilon^2 + \dots \right] \\
 & + a_1(\varepsilon, N) \left[ F_0(N+1) + F_1(N+1)\varepsilon + F_2(N+1)\varepsilon^2 + \dots \right] \\
 & + \\
 & \vdots \\
 & + a_d(\varepsilon, N) \left[ F_0(N+d) + F_1(N+d)\varepsilon + F_2(N+d)\varepsilon^2 + \dots \right] \\
 & \qquad \qquad \qquad = h_0(N) + h_1(N)\varepsilon + h_2(N)\varepsilon^2 + \dots
 \end{aligned}$$

↓ constant terms must agree

$$a_0(0, N)F_0(N) + a_1(0, N)F_0(N+1) + \dots + a_d(0, N)F_0(N+d) = h_0(N)$$

$$\begin{aligned} & a_0(\varepsilon, N) \left[ F_1(N)\varepsilon + F_2(N)\varepsilon^2 + \dots \right] \\ & + a_1(\varepsilon, N) \left[ F_1(N+1)\varepsilon + F_2(N+1)\varepsilon^2 + \dots \right] \\ & + \\ & \vdots \\ & + a_d(\varepsilon, N) \left[ F_1(N+d)\varepsilon + F_2(N+d)\varepsilon^2 + \dots \right] \\ & \qquad \qquad \qquad = h'_0(N) + h'_1(N)\varepsilon + h'_2(N)\varepsilon^2 + \dots \end{aligned}$$

$$\begin{aligned}
 & a_0(\varepsilon, N) \left[ F_1(N)\varepsilon + F_2(N)\varepsilon^2 + \dots \right] \\
 & + a_1(\varepsilon, N) \left[ F_1(N+1)\varepsilon + F_2(N+1)\varepsilon^2 + \dots \right] \\
 & + \\
 & \vdots \\
 & + a_d(\varepsilon, N) \left[ F_1(N+d)\varepsilon + F_2(N+d)\varepsilon^2 + \dots \right] \\
 & \qquad \qquad \qquad = \underbrace{h'_0(N)}_{=0} + h'_1(N)\varepsilon + h'_2(N)\varepsilon^2 + \dots
 \end{aligned}$$

Devide by  $\varepsilon$

$$\begin{aligned}
& a_0(\varepsilon, N) \left[ F_1(N) + F_2(N)\varepsilon + \dots \right] \\
& + a_1(\varepsilon, N) \left[ F_1(N+1) + F_2(N+1)\varepsilon + \dots \right] \\
& + \\
& \vdots \\
& + a_d(\varepsilon, N) \left[ F_1(N+d) + F_2(N+d)\varepsilon + \dots \right] = h'_1(N) + h'_2(N)\varepsilon + \dots
\end{aligned}$$

**Now repeat for**  $F_1(N), F_2(N), \dots$

Remark: Works the same for Laurent series.

Blümlein, Klein, CS, Stan, J. Symbol. Comput. 2012; arXiv:1011.2656[cs.SC]

Ablinger, Blümlein, Round, CS, LL2012, arXiv:1210.1685 [cs.SC]



$$F(N) = \sum_{k=1}^N (-1)^k e^{-\frac{3\varepsilon\gamma}{2}} \Gamma\left(-1 - \frac{3\varepsilon}{2}\right) B\left(2+k, \frac{\varepsilon}{2}\right) B(-\varepsilon+k, -\varepsilon) B\left(1 - \frac{\varepsilon}{2} + k, 1 + \frac{\varepsilon}{2}\right) \binom{N}{k}$$

↓ (summation package Sigma.m)

$$2(N+1)^2 F(N) + (3\varepsilon^2 + 3\varepsilon N + 9\varepsilon - 4N^2 - 12N - 8) F(N+1) - (2\varepsilon - N - 1)(\varepsilon + 2N + 6) F(N+2) = 0\varepsilon^{-3} - \frac{16}{3}\varepsilon^{-2} + \frac{40}{3}\varepsilon^{-1} - (2\zeta_2 - \frac{68}{3})\varepsilon^0 + \dots$$

$$F(1) = \frac{2}{3}\varepsilon^{-3} - \frac{11}{6}\varepsilon^{-2} + \left(\frac{\zeta_2}{4} + \frac{79}{24}\right)\varepsilon^{-1} + \dots$$

$$F(2) = \frac{8}{9}\varepsilon^{-3} - \frac{73}{27}\varepsilon^{-2} + \left(\frac{\zeta_2}{3} + \frac{1415}{324}\right)\varepsilon^{-1} + \dots$$

↓

$$F(N) = F_{-3}(N) \varepsilon^{-3} + F_{-2}(N) \varepsilon^{-2} + F_{-1}(N) \varepsilon^{-1} + \dots$$

$$F(N) = \sum_{k=1}^N (-1)^k e^{-\frac{3\varepsilon\gamma}{2}} \Gamma\left(-1 - \frac{3\varepsilon}{2}\right) B\left(2+k, \frac{\varepsilon}{2}\right) B(-\varepsilon+k, -\varepsilon) B\left(1 - \frac{\varepsilon}{2} + k, 1 + \frac{\varepsilon}{2}\right) \binom{N}{k}$$

↓ (summation package Sigma.m)

$$2(N+1)^2 F(N) + (3\varepsilon^2 + 3\varepsilon N + 9\varepsilon - 4N^2 - 12N - 8) F(N+1) - (2\varepsilon - N - 1)(\varepsilon + 2N + 6) F(N+2) = 0\varepsilon^{-3} - \frac{16}{3}\varepsilon^{-2} + \frac{40}{3}\varepsilon^{-1} - (2\zeta_2 - \frac{68}{3})\varepsilon^0 + \dots$$

$$F(1) = \frac{2}{3}\varepsilon^{-3} - \frac{11}{6}\varepsilon^{-2} + \left(\frac{\zeta_2}{4} + \frac{79}{24}\right)\varepsilon^{-1} + \dots$$

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↓ (summation package Sigma.m)

$$F(N) = \frac{4N}{3(N+1)}\varepsilon^{-3} - \left(\frac{2(2N+1)}{3(N+1)}S_1(N) + \frac{2N(2N+3)}{3(N+1)^2}\right)\varepsilon^{-2}$$

$$\left(\frac{(1-4N)}{6(N+1)}S_1(N)^2 - \frac{N(N^2-2)}{3(N+1)^3} + \frac{(3N+2)(4N+5)}{3(N+1)^2}S_1(N) + \frac{(1-4N)}{6(N+1)}S_2(N) + \frac{N\zeta_2}{2(N+1)}\right)\varepsilon^{-1} + \dots$$

## Find a recurrence for the integral/sum

$$F(N) = \int_0^1 \cdots \int_0^1 \Phi(\varepsilon, N, x_1, x_2, \dots, x_7) dx_1 dx_2 \cdots dx_7$$
$$\stackrel{?}{=} F_{-3}(N)\varepsilon^{-3} + F_{-2}(N)\varepsilon^{-2} + F_{-1}(N)\varepsilon^{-1} + \dots$$

multivariate  
Almquist/Zeilberger  
(J. Ablinger)

$$a_0(\varepsilon, N)F(N) + \dots + a_d(\varepsilon, N)F(N + d) = h(\varepsilon, N)$$

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 $\varepsilon$ -recurrence solver

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$$\sum_{i_1} \cdots \sum_{i_7} f(\varepsilon, N, i_1, i_2, \dots, i_7)$$

Wegschaider's MultiSum  
Package (F. Stan)

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multivariate  
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Wegschaider's MultiSum  
Package (F. Stan)

Holonomic/difference field  
approach (M. Round)

$$a_0(\varepsilon, N)F(N) + \dots + a_d(\varepsilon, N)F(N + d) = h(\varepsilon, N)$$

## The general tactic

Feynman integrals

↓ non-trivial transformations (DESY)

multiple sums

↓ symbolic summation

compact expression in terms  
of special functions

## Tactic 2: Expand a recurrence in $\epsilon$

Blümlein, Klein, CS, Stan, J. Symbol. Comput. 2012; arXiv:1011.2656 [cs.SC]

Ablinger, Blümlein, Round, CS, LL2012, arXiv:1210.1685 [cs.SC]

## The general tactic

Feynman integrals

↓ non-trivial transformations (DESY)

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compact expression in terms  
of special functions

Tactic 3: Guess a recurrence and solve it



In the non-singlet (3-loop, massless) case  $\sim 360$  diagrams contribute. The integrals are of the form:

$$F(n, \varepsilon) = \int_0^1 dx_1 \cdots \int_0^1 dx_7 \sum_{i=1}^K \frac{p_i(x_1, x_2, \dots, x_7)^{n+\dots+r_i\varepsilon+\dots}}{q_i(x_1, x_2, \dots, x_7)^{\dots+s_i\varepsilon+\dots}}$$

where  $K \in \mathbb{N}$ ,  $r_i, s_i \in \mathbb{Q}$ , and  $p_i, q_i$  are polynomials in  $x_1, \dots, x_7$ .

Vermaseren, Moch: 3-5 CPU years (2004)

In the non-singlet (3-loop, massless) case  $\sim 360$  diagrams contribute. The integrals are of the form:

$$\begin{aligned}
 F(n, \varepsilon) &= \int_0^1 dx_1 \cdots \int_0^1 dx_7 \sum_{i=1}^K \frac{p_i(x_1, x_2, \dots, x_7)^{n+\dots+r_i\varepsilon+\dots}}{q_i(x_1, x_2, \dots, x_7)^{\dots+s_i\varepsilon+\dots}} \\
 &= F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + F_{-1}(n)\varepsilon^{-1} + \boxed{F_0(n)}\varepsilon^0 + \dots \\
 &\quad \downarrow
 \end{aligned}$$

Initial values  $F_0(i)$ ,  $i = 1, \dots, 5114$

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 &\quad \downarrow
 \end{aligned}$$

Initial values  $F_0(i)$ ,  $i = 1, \dots, 5114$

$\downarrow$  Recurrence guesser (M. Kauers)

$$a_0(n)F_0(n) + a_1(n)F_0(n+1) + \dots + a_{35}(n)F_0(n+35) = 0$$

$$a_0(n)F_0(n) + a_1(n)F_0(n + 1) + \cdots + \boxed{a_{35}(n)}F_0(n + 35) = 0$$

$$a_0(n)F_0(n) + a_1(n)F_0(n+1) + \cdots + \boxed{a_{35}(n)}F_0(n+35) = 0$$

$$a_{35}(n) = \boxed{A_0} + A_1n + A_2n^2 + \cdots + A_{938}n^{983} \in \mathbb{Z}[n]$$

$$a_0(n)F_0(n) + a_1(n)F_0(n+1) + \dots + \boxed{a_{35}(n)}F_0(n+35) = 0$$

$$a_{35}(n) = \boxed{A_0} + A_1n + A_2n^2 + \dots + A_{938}n^{983} \in \mathbb{Z}[n]$$

$$A_0 = 4640944309211313672503980223716264124200407085993854002412460315194$$

$$95765021269344971048446299722216293405285738333200767150194016391501666$$

$$27950213807356109710952045603966273388757782697588602201277983560532017$$

$$37487592671445911325765145271945214255462153147308420597210761595329365$$

$$51563452998613135384718911305253299053198893606401464021608911620974192$$

$$09001668029951620780182947258262939450801154511774527832503874341661898$$

$$89167522107378468797979810265385510643937043867557563467523740406094658$$

$$99100467933353731959645624977524424672990654427732309881685346483771128$$

$$69020837147452024401528169079406933665344476181260243344172097691636706$$

$$62803059675535809027169693064474147719610219849628486896079642312975136$$

$$20776876867741883488363846944854496482629372436829699055391369178850397$$

$$00381638011612302679580897488076647721311930634735316787779620757659951$$

$$5202809978299053753901432067359626151$$

(885 decimal digits)

In the non-singlet (3-loop, massless) case  $\sim 360$  diagrams contribute. The integrals are of the form:

$$\begin{aligned}
 F(n, \varepsilon) &= \int_0^1 dx_1 \cdots \int_0^1 dx_7 \sum_{i=1}^K \frac{p_i(x_1, x_2, \dots, x_7)^{n+\dots+r_i\varepsilon+\dots}}{q_i(x_1, x_2, \dots, x_7)^{\dots+s_i\varepsilon+\dots}} \\
 &= F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + F_{-1}(n)\varepsilon^{-1} + \boxed{F_0(n)}\varepsilon^0 + \dots \\
 &\quad \downarrow
 \end{aligned}$$

Initial values  $F_0(i)$ ,  $i = 1, \dots, 5114$

$\downarrow$  Recurrence guesser (M. Kauers)

$$a_0(n)F_0(n) + a_1(n)F_0(n+1) + \dots + a_{35}(n)F_0(n+35) = 0$$

$\downarrow$  Sigma

**CLOSED FORM**

## Tactic 4: Solve coupled systems of differential equations



A coupled differential system for  $\hat{I}_1(x)$ ,  $\hat{I}_2(x)$ ,  $\hat{I}_3(x)$

(produced by IBP [extension of REDUZE\_2, A.v. Manteuffel])

$$D_x \begin{pmatrix} \hat{I}_1(x) \\ \hat{I}_2(x) \\ \hat{I}_3(x) \end{pmatrix} = \begin{pmatrix} -\frac{-1-\varepsilon+x}{(x-1)x} & -\frac{2}{(x-1)x} & 0 \\ \frac{\varepsilon(3\varepsilon+2)}{4(x-1)} & -\frac{-2-\varepsilon+3x+3\varepsilon x}{2(x-1)x} & -\frac{\varepsilon+1}{2(x-1)} \\ -\frac{\varepsilon(3\varepsilon+2)(x-2)}{4(x-1)x} & \frac{-2-5\varepsilon+x+3\varepsilon x}{2(x-1)x} & \frac{(-2\varepsilon-x+\varepsilon x)}{2(x-1)x} \end{pmatrix} \begin{pmatrix} \hat{I}_1(x) \\ \hat{I}_2(x) \\ \hat{I}_3(x) \end{pmatrix} + \begin{pmatrix} \hat{R}_1(x) \\ \hat{R}_2(x) \\ -\hat{R}_2(x) \end{pmatrix}$$

where

$$\hat{R}_1(x) = \frac{\hat{B}_4(x)}{(x-1)x},$$

$$\begin{aligned} \hat{R}_2(x) &= \frac{-(\varepsilon+2)^3}{16(\varepsilon+1)(x-1)x} \hat{B}_1(x) + \frac{(\varepsilon+2)(3\varepsilon+4)(19\varepsilon^2+36\varepsilon+16)}{16\varepsilon(5\varepsilon+6)(x-1)x} \hat{B}_2(x) \\ &+ \frac{(\varepsilon+1)^2(3\varepsilon+4)^2}{2\varepsilon(5\varepsilon+6)x} \hat{B}_3(x) + \frac{-24-50\varepsilon-25\varepsilon^2+8x+14\varepsilon x+6\varepsilon^2 x}{4(5\varepsilon+6)(x-1)x} \hat{B}_4(x) \end{aligned}$$

$\hat{B}_1(x)$ ,  $\hat{B}_2(x)$ ,  $\hat{B}_3(x)$  have been solved with symbolic summation.

## Tactic 4: the DE-REC approach

DE system

$$D\hat{I}(x) = A\hat{I}(x) + \hat{R}(x)$$

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DE system

$$D\hat{I}(x) = A\hat{I}(x) + \hat{R}(x)$$

OreSys package (S. Gerhold)

uncoupling algorithm

uncoupled DE system

$$\sum_i a_i(x) D^i \hat{I}_1(x) = r(x)$$
$$\hat{I}_k(x) = \text{expr}_k(\hat{I}_1(x)), k > 1$$

## Tactic 4: the DE-REC approach

$$\begin{array}{c} \text{DE system} \\ D\hat{I}(x) = A\hat{I}(x) + \hat{R}(x) \end{array}$$

OreSys package (S. Gerhold)  
uncoupling algorithm

$$\begin{array}{c} \text{uncoupled DE system} \\ \sum_i a_i(x) D^i \hat{I}_1(x) = r(x) \\ \hat{I}_k(x) = \text{expr}_k(\hat{I}_1(x)), k > 1 \end{array}$$

$$\hat{I}_1(x) = \sum_{N=0}^{\infty} I_1(N) x^N$$

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$$\text{DE system} \\ D\hat{I}(x) = A\hat{I}(x) + \hat{R}(x)$$

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holonomic closure prop.

$$\text{linear recurrence} \\ \sum_i a'_i(N) I_1(N+i) = r'(N)$$

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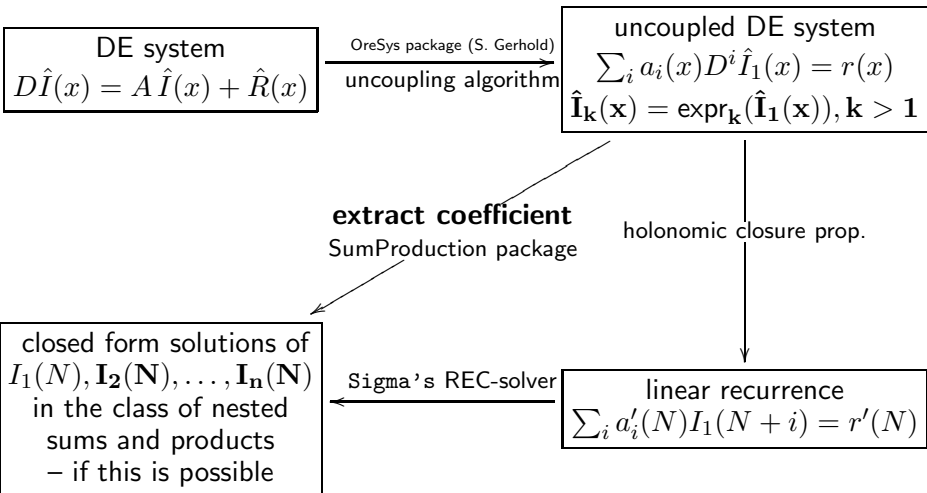
holonomic closure prop.

closed form solutions of  
 $I_1(N)$   
in the class of nested  
sums and products  
– if this is possible

Sigma's REC-solver

$$\text{linear recurrence} \\ \sum_i a'_i(N) I_1(N + i) = r'(N)$$

## Tactic 4: the DE-REC approach (SolveCoupledSystem package)



## Solving a coupled differential system

In[8]:= << OreSys.m

OreSys by Stefan Gerhold (optimized by C. Schneider) © RISC-Linz

In[9]:= << SolveCoupledSystem.m

SolveCoupledSystem by Carsten Schneider © RISC-Linz

In[10]:= **coupledDESys** = D[{ $\hat{I}_1(x)$ ,  $\hat{I}_2(x)$ ,  $\hat{I}_3(x)$ }, x] - A. $\{\hat{I}_1(x)$ ,  $\hat{I}_2(x)$ ,  $\hat{I}_3(x)\}$ ;

In[11]:= **rhs** =  $\{\hat{R}_1(x)$ ,  $\hat{R}_2(x)$ ,  $-\hat{R}_2(x)\}$  in power series representation;

In[12]:= **SolveCoupledDESys**[coupledDESys,  $\{I_1[x]$ ,  $I_2[x]$ ,  $I_3[x]\}$ ,  $\epsilon$ , -3,  
 $\{-2, -2, -2\}$ , rhs, ...]



## Solving a coupled differential system

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OreSys by Stefan Gerhold (optimized by C. Schneider) © RISC-Linz

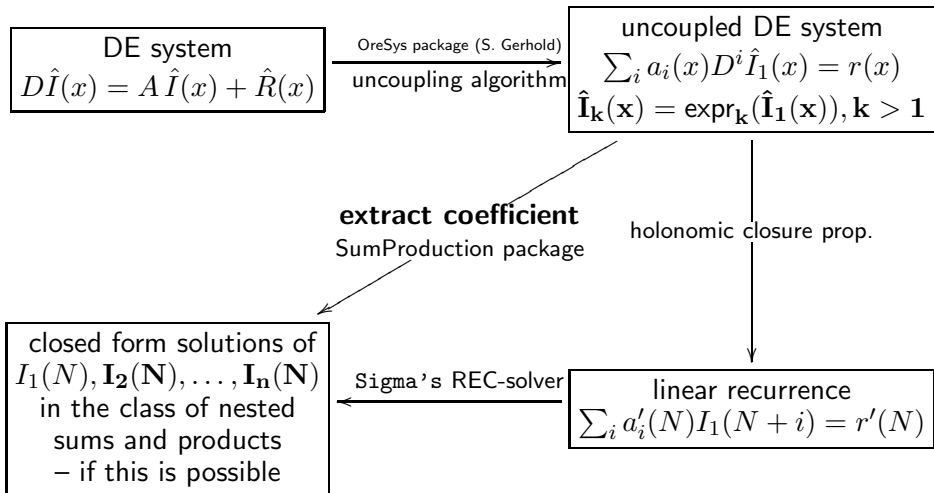
In[9]:= &lt;&lt; SolveCoupledSystem.m

SolveCoupledSystem by Carsten Schneider © RISC-Linz

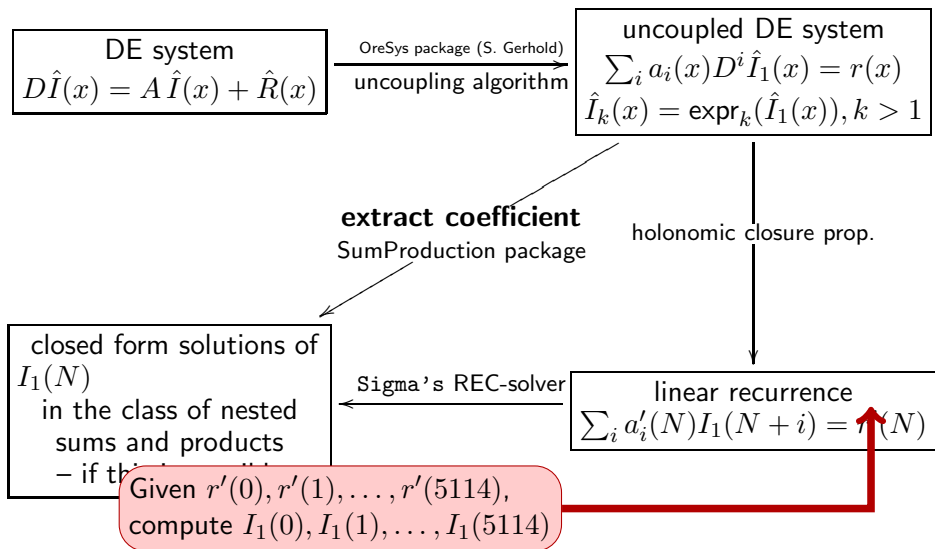
In[10]:= coupledDESys = D[{ $\hat{I}_1(x)$ ,  $\hat{I}_2(x)$ ,  $\hat{I}_3(x)$ }, x] - A. $\{\hat{I}_1(x)$ ,  $\hat{I}_2(x)$ ,  $\hat{I}_3(x)\}$ ;In[11]:= rhs = { $\hat{R}_1(x)$ ,  $\hat{R}_2(x)$ ,  $-\hat{R}_2(x)$ } in power series representation;In[12]:= SolveCoupledDESys[coupledDESys, { $I_1[x]$ ,  $I_2[x]$ ,  $I_3[x]$ },  $\epsilon$ , -3,  
{-2, -2, -2}, rhs, ...]Out[12]= 
$$\left\{ \frac{1}{\epsilon^3} \left( \frac{4(3N^2 + 6N + 4)}{3(N+1)^2} + \frac{4S_1[N]}{3(N+1)} \right) + \frac{1}{\epsilon^2} \left( -\frac{2(20N^3 + 58N^2 + 57N + 22)}{3(N+1)^3} + \frac{2(N+2)(2N-1)S_1[N]}{3(N+1)^2} - \frac{S_1[N]^2}{N+1} - \frac{S_2[N]}{N+1} \right), \right.$$

$$\left. \frac{4}{3\epsilon^3} - \frac{2}{\epsilon^2}, \frac{8}{3\epsilon^3} + \frac{1}{\epsilon^2} \left( -\frac{4(4N^2 + 7N + 2)}{3(N+1)^2} + \frac{4(N+2)S_1[N]}{3(N+1)} \right) \right\}$$

## Tactic 4: the DE-REC approach (SolveCoupledSystem package)



## Tactic 5: compute large moments (SolveCoupledSystem package)



In the non-singlet (3-loop, massless) case  $\sim 360$  diagrams contribute. The integrals are of the form:

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 F(n, \varepsilon) &= \int_0^1 dx_1 \cdots \int_0^1 dx_7 \sum_{i=1}^K \frac{p_i(x_1, x_2, \dots, x_7)^{n+\dots+r_i\varepsilon+\dots}}{q_i(x_1, x_2, \dots, x_7)^{\dots+s_i\varepsilon+\dots}} \\
 &= F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + F_{-1}(n)\varepsilon^{-1} + \boxed{F_0(n)}\varepsilon^0 + \dots \\
 &\quad \downarrow
 \end{aligned}$$

Initial values  $F_0(i)$ ,  $i = 1, \dots, 5114$

$\downarrow$  Recurrence guesser (M. Kauers)

$$a_0(n)F_0(n) + a_1(n)F_0(n+1) + \dots + a_{35}(n)F_0(n+35) = 0$$

$\downarrow$  Sigma

**CLOSED FORM**

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 &= F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + F_{-1}(n)\varepsilon^{-1} + \boxed{F_0(n)}\varepsilon^0 + \dots
 \end{aligned}$$

↓ a new method for large moments

Initial values  $F_0(i)$ ,  $i = 1, \dots, 5114$

↓ Recurrence guesser (M. Kauers)

$$a_0(n)F_0(n) + a_1(n)F_0(n+1) + \dots + a_{35}(n)F_0(n+35) = 0$$

↓ Sigma

**CLOSED FORM**

## SUMMARY (of RISC packages)

Backbone: a new difference ring/field approach implemented in Sigma.m

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- ▶ solving coupled systems
  - ▶ S. Gerhold's `OreSys.m`
  - ▶ CS's `SolveCoupledSystem.m`

## SUMMARY (of RISC packages)

Backbone: a new difference ring/field approach implemented in `Sigma.m`

- ▶ symbolic summation
  - ▶ CS's `EvaluateMultiSum.m`
  - ▶ M. Round's `RhoSum.m` (difference ring/holonomic approach)
  - ▶ K. Wegschaider's `MultiSum.m` (enhanced Sister Celine/WZ approach, utilized by F. Stan)
- ▶ symbolic integration
  - ▶ J. Ablinger's `MultiIntegrate.m` (refined multivariate Almkvist-Zeilberger approach)
- ▶ solving coupled systems
  - ▶ S. Gerhold's `OreSys.m`
  - ▶ CS's `SolveCoupledSystem.m`
- ▶ special function algorithms
  - ▶ J. Ablinger's `HarmonicSums.m`