Rogers-Ramanujan Functions, Modular Functions, and Computer Algebra *

Dedicated to the symbolic summation pioneer Sergei Abramov who concretely passed milestone 70

Peter Paule and Silviu Radu

Abstract Many generating functions for partitions of numbers are strongly related to modular functions. This article introduces such connections using the Rogers-Ramanujan functions as key players. After exemplifying basic notions of partition theory and modular functions in tutorial manner, relations of modular functions to q-holonomic functions and sequences are discussed. Special emphasis is put on supplementing the ideas presented with concrete computer algebra. Despite intended as a tutorial, owing to the algorithmic focus the presentation might contain aspects of interest also to the expert. One major application concerns an algorithmic derivation of Felix Klein's classical icosahedral equation.

1 Introduction

The main source of inspiration for this article was the truly wonderful paper [14] by William Duke. When reading Duke's masterly exposition, the first named author started to think of writing kind of a supplement which relates the beautiful ingredients of Duke's story to computer algebra. After starting, the necessity to connect to readers with diverse backgrounds soon became clear. As a consequence, this tutorial grew longer than originally intended. As a compensation for its length, we hope some readers will find it useful to find various things presented together at one place the first time. Owing to the algorithmic focus, some aspects might have a new appeal also to the expert.

Starting with partition generating functions and using the Omega package, in Section 2 the key players of this article are introduced, the Rogers-Ramanujan functions $F(1)$ and $F(q)$.

To prove non-holonomicity, in Section 3 the series presentations of $F(1)$ and $F(q)$ are converted into infinite products. Viewing things analytically, the Dedekind eta function, also defined via an infinite product on the upper half complex plane $\mathbb{H}$, is of fundamental importance, in particular, owing to its modular transformation properties.

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Section 4 presents basic notions and definitions for modular functions associated to congruence subgroups Γ of the modular group $\mathrm{SL}_2(\mathbb{Z})$. These groups are acting on $\mathbb{H}$ and, more generally, also on $\mathbb{H}\cup\mathbb{Q}\cup\{\infty\}$. When restricting this extended action of Γ to $\mathbb{Q}\cup\{\infty\}$ the resulting orbits are called cusps; cusps play a crucial role for zero recognition of modular functions.

Section 5 presents concrete examples of modular functions for congruence subgroups Γ like the Klein j-invariant for $\Gamma := \mathrm{SL}_2(\mathbb{Z})$, the modular discriminant quotient $\Phi_2(\tau) := \Delta(2\tau)/\Delta(\tau)$ for $\Gamma := \Gamma_0(2)$, the (modified) Rogers-Ramanujan functions $G(\tau)$ and $H(\tau)$ being quasi-modular functions for $\Gamma := \Gamma_1(5)$, and the Rogers-Ramanujan quotient $r(\tau) := H(\tau)/G(\tau)$ for $\Gamma(5)$. To obtain information about congruence subgroups the computer algebra system SAGE [36] is used.

Section 6 introduces basic ideas of zero recognition of modular functions. To this end, one passes from modular functions g defined on the upper half complex plane to induced functions g^* defined on compact Riemann surfaces. By transforming problems into settings which involve modular functions with a pole at ∞ only, zero recognition turns into a finitary algorithmic procedure.

In Section 7 we present examples for zero recognition which despite being elementary should illustrate how to prove relations between q-series/q-products using modular function machinery. Among other tools, "valence formulas" are used which describe relations between orders of Laurent series expansions.

The example given in Section 8 shows that by transforming zero recognition problems into ones involving solely modular functions with a pole at ∞ only, one gets an "algorithmic bonus": a method to derive identities algorithmically.

Many modular functions connected to partition generating functions are not holonomic. But there are strong connections to q-holonomic sequences and series which are briefly discussed in Section 9. Again the Rogers-Ramanujan functions serve as illustrating examples; here also q-hypergeometric summation theory comes into play.

Section 10 is devoted to another classical theme, the presentation of the Rogers-Ramanujan quotient $r(\tau)$ as a continued fraction. Evaluations at real or complex arguments are briefly discussed: most prominently, Ramanujan's presentation of $r(i)$ in terms of nested radicals.

Finally, Section 11 returns to a main theme of Duke's beautiful exposition [14]. Namely, there is a stunning connection, first established by Felix Klein, between the fixed field of the icosahedral group and modular functions. In the latter context Ramanujan's evaluation of $r(i)$ finds a natural explanation as a root of Klein's icosohedral polynomial. An algorithmic derivation of this polynomial is given.

In Section 12 (Appendix 1) we briefly discuss general types of function families the Rogers-Ramanujan functions belong to. One such class are generalized Dedekind eta functions which were studied by Meyer [24], Dieter [12], and Schoeneberg [34, Ch. VIII] in connection with work of Felix Klein. These functions form a subfamily of an even more general class, the theta functions studied extensively by Farkas and Kra [15].

In Section 7 "valence formulas" for $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ and $\Gamma = \Gamma_0(2)$ were used. The Rogers-Ramanujan function setting used in Section 11 connects to a "valence formula" for $\Gamma = \Gamma_1(5)$. For the sake of completeness, in Section 13 (Appendix 3) we present a "valence formula", Thm. 13.2, which contains all these instances as special cases. Being not relevant to the main themes of this article, we state this theorem without proof.

Concerning computer algebra packages: In addition to the SAGE examples and RISC packages used in our exposition, we want to point to Frank Garvan who has developed various software relevant to the themes discussed in this tutorial; see, for example, [16] and Garvan's web page for other packages.

2 Partition Generating Functions

Problem. Given $n, k \in \mathbb{Z}_{>0}$, determine

$$r_k(n) := \#\{(a_1,\dots,a_k) \in \mathbb{Z}_{>0}^k : a_1 + \cdots + a_k = n \text{ and } a_j - a_{j+1} \geq 2 \text{ for } 1 \leq j \leq k-1\}.$$

Example. $r_2(8) = 3$ because 8 equals $7+1, 6+2$, and $5+3$. For convenience we define $r_k(0) := 1$ for $k \geq 0$.

To solve the problem we consider the generating function of such partitions,

$$R_k := \sum_{n \geq 0} r_k(n) q^n = \sum_{\substack{a_1, a_2, \dots, a_k \geq 1 \\ a_1 - a_2 \geq 2, a_2 - a_3 \geq 2, \dots, a_{k-1} - a_k \geq 2}} q^{a_1 + a_2 + \cdots + a_k}.$$

To compute such generating functions one can use the Omega package which implements MacMahon's method of partition analysis; see the references in [29].

In[1]:= << **RISC‘Omega‘**

Omega Package V2.49 written by Axel Riese
(in cooperation with George E. Andrews and Peter Paule) © RISC-JKU

To compute R_4 one calls

In[2]:= $\mathbf{OR[OSum[q^{a1+a2+a3+a4}, \{a1 - a2 \geq 2, a2 - a3 \geq 2, a3 - a4 \geq 2, a4 \geq 1\}, \lambda]]}$

Out[2]= $\frac{q^{16}}{(1-q)(1-q^2)(1-q^3)(1-q^4)}$

In view of the instances for $k = 0, 1, 2, 3$:

$$\{R_0, R_1, R_2, R_3\} = \left\{1, \frac{q}{1-q}, \frac{q^4}{(1-q)(1-q^2)}, \frac{q^9}{(1-q)(1-q^2)(1-q^3)}\right\},$$

the general pattern

$$R_k = \frac{q^{k^2}}{(1-q)(1-q^2)\dots(1-q^k)}, \quad k \geq 0,$$

becomes obvious. Its proof is by elementary partition reasoning.

Next we consider all such partitions with parts greater or equal to 2:

$$S_k := \sum_{n \geq 0} s_k(n) q^n = \sum_{\substack{a_1, a_2, \dots, a_k \geq 2 \\ a_1 - a_2 \geq 2, a_2 - a_3 \geq 2, \dots, a_{k-1} - a_k \geq 2}} q^{a_1 + a_2 + \cdots + a_k}.$$

For $k = 4$ the Omega package gives:

In[3]:= $\mathbf{OR[OSum[q^{a1+a2+a3+a4}, \{a1 - a2 \geq 2, a2 - a3 \geq 2, a3 - a4 \geq 2, a4 \geq 2\}, \lambda]]}$

Out[3]= $\frac{q^{20}}{(1-q)(1-q^2)(1-q^3)(1-q^4)}$

In view of the instances for $k = 0, 1, 2, 3$:

$$\{S_0, S_1, S_2, S_3\} = \left\{1, \frac{q^2}{1-q}, \frac{q^6}{(1-q)(1-q^2)}, \frac{q^{12}}{(1-q)(1-q^2)(1-q^3)}\right\},$$

the general pattern

$$S_k = \frac{q^{k^2+k}}{(1-q)(1-q^2)\dots(1-q^k)}, \quad k \geq 0,$$

becomes obvious. Again the proof is by elementary partition reasoning.

We will use the standard q-notation

$$(q;q)_k := (1-q)(1-q^2)\dots(1-q^k), k \geq 1, \quad (q;q)_0 := 1, \tag{1}$$

and

$$(q;q)_\infty := \prod_{k=1}^{\infty}\left(1-q^k\right). \tag{2}$$

For example, the generating function for $p(n)$, the number all partitions of n, is

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty}.$$

This infinite product representation implies that the sequence $(p(n))_{n\geq 0}$ is not holonomic,[2] because otherwise its generating function would have at most finitely many singularities; see, for instance, [19].

In connection with R_k and S_k, the alternative notation

$$f_k(z) := \frac{q^{k^2} z^k}{(q;q)_k}$$

will be useful: $R_k = f_k(1)$ and $S_k = f_k(q)$. Defining

$$F(z) := \sum_{k=0}^{\infty} f_k(z) = \sum_{k=0}^{\infty} \frac{q^{k^2} z^k}{(q;q)_k}, \tag{3}$$

the key players of this article will be

$$F(1) = \sum_{k=0}^{\infty} \frac{q^{k^2}}{(q;q)_k} \text{ and } F(q) = \sum_{k=0}^{\infty} \frac{q^{k^2+k}}{(q;q)_k}, \tag{4}$$

called *Rogers-Ramanujan functions*; see, for instance, [7, Ch. 8].

From above it is clear that $F(1) = \sum_{n\geq 0} r(n)q^n$, resp. $F(q) = \sum_{n\geq 0} s(n)q^n$, are the generating functions for the number $r(n)$, resp. $s(n)$, of partitions into parts with minimal difference 2 with all parts greater than 0, respectively 1. For combinatorial purposes it is absolutely sufficient to view them as formal power series in the indeterminate q. But as we shall see, when interpreting them in the context of complex analysis — citing Zagier [40] — there is also a "hidden non-abelian symmetry" which can be used as the "magic principle of modular forms."

3 q-Products and Dedekind's eta function

As with $(p(n))_{n\geq 0}$, to decide whether the sequences $(r(n))_{n\geq 0}$ and $(s(n))_{n\geq 0}$ are holonomic, one could try to convert their generating functions $F(1)$ and $F(q)$ into infinite product form.

[2] I.e., it does not satisfy a linear recurrence with polynomial coefficients.

To do so, there is a tool already known to Euler and popularized by Andrews [4]; we state it in a (slightly modified) version taken from [22, Theorem 2.9].

Theorem 3.1. *Let $\varphi(q)$ be an analytic function without zeros in the disk $|q| < R$ for some $R \leq 1$, and let $(\varepsilon_n)_{n\geq 1}$ be a sequence containing only the numbers 1 and -1. Then there exists a unique sequence $(a_n)_{n\geq 1}$ of complex numbers such that the product $\varphi(0)\prod_{n=1}^{\infty}(1+\varepsilon_n q^n)^{a_n}$ converges to φ uniformly on compact subsets of the disk $|q| < R$. Moreover, if $\varphi(q) = 1+\sum_{n=1}^{\infty} b_n q^n$ and $\varepsilon_n = -1$ for all $n \geq 1$, then*

$$-n\,a_n = n\,b_n + \sum_{\substack{d|n\\ d<n}} d\,a_d + \sum_{j=1}^{n-1}\sum_{d|j} d\;a_d\,b_{n-j}, \quad n \geq 1. \tag{5}$$

Taking as input the Taylor series coefficients b_n, with recurrence (5) one can compute the exponents a_n. For example, for the truncated $F(1)$ series:

$$\begin{aligned}\varphi(q) &:= \sum_{k=0}^{30} f_k(1) = \sum_{k=0}^{30}\frac{q^{k^2}z^k}{(q;q)_k} = 1+\sum_{n=1}^{\infty} b_n q^n = 1+q+q^2+q^3+2q^4+2q^5\\ &+3q^6+3q^7+4q^8+5q^9+6q^{10}+7q^{11}+9q^{12}+10q^{13}+12q^{14}+14q^{15}\\ &+17q^{16}+19q^{17}+23q^{18}+26q^{19}+31q^{20}+35q^{21}+41q^{22}+46q^{23}+54q^{24}\\ &+61q^{25}+70q^{26}+79q^{27}+91q^{28}+102q^{29}+117q^{30}+\dots,\end{aligned}$$

one obtains as output

$$\begin{aligned}(a_n)_{n\geq 1} = (&-1,0,0,-1,0,-1,0,0,-1,0,-1,0,0,-1,0,\\ &-1,0,0,-1,0,-1,0,0,-1,0,-1,0,0,-1,0,\dots).\end{aligned}$$

The pattern suggests that

$$F(1) = \sum_{n=0}^{\infty} r(n)q^n = \sum_{k=0}^{\infty}\frac{q^{k^2}}{(q;q)_k} = \prod_{m=0}^{\infty}\frac{1}{(1-q^{5m+1})(1-q^{5m+4})}, \tag{6}$$

and, after carrying out an analogous computation for $F(q)$,

$$F(q) = \sum_{n=0}^{\infty} s(n)q^n = \sum_{k=0}^{\infty}\frac{q^{k^2+k}}{(q;q)_k} = \prod_{m=0}^{\infty}\frac{1}{(1-q^{5m+2})(1-q^{5m+3})}. \tag{7}$$

As in the case of $p(n)$ the non-holonomicity of the partition number sequences $r(n)$ and $s(n)$ follows immediately from the infinite product representations (6) and (7). But in contrast to the simple derivation of the generating function for the $p(n)$, these product expansions are substantially more difficult to prove. In fact, (6) and (7) are the celebrated Rogers-Ramanujan identities, also called Rogers-Ramanujan-Schur identities owing to the fact that Issai Schur independently discovered and proved them. There is a vast literature on background, proofs, and history; [4] is a reference which also connects to computer algebra and to applications in the frame of Baxter's hard hexagon model in statistical mechanics.

In the general setting of Theorem 3.1, q is interpreted as a complex variable. Nevertheless, to compute the exponent sequence (a_n) we can apply the recurrence (5) in the case of a given *formal* power series $\varphi(q)$; i.e.; taking q as an indeterminate. But, in order to consider the announced "magic principle of modular forms", one again turns to complex analysis by setting

$$q = q(\tau) := \exp(2\pi i\tau) = e^{2\pi i\tau},$$

where τ is taken from the upper half complex plane $\mathbb{H} := \{\tau \in \mathbb{C} : \mathrm{Im}(\tau) > 0\}$. In this setting, the analytic counterpart of the multiplicative inverse of the formal power series $\sum_{n=0}^{\infty} p(n)q^n$ is the Dedekind eta function,

$$\eta : \mathbb{H} \to \mathbb{C}, \tau \mapsto \eta(\tau) := q(\tau)^{\frac{1}{24}} \prod_{k=1}^{\infty} (1 - q(\tau))^k .$$

The above mentioned "hidden non-abelian symmetry" is with respect to modular transformations of τ under elements of the non-abelian modular group,

$$\mathrm{SL}_2(\mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{Z}^{2\times 2} : a\,d - b\,c = 1 \right\},$$

which acts on $\mathbb{H}$ by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \tau := \frac{a\tau+b}{c\tau+d}$, and which is generated by the matrices $S := \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $T := \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Under modular transformations η behaves as follows [13, 23.18.5]:

$$\eta(\tau+1) = e^{2\pi i/24}\eta(\tau) \text{ and } \eta\left(-\frac{1}{\tau}\right) = \sqrt{\frac{\tau}{i}} \cdot \eta(\tau), \tag{8}$$

and w.l.o.g. assuming that $c > 0$:

$$\eta\left(\frac{a\tau+b}{c\tau+d}\right) = e^{2\pi i \rho(a,b,c,d)/24} \cdot \sqrt{\frac{c\tau+d}{i}} \cdot \eta(\tau).$$

Here $\rho(a,b,c,d)$ is a complicated but integer-valued expression depending on $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$; the complex-valued square root is taken to have positive real part.

As a consequence, the modular discriminant[3]

$$\Delta(\tau) := \eta(\tau)^{24} = q\,(q;q)_\infty^{24}, \quad q = q(\tau), \tag{9}$$

behaves under modular transformation as

$$\Delta\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^{12} \cdot \Delta(\tau). \tag{10}$$

This, in view of $(c\tau+d)^{12}$ as the "factor of automorphy", makes Δ a modular form of weight 12 for $\mathrm{SL}_2(\mathbb{Z})$. However, in this article we will mostly deal with modular functions having 1 as the "factor of automorphy"; i.e., modular forms of weight 0.

For later we note that the Dedekind eta function $\eta(\tau)$ and hence the modular discriminant $\Delta(\tau)$ are non-zero analytic functions on $\mathbb{H}$. This is implied by

Lemma 3.2. *Let* $f(\tau) := \prod_{m=0}^{\infty}\left(1 - e^{2\pi i\tau(am+b)}\right)$ *where* $\tau \in \mathbb{H}$ *and* $a, b \in \mathbb{Z}$ *such* $0 \le b < a$. *Then* $f(\tau)$ *is an analytic function on* $\mathbb{H}$ *with* $f(\tau) \neq 0$ *for all* $\tau \in \mathbb{H}$.

Proof (sketch). The statement follows from convergence properties of the infinite product form of f; see, e.g., [5, Appendix A] for details.

[3] In our context it is convenient to normalize as in (9) instead of using the version $\Delta(\tau) := (2\pi)^{12}\eta(\tau)^{24}$.

4 Modular Functions: Definitions

We restrict our discussion to basic definitions and very few notions. For further details on modular forms and modular functions see, for instance, the classical monograph [11].

Besides the full modular group $\mathrm{SL}_2(\mathbb{Z})$ the following subgroups for $N \in \mathbb{Z}_{>0}$ will be relevant:

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} \star & \star \\ 0 & \star \end{pmatrix} \pmod{N} \right\},$$
$$\Gamma_1(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & \star \\ 0 & 1 \end{pmatrix} \pmod{N} \right\},$$
$$\Gamma(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}.$$

Here $\star$ serves as a placeholder for an integer; the congruence relation $\equiv$ between matrices has to be taken entrywise. Sometimes we write I for the identity matrix; i.e., $I := \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. To indicate subgroup relations we use $\leq$; hence

$$\Gamma(N) \leq \Gamma_1(N) \leq \Gamma_0(N) \leq \mathrm{SL}_2(\mathbb{Z}) = \Gamma(1).$$

A subgroup $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$ with $\Gamma(N) \leq \Gamma$ for some fixed $N \in \mathbb{Z}_{>0}$ is called congruence subgroup. For the subgroup $\Gamma(N)$, called principal congruence subgroup of level N, one has

Proposition 4.1. *The principal congruence subgroup $\Gamma(N)$ is normal in $\mathrm{SL}_2(\mathbb{Z})$; its index $[\mathrm{SL}_2(\mathbb{Z}) : \Gamma(N)]$ is finite for all N.*

Proof. Considering the entries of $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ modulo N induces a group isomorphism $\mathrm{SL}_2(\mathbb{Z})/\Gamma(N) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$. This implies the statement.

Definition 4.2. *An analytic (resp. meromorphic) modular function $g : \mathbb{H} \to \hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ for a congruence subgroup Γ is defined by the following three properties:*

- *$g : \mathbb{H} \to \hat{\mathbb{C}}$ is analytic on $\{\tau \in \mathbb{H} : \mathrm{Im}(\tau) > M\}$ for some $M > 0$;*
- *for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$,*
$$g\left(\frac{a\tau + b}{c\tau + d}\right) = g(\tau), \quad \tau \in \mathbb{H}; \tag{11}$$
- *for each $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ there exists a Laurent series expansion of $g(\gamma\tau)$ with finite principal part. This means, for all $\tau \in \mathbb{H}$ (resp. for all $\tau \in \mathbb{H}$ with $\mathrm{Im}(\tau) > M$ in case g is meromorphic),*
$$g\left(\frac{a\tau + b}{c\tau + d}\right) = g(\gamma\tau) = \sum_{n=-M_\gamma}^{\infty} c_n(\gamma) e^{2\pi i n \tau / w_\gamma^\Gamma} \tag{12}$$
where $M_\gamma \in \mathbb{Z}$ and
$$w_\gamma^\Gamma := \min\left\{ h \in \mathbb{Z}_{>0} : \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \in \gamma^{-1}\Gamma\gamma \text{ or } \begin{pmatrix} -1 & h \\ 0 & -1 \end{pmatrix} \in \gamma^{-1}\Gamma\gamma \right\}. \tag{13}$$

Note 1. It is possible that for $h \in \mathbb{Z}_{>0}$,

$$\begin{pmatrix} -1 & h \\ 0 & -1 \end{pmatrix} \in \gamma^{-1}\Gamma\gamma \text{ but } \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \notin \gamma^{-1}\Gamma\gamma;$$

take, for instance, $\Gamma = \Gamma_1(4), h = 1$ and $\gamma = \left(\begin{smallmatrix} 1 & -1 \\ 2 & -1 \end{smallmatrix}\right)$.

The condition (12) has a strong technical flavour. Hence some background motivation is in place. The fundamental underlying observation is a basic fact concerning Fourier expansions:

Lemma 4.3. *For $M > 0$ let $f : \mathbb{H} \to \hat{\mathbb{C}}$ be meromorphic such that* $\mathrm{Im}(p) \leq M$ *for each of its poles $p \in \mathbb{H}$. Suppose f is periodic with period 1. Then there exists a unique analytic function*

$$h : \{z \in \mathbb{C} : 0 < |z| < R\} \to \mathbb{C} \textit{ for some } R \leq 1,$$

such that

$$f(\tau) = h(e^{2\pi i\tau}), \quad \tau \in \mathbb{H}. \tag{14}$$

Moreover, if f has no poles, one can choose $R = 1$.

Since h is analytic on a punctured open disk, there exists a Laurent expansion $h(z) = \sum_{n=-\infty}^{\infty} h_n z^n$ about 0 with coefficients in $\mathbb{C}$; i.e., for $\mathrm{Im}(\tau)$ large enough,

$$f(\tau) = \sum_{n=-\infty}^{\infty} h_n \left(e^{2\pi i\tau}\right)^n. \tag{15}$$

Suppose f is as in Lemma 4.3 but with period $w \in \mathbb{Z}_{>0}$ greater than 1.Then $F(\tau) := f(w\tau)$ has period 1, and f has an expansion of the form

$$f(\tau) = F\left(\frac{\tau}{w}\right) = \sum_{n=-\infty}^{\infty} h_n \left(e^{2\pi i\tau}\right)^{\frac{n}{w}}. \tag{16}$$

Now let $g : \mathbb{H} \to \hat{\mathbb{C}}$ be a meromorphic function satisfying the same conditions as f in Lemma 4.3 and, in addition, the modular invariance property (11). Then for any $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ the function $g \circ \gamma$ has period w_γ^Γ. Namely, according to the definition of w_γ^Γ,

$$\begin{pmatrix} 1 & w_\gamma^\Gamma \\ 0 & 1 \end{pmatrix} \in \gamma^{-1}\Gamma\gamma \text{ or } \begin{pmatrix} -1 & w_\gamma^\Gamma \\ 0 & -1 \end{pmatrix} \in \gamma^{-1}\Gamma\gamma.$$

Hence, in any case, there is a $\rho \in \Gamma$ such that $\tau + w_\gamma^\Gamma = \gamma^{-1}\rho\gamma\tau$, and thus

$$(g \circ \gamma)(\tau + w_\gamma^\Gamma) = g(\gamma(\tau + w_\gamma^\Gamma)) = g(\gamma\gamma^{-1}\rho\gamma\tau) = g(\rho\gamma\tau) = g(\gamma\tau) = (g \circ \gamma)(\tau).$$

As a consequence, $f := g \circ \gamma$ has the period w_γ^Γ, and an expansion as in (16) exists. Condition (12) now requires that this expansion has *finite principal part*. As we shall see this requirement is needed to extend $g \circ \gamma : \mathbb{H} \to \hat{\mathbb{C}}$ to a function $g \circ \gamma : \mathbb{H} \cup \mathbb{Q} \cup \{\infty\} \to \hat{\mathbb{C}}$.

We want to emphasize that representations of infinity as $\infty = \frac{a}{0}$, $a \in \mathbb{Z} \setminus \{0\}$, are explicitly included in our setting which formally is done by including the obvious arithmetical rules and by the natural extension of the group action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$ to an action on $\hat{\mathbb{H}} := \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$. Note that the extended action maps elements from $\mathbb{Q} \cup \{\infty\}$ to $\mathbb{Q} \cup \{\infty\}$.

Further remarks on Def. 4.2 are in place. In view of $q^{1/w_\gamma^\Gamma} = e^{2\pi i\tau/w_\gamma^\Gamma}$ and $\frac{a}{c} = \gamma\infty$ $(= \lim_{\mathrm{Im}(\tau)\to\infty} \gamma\tau)$, expansions as in (12) are called q-expansions of g at $\frac{a}{c}$ for Γ. Taking $\gamma =$

$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \gamma' = \left(\begin{smallmatrix} a & b' \\ c & d' \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ such that $\frac{a}{c} = \gamma\infty = \gamma'\infty$, it is a natural question to ask in which way the corresponding q-expansions differ. The answer is given by the following fact that is straightforward to verify.

Proposition 4.4. *Let g be a meromorphic modular function for a congruence subgroup Γ. Let $\gamma, \gamma' \in \mathrm{SL}_2(\mathbb{Z})$, $\rho \in \Gamma$, and $m \in \mathbb{Z}$ such that*

$$\gamma' = \rho\,\gamma \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \;\; \textit{or} \;\; \gamma' = \rho\,\gamma \begin{pmatrix} -1 & m \\ 0 & -1 \end{pmatrix} \tag{17}$$

Suppose the q-expansion of g at $\frac{a}{c} := \gamma\infty$ is

$$g(\gamma\tau) = \sum_{n=-M}^{\infty} c_n q^{\frac{n}{w}} \tag{18}$$

with $w := w_\gamma^\Gamma$. Then

$$w = w_{\gamma'}^\Gamma \tag{19}$$

and

$$g(\gamma'\tau) = g(\gamma(\tau+m)) = \sum_{n=-M}^{\infty} e^{2\pi i m n/w} c_n q^{\frac{n}{w}}. \tag{20}$$

This means, the q-expansions (18) and (20) at $\frac{a}{c}$ differ in their coefficients only by the factor $e^{2\pi i m n/w}$. This in particular holds if besides $\frac{a}{c} = \gamma\infty$ also $\frac{a}{c} = \gamma'\infty$. Because then $\gamma^{-1}\gamma'\infty = \infty$, which implies that $\gamma^{-1}\gamma' = \left(\begin{smallmatrix} \pm 1 & m \\ 0 & \pm 1 \end{smallmatrix}\right)$ for some $m \in \mathbb{Z}$ and we are in the case $\rho = I$.

This observation also enables us to extend the domain of the modular function g to $\hat{\mathbb{H}}$. This extension is of particular relevance for the zero recognition of modular functions; see Section 6.

Definition 4.5. *Let $g : \mathbb{H} \to \hat{\mathbb{C}}$ be a meromorphic modular function for a congruence subgroup Γ with q-expansion at $\frac{a}{c} := \gamma\infty$ as in (18). Then g extends to $\hat{g} : \hat{\mathbb{H}} \to \hat{\mathbb{C}}$ as follows: $\hat{g}(\tau) := g(\tau)$ for $\tau \in \mathbb{H}$, and*

$$\hat{g}\left(\frac{a}{c}\right) := \begin{cases} \infty, & \textit{if } M > 0, \\ c_0, & \textit{if } M = 0, \\ 0, & \textit{if } M > 0. \end{cases}$$

Convention. Since each modular function has such an extension, we will also write g for the extension $\hat{g}$.

Using Prop 4.4 one can verify that the Γ-invariance (11) of g on points $\tau \in \mathbb{H}$ carries over to the points $\frac{a}{c} \in \hat{\mathbb{Q}} := \mathbb{Q} \cup \{\infty\}$:

Proposition 4.6. *Let g be a meromorphic modular function for a congruence subgroup Γ. Then for any $\frac{a}{c} \in \hat{\mathbb{Q}}$:*

$$g\left(\rho\frac{a}{c}\right) = g\left(\frac{a}{c}\right) \;\; \textit{for all} \;\; \rho \in \Gamma. \tag{21}$$

Property (11) says that g is invariant on the orbits of the Γ-action on $\mathbb{H}$; Prop. 4.6 says that g is invariant on the orbits of the extended Γ-action on $\hat{\mathbb{Q}}$. The latter orbits got a special name.

Definition 4.7. *Let Γ be a congruence subgroup. The Γ-orbits*

$$\left[\frac{a}{c}\right]_{\Gamma} := \left\{\rho\frac{a}{c} : \rho \in \Gamma\right\}, \quad \frac{a}{c} \in \hat{\mathbb{Q}},$$

of the action of Γ on $\hat{\mathbb{Q}}$ are called cusps (of Γ).

Convention. If Γ is clear from the context, we write $\left[\frac{a}{c}\right]$ instead of $\left[\frac{a}{c}\right]_{\Gamma}$.

Proposition 4.8. *Let Γ be a congruence subgroup. Then the number of cusps, this means, the number of orbits of the action of Γ on $\hat{\mathbb{Q}}$, is finite.*

Proof. The statement is true because congruence subgroups have finite index (Prop. 4.1) and any coset decomposition

$$\mathrm{SL}_2(\mathbb{Z}) = \Gamma\gamma_0 \cup \Gamma\gamma_1 \cup \ldots \cup \Gamma\gamma_k$$

implies

$$\hat{\mathbb{Q}} = \mathrm{SL}_2(\mathbb{Z})(\infty) = \Gamma(\gamma_0\infty) \cup \Gamma(\gamma_1\infty) \cup \cdots \cup \Gamma(\gamma_k\infty).$$

Definition 4.9. *Let Γ be a congruence subgroup. Recalling* (13)*, for $\frac{a}{c} \in \hat{\mathbb{Q}}$ define the width of the cusp $\left[\frac{a}{c}\right]_{\Gamma}$ as*

$$w^{\Gamma}_{[a/c]} := w^{\Gamma}_{\gamma} \ \text{ with } \ \gamma \in \mathrm{SL}_2(\mathbb{Z}) \ \text{ such that } \ \gamma\infty = \frac{a}{c}. \tag{22}$$

The width is well-defined: Suppose $\frac{a'}{c'} = \gamma'\infty \in \left[\frac{a}{c}\right]_{\Gamma}$. Then $\frac{a'}{c'} = \rho\frac{a}{c}$ for some $\rho \in \Gamma$. Hence $\gamma'\infty = \rho\gamma\infty$; i.e., $(\gamma')^{-1}\rho\gamma = \left(\begin{smallmatrix} 1 & m \\ 0 & 1 \end{smallmatrix}\right)$. The rest follows from (19).

Convention. If Γ is clear from the context, we will write $w_{[a/c]}$ instead of $w^{\Gamma}_{[a/c]}$.

Another fact implied by Prop. 4.4 is that one has to consider only *finitely* many cases to check the finite principal-part-property (12). But more is true. Define the stabilizer subgroup

$$\mathrm{Stab}(\infty) := \{\gamma \in \mathrm{SL}_2(\mathbb{Z}) : \gamma\infty = \infty\} = \left\{\pm\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} : m \in \mathbb{Z}\right\} \leq \mathrm{SL}_2(\mathbb{Z}).$$

Given a coset decomposition $\mathrm{SL}_2(\mathbb{Z}) = \Gamma\gamma_0 \cup \cdots \cup \Gamma\gamma_k$, it is obvious that the set of all cusps of Γ is formed by $\{[\gamma_j\infty]_{\Gamma} : j = 0, \ldots, k\}$.[4] The following lemma is important but straighforward to check.

Lemma 4.10.

$$[\gamma_i\infty]_{\Gamma} = [\gamma_j\infty]_{\Gamma} \quad \Leftrightarrow \quad \gamma_j = \rho\gamma_i\sigma \ \textit{ for some } \ \rho \in \Gamma \ \textit{ and } \ \sigma \in \mathrm{Stab}(\infty).$$

This lemma puts us into the position to verify that to establish the finite principal-part-property (12), it is sufficient to check (12) at the cusps:

Let $g : \mathbb{H} \to \hat{\mathbb{C}}$ be a meromorphic function which is analytic on $\{\tau \in \mathbb{H} : \mathrm{Im}(\tau) > M\}$ for some $M > 0$. Suppose that g satisfies the modular transformation property (11) for a

[4] Despite the cosets being assumed to be pairwise different, it may well be that $[\gamma_i\infty]_{\Gamma} = [\gamma_j\infty]_{\Gamma}$ for $i \neq j$.

congruence subgroup Γ. Then by the same reasoning as to obtain (16) we know that for each $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ there exists a Laurent series expansion of $g(\gamma\tau)$. This means, for all $\tau \in \mathbb{H}$ with $\mathrm{Im}(\tau) > M$,

$$g\left(\frac{a\tau+b}{c\tau+d}\right) = g(\gamma\tau) = \sum_{n=-\infty}^{\infty} c_n(\gamma) e^{2\pi i n\tau/w_\gamma^\Gamma}. \tag{23}$$

Lemma 4.11. *In the given setting, let* $\{[\delta_1\infty]_\Gamma, \ldots, [\delta_m\infty]_\Gamma\}$ *with* $\delta_\ell \in \mathrm{SL}_2(\mathbb{Z})$ *be a complete set of different cusps of* Γ. *Suppose the q-expansions at all these cusps have a finite principal part; i.e.,*

$$g(\delta_\ell\tau) = \sum_{n=-M_{\delta_\ell}}^{\infty} c_n(\delta_\ell) e^{2\pi i n\tau/w_{\delta_\ell}^\Gamma}, \quad \ell = 1, \ldots, m.$$

Then the q-expansions (23) *have finite principal parts for all* $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$.

Proof. Let $\mathrm{SL}_2(\mathbb{Z}) = \Gamma\gamma_0 \cup \cdots \cup \Gamma\gamma_k$ be a coset decomposition. Then for a given $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ there is a $j \in \{0, \ldots, k\}$ such that $\gamma \in \Gamma\gamma_j$. For the respective cusp we have $[\gamma_j\infty]_\Gamma = [\delta_\ell\infty]_\Gamma$ for some $\ell \in \{1, \ldots, m\}$. By Lemma 4.10, $\gamma_j = \rho_1\delta_\ell\sigma$ for some $\rho_1 \in \Gamma$ and $\sigma \in \mathrm{Stab}(\infty)$. By assumption, $\gamma_j = \rho_2^{-1}\gamma$ for some $\rho_2 \in \Gamma$, and thus $\gamma = (\rho_2\rho_1)\delta_\ell\sigma$. But now Prop. 4.4 says: if the q-expansion of $g(\delta_\ell\tau)$ has finite principal part, this is also true for $g(\gamma\tau)$.

5 Modular Functions: Examples

In this section we present examples to illustrate the notions of Section 4 and which are of relevance for later sections.

Example 1. Consider the Φ_2 function which we will use also in Example 2,

$$\Phi_2 : \mathbb{H} \to \mathbb{C},\ \tau \mapsto \Phi_2(\tau) := \left(\frac{\eta(2\tau)}{\eta(\tau)}\right)^{24} = \frac{\Delta(2\tau)}{\Delta(\tau)}. \tag{24}$$

It is an analytic modular function for $\Gamma_0(2)$: by Lemma 3.2 it is analytic on $\mathbb{H}$ and, as verified below, it satisfies for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(2)$,

$$\Phi_2\left(\frac{a\tau+b}{c\tau+d}\right) = \Phi_2(\tau). \tag{25}$$

The disjoint coset decomposition

$$\mathrm{SL}_2(\mathbb{Z}) = \Gamma_0(2)\gamma_0 \dot\cup \Gamma_0(2)\gamma_1 \dot\cup \Gamma_0(2)\gamma_2 \text{ with } \gamma_0 = I, \gamma_1 = T, \gamma_2 = TS,^5$$

is straightforward to verify; hence $[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(2)] = 3$. Owing to $0 = \gamma_1\infty = \gamma_2\infty$,

$$\hat{\mathbb{Q}} = \mathrm{SL}_2(\mathbb{Z})(\infty) = \Gamma_0(2)(\infty) \cup \Gamma_0(2)(0);$$

[5] Recall, $T = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, $S = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$.

i.e., $\Gamma_0(2)$ has the two cusps $[\infty]_{\Gamma_0(2)}$ and $[0]_{\Gamma_0(2)}$ with widths $w_{[\infty]}^{\Gamma_0(2)} = w_I^{\Gamma_0(2)} = 1$ and $w_{[0]}^{\Gamma_0(2)} = w_T^{\Gamma_0(2)} = 2$.[6] By Lemma 4.11, to check the finite principal part property (12) it is sufficient to inspect the q-expansions at $\frac{a}{c} = \infty = I\infty$ and $\frac{a}{c} = 0 = T\infty$:

$$\begin{aligned}\Phi_2(I\tau) = \Phi_2(\tau) &= \sum_{n=-M_I}^{\infty} c_n(I) e^{2\pi i n\tau/w_I^{\Gamma_0(2)}} = \sum_{n=-M_I}^{\infty} c_n(I) q^n \\ &= q + 24q^2 + 300q^3 + 2624q^4 + 18126q^5 + 105504q^6 + 538296q^7 + \dots, \quad (26)\end{aligned}$$

$$\begin{aligned}\Phi_2(T\tau) = \Phi_2\left(-\frac{1}{\tau}\right) &= \sum_{n=-M_T}^{\infty} c_n(T) e^{2\pi i n\tau/w_T^{\Gamma_0(2)}} = \sum_{n=-M_T}^{\infty} c_n(T) q^{\frac{n}{2}} \\ &= \frac{1}{2^{12}} \frac{1}{\Phi_2(\tau/2)} = \frac{1}{2^{12}} (q^{-1/2} - 24 + 276q^{1/2} - 2048q + 11202q^{3/2} - \dots). \quad (27)\end{aligned}$$

The first equality in (27) comes from (10), the second equality from (26).

Proof (Proof of (25)*).* The proof is a consequence of the following observation. For $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(2)$,

$$\Delta\left(2\frac{a\tau+b}{c\tau+d}\right) = \Delta\left(\frac{a(2\tau)+2b}{\frac{c}{2}(2\tau)+d}\right) = \Delta\left(\begin{pmatrix} a & 2b \\ \frac{c}{2} & d \end{pmatrix}(2\tau)\right) = \left(\frac{c}{2}(2\tau)+d\right)^{12} \Delta(2\tau).$$

Example 2. Consider the Klein j function (also called: j-invariant),

$$\mathbb{H} \to \mathbb{C},\ \tau \mapsto j(\tau) := \frac{\left(1 + 2^8 \Phi_2(\tau)\right)^3}{\Phi_2(\tau)} \text{ with } \Phi_2(\tau) := \left(\frac{\eta(2\tau)}{\eta(\tau)}\right)^{24}. \quad (28)$$

By Lemma 3.2 it is analytic on $\mathbb{H}$. Hence, by Ex. 1, it is an analytic modular function for $\Gamma = \Gamma_0(2)$. But more is true: it is a well-known classical fact that j is a modular function for the full modular group $\mathrm{SL}_2(\mathbb{Z})$. Nevertheless, this $\mathrm{SL}_2(\mathbb{Z})$-modularity cannot be directly deduced from the function presentation (28). To this end, one better uses one of the classical presentations like

$$j(\tau) = \frac{E_4(\tau)^3}{\Delta(\tau)} \text{ with } E_4(\tau) := 1 + 240\sum_{n=1}^{\infty}\left(\sum_{1\le d|n} d^3\right) q^n. \quad (29)$$

We will prove this identity in Example 6. Assuming the $\mathrm{SL}_2(\mathbb{Z})$-modularity of j as proven, in view of (12), at *all* points $\frac{a}{c} = \gamma\infty \in \hat{\mathbb{Q}}$, $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, one can use one and the same q-expansion:

$$\begin{aligned} j\left(\frac{a\tau+b}{c\tau+d}\right) = j(\gamma\tau) = j(\tau) = j(I\tau) &= \sum_{n=-M_I}^{\infty} c_n(I) e^{2\pi i n\tau/w_I^{\mathrm{SL}_2(\mathbb{Z})}} \\ &= \frac{1}{q} + 744 + 196884q + 21493760q^2 + 864299970q^3 + \dots \quad (30)\end{aligned}$$

We point to the (elementary) fact that $\mathrm{SL}_2(\mathbb{Z})$ has only one cusp; namely, $\hat{\mathbb{Q}} = \mathrm{SL}_2(\mathbb{Z})(\infty) = [\infty] = [\gamma\infty]$ for any $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. Obviously, its width is $w_I^{\mathrm{SL}_2(\mathbb{Z})} = 1 = w_{[a/c]}^{\mathrm{SL}_2(\mathbb{Z})}$ for any $\frac{a}{c} \in \hat{\mathbb{Q}}$.

6 $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} a & b \\ 2c' & d \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} d & -2c' \\ -b & a \end{smallmatrix}\right)$ implies $\left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right) \in T^{-1}\Gamma_0(2)T$.

Concerning the presentation (28) of j in terms of eta products, in Section 8 we shall not only give a proof but also *derive* it algorithmically. — We also note that $E_4(\tau)$ belongs to the sequence of Eisenstein series defined as[7]

$$E_{2k}(\tau) := 1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \Big(\sum_{1 \le d | n} d^{2k-1} \Big) q^n, \quad k \ge 2,$$

and which under modular transformations behave similarly to Δ:

$$E_{2k}\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^{2k} \cdot E_{2k}(\tau), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}). \tag{31}$$

Now, this transformation property of E_4 together with that of Δ in a *direct fashion* yields that j satisfies (11) for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. As for j there are various presentations of the E_{2k}. For example, presenting E_4 in terms of the eta function as in [26, (1.28)] implies (28).

Note 2. As we will see below, information about cusps, widths of cusps, etc. can be essential also for computational reasons. With computer algebra systems like SAGE such kind of data can be also obtained algorithmically; see, e.g., [35]. For example,

```
beagle:~> sage
 SageMath version 7.6, Release Date: 2017-03-25
 Type "notebook()" for the browser-based notebook interface.
 Type "help()" for help.
sage: sage.modular.arithgroup.arithgroup_generic.ArithmeticSubgroup.coset_reps(Gamma0(2))

[
[1 0]  [ 0 -1]  [ 0 -1]
[0 1], [ 1  0], [ 1  1]
]
sage: Cusps=Gamma0(2).cusps()
sage: Cusps
[0, Infinity]
sage: [Gamma0(2).cusp_width(c) for c in Cusps]
[2, 1]
```

The first command loads SAGE; the second computes the coset representatives I, T and TS; the third and fourth commands tell us that $\Gamma_0(2)$ has the two cusps $[0]$ and $[\infty]$ with widths 2 and 1, respectively.

Example 3. Using again $q = q(\tau) = e^{2\pi i\tau}$, the slightly modified Rogers-Ramanujan functions,

$$G(\tau) := q^{-\frac{1}{60}} F(1) = q^{-\frac{1}{60}} \prod_{m=0}^{\infty} \frac{1}{(1-q^{5m+1})(1-q^{5m+4})} \tag{32}$$

and

$$H(\tau) := q^{\frac{11}{60}} F(q) = q^{\frac{11}{60}} \prod_{m=0}^{\infty} \frac{1}{(1-q^{5m+2})(1-q^{5m+3})} \tag{33}$$

behave well under the action of $\Gamma_1(5)$: for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1(5)$ with $\gcd(a,6) = 1$:

$$G\left(\frac{a\tau+b}{c\tau+d}\right) = e^{2\pi i\,\alpha(a,b,c)/60}\, G(\tau) \tag{34}$$

and

$$H\left(\frac{a\tau+b}{c\tau+d}\right) = e^{2\pi i\,\beta(a,b,c)/60}\, H(\tau), \tag{35}$$

[7] The B_n are the Bernoulli numbers; as for Δ, also for the Eisenstein series we prefer the normalized versions.

where

$$\alpha(a,b,c) = a(9-b+c)-9 \text{ and } \beta(a,b,c) = a(3+11b+c)-3.$$

Proof (Proof (sketch).). As a general note, all known proofs of the modular transformation property (11) of $G(\tau)$ and $H(\tau)$, and of functions of similar type, rely on product representations like (32) and (33). — In [33] Robins considered an important special case of a very general formula established by Schoeneberg [34]. For the crucial ingredient $\mu_{\delta,g}$ of this formula [33, (9)], Robins derived a very compact expression [33, Theorem 2] which for $\delta = 5$ and $g = 1,2$ gives the stated versions of $\alpha(a,b,c)$ and $\beta(a,b,c)$. As a note, Robins' formula is correct provided $c > 0$. But in view of $(a\tau+b)/(c\tau+d) = (-a\tau-b)/(-c\tau-d)$ this is no problem; moreover, the special case $c = 0$ is a trivial check, since then we can assume $a = 1$.

In Section 12 we will discuss generalized Dedekind eta functions and their transformation behaviour (66) under elements $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. As stated in Cor. 12.3, this implies that the Rogers-Ramanujan functions $G(\tau)$ and $H(\tau)$ satisfy property (12) concerning the finiteness of the principal part. In addition we have (34) and (35), hence $G(\tau)^{60}$ and $H(\tau)^{60}$ are modular functions for $\Gamma_1(5)$. In view of Def. 5.1 we say that G and H are *quasi-modular* functions for $\Gamma_1(5)$.

Definition 5.1. *Let $f : \mathbb{H} \to \hat{\mathbb{C}}$ be a meromorphic function. If there is an $\ell \in \mathbb{Z}_{>0}$ such that f^ℓ is a modular function for a congruence subgroup Γ, we say that f is a quasi-modular function for Γ.*

Remark 1. Our notion of quasi-modular function should not be confused with some authors usage of the notion of quasi-modular form which applies for functions that are derivatives of modular forms, like for example the Eisenstein series of weight 2.

Remark 2. One can show that the Rogers-Ramanujan functions $G(\tau)$ and $H(\tau)$ are modular functions for a subgroup Γ_{RR} of $\Gamma(5)$ of index $[\Gamma(5) : \Gamma_{\mathrm{RR}}] = 12$. Moreover, Γ_{RR} has $\Gamma(60)$ as a subgroup with index $[\Gamma_{\mathrm{RR}} : \Gamma(60)] = 96$. So one explicit way to present the Rogers-Ramanujan group Γ_{RR} is by its disjoint coset decomposition with respect to $\Gamma(60)$. This can be done without any effort using a computer algebra system.

Example 4. The Rogers-Ramanujan quotient

$$r(\tau) := \frac{H(\tau)}{G(\tau)} = q^{\frac{1}{5}} \prod_{m=0}^{\infty} \frac{(1-q^{5m+1})(1-q^{5m+4})}{(1-q^{5m+2})(1-q^{5m+3})}$$

is an analytic[8] modular function for $\Gamma(5)$. Moreover, for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1(5)$,

$$r\left(\frac{a\tau+b}{c\tau+d}\right) = e^{2\pi i b/5} r(\tau). \tag{36}$$

Proof. Cor. 12.3 implies that $r(\tau)$ satisfies property (12) concerning the finiteness of the principal part.

Concerning the modularity property (11), we first prove this property for matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(5)$ with $\gcd(a,6) = 1$. With this assumption we have $5|b$ and $a \equiv 1 \pmod 5$. The latter together with $\gcd(a,6) = 1$ gives $a = 30m+1$ or $a = 30m+11$ for some $m \in \mathbb{Z}$. Hence the exponent resulting from Ex. 3, $(\beta(a,b,c) - \alpha(a,b,c))/6 = 2ab - a + 1$, reduces to $2b$ modulo 10. To extend the statement to arbitrary matrices in $\Gamma(5)$, apply Lemma 5.2(b). To prove the extended modular transformation property (36), apply Lemma 5.2(c).

[8] By Lemma 3.2.

Lemma 5.2. *(a) Each matrix in $\Gamma(10)$ can be written as a product of matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(10)$ with* $\gcd(a,6)=1$. *(b) Each matrix in $\Gamma(5)$ can be written as a product of matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(5)$ with* $\gcd(a,6)=1$. *(c) Each matrix in $\Gamma_1(5)$ can be written as a product of matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1(5)$ with* $\gcd(a,6)=1$.

Proof. First of all, note that in the subgroup $\Gamma(30)$ of $\Gamma(5)$ all matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ have $\gcd(a,6)=1$. To prove (a), the first observation is that $\Gamma(30)$ is a normal subgroup of $\Gamma(10)$ with index 24. Hence there are $g_j \in \Gamma(10)$ which generate 24 (right) cosets such that

$$\Gamma(10)/\Gamma(30) = \{\Gamma(30)g_1, \dots, \Gamma(30)g_{24}\}.$$

Consider the following 13 matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(10)$, all satisfying $\gcd(a,6)=1$. To save space we use (a,b,c,d) instead of matrix notation $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$:

$$\begin{aligned}
&(1,0,0,1),(1,0,10,1),(1,0,20,1),(11,50,20,91),(11,50,130,591),\\
&(11,50,240,1091),(11,100,10,91),(11,100,120,1091),(11,100,230,2091),\\
&(1121,10200,1020,9281),(2231,20190,2030,18371),(3421,170,1630,81),\\
&(3631,16520,1730,7871).
\end{aligned}$$

One can verify that these 13 matrices give rise to 13 pairwise disjoint (right) cosets in $\Gamma(10)/\Gamma(30)$. These cosets must generate the whole group $\Gamma(10)/\Gamma(30)$, because any proper subgroup would consist of maximally 12 elements. Hence, for any $j \in \{1,\dots,24\}$:

$$\Gamma(30)g_j = \Gamma(30)h_1\Gamma(30)h_2\cdots = \Gamma(30)h_1h_2\dots$$

with particular h_k chosen from the 13 matrices. This proves (a).

To prove (b), we apply the same strategy observing that $\Gamma(10)$ is a normal subgroup of $\Gamma(5)$ with index 6. Hence there are $G_j \in \Gamma(5)$ which generate 6 (right) cosets such that

$$\Gamma(5)/\Gamma(10) = \{\Gamma(10)G_1, \dots, \Gamma(10)G_6\},$$

Consider the following 4 matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(5)$, all satisfying $\gcd(a,6)=1$:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 5 \\ 5 & 26 \end{pmatrix}, \begin{pmatrix} 31 & 285 \\ 160 & 2471 \end{pmatrix}, \begin{pmatrix} 281 & 1460 \\ 235 & 1221 \end{pmatrix}.$$

One can verify that these 4 matrices give rise to 4 pairwise disjoint (right) cosets which thus generate the full group $\Gamma(5)/\Gamma(10)$. Using the same argument as in (a), this proves (b).

To prove (c), we apply again the same strategy observing that $\Gamma(5)$ is a normal subgroup of $\Gamma_1(5)$ of index 5. One can verify that

$$\Gamma_1(5)/\Gamma(5) = \left\{\Gamma(5)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \Gamma(5)\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \dots, \Gamma(5)\begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}\right\}.$$

Hence for each element $\gamma \in \Gamma_1(5)$ we have $\gamma = \xi \left(\begin{smallmatrix} 1 & h \\ 0 & 1 \end{smallmatrix}\right)$ with $\xi \in \Gamma(5)$. By (b), ξ can be written as a product of matrices in $\Gamma(5)$ with $\gcd(a,6)=1$. Moreover, the matrices $\left(\begin{smallmatrix} 1 & h \\ 0 & 1 \end{smallmatrix}\right)$ also have $\gcd(a,6)=1$. Consequently, every matrix in $\Gamma_1(5)$ can be written as a product of matrices in $\Gamma_1(5)$ with $\gcd(a,6)=1$.

Note 3. For $\Gamma(5)$ SAGE computes 12 inequivalent cusps each of width 5:

```
sage: Cusps=Gamma(5).cusps(); Cusps
[0, 2/5, 1/2, 1, 3/2, 2, 5/2, 3, 7/2, 4, 9/2, Infinity]
sage: [Gamma(5).cusp_width(c) for c in Cusps]
[5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5]
```

Example 5. The 5th power of the Rogers-Ramanujan quotient

$$R(\tau) := r(\tau)^5 = q\prod_{m=0}^{\infty}\frac{(1-q^{5m+1})^5(1-q^{5m+4})^5}{(1-q^{5m+2})^5(1-q^{5m+3})^5} \tag{37}$$

is an analytic [9] modular function for $\Gamma_1(5)$: for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1(5)$,

$$r\left(\frac{a\tau+b}{c\tau+d}\right)^5 = e^{2\pi i b} r(\tau)^5 = r(\tau)^5.$$

Proof. Immediate from Ex. 4.

Remark 3. To prove property (12) of modular functions, i.e., the finiteness of the principal part of q-expansions, in Ex. 4 we were relying on the modular transformation property of generalized Dedekind eta functions; see Cor. 12.3 in connection with Prop. 12.2.

Note 4. For $\Gamma_1(5)$ SAGE computes 4 inequivalent cusps, two of them of width 1 and two of them of width 5:

```
sage: Cusps=Gamma1(5).cusps(); Cusps
[0, 2/5, 1/2, Infinity]
sage: [Gamma1(5).cusp_width(c) for c in Cusps]
[5, 1, 5, 1]
```

6 Zero Recognition of Modular Functions: Basic Ideas

Zagier's "magic principle" cited at the end of Section 2 enables algorithmic zero recognition of q-series/q-products which present meromorphic modular functions for congruence subgroups $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$. Obviously, for fixed Γ such functions form a field. But for various (computational) reasons, in particular when working with the q-expansions, it can be useful to view these functions as elements from a $\mathbb{C}$-algebra.[10] In Section 8 we shall come back to this aspect.

We will denote such modular function fields, resp. $\mathbb{C}$-algebras, by $M(\Gamma)$; i.e.,

$$M(\Gamma) := \{g : \hat{\mathbb{H}} \to \hat{\mathbb{C}} \,|\, g \ \text{ is a meromorphic modular function for } \Gamma\}.$$

Recall that by extending the group action of Γ on $\mathbb{H}$ to an action on $\hat{\mathbb{H}} = \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$, we extended modular functions $g : \mathbb{H} \to \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ to functions $g : \hat{\mathbb{H}} \to \hat{\mathbb{C}}$. For zero recognition we generalize further. Let

$$X(\Gamma) := \ \text{set of orbits of } \ \Gamma \ \text{ on } \ \hat{\mathbb{H}} = \left\{[\tau]_\Gamma : \tau \in \hat{\mathbb{H}}\right\},$$

[9] By Lemma 3.2.

[10] I.e., a commutative ring with 1 which is also a vector space over $\mathbb{C}$.

where we use the notation $[\tau]_\Gamma := \{\gamma\tau : \gamma \in \Gamma\}$ for orbits. Then to any meromorphic modular function $g \in M(\Gamma)$ we can assign a function $g^* : X(\Gamma) \to \hat{\mathbb{C}}$ (we say, g^* is *induced* by g) defined by

$$g^*([\tau]_\Gamma) := g(\tau), \quad \tau \in \hat{\mathbb{H}}.$$

Notice that the function values of g^* are well-defined owing to Prop. 4.4 and the related discussions in Section 4.[11]

It is important to note that by defining a suitable topology on $\hat{\mathbb{H}}$ one can introduce a topology on $X(\Gamma)$ that makes $X(\Gamma)$ Hausdorff and compact. To this end, for any $M > 0$ one defines open neighborhoods of $\infty \in \hat{\mathbb{H}}$ as

$$U_M(\infty) := \{\tau \in \mathbb{H} : \mathrm{Im}(\tau) > M\} \cup \{\infty\}.$$

The desired topology on $\hat{\mathbb{H}}$ then is defined to be generated by all finite intersections and all arbitrary unions built from the standard open sets in $\mathbb{H}$ and the sets $\gamma(U_M(\infty))$, $M > 0$, $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. Finally, the topology on $X(\Gamma)$ is defined as the quotient topology of the projection map $\pi : \hat{\mathbb{H}} \to X(\Gamma), \tau \mapsto [\tau]_\Gamma$.

The final step towards building the framework for zero recognition is the observation that the connected and compact Hausdorff space $X(\Gamma)$ can be equipped with the structure of a Riemann surface as explained in detail in [11]. It is straightforward to verify that, given a meromorphic function $g \in M(\Gamma)$, the induced function $g^* : X(\Gamma) \to \hat{\mathbb{C}}$ turns into a meromorphic function on the compact Riemann surface $X(\Gamma)$.

In this setting, the expansions (12) correspond to the local (Laurent) series expansions for g^* about the cusp $\left[\frac{a}{c}\right]_\Gamma$ with respect to local charts φ. To be more precise, suppose $\frac{a}{c} = \gamma\infty$ for $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, and consider g^* in an open neighborhood of $\left[\frac{a}{c}\right]_\Gamma$ of the form $V_M := \{[t]_\Gamma : t = \gamma\tau \text{ for } \tau \in U_M(\infty)\}$.[12] If M is sufficiently large, we have for $t \in \mathbb{H}$ such that $[t]_\Gamma \in V_M$,

$$g^*([t]_\Gamma) = g(t) = g(\gamma\tau) = h\left(e^{2\pi i \gamma^{-1} t / w_\gamma^\Gamma}\right),$$

where, according to (12), $h : \{z \in \mathbb{H} : |z| < m\} \to \hat{\mathbb{C}}$, $m > 0$ suitably chosen, is a meromorphic function with Laurent expansion of the form

$$h(z) = \sum_{n=-m_\gamma}^{\infty} c_n(\gamma) z^n.$$

In fact, one can verify that for suitably chosen $m, M > 0$,

$$\varphi : V_M \to \{z \in \mathbb{C} : |z| < m\},\ [t]_\Gamma \mapsto e^{2\pi i \gamma^{-1} t / w_\gamma^\Gamma} \tag{38}$$

is a coordinate chart at $\left[\frac{a}{c}\right]_\Gamma$;[13] i.e., a homeomorphism such that

$$g^*([t]_\Gamma) = h(\varphi([t]_\Gamma)) = \sum_{n=-m_\gamma}^{\infty} c_n(\gamma) \big(\varphi([t]_\Gamma) - \varphi([a/c]_\Gamma)\big)^n.$$

Notice that $\varphi([a/c])_\Gamma = e^{2\pi i \gamma^{-1} \frac{a}{c} / w_\gamma^\Gamma} = e^{-2\pi\infty / w_\gamma^\Gamma} = 0$.

---

[11] In fact one can use the observation (75) from Section 13.

[12] $\pi^{-1}(V_M)$ contains $\frac{a}{c}$ $(= \gamma\infty)$, and $\pi^{-1}(V_M) \setminus \{\frac{a}{c}\}$ is an open disc in $\mathbb{H}$ tangent to the real line at $\frac{a}{c}$.

[13] Apart from the requirement to be a homeomorphism, a second property one needs to verify is that such φ also satisfy the Riemann surface compatibility conditions; see, e.g., [25] or [11].

Definition 6.1. *Let $g \in M(\Gamma)$, Γ a congruence subgroup. Let $\frac{a}{c} = \gamma\infty \in \hat{\mathbb{Q}}$ for $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. Suppose the q-expansion of g at $\frac{a}{c}$ is*

$$g(\gamma\tau) = \sum_{n=-M}^{\infty} c_n q^{n/w_\gamma^\Gamma} \tag{39}$$

with $c_{-M} \neq 0$. Then the order of g at $\frac{a}{c}$ is defined as

$$\mathrm{ord}_{a/c}^{\Gamma}(g) := -M.$$

Moreover, we say that (39) *is the "local q-expansion of the induced function g^* at the cusp $\left[\frac{a}{c}\right]_\Gamma$" (with respect to the chart φ defined as in* (38)*).*

Because of Prop. 4.4, this order notion is well-defined and two local expansions at the same cusp differ in their coefficients only by an exponential factor.

A slightly more general implication of Prop. 4.4 is the fact that $\mathrm{ord}_{a/c}(g)$ for $g \in M(\Gamma)$ is invariant on the elements of cusps of Γ:

Corollary 6.2. *Let $g \in M(\Gamma)$. Then for $\frac{a}{c} \in \hat{\mathbb{Q}}$ and $\rho \in \Gamma$:*

$$\mathrm{ord}_{\rho\frac{a}{c}}^{\Gamma}(g) = \mathrm{ord}_{\frac{a}{c}}^{\Gamma}(g). \tag{40}$$

Proof. Let $\frac{a'}{c'} := \rho\frac{a}{c}$. There are $\gamma, \gamma' \in \mathrm{SL}_2(\mathbb{Z})$ such that $\gamma\infty = \frac{a}{c}$ and $\gamma'\infty = \frac{a'}{c'}$. Hence $\frac{a'}{c'} = \rho\frac{a}{c}$ translates into $\infty = \gamma^{-1}\rho^{-1}\gamma'\infty$; i.e., $\gamma^{-1}\rho^{-1}\gamma' = \left(\begin{smallmatrix} 1 & m \\ 0 & 1 \end{smallmatrix}\right)$ for some $m \in \mathbb{Z}$. The rest follows from Prop. 4.4.

In addition to having orders at cusps, we also need the usual notion of order for Laurent series with finite principal part:

Definition 6.3. *Let $f : U \to \hat{\mathbb{C}}$ be meromorphic in an open neighborhood $U \subseteq \mathbb{C}$ of z_0 containing no pole except possibly z_0 itself. Then, by assumption, f is analytic in $U \setminus \{z_0\}$ and can be expanded in a Laurent series about z_0,*

$$f(z) = \sum_{n=-M}^{\infty} c_n (z - z_0)^n.$$

Assuming that $c_{-M} \neq 0$, one defines $\mathrm{ord}_p(f) := -M$.

The following theorem is folklore, e.g. [25, Thm. 1.37], but of fundamental importance: it lies at the bottom of the "magic principle" for modular functions.

Theorem 6.4. *Let X be a compact Riemann surface. Suppose that $f : X \to \mathbb{C}$ is an analytic function on all of X. Then f is a constant function.*

Example 6. From Ex. 1 we know that Φ_2 is an analytic modular function in $M(\Gamma_0(2))$ with no zeros in $\mathbb{H}$. Therefore $j(\tau) = \frac{(1+2^8\Phi_2(\tau))^3}{\Phi_2(\tau)}$ is an analytic modular function in $M(\Gamma_0(2))$. Since it is non-constant, its induced function j^*, which is a meromorphic function on $X(\Gamma_0(2))$, according to Thm. 6.4 we must have at least one pole. Indeed, it has a pole at $[\infty]_{\Gamma_0(2)}$ which is made explicit by the local q-expansion (Def. 6.1)

$$\begin{aligned}j(I\tau) &= \sum_{n=-M_I}^{\infty} c_n(I) e^{2\pi i n\tau/w_I^{\Gamma}} = \sum_{n=-M_I}^{\infty} c_n(I) e^{2\pi i n\tau} \\ &= j(\tau) = \frac{1}{q} + 744 + 196884\,q + 21493760\,q^2 + 864299970\,q^3 + \ldots\end{aligned}$$

Recall from Ex. 1 that $X(\Gamma_0(2))$ has two cusps, $[\infty]_{\Gamma_0(2)}$ and $[0]_{\Gamma_0(2)}$, with widths $w_{[\infty]}^{\Gamma_0(2)} = w_I^{\Gamma_0(2)} = 1$ and $w_{[0]}^{\Gamma_0(2)} = w_T^{\Gamma_0(2)} = 2$. Accordingly, j^* has another pole—of multiplicity 2—at $[0]_{\Gamma_0(2)}$, which is made explicit by the local q-expansion (Def. 6.1)

$$j(T\tau) = \sum_{n=-M_T}^{\infty} c_n(T) e^{2\pi i n\tau/w_T^{\Gamma}} = \sum_{n=-M_T}^{\infty} c_n(T) e^{2\pi i n\tau/2}$$

$$= j(\tau) = \left(\frac{1}{q^{1/2}}\right)^2 + 744 + 196884\,\left(q^{1/2}\right)^2 + 21493760\,\left(q^{1/2}\right)^4 + \ldots$$

Summarizing, j^* is a meromorphic function on $X(\Gamma_0(2))$ having no other poles than a single pole at $[\infty]_{\Gamma_0(2)}$ and a double pole at $[0]_{\Gamma_0(2)}$; i.e.,

$$\mathrm{ord}_{\infty}^{\Gamma_0(2)}(j) = -1 \ \text{ and } \ \mathrm{ord}_{0}^{\Gamma_0(2)}(j) = -2.$$

Proof of (29). The properties of j^* as a meromorphic function of $X(\Gamma_0(2))$ as exhibited in Ex. 6, put us into the position to prove the presentation of j in (29),

$$j(\tau) = \frac{E_4(\tau)^3}{\Delta(\tau)}, \quad \tau \in \mathbb{H};$$

as announced in Ex. 2. The definitions (9) and (29) tell us that Δ and E_4 are analytic functions on $\mathbb{H}$. Their modular symmetries (10) and (31) together with their q-expansions at ∞ imply that $g := E_4^3/\Delta \in M(\mathrm{SL}_2(\mathbb{Z})) \subseteq M(\Gamma_0(2))$ and also, by Lemma 3.2, that g^* viewed as a meromorphic function on $X(\Gamma_0(2))$ has its only poles at the cusps $[\infty]_{\Gamma_0(2)}$ and $[0]_{\Gamma_0(2)}$, which are of orders 1 and 2, respectively. By taking the difference $h(\tau) := j(\tau) - g(\tau)$, one obtains a function $h \in M(\Gamma_0(2)$ with no poles, which can be verified by the q-expansions

$$h(\tau) = 0\cdot\frac{1}{q} + 0\cdot q^0 + \text{etc.} \ \text{ and } \ h(T\tau) = 0\cdot\left(\frac{1}{q^{1/2}}\right)^2 + \text{etc.}$$

Consequently, h^* is an analytic function on $X(\Gamma_0(2))$, and Thm. 6.4 implies that h^* is a constant function. From $h^*([\infty_{\Gamma_0(2)}]) = h(\infty) = 0$ we conclude $h^* = 0$, and thus $h = 0$. This means, we have proved (29). □

In view of Ex. 6, consider the following subalgebra of $M(\Gamma)$:

$$M^{\infty}(\Gamma) := \{g \in M(\Gamma) : g \ \text{ has no pole except at } \ [\infty]_{\Gamma}\}.$$

The q-expansions at the cusps $[\infty]_{\Gamma}$ give finitary normal form presentations for the modular functions in $M^{\infty}(\Gamma)$. More precisely, despite the analytic setting, to decide equality of two functions in $M^{\infty}(N)$ can be done in purely algebraic and finitary fashion:

Lemma 6.5. *Let g and h be in $M^{\infty}(\Gamma)$ with q-expansions*

$$g(\tau) = \sum_{n=\mathrm{ord}_{\infty}^{\Gamma}(g)}^{\infty} a_n q^n \ \textit{ and } \ h(\tau) = \sum_{n=\mathrm{ord}_{\infty}^{\Gamma}(h)}^{\infty} b_n q^n.$$

Then $g = h$ *if and only if* $\mathrm{ord}_\infty^\Gamma(g) = \mathrm{ord}_\infty^\Gamma(h) =: \ell$ *and*

$$(a_\ell, \dots, a_{-1}, a_0) = (b_\ell, \dots, b_{-1}, b_0).$$

Proof. Apply Thm. 6.4.

In other words, if $g(\tau) = \sum_{n \geq \mathrm{ord}_\infty^\Gamma(g)} a_n q^n \in M^\infty(\Gamma)$, the coefficients a_n, $n \geq 1$, are uniquely determined by those of the principal part and a_0. Algebraically this corresponds to an isomorphic embedding of $\mathbb{C}$-algebras:

$$\begin{aligned} \varphi &: M^\infty(N) \to \mathbb{C}[z], \\ g = \sum_{n=\mathrm{ord}_\infty^\Gamma(g)}^{\infty} a_n q^n &\mapsto a_{\mathrm{ord}_\infty^\Gamma(g)} z^{-\mathrm{ord}_\infty^\Gamma(g)} + \cdots + a_{-1} z + a_0. \end{aligned} \tag{41}$$

In computationally feasible cases the zero test for $g - h \stackrel{?}{=} 0$ according to Lemma 6.5 trivializes the task of proving identities between modular functions.

In order to invoke this zero test for $G - H \stackrel{?}{=} 0$ with given $G, H \in M(\Gamma)$, in a preprocessing step one has to transform the problem into the form $g - h \stackrel{?}{=} 0$, where g and h are elements in $M^\infty(\Gamma)$. Computational examples of this strategy are given in the Sections 8 and 11.2. As shown in these sections, when reducing things to $M^\infty(\Gamma)$ there is an "algorithmic bonus" which enables the algorithmic derivation of identities. For the single purpose of zero recognition, other variants of applying Thm. 6.4 can be used. This is illustrated by examples in Section 7.

7 Zero Recognition of Modular Functions: Examples

In this section we present various examples which despite being elementary should illustrate how to prove relations between q-series/q-products using modular function machinery.

For a better understanding of the "valence formulas" used in the concrete examples, we stress that the invariance property stated in Cor. 6.2 also holds for the usual order $\mathrm{ord}_\tau(g)$ when $\tau \in \mathbb{H}$:

Corollary 7.1. *Let* $g \in M(\Gamma)$. *Then for* $\tau \in \mathbb{H}$ *and* $\rho \in \Gamma$*:*

$$\mathrm{ord}_{\rho\tau}(g) = \mathrm{ord}_\tau(g). \tag{42}$$

Proof (Proof of Cor. 6.2). The statement follows from a general fact which can be verified in a straighforward manner: Let f be a meromorphic function on $\mathbb{H}$ and $\tau_0 \in \mathbb{H}$ then $\mathrm{ord}_{\gamma\tau_0}(f(\tau)) = \mathrm{ord}_{\tau_0}(f(\gamma\tau))$ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$.

In other words, $\mathrm{ord}_\tau(g)$ for $g \in M(\Gamma)$ is invariant on the elements of the orbits $[\tau]_\Gamma$ of Γ, and the "valence formulas" below should be read having this invariance property in mind.

For the first example in this section, we recall the "valence formula" for the full modular group $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ which can be found at many places in the literature; e.g. [26].

Corollary 7.2 ("valence formula" for $\mathrm{SL}_2(\mathbb{Z})$). *If $g \in M(\mathrm{SL}_2(\mathbb{Z}))$ then*

$$\frac{\mathrm{ord}_i(g)}{2} + \frac{\mathrm{ord}_\omega(g)}{3} + \mathrm{ord}_\infty^{\mathrm{SL}_2(\mathbb{Z})}(g) + \sum_{\substack{\tau \in H(\mathrm{SL}_2(\mathbb{Z})) \\ [\tau]\neq[i],[\tau]\neq[\omega]}} \mathrm{ord}_\tau(g) = 0, \tag{43}$$

where $H(\mathrm{SL}_2(\mathbb{Z})) \subseteq \mathbb{H}$ is a complete set of representatives of the orbits $[\tau]_{\mathrm{SL}_2(\mathbb{Z})}$ with $\tau \in \mathbb{H}$, $\mathrm{ord}_\tau(g)$ is the usual order as in Def. 6.3, and $\omega := e^{2\pi i/3}$.

In Section 13 (Appendix 3) we state—without proof—Thm. 13.2 which presents a valence formula that holds for any congruence subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$. Formula (43) is an immediate corollary setting $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ there.

Example 7. We consider (43) for $g = E_4^3/\Delta = j \in M(\Gamma)$ where $\Gamma = \mathrm{SL}_2(\mathbb{Z})$.[14] Noting that $[\infty]_{\mathrm{SL}_2(\mathbb{Z})}$ is the only cusp of $X(\mathrm{SL}_2(\mathbb{Z}))$ and that $\mathrm{ord}_\infty^{\mathrm{SL}_2(\mathbb{Z})}(g) = -1$ (by inspection of the q-expansion at ∞), relation (43) turns into

$$\frac{\mathrm{ord}_i(g)}{2} + \frac{\mathrm{ord}_\omega(g)}{3} + \sum_{\substack{\tau \in H(\mathrm{SL}_2(\mathbb{Z})) \\ [\tau]\neq[i],[\tau]\neq[\omega]}} \mathrm{ord}_\tau(g) = 1.$$

Therefore, and also in view of Cor. 7.1, there remain three possibilities for $g = j$ having a zero in $\mathbb{H}$:[15]

(a) j has triple zeros at the points in $[\omega]$ and nowhere else;
(b) j has double zeros at the points in $[i]$ and nowhere else;
(c) the only zeros of j are single zeros at the points of some orbit $[\tau] \neq [i], [\omega]$.

By Lemma 3.2, $\Delta(\tau) \neq 0$ for all $\tau \in \mathbb{H}$. In view of this and of the third power of E_4, only alternative (a) can apply. In particular, we obtain:

$\star\, E_4(\tau_0) = 0$ for $\tau_0 \in [\omega] = \{\gamma\omega : \gamma \in \mathrm{SL}_2(\mathbb{Z})\}$;
$\star$ the elements of $[\omega]$ are the only zeros of E_4;
$\star$ each of these zeros has multiplicity 3.

Example 8. Again $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. Consider (43) for

$$g = \frac{E_6^2}{\Delta} = \frac{1}{q} - 984 + 196884\,q + \cdots \in M(\mathrm{SL}_2(\mathbb{Z})).$$

With an argument analogous to Ex. 7 we conclude that

$$\frac{E_6^2}{\Delta} \text{ has double zeros at the points in } [i] \text{ and nowhere else.}$$

In particular:

$\star\, E_6(\tau_0) = 0$ for $\tau_0 \in [i] = \{\gamma i : \gamma \in \mathrm{SL}_2(\mathbb{Z})\}$;
$\star$ the elements of $[i]$ are the only zeros of E_6;
$\star$ each of these zeros has multiplicity 2.

[14] Cf. Ex. 6.

[15] It is important to note that the orbit sets of modular transformations are discrete; i.e., they do not have a limit point.

Example 9. The transformation property (31) implies

$$\frac{E_4^3 - E_6^2}{\Delta} = 1728 + \cdots \in M(\mathrm{SL}_2(\mathbb{Z})).$$

This quotient is an analytic modular function, hence Theorem 6.4 gives

$$\frac{E_4(\tau)^3 - E_6(\tau)^2}{\Delta(\tau)} = 1728, \quad \tau \in \mathbb{H}. \tag{44}$$

Hence (6) implies

$$j(\tau) = \frac{E_6(\tau)^2}{\Delta(\tau)} + 1728, \quad \tau \in \mathbb{H}, \tag{45}$$

and by Ex. 8 we obtain the evaluation,

$$j(\tau_0) = 1728 \text{ for all } \tau_0 \in [i] = \{\gamma i : \gamma \in \mathrm{SL}_2(\mathbb{Z})\}. \tag{46}$$

The next example presents an alternative derivation of the equalities (45) and (46).

Example 10. From Ex. 8 we know that

$$g(\tau) = \frac{E_6(\tau)^2}{\Delta(\tau)} = \frac{1}{q} - 984 + 196884q + \cdots \in M(\mathrm{SL}_2(\mathbb{Z}))$$

has a single pole at the points in $[\infty]$ and no pole elsewhere, and double zeros at the points in $[i]$ and nowhere else.[16] Applying the same reasoning as in Ex. 7 one obtains that also

$$f(\tau) := j(\tau) - j(i) = \frac{1}{q} + (744 - j(i)) + 196884q + \cdots \in M(\mathrm{SL}_2(\mathbb{Z}))$$

has a single pole at the points in $[\infty]$ and no pole elsewhere, and double zeros at the points in $[i]$ and nowhere else. Hence Theorem 6.4 implies that the induced function $(g/f)^*$ is constant and hence $g = c \cdot f$ for some $c \in \mathbb{C}$. Finally, the comparison of the q-expansions at ∞ of both sides gives:

$$\frac{1}{q} - 984 + 196884q + \cdots = \frac{c}{q} + c(744 - j(i)) + c\,196884q + \ldots$$

Consequently, $c = 1$ and $j(i) = 744 + 984 = 1728$, which proves (45) and (46).

More generally, the reasoning used in Ex. 10 can be easily extended to prove the

Theorem 7.3. *Suppose $g \in M(\mathrm{SL}_2(\mathbb{Z}))$. Then $g \in \mathbb{C}(j)$; i.e., g is a rational function in the Klein j function.*

For the next example we need

Corollary 7.4 ("valence formula" for $\Gamma_0(2)$). *Let $g \in M(\Gamma)$. If $\Gamma = \Gamma_0(2)$ then*

---

[16] Often one restricts to consider such functions only on a complete set of orbit representatives; for example, in the case of $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ to $\{\tau \in \mathbb{H} : -1/2 \le \mathrm{Re}(\tau) \le 0 \text{ and } \mathrm{Im}(\tau) \ge \mathrm{Im}(e^{i\tau})\} \cup \{0 < \mathrm{Re}(\tau) < 1/2 \text{ and } \mathrm{Im}(\tau) > \mathrm{Im}(e^{i\tau})\}$.

$$\mathrm{ord}_i(g)+\frac{\mathrm{ord}_{TSi}(g)}{2}+\mathrm{ord}_\omega(g)+\mathrm{ord}_\infty^\Gamma(g)+\mathrm{ord}_0^\Gamma(g)+\sum_{\substack{\tau\in H(\mathrm{SL}_2(\mathbb{Z}))\\ [\tau]\neq[i],[\tau]\neq[\omega]}}\sum_{\gamma\in\{I,T,TS\}}\mathrm{ord}_{\gamma\tau}(g)=0, \tag{47}$$

where $H(\mathrm{SL}_2(\mathbb{Z}))\subseteq\mathbb{H}$ *is a complete set of representatives of the orbits* $[\tau]_{\mathrm{SL}_2(\mathbb{Z})}$ *with* $\tau\in\mathbb{H}$, $\mathrm{ord}_{\gamma\tau}(g)$ *is the usual order as in Def. 6.3, and* $\omega:=e^{2\pi i/3}$.

As formula (43), also formula (47) is an immediate corollary of Thm. 13.2 setting $\Gamma=\Gamma_0(2)$ there.

Example 11. By Ex. 1, $\Psi_2:=\frac{1}{\Phi_2}$ is an analytic modular function for $\Gamma_0(2)$. By Lemma 3.2, $\Psi_2\in M(\Gamma_0(2))$ has no zero in $\mathbb{H}$. By (26) we have the q-expansion

$$\Psi_2(\tau)=\frac{\Delta(\tau)}{\Delta(2\tau)}=\frac{1}{\Phi_2(\tau)}=\frac{1}{q}-24+276q-2048q^2+11202q^3-\dots,$$

which tells that Ψ_2^* has a single pole at $[\infty]_{\Gamma_0(2)}$.[17] Hence Cor. 7.4 implies that Ψ_2^* must also have a single zero which must sit at $[0]_{\Gamma_0(2)}$. Why? Because Ψ_2 has no zero in $\mathbb{H}$, so $[0]_{\Gamma_0(2)}$ is the only remaining option. This is in accordance with Ex. 1; namely, with the fact that $X(\Gamma_0(2))$ has two cusps $[\infty]_{\Gamma_0(2)}$ and $[0]_{\Gamma_0(2)}$ with widths $w_{[\infty]}^{\Gamma_0(2)}=1$ and $w_{[0]}^{\Gamma_0(2)}=2$. Consequently, the single zero of Ψ_2^* must be at $[0]_{\Gamma_0(2)}$, which using (27) is confirmed by the expansion

$$\Psi_2\left(-\frac{1}{\tau}\right)=2^{12}\Phi_2\left(\frac{\tau}{2}\right)=2^{12}(q^{1/2}+24q+300q^{3/2}+2624q^2+18126q^{5/2}+\dots).$$

The properties of Ψ_2 made explicit in Ex. 11 allow to apply the same argument as used in Ex. 10, resp. Thm. 7.3, to derive

Theorem 7.5. *Suppose* $g\in M(\Gamma_0(2))$. *Then* $g\in\mathbb{C}(\Psi_2)$*; i.e., g is a rational function in* $\Psi_2(=\frac{1}{\Phi_2})$.

8 Zero Recognition: Computing Modular Function Relations

In view of Theorem 7.5 we consider the

TASK. Compute a rational function $\mathrm{rat}(x)\in\mathbb{C}(x)$ such that

$$j=\mathrm{rat}(\Psi_2).$$

From Examples 1 and 11 we know that Ψ_2^* as a meromorphic function on $X(\Gamma_0(2))$

- has at $[\infty]_{\Gamma_0(2)}$ its only pole which is of order 1,
- and at $[0]_{\Gamma_0(2)}$ its only zero, also of order 1.

From Example 6 we know that j^* as a meromorphic function on $X(\Gamma_0(2))$

[17] Equivalently, Ψ_2 has single poles at all the elements of the orbit $[\infty]_{\Gamma_0(2)}$.

- has at $[\infty]_{\Gamma_0(2)}$ a pole of order 1, at $[0]_{\Gamma_0(2)}$ a pole of order 2, and no pole elsewhere,
- has at $[\omega]_{\Gamma_0(2)}$ a triple zero and no zero elsewhere.

To solve our TASK the decisive observation is that for $F := j \cdot \Psi_2^2 \in M(\Gamma_0(2))$ the induced function F^* as a meromorphic function on $X(\Gamma_0(2))$

- has a possible pole only at $[\infty]_{\Gamma_0(2)}$.

From our knowledge about poles and zeros of j^* and Ψ_2^* we expect a pole of order 3, which is confirmed by computing the q-expansion of F. We take as input the q-expansions (30) and (27):

In[4]:= $\mathbf{j = \frac{1}{q} + 744 + 196884q + 21493760q^2 + O[q]^3;}$

$\Psi = \mathbf{\frac{1}{q} - 24 + 276q - 2048q^2 + O[q]^3;}$

In[5]:= $\mathbf{F = j * \Psi^2}$

Out[5]= $\frac{1}{q^3} + \frac{696}{q^2} + \frac{162300}{q} + 12865216 + O[q]^1$

Since F^* has the only pole at $[\infty]_{\Gamma_0(2)}$, we can successively reduce its local q-expansion (in the sense of Def. 6.1) using only powers of Ψ_2 ($= \Psi$ in the computation) until we reach a constant:

In[6]:= $\mathbf{F - \Psi^3}$

Out[6]= $\frac{768}{q^2} + \frac{159744}{q} + 12924928 + O[q]^1$

In[7]:= $\mathbf{F - \Psi^3 - 768\Psi^2}$

Out[7]= $\frac{196608}{q} + 12058624 + O[q]^1$

In[8]:= $\mathbf{F - \Psi^3 - 768\Psi^2 - 196608\Psi}$

Out[8]= $16777216 + O[q]^1$

In[9]:= $\mathbf{Factor[x^3 + 768x^2 + 196608x + 16777216]}$

Out[9]= $(256 + x)^3$

Consequently, $F = j \cdot \Psi_2^2 = (\Psi_2 + 2^8)^3$; i.e., we derived that

$$j = \Psi_2 \left(\frac{\Psi_2 + 2^8}{\Psi_2} \right)^3 = \frac{(1 + 2^8 \Phi_2)^3}{\Phi_2},$$

which is (28).

Despite the simplicity of this example, the underlying idea of algorithmic reduction is quite powerful. For example, it is used in Radu's Ramanujan-Kolberg algorithm [31]. We want to stress that computationally one works with the coefficients of the principal parts of Laurent series as finitary representations of the elements in $M(\Gamma)$. Consequently, the underlying structural aspect relevant to such methods is that of a $\mathbb{C}$-algebra rather than a field. Other recent applications of this reduction strategy can be found in [29], [30], and [17].

In Section 11.2 this algorithmic reduction is used to derive Felix Klein's icosahedral relation for the Rogers-Ramanujan quotient $r(\tau)$. Before returning to this theme in Section 10, in Section 9 we briefly discuss some connections between modular functions and holonomic functions.

9 Interlude: q-Holonomicity and Modular Functions

As mentioned in Section 2, the sequence $(p(n))_{n\geq 0}$ is not holonomic since its generating function $\prod_{k\geq 1}(1-q^k)^{-1}$ is not holonomic. The same applies to the other modular forms and functions we have presented so far. Nevertheless, there are several tight connections to *q-holonomic* sequences and series which we will briefly indicate in this section.

Let $\mathbb{F} = \mathbb{Q}(z_1,\dots,z_\ell)$ be a rational function field over $\mathbb{Q}$ with parameters $z_1,\dots,z_\ell$. Set $\mathbb{K} = \mathbb{F}(q)$ where q is taken to be an indeterminate.[18] Let $\mathbb{K}[[x]]$ denote the ring of formal power series with coefficients in $\mathbb{K}$. The q-derivative D_q on $\mathbb{K}[[x]]$ is defined as

$$D_q \sum_{n=0}^{\infty} a_n x^n := \sum_{n=0}^{\infty} a_n \frac{q^n-1}{q-1} x^{n-1}.$$

A sequence $(a_n)_{n\geq 0}$ with values in $\mathbb{K}$ is called q-holonomic (over $\mathbb{K}$), if there exist polynomials $p, p_0, \dots, p_r$ in $\mathbb{K}[x]$, not all zero, such that

$$p_r(q^n)\, a_{n+r} + p_{r-1}(q^n)\, a_{n+r-1} + \dots + p_0(q^n)\, a_n = p(q^n), \quad n \geq 0. \tag{48}$$

If $r = 1$ and $p = 0$, the sequence $(a_n)_{n\geq 0}$ is called q-hypergeometric. A finite sum, resp. infinite series, over a q-hypergeometric summand sequence is said to be a q-hypergeometric sum, resp. series. For example, the Rogers-Ramanujan functions $F(1)$ and $F(q)$ are q-hypergeometric series.

A formal power series $f(x) \in \mathbb{K}[[x]]$ is called q-holonomic, if there exist polynomials $p, p_0, \dots, p_r \in \mathbb{K}[x]$, not all zero, such that

$$p_r(x)\, D_q^r f(x) + p_{r-1}(x)\, D_q^{r-1} f(x) + \dots + p_0(x)\, f(x) = p(x). \tag{49}$$

There are several variations of these definitions. We are following the setting of Kauers and Koutschan [18], who developed a computer algebra package for q-holonomic sequences and series which assists the manipulation of such objects, including the execution of closure properties. In our context we do not need to go into further q-holonomic details. We only remark that, as in the standard holonomic case "$q = 1$", a sequence $(a_n)_{n\geq 0}$ is q-holonomic if and only if its generating function $f(x) = \sum_{n=0}^{\infty} a_n x^n$ is q-holonomic. For example, a q-hypergeometric series is also a q-holonomic series since its summand sequence is q-hypergeometric, i.e., it satisfies a q-holonomic recurrence (48) of order 1.

9.1 q-*Holonomic approximations of modular forms*

Despite being neither holonomic nor q-holonomic, modular forms and (quasi-)modular functions often find q-holonomic approximations; i.e., a presentation as a limit of a q-holonomic sequence.

There is theoretical and algorithmic framework for q-holonomic functions and sequences described in the literature; see [38] or [18]. Nevertheles, to our knowledge no systematic account of q-holonomic approximation of modular forms or modular forms as projections

[18] I.e., q is transcendental over $\mathbb{F}$.

of q-holonomic functions has been given so far. In this and in the next subsection we present illustrating examples for which we use the notation introduced in (1) and (2).

Set

$$a_n := \sum_{k=0}^{n} c(n,k) \text{ with } c(n,k) := \frac{q^{k^2}}{(q;q)_k (q;q)_{n-k}}, \quad n \geq 0.$$

Using the notion of convergence in the formal power series ring $\mathbb{Q}[[q]]$[19], or proceeding analytically with $|q| < 1$, it is straightforward to verify that

$$\lim_{n\to\infty} a_n = \prod_{\ell=1}^{\infty} \frac{1}{1-q^{\ell}} \sum_{k=0}^{\infty} \frac{q^{k^2}}{(q;q)_k} = \frac{F(1)}{(q;q)_\infty}. \tag{50}$$

The summand $c(n,k)$ of the definite q-hypergeometric sum a_n is q-hypergeometric in both variables n and k:

$$\frac{c(n+1,k)}{c(n,k)} = \frac{1}{1-q^{n+1-k}} \text{ and } \frac{c(n,k+1)}{c(n,k)} = q^{2k+1}\frac{1-q^{n-k}}{1-q^{k+1}}.$$

By applying a q-version of Zeilberger's algorithm one obtains a q-holonomic recurrence (48) for the sequence a_n. We use the implementation [28]. With respect to the input "`qZeil[f(n,k),{k,a(n),b(n)},n,order]`" the output symbol `SUM[n]` refers to the sum $\sum_{k=a(n)}^{b(n)} f(n,k)$; for instance, in `Out[12]` to a_n:

In[10]:= $<<$ **RISC`qZeil`**

Package q-Zeilberger version 4.50
written by Axel Riese © RISC-JKU

In[11]:= **qP = qPochhammer**;

In[12]:= $\mathbf{qZeil}\left[\frac{\mathbf{q^{k^2}}}{\mathbf{qP[q,q,k]qP[q,q,n-k]}}, \{\mathbf{k},\mathbf{0},\mathbf{n}\},\mathbf{n},\mathbf{2}\right]$

Out[12]= $\mathrm{SUM}[n] = \frac{(q^{2n}-q^{n+1}+q^2+q)\,\mathrm{SUM}[n-1]}{q(1-q^n)} - \frac{q\mathrm{SUM}[n-2]}{1-q^n}$

Summarizing, the q-holonomic sequence[20] $((q;q)_n a_n)_{n\geq 0}$ is a q-holonomic approximation of $F(1)$ in the sense that

$$\lim_{n\to\infty} (q,q)_n a_n = \sum_{k=0}^{\infty} \frac{q^{k^2}}{(q;q)_k} = F(1).$$

Also the product side of (6) has a q-holonomic approximation: set

$$b_n = \sum_{k=-n}^{n} \frac{(-1)^k q^{(5k^2-k)/2}}{(q;q)_{n+k}(q;q)_{n-k}}, \quad n \geq 0.$$

Using the notion of convergence in the formal powers series ring $\mathbb{Q}[[q]]$[21], or proceeding analytically with $|q| < 1$, it is straightforward to verify that

[19] See, for instance, [19].

[20] That $((q,q)_n a_n)_{n\geq 0}$ is q-holonomic is immediate by q-holonomic closure properties.

[21] See, for instance, [19].

$$\begin{aligned}\lim_{n\to\infty} b_n &= \frac{1}{(q;q)_\infty^2}\sum_{k=-\infty}^{\infty}(-1)^k q^{(5k^2-k)/2}\\ &= \frac{1}{(q;q)_\infty^2}\prod_{m=0}^{\infty}(1-q^{5m+2})(1-q^{5m+3})(1-q^{5m+5})\\ &= \frac{1}{(q;q)_\infty}\prod_{m=0}^{\infty}\frac{1}{(1-q^{5m+1})(1-q^{5m+4})}.\end{aligned}$$

The last equality is immediate, the second is by Jacobi's triple product identity [5],

$$\sum_{k=-\infty}^{\infty} q^{\binom{k}{2}}x^k = \prod_{m=0}^{\infty}(1+xq^m)(1+\frac{q}{x}q^m)(1-q^{m+1}). \tag{51}$$

Again one can use the q-version of Zeilberger's algorithm to derive a q-holonomic recurrence for b_n. Nevertheless, doing so results in a surprise:

In[13]:= $\mathbf{qZeil}\left[\frac{(-1)^k q^{(5k^2-k)/2}}{\mathbf{qP}[q,q,n+k]\mathbf{qP}[q,q,n-k]},\{k,-n,n\},n,5\right]$

Out[13]= $\mathrm{SUM}[n] = \frac{q^{10}\mathrm{SUM}[n-5]}{(1-q^{2n})(1-q^{2n-1})} + r_4\,\mathrm{SUM}[n-4] + r_3\,\mathrm{SUM}[n-3] + r_2\,\mathrm{SUM}[n-2] + r_1\,\mathrm{SUM}[n-1]$

When choosing instead of 5 the orders 1 to 4, the output will be empty. In other words, the first non-trivial recurrence the algorithm returns is `Out[13]` of order 5 (!), where the r_j are rational functions in q and q^n — too big to be displayed here. But the q-holonomic approximations coincide; i.e.,

$$a_n = b_n, \quad n \geq 0, \tag{52}$$

as proven by Andrews—inspired by Watson.[22] Hence, in order to prove (52) algorithmically, viewing the recurrences also as shift operators, one has to identify the order 2 recurrence for a_n as a right factor of the order 5 recurrence for b_n. Alternatively, as described in [27], one can apply "symmetrization" which results in the following modification of the b_n sum,

$$b_n = \frac{1}{2}\sum_{k=-n}^{n}\frac{(-1)^k(1+q^k)q^{(5k^2-k)/2}}{(q;q)_{n+k}(q;q)_{n-k}}.$$

Remarkably, for this version of b_n the q-Zeilberger algorithm outputs the same recurrence as for a_n:

In[14]:= $\mathbf{qZeil}\left[\frac{(-1)^k(1+q^k)q^{(5k^2-k)/2}}{\mathbf{qP}[q,q,n+k]\mathbf{qP}[q,q,n-k]},\{k,-n,n\},n,5\right]$

Out[14]= $\mathrm{SUM}[n] = \frac{(q^{2n}-q^{n+1}+q^2+q)\,\mathrm{SUM}[n-1]}{q(1-q^n)} - \frac{q\mathrm{SUM}[n-2]}{1-q^n}$

Note. There are also situations where for a given definite hypergeometric sum $S(n) := \sum_{k=0}^{n} f(n,k)$ the q-version of Zeilberger's algorithm returns a recurrence of order greater than one, but where, in fact, $S(n)$ is a q-hypergeometric sequence. Such situations can be resolved by applying a q-version q-version of Petkovšeks algorithm Hyper; see [2, 1].

For the second Rogers-Ramanujan identity (7) all these observations work completely the same.

[22] See [27] for more information about such finite versions of the Rogers-Ramanujan identities.

9.2 Modular functions as projections of q-holonomic series

Despite being neither holonomic nor q-holonomic, modular forms and (quasi-)modular functions can arise as projections of q-holonomic series; i.e., can be obtained by specifying a parameter in a q-holonomic series. Instead of setting up a theoretical framework, we take again the Rogers-Ramanujan functions as an illustrating example. Recalling their common setting (3), they are the projections $z=1$ and $z=q$ of

$$F(z)=\sum_{k=0}^{\infty}\frac{q^{k^2}z^k}{(q;q)_k}.$$

Using computer algebra, we now show that $F(z)$ is a q-holonomic series in $\mathbb{K}[[z]]$ with $\mathbb{K}=\mathbb{Q}(q)$.[23] Concretely, we derive a q-difference equation for $F(z)$; i.e., determine $r_j(z)\in\mathbb{K}[z]$ such that

$$r_0(z)\,F(z)+r_1(z)\,F(q\,z)+r_2(z)\,F(q^2\,z)=0. \tag{53}$$

Owing to $D_qF(z)=\frac{F(qz)-F(z)}{(q-1)z}$ such q-shift equations are equivalent to q-differential equations (49).

The summand sequence of $F(z)$, $(f_k(z))_{k\geq 0}=\left(\frac{q^{k^2}z^k}{(q;q)_k}\right)_{k\geq 0}$ is q-hypergeometric:

$$\frac{f_{k+1}(z)}{f_k(z)}=\frac{q^{2k+1}z}{1-q^{k+1}}\in\mathbb{K}(z)(q^k).$$

Consequently, one can apply *parametrized telescoping* to compute a q-hypergeometric sequence $(g_k(z))_{k\geq 0}$ and $r_j(z)\in\mathbb{K}[z]$ such that

$$r_0(z)\,f_k(z)+r_1(z)\,f_k(qz)+r_2(z)\,f_k(q^2z)=g_{k+1}(z)-g_k(z),\quad k\geq 0.^{24} \tag{54}$$

Then summing (54) over k from 0 to ∞ gives

$$r_0(z)\,F(z)+r_1(z)\,F(q\,z)+r_2(z)\,F(q^2\,z)=g_\infty(z)-g_0(z),$$

provided the limit $\lim_{k\to\infty}g_k(z)=g_\infty(z)$ exists. More precisely, the algorithm runs parameterized telescoping on the summand

$$f_k(z)\left(r_0(z)+r_1(z)\frac{f_k(qz)}{f_k(z)}+r_2(z)\frac{f_k(q^2z)}{f_k(z)}\right)=f_k(z)\left(r_0(z)+r_1(z)\,q^k+r_2(z)\,q^{2k}\right)$$

with unknown $r_j(z)$. Using the RISC package `qZeil` this is executed as follows:

In[15]:= $\mathbf{qTelescope}\left[\frac{\mathbf{q^{k^2}z^k}}{\mathbf{qP[q,q,k]}},\{\mathbf{k},\mathbf{0},\mathbf{N}\},\mathbf{qParameterized}\to\{\mathbf{1},\mathbf{q^k},\mathbf{q^{2k}}\}\right]$

Out[15]= $\mathtt{Sum}[\mathtt{q\,z\,F_2[k]-F_0[k]+F_1[k],\{k,0,N\}}]=\frac{\mathtt{q^{N^2+2N+1}z^{N+1}}}{\mathtt{(q;q)_N}}$

The output corresponds to summing (54) over k from 0 to N where $F_j[k]=f_k(q^jz)$ and with the following ingredients computed in the steps of the algorithm:

$$r_0(z)=-1,r_1(z)=1,r_2(z)=q\,z,\ \text{ and }\ g_N(z)=(1-q^N)\,\frac{q^{N^2}z^N}{(q;q)_N},N\geq 0.$$

---

[23] $F(z)$ is also a q-hypergeometric series; its summand sequence $(f_k(z))_{k\geq 0}$ is q-hypergeometric over $\mathbb{K}$ with $\mathbb{K}=\mathbb{Q}(z)(q)$.

[24] In case no such order 2 equation exists, one proceeds with incrementing the order by one.

Obviously, $g_0(z) = 0$ and $\lim_{N\to\infty} g_N(z) = 0$, and we thus obtained

$$F(qz) + qzF(q^2z) = F(z). \tag{55}$$

In the next section we will see how (55) will be used to obtain a continued fraction representation of $r(\tau)$.

10 The Rogers-Ramanujan Continued Fraction

After deriving the functional relation (55) we unfold it as a continued fraction — following Ramanujan. Divide both sides of (55) by $F(z)$,

$$\frac{F(qz)}{F(z)} + qz\frac{F(q^2z)}{F(z)} = 1,$$

and rewrite

$$\left(1 + qz\frac{F(q^2z)}{F(qz)}\right)\frac{F(qz)}{F(z)} = 1$$

such that

$$\frac{F(qz)}{F(z)} = \frac{1}{1 + qz\dfrac{F(q^2z)}{F(qz)}}.$$

Then iterate,

$$\frac{F(qz)}{F(z)} = \cfrac{1}{1 + \cfrac{qz}{1 + \cfrac{q^2z}{1 + q^3z\dfrac{F(q^4z)}{F(q^3z)}}}}. \tag{56}$$

This connects to the Rogers-Ramanujan quotient from Example 4,

$$r(\tau) = \frac{H(\tau)}{G(\tau)} = q^{\frac{1}{5}}\frac{F(q)}{F(1)} = q^{\frac{1}{5}}\prod_{m=0}^{\infty}\frac{(1-q^{5m+1})(1-q^{5m+4})}{(1-q^{5m+2})(1-q^{5m+3})},$$

which, as we noted, is a modular function for $\Gamma(5)$. Namely, taking $z = 1$ in (56) and iterating ad infinitum, one obtains

$$r(\tau) = q^{\frac{1}{5}}\cfrac{1}{1 + \cfrac{q}{1 + \cfrac{q^2}{1 + \cfrac{q^3}{1 + \cdots}}}}. \tag{57}$$

By Worpitzky's theorem the continued fraction (57) converges for $\tau \in \mathbb{H}$, i.e., for $|q| < 1$ when $q = q(\tau) = \exp(2\pi i\tau)$.[25] It converges also for some $\tau \in \mathbb{R}$. For example, for $\tau = 0$ one has $q = 1$ and thus

[25] An excellent account on convergence questions related to the Rogers-Ramanujan continued fraction is [10].

$$r(0) = \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cdots}}}} \tag{58}$$

$$= \frac{1}{\phi}, \tag{59}$$

where

$$\frac{1}{\phi} = -\frac{1}{2} + \frac{\sqrt{5}}{2} \text{ and } \phi = \frac{1}{2} + \frac{\sqrt{5}}{2}.$$

This evaluation of $r(\tau)$ for $\tau = 0$ is made plausible by rewriting (58) as $r(0) = \frac{1}{1+r(0)}$. From a rigorous point of view, the situation is this: For $u_k, v_k \in \mathbb{C}$ the continued fraction

$$\cfrac{u_1}{v_1+\cfrac{u_2}{v_2+\cfrac{u_3}{v_3+\ldots}}}$$

converges to $c \in \mathbb{C}$, if there exists a $d \in \mathbb{Z}_{\geq 0}$ such $\lim_{n\to\infty} A_{n+d}(0) = c$, where the approximants A_n are defined as

$$A_n(z) := (a_1 \circ a_2 \cdots \circ a_n)(z) \text{ with } a_k(z) := \frac{u_k}{v_k + z}.$$

With respect to (58) the approximants turn out to be quotients of successive Fibonacci numbers,

$$A_1(0) = \frac{1}{1}, A_2(0) = \frac{1}{2}, A_2(0) = \frac{2}{3}, A_3(0) = \frac{3}{5}, \text{ a.s.o.}$$

Similarly one obtains the evaluation of $r(\tau)$ for $\tau = \frac{1}{2}$:

$$r(1/2) = e^{\pi i/5} \cfrac{1}{1-\cfrac{1}{1+\cfrac{1}{1-\cfrac{1}{1+\cdots}}}}$$

$$= e^{\pi i/5}\phi. \tag{60}$$

Here the approximants are of the form,

$$A_1(z) = \frac{1}{1+z}, A_3(z) = 2+z, A_5(z) = \frac{3+z}{2+z}, A_7(z) = \frac{5+z}{3+z}, \text{ a.s.o.},$$

and

$$A_2(z) = \frac{1+z}{z}, A_4(z) = \frac{1+2z}{1+z}, A_6(z) = \frac{2+3z}{1+2z}, A_8(z) = \frac{3+5z}{2+3z}, \text{ a.s.o.}$$

For $\tau \in \mathbb{H}$ Ramanujan gave several beautiful evaluations; for example,

$$
\begin{aligned}
r(i) &= e^{-\frac{2\pi}{5}} \cfrac{1}{1+\cfrac{e^{-2\pi}}{1+\cfrac{e^{-4\pi}}{1+\cfrac{e^{-6\pi}}{1+\cdots}}}} \\
&= \sqrt{\frac{5+\sqrt{5}}{2}} - \phi \qquad (61)
\end{aligned}
$$

There is much history and literature connected to these evaluations of Ramanujan. Besides the pointers given in [14], see, for instance, the extensive survey [9] which presents many formulas related to $r(\tau)$ and discusses also analytic questions like convergence of the Rogers-Ramanujan continued fraction.

In the next section, using an algorithmic approach to Klein's icosahedral equation, we give a compact proof of the evaluation (61) using modular function machinery.

11 Klein's Icosahedron and Ramanujan's Evaluation

In this section we describe a beautiful connection, first established by Felix Klein [20] between the fixed field of the icosahedral group and modular functions. In the latter context Ramanujan's evaluation (61) finds a natural explanation.

11.1 Klein's icosahedral function and Ramanujan's evaluation

Consider the following subfield of $\mathbb{C}(z)$, the field of rational functions with complex coefficients,

$$\mathbb{K} = \left\{ f(z) \in \mathbb{C}(z) : f(z) = f\left(\frac{1}{z}\right) \right\}.$$

It is not too much a surprise that $\mathbb{K} = \mathbb{C}\left(z + \frac{1}{z}\right)$; this means, $\mathbb{K} = \mathbb{C}(f)$ is generated as a rational function field over $\mathbb{C}$ by one element, $f = z + \frac{1}{z} \in \mathbb{C}(z)$. By Lüroth's theorem this is true for all non-trivial subfields of $\mathbb{C}(z)$.

In order to produce such non-trivial subfields one can take fixed fields with respect to groups. For instance, $\mathbb{K}$ is the fixed field of the group $G = \left\{z \mapsto z, z \mapsto \frac{1}{z}\right\}$ acting on $\mathbb{C}(z)$. Felix Klein [20] considered finite subgroups G of the three-dimensional rotation group which turn out to be the finite dihedral groups, and the symmetry groups of the Platonic solids up to conjugation by a rotation. For further details see [23] for Riemann surfaces aspects or [37] for the underlying geometry.

Of particular interest for our context is the case where G is the group induced by the symmetry group of the icosahedron.[26] Defining the *icosahedral function* $I(z) \in \mathbb{C}(z)$ as

$$I(z) := -\frac{(z^{20} - 228z^{15} + 494z^{10} + 228z^5 + 1)^3}{z^5(z^{10} + 11z^5 - 1)^5}, \qquad (62)$$

[26] By stereographic projection the rotations of the sphere turn into Möbius transformations $z \mapsto \frac{az+b}{cz+d}$ of the complex plane.

the subfield $\mathbb{K}$ of $\mathbb{C}(z)$ whose elements are invariant under the icosahedral mappings from G is generated by $I(z)$, as computed by Klein [20]. This means, $\mathbb{K} = \mathbb{C}(I(z))$. As mentioned, such groups G are determined up to conjugation by a rotation. Geometrically, the icosahedral function $I(z)$ emerges from inscribing an icosahedron into a sphere in a natural way; see [23, Sect. 1.7].

There is a beautiful connection between the icosahedral fixed field and modular functions which traces back to Felix Klein [20], namely

Theorem 11.1 ("icosahedral key relation"). *The Klein j function and the Rogers-Ramanujan continued fraction r are related via the icosahedral function as*

$$j(\tau) = I(r(\tau)), \quad \tau \in \mathbb{H}. \tag{63}$$

As a "by-product", for $\tau = i$ this gives Ramanujan's evaluation in a straightforward manner: by (46) we have

$$1728 = I(r(i));$$

this means, $r(i)$ is the root of a polynomial over the integers of degree 60. Despite the high degree it is a polynomial with underlying rich mathematical structure. For instance, its roots are related in geometrical fashion to Klein's icosahedron; see [20], [14] or [23]. Computationally, using a computer algebra system like Mathematica gives

In[16]:= $\mathbf{Factor}[(\mathbf{z}^{20} - 228\mathbf{z}^{15} + 494\mathbf{z}^{10} + 228\mathbf{z}^5 + 1)^3 + 1728\mathbf{z}^5(\mathbf{z}^{10} + 11\mathbf{z}^5 - 1)^5]$

Out[16]= $(z^2+1)^2(z^4+2z^3-6z^2-2z+1)^2(z^8-z^6+z^4-z^2+1)^2$
$(z^8-6z^7+17z^6-18z^5+25z^4+18z^3+17z^2+6z+1)^2$
$(z^8+4z^7+17z^6+22z^5+5z^4-22z^3+17z^2-4z+1)^2$

In[17]:= $\mathbf{Solve}[\mathbf{1} - \mathbf{2z} - \mathbf{6z}^2 + \mathbf{2z}^3 + \mathbf{z}^4 == \mathbf{0}, \mathbf{z}]$

Out[17]= $\left\{z \to -\frac{1}{2} + \frac{\sqrt{5}}{2} + \sqrt{\frac{1}{2}\left(5-\sqrt{5}\right)}\right\}, \left\{z \to \frac{1}{2}\left(-1+\sqrt{5}-\sqrt{2\left(5-\sqrt{5}\right)}\right)\right\},$
$\left\{z \to \frac{1}{2}\left(-1-\sqrt{5}-\sqrt{2\left(5+\sqrt{5}\right)}\right)\right\}, \left\{z \to \frac{1}{2}\left(-1-\sqrt{5}+\sqrt{2\left(5+\sqrt{5}\right)}\right)\right\}$

The fourth root is Ramanujan's evaluation for $r(i)$; it can be picked by the numerics of the continued fraction on the left side of (61).

Finally we show that modular function machinery not only enables to prove but also to *derive* the icosahedral key relation (63) in an algorithmic fashion.

11.2 Algorithmic derivation of Klein's icosahedral key relation

Our task in this section is to derive the icosahedral key relation (63). To this end, notice that in (63) only powers of $r(\tau)^5$ arise. From Example 5 we know that $R(\tau) := r(\tau)^5$ is an analytic modular function for $\Gamma_1(5)$ which is non-zero on $\mathbb{H}$. By Note 4, $X(\Gamma_1(5))$ has 4 inequivalent cusps:

$[0]_{\Gamma_1(5)}$ and $[1/2]_{\Gamma_1(5)}$ of width 5, and $[2/5]_{\Gamma_1(5)}$ and $[\infty]_{\Gamma_1(5)}$ of width 1.

Analogous to the example treated in Section 8 we consider the

TASK. Compute a rational function $\mathrm{rat}(x) \in \mathbb{C}(x)$ such that

$$j = \mathrm{rat}(R).$$

Owing to the fact that $R(\tau) \in M(\Gamma_1(5))$ is an analytic modular function which, by Lemma 3.2, is non-zero on $\mathbb{H}$, $R^* : X(\Gamma_1(5)) \to \hat{\mathbb{C}}$ must have all its zeros and poles at the cusps. Indeed, formula (18) in [33, Thm. 4] gives that R^*

- has a zero of order 1 at $[\infty]_{\Gamma_1(5)}$;[27]
- has a pole of order 1 at $[2/5]_{\Gamma_1(5)}$;
- has order zero at the cusps $[0]_{\Gamma_1(5)}$ and $[1/2]_{\Gamma_1(5)}$.

Analogous to Example 6 we have that j^* as a meromorphic function on $X(\Gamma_1(5))$

- has poles of order 1 at the cusps $[2/5]_{\Gamma_1(5)}$ and $[\infty]_{\Gamma_1(5)}$;
- has poles of order 5 at the cusps $[0]_{\Gamma_1(5)}$ and $[1/2]_{\Gamma_1(5)}$.

Note 5. Not being relevant to our derivation of (63), we only remark that as a consequence of the pole count and the "valence formula" (13.3),[28] j^* must have 12 zeros (including multiplicities) at orbits $[\gamma\omega]_{\Gamma_1(5)}$ with $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. In fact, as sketched in Ex. 16, one can verify that j^* has a zero of order 3 at each of the orbits

$$[\gamma_1\omega]_{\Gamma_1(5)}, [\gamma_4\omega]_{\Gamma_1(5)}, [\gamma_5\omega]_{\Gamma_1(5)}, [\gamma_7\omega]_{\Gamma_1(5)},$$

with γ_j as in Ex. 16, and no other zero elsewhere.

Analogous to Section 8, to solve our TASK, the decisive observation is that for

$$F(\tau) := j(\tau)\frac{(R(\tau)-R(0))^5\,(R(\tau)-R(1/2))^5}{R(\tau)^{11}} \in M(\Gamma_1(5)) \tag{64}$$

the induced function F^* as a meromorphic function on $X(\Gamma_1(5))$

- has a possible pole only at $[\infty]_{\Gamma_1(5)}$.

Namely, the factors $(R(\tau)-R(0))^5$ and $(R(\tau)-R(1/2))^5$ cancel the poles of F^* at $[0]_{\Gamma_1(5)}$ and $[1/2]_{\Gamma_1(5)}$, but altogether introduce a pole of order 10 at $[2/5]_{\Gamma_1(5)}$. Since j^* has a pole of order 1 also at $[2/5]_{\Gamma_1(5)}$, we cancel this pole by dividing with $R(\tau)^{11}$. Hence the only remaining pole of F^* is located at $[\infty]_{\Gamma_1(5)}$. Counting the pole order on the right side of (64) gives $1+0+0+11 = 12$, since j^* has a pole of order 1 and $(1/R)^*$ of order 11 at $[\infty]_{\Gamma_1(5)}$.

Set $S := 1/R$. As F^*, also S^* has its only pole at $[\infty]_{\Gamma_1(5)}$. Owing to the fact that this only pole of S^* is of order 1, we can proceed as in Section 8. This means, we will reduce F successively using only powers of S until we reach a constant.

To input F we need to know the values $R(0)$ and $R(1/2)$. In general, to determine such specific values could be a serious problem. But in our case, the required evaluations are

[27] This is also immediate from the q-expansion (37) of $R(\tau)$ at ∞.

[28] And also taking into account the fact that j has zeros of multiplicity 3 at each element of the orbit $[\omega]_{\mathrm{SL}_2(\mathbb{Z})}$, and no zero elsewhere; see Ex. 7.

the limits of the Rogers-Ramanujan continued fraction, (59) and (60), which we found by using elementary means only:

$$R(0) = \frac{1}{\phi^5} = -\frac{11}{2} + \frac{5\sqrt{5}}{2} \text{ and } R(1/2) = e^{\pi i}\phi^5 = -\frac{11}{2} - \frac{5\sqrt{5}}{2}.$$

In view of the denominator in (62), we note explicitly that these values give

$$(R(\tau) - R(0))\,(R(\tau) - R(1/2)) = R(\tau)^2 + 11R(\tau) - 1.$$

We take as input the q-expansions (30) and (37), and compute the expansions for F and $S = 1/R$:

In[18]:= $\mathbf{j = \frac{1}{q} + 744 + 196884q + 21493760q^2 + 864299970q^3} + \cdots + \mathbf{O[q]^{14}}$;
$\mathbf{R = q - 5q^2 + 15q^3 - 30q^4 + 40q^5 - 26q^6} + \cdots + \mathbf{O[q]^{14}}$;

In[19]:= $\mathbf{F = Series[j \frac{(R^2 + 11R - 1)^5}{R^{11}}, q, 0, 2]}$

Out[19]= $-\frac{1}{q^{12}} - \frac{744}{q^{11}} - \frac{196824}{q^{10}} - \frac{21449060}{q^9} - \frac{852444060}{q^8} - \frac{18945738096}{q^7} - \frac{280406147430}{q^6} - \frac{3024142415076}{q^5} - \frac{25050805181610}{q^4} - \frac{164605868039100}{q^3} - \frac{874299071995668}{q^2} - \frac{3783906304850712}{q} - 13295075401691261 + O[q]^1$

In[20]:= $\mathbf{S = Series[\frac{1}{R}, q, 0, 12]}$

Out[20]= $\frac{1}{q} + 5 + 10q + 5q^2 - 15q^3 - 24q^4 + 15q^5 + 70q^6 + 30q^7 - 125q^8 - 175q^9 + 95q^{10} + 420q^{11} + O[q]^{12}$

Then we reduce F successively using powers of S times a suitable constant, until the coefficient of q^0 vanishes:

In[21]:= $\mathbf{F + S^{12}}$

Out[21]= $-\frac{684}{q^{11}} - \frac{195054}{q^{10}} - \frac{21414900}{q^9} - \frac{851959965}{q^8} - \frac{18940379184}{q^7} - \frac{280358028740}{q^6} - \frac{3023783306916}{q^5} - \frac{25048541972115}{q^4} - \frac{164593704824480}{q^3} - \frac{874243078444152}{q^2} - \frac{3783685739309592}{q} - 13294338120698731 + O[q]^1$

In[22]:= $\mathbf{F + S^{12} + 684S^{11}}$

Out[22]= $-\frac{157434}{q^{10}} - \frac{20399160}{q^9} - \frac{834052845}{q^8} - \frac{18709129044}{q^7} - \frac{278032352816}{q^6} - \frac{3004884410856}{q^5} - \frac{24921527108535}{q^4} - \frac{163877480555060}{q^3} - \frac{870829109716752}{q^2} - \frac{3769918758959172}{q} - 13247719680862951 + O[q]^1$

In[23]:= ...

In[24]:= $\mathbf{F + S^{12} + 684S^{11} + 157434S^{10} + 12527460S^9 + 77460495S^8 + 130689144S^7 - 33211924S^6 - 130689144S^5 + 77460495S^4 - 12527460S^3 + 157434S^2 - 684S + 1}$

Out[24]= $O[q]^1$

In[25]:= $\mathbf{Clear[S]}$

In[26]:= $\mathbf{Factor[S^{12} + 684S^{11} + 157434S^{10} + 12527460S^9 + 77460495S^8 + 130689144S^7 - 33211924S^6 - 130689144S^5 + 77460495S^4 - 12527460S^3 + 157434S^2 - 684S + 1]}$

Out[26]= $(S^4 + 228S^3 + 494S^2 - 228S + 1)^3$

This means, we obtained the relation

$$F(\tau) = j(\tau)\frac{(R(\tau)^2 + 11R(\tau) - 1)^5}{R(\tau)^{11}} = -\left(\frac{1}{R(\tau)^4} + \frac{228}{R(\tau)^3} + \frac{494}{R(\tau)^2} - \frac{228}{R(\tau)} + 1\right)^3$$

which completes the algorithmic derivation of the icosahedral key relation (63).

12 Appendix 1: Generalized Dedekind Eta-Functions

We give the definition of generalized Dedekind eta functions $\eta_{g,h}(\tau;N)$ following the notation of Berndt [8] and Schoeneberg [34, Ch. VIII]. Again we put $q = e^{2\pi i\tau}$. For the Bernoulli polynomials $B_1(x) = x - \frac{1}{2}$ and $B_2(x) = x^2 - x + \frac{1}{6}$ let

$$b_1(x) := B_1(\{x\}) \text{ and } b_2(x) := B_2(\{x\}), \quad x \in \mathbb{R},$$

where $\{x\} := x - \lfloor x \rfloor$ is the fractional part. Furthermore, for $g, h \in \mathbb{Z}$ define

$$\alpha(g,h) := \begin{cases} (1 - e^{-2\pi i h})e^{\pi i b_1(h)}, & \text{if } g \in \mathbb{Z} \text{ and } h \notin \mathbb{Z}, \\ 1, & \text{otherwise.} \end{cases}$$

Definition 12.1 (generalized Dedekind eta functions). *Let $g, h \in \mathbb{Z}$, $N \in \mathbb{Z}_{>0}$, and $\zeta_N := e^{2\pi i/N}$. For $\tau \in \mathbb{H}$:*

$$\eta_{g,h}(\tau;N) := \alpha(g/N, h/N) q^{b_2(g/N)/2} \prod_{\substack{m \geq 1 \\ m \equiv g \,(\mathrm{mod} N)}} \left(1 - \zeta_N^h q^{\frac{m}{N}}\right) \prod_{\substack{m \geq 1 \\ m \equiv -g \,(\mathrm{mod} N)}} \left(1 - \zeta_N^{-h} q^{\frac{m}{N}}\right).$$

If $g \not\equiv 0 \pmod N$ one can write this as

$$\eta_{g,h}(\tau;N) = \alpha(g/N, h/N) q^{b_2(g/N)/2} (\zeta_N^h q^{\frac{G}{N}}; q)_\infty (\zeta_N^{-h} q^{\frac{N-G}{N}}; q)_\infty, \tag{65}$$

where $G \in \{0, \ldots, N-1\}$ such that $G \equiv g \pmod N$.

Example 12. If $g = h = 0$: $\alpha(0,0) = 1$, $b_2(0)/2 = 1/12$, and

$$\eta_{0,0}(\tau;N) = q^{\frac{1}{12}} \prod_{k \geq 1} \left(1 - q^k\right)^2 = \eta(\tau)^2.$$

This motivates to call the $\eta_{g,h}(\tau;N)$ *generalized Dedekind eta functions*.

The Rogers-Ramanujan functions $G(\tau)$ and $H(\tau)$ from Ex. 3 are obtained as follows.

Example 13. According to (65),

$$\eta_{1,0}(\tau;5) = q^{\frac{1}{300}} (q^{\frac{1}{5}}; q)_\infty (q^{\frac{4}{5}}; q)_\infty \text{ and } \eta_{2,0}(\tau;5) = q^{-\frac{11}{300}} (q^{\frac{2}{5}}; q)_\infty (q^{\frac{3}{5}}; q)_\infty.$$

Hence

$$G(\tau) = q^{-\frac{1}{60}} F(1) = \frac{1}{\eta_{1,0}(5\tau;5)} \text{ and } H(\tau) = q^{\frac{11}{60}} F(q) = \frac{1}{\eta_{2,0}(5\tau;5)}.$$

Note 6. If one expands the products in Def. 12.1 one obtains a Laurent series with *finite* principal part. The explicit expansion can be obtained with Jacobi's triple product identity (51); for example, if $g \not\equiv 0 \pmod N$,

$$\left(\frac{\alpha(g/N, h/N) q^{b_2(g/N)/2}}{(q;q)_\infty}\right)^{-1} \eta_{g,h}(\tau;N) = \sum_{n=-\infty}^{\infty} (-1)^n q^{\binom{n}{2}} \left(\zeta_N^h q^{\frac{G}{N}}\right)^n$$
$$= 1 + \sum_{n=1}^{\infty} (-1)^n q^{\binom{n}{2}} \zeta_N^{hn} q^{\frac{G}{N}n} + \sum_{n=1}^{\infty} (-1)^n q^{\binom{n}{2}} \zeta_N^{-hn} q^{\frac{N-G}{N}n},$$

where $G \in \{0, \ldots, N-1\}$ such that $G \equiv g \pmod N$.

The following transformation behaviour, respectively variants of it, has been studied and derived by Curt Meyer [24], Ulrich Dieter [12], and Bruno Schoeneberg [34, Ch. VIII].

Proposition 12.2. *Let $N \in \mathbb{Z}_{>0}$ and $g, h \in \mathbb{Z}$ such that g and h are not both $\equiv 0 \pmod{N}$. Then for any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$:*

$$\eta_{g,h}(\gamma\tau; N) = e^{\pi i \mu(\gamma, g', h'; N)} \eta_{g',h'}(\tau; N) \tag{66}$$

where

$$\begin{pmatrix} g' \\ h' \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} g \\ h \end{pmatrix},$$

and where the rational number $\mu(\gamma, g', h'; N) \in \mathbb{Q}$ is produced by a complicated expression.[29]

In [8] Bruce Berndt succeeded to streamline work of Joseph Lewittes [21] and obtained (66) as a special case of his setting. For a different approach to transformation formulas for generalized Dedekind eta functions see Yifan Yang [39].

Corollary 12.3. *The Rogers-Ramanujan functions $G(\tau)$ and $H(\tau)$ from Ex. 3 satisfy property* (12) *concerning the finiteness of the principal part.*

Proof. This is a straightforward consequence of Ex. 13, Note 6, and the transformation formula (66).

We conclude Appendix 1 by mentioning some connections to theta functions.[30] Classically, there are four Jacobi theta functions $\theta_1, \ldots, \theta_4$, [5, (10.7.1)–(10.7.4)], but which, as stated in [5], "are really the same function." For example, noting that for $q = e^{2\pi i\tau}$, $\tau \in \mathbb{H}$, and $z \in \mathbb{C}$,

$$\theta_1(z, \tau) = (-i) q^{1/8} \sum_{n=-\infty}^{\infty} (-1)^n q^{\binom{n}{2}} e^{-(2n-1)iz}. \tag{67}$$

Using Jacobi's triple product identity (51) one obtains for the Rogers-Ramanujan functions from Ex. 3,

$$iq^{2/5}\theta_1(-2\pi\tau, 5\tau) = \eta(\tau)G(\tau) \text{ and } iq^{1/10}\theta_1(-\pi\tau, 5\tau) = \eta(\tau)H(\tau). \tag{68}$$

More generally, for $q = e^{2\pi i\tau}$, $\tau \in \mathbb{H}$, and $z = e^{2\pi i\zeta}$, $\zeta \in \mathbb{C}$, consider the theta function studied extensively by Farkas and Kra [15, (2.53)]:

$$\theta \begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix} (\zeta, \tau) = e^{\frac{\pi i \varepsilon\varepsilon'}{2}} z^{\frac{\varepsilon}{2}} q^{\frac{\varepsilon^2}{8}} (q;q)_\infty \left(-z e^{\pi i \varepsilon'} q^{\frac{1+\varepsilon}{2}}; q\right)_\infty \left(-z^{-1} e^{-\pi i \varepsilon'} q^{\frac{1-\varepsilon}{2}}; q\right)_\infty, \tag{69}$$

where ε and ε' are real parameters. One can verify, again by using the triple product identity (51), that generalized Dedekind eta functions are a subfamily of these functions. For instance, if $g, h \in \{1, \ldots, N-1\}$,

$$\theta \begin{bmatrix} 1 - 2g_N \\ 1 \end{bmatrix} (-h_N, \tau) = \frac{i e^{\pi i (2 g_N h_N - g_N - h_N)}}{2 \sin(h_N)} \eta(\tau) \eta_{g,h}(\tau; N), \tag{70}$$

where $g_N := g/N$ and $h_N := h/N$. For $h = 0$ and $g \in \{1, \ldots, N-1\}$ one has,

[29] To obtain an explicit form of this expression set, for instance, $b_{g,h}(N) = 0$ on the right side of [8, (24)].

[30] Warning: in many texts on Jacobi theta functions $q = e^{\pi i\tau}$, in contrast to $q = e^{2\pi i\tau}$ as throughout this article.

$$\theta\begin{bmatrix}1-2g_N\\1\end{bmatrix}(0,\tau) = ie^{-\pi i g_N}\eta(\tau)\eta_{g,0}(\tau;N). \tag{71}$$

Theta functions with $z=0$, i.e., $\theta\left[\begin{smallmatrix}\varepsilon\\\varepsilon'\end{smallmatrix}\right](0,\tau)$, are called theta constants. As studied in detail in[15], already this subfamily satisfies a rich variety of transformation formulas. For example, Duke uses this tool-box to derive the following modular transformation [14, (4.10)] for the Rogers-Ramanujan quotient $r(\tau)$ from Ex. 4:

$$r\left(-\frac{1}{\tau}\right) = \frac{-(1+\sqrt{5})r(\tau)+2}{2r(\tau)+1+\sqrt{5}}.$$

As revealed also by other applications in [14], the Farkas-Kra theta function calculus is providing computational alternatives to some of the methods presented in this tutorial.

13 Appendix 3: Valence Formula

For zero recognition of modular functions and, more generally, of modular forms "valence formulas" are often very useful. Such formulas describe relations between the orders at points $\tau \in \mathbb{H}$ corresponding to orbits $[\tau]_\Gamma$ in the sense of Cor. 7.1, and at points $\frac{a}{c} \in \hat{\mathbb{Q}}$ corresponding to cusps in the sense of Def. 6.1. In our context we only need to discuss "valence formulas" for modular functions which can be viewed as specializations of another "folklore theorem" from Riemann surfaces, e.g., [25, Prop. 4.12]:[31]

Theorem 13.1. *Let $f: X \to \hat{\mathbb{C}}$ be a non-constant meromorphic function on a compact Riemann surface X. Then*

$$\sum_{p\in X} \mathrm{Ord}_p(f) = 0. \tag{72}$$

For functions on Riemann surfaces the orders $\mathrm{Ord}_p(f)$ are defined via the orders of local (Laurent) series expansions about $p \in X$ with respect to charts φ. Concretely, let $U \subseteq X$ be an open neighborhood of p containing no pole except possibly p itself, and let $\varphi: U \to V \subseteq \mathbb{C}$ be a homeomorphism.[32] Then, by assumption, $f\circ\varphi^{-1}$ is analytic in $V\setminus\{\varphi(p)\}$ and can be expanded in a Laurent series about $z_0 := \varphi(p)$,

$$f(\varphi^{-1}(z)) = \sum_{n=-M}^{\infty} c_n(z-z_0)^n.$$

Assuming that $c_{-M} \neq 0$, one defines $\mathrm{Ord}_p(f) := -M$.

Note 7. Obviously, when taking the standard open sets as neighborhoods and as charts the identity maps, the complex plane can be turned into a Riemann surface. In this case, the order is the usual order $\mathrm{ord}_p(f)$ from Def. 6.3 for Laurent series with finite principal part; i.e., for $p \in X := \mathbb{C}$ and a function $f: U \to \hat{\mathbb{C}}$ being meromorphic in a neighborhood U of $p \in U \subseteq \mathbb{C}$,

$$\mathrm{Ord}_p(f) = \mathrm{ord}_p(f). \tag{73}$$

[31] The first such "folklore theorem" we considered was Theorem 6.4.

[32] In addition, φ is supposed to be compatible with the other charts; see e.g. [25].

Finally we connect (72) to our context; namely, where $X := X(\Gamma) = \{[\tau]_\Gamma : \tau \in \hat{\mathbb{H}}\}$ for some congruence subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$ and where the $[\tau]_\Gamma = \{\gamma\tau : \gamma \in \Gamma\}$ are the orbits of the action of Γ on $\hat{\mathbb{H}}$. Here as meromorphic functions $f : X \to \hat{\mathbb{C}}$ we have the induced functions $g^* : X(\Gamma) \to \hat{\mathbb{C}}$ of meromorphic $g \in M(\Gamma)$. If $p = \left[\frac{a}{c}\right]_\Gamma \in X(\Gamma)$ is a cusp, then in view of the remarks leading up to Def. 6.1 we have

$$\mathrm{Ord}_p(f) = \mathrm{Ord}_{[a/c]}(g^*) = \mathrm{ord}^\Gamma_{a/c}(g).$$

For orbits $p = [\tau]_\Gamma$ with $\tau \in \mathbb{H}$, the discussion of how to define Ord_p is more involved. Therefore we refrain from doing so, and state our modular function adaptation (77) of (72) without proof.

Nevertheless, we present a version of a "valence formula" which is sufficiently flexible for many (algorithmic) applications we have in mind.[33] We also note that our version is different from the many versions of "valence formulas" one finds in the literature in the following sense. The formula applied to a given group Γ can be made explicit directly by knowing a complete set of representatives of the right cosets of Γ in $\mathrm{SL}_2(\mathbb{Z})$. One basically lifts the formula valid form $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ to any Γ in a natural way from our point of view. We view this as natural because we only need to consider how the orbit $[\tau]_{\mathrm{SL}_2(\mathbb{Z})}$ splits into smaller orbits under the action of Γ for every τ going throw a complete set of representatives of the orbits of the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}^*$. So we can split our analysis into four cases: the orbits $[\tau]_{\mathrm{SL}_2(\mathbb{Z})}$ different from $[i]$, $[\omega]$ and $[\infty]$ and these remaining three orbits. This idea will be seen clearly from the examples where we apply the formula on the group $\Gamma_0(2)$ and $\Gamma(5)$. This gives, in particular, a more pragmatic flavour to our formula when compared to the classic versions that talk about elliptic points, parabolic points without making them more explicit. The transition from the formal statement to the concrete application can be tedious, at least from our experience.

Before stating it, we need some preparations.

Suppose $\gamma_1, \dots, \gamma_m \in \mathrm{SL}_2(\mathbb{Z})$ is a complete set of right coset representatives of Γ in $\mathrm{SL}_2(\mathbb{Z})$; i.e., as a disjoint union,

$$\mathrm{SL}_2(\mathbb{Z}) = \Gamma\gamma_1 \dot\cup \dots \dot\cup \Gamma\gamma_m. \tag{74}$$

Then for any $\tau \in \mathbb{H}$ the $\mathrm{SL}_2(\mathbb{Z})$-orbit of τ splits into Γ-orbits accordingly,

$$[\tau]_{\mathrm{SL}_2(\mathbb{Z})} = [\gamma_1\tau]_\Gamma \cup \dots \cup [\gamma_m\tau]_\Gamma.$$

We note explicitly that, in contrast to (74) it might well happen that $[\gamma_k\tau]_\Gamma = [\gamma_\ell\tau]_\Gamma$ for $k \neq \ell$. Actually it is true that

$$[\gamma_k\tau]_\Gamma = [\gamma_\ell\tau]_\Gamma \Leftrightarrow \gamma_\ell \in \Gamma\gamma_k/\mathrm{Stab}(\tau) \tag{75}$$

with

$$\mathrm{Stab}(\tau) := \{\gamma \in \mathrm{SL}_2(\mathbb{Z}) : \gamma\tau = \tau\},$$

and where $\Gamma\gamma_k/\mathrm{Stab}(\tau)$ is a particular subset of the right cosets of Γ in $\mathrm{SL}_2(\mathbb{Z})$ defined as an orbit of an action of $\mathrm{Stab}(\tau)$ which permutes cosets:

$$\Gamma\gamma_j/\,\mathrm{Stab}(\tau) := \{\Gamma\gamma_j\gamma : \gamma \in \mathrm{Stab}(\tau)\}.$$

For fixed $\tau \in \mathbb{H}$, the set of different Γ-orbits is denoted by

$$S_\Gamma(\tau) := \{[\gamma_j\tau]_\Gamma : j = 1, \dots, m\}.$$

[33] From modular *forms* point of view, (77) deals with the case of forms of weight zero only.

Note that in general, $|S_\Gamma(\tau)| \leq m$. One can verify in a straightforward manner that for fixed $\tau \in \mathbb{H}$ the following map is bijective:

$$\phi : \{\Gamma\gamma_j/\mathrm{Stab}(\tau) : j = 1,\ldots,m\} \to S_\Gamma(\tau), \quad \phi\left(\Gamma\gamma_j/\mathrm{Stab}(\tau)\right) := [\gamma_j\tau]_\Gamma. \tag{76}$$

The stabilizer subgroup $\mathrm{Stab}(\tau)$ comes in because special care has to be taken of "elliptic" points; cf. [11, Ch. 2.3 and 2.4]. These are points $\tau_0 \in \mathbb{H}$, resp. orbits $[\tau_0]_\Gamma$, which are fixed by non-trivial elements from $\mathrm{SL}_2(\mathbb{Z})$. To handle this matter technically, it is convenient to introduce a special notation for the map induced by the action of $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$:

$$\overline{\gamma} : \hat{\mathbb{H}} \to \hat{\mathbb{H}}, \tau \mapsto \overline{\gamma}(\tau) := \gamma\tau = \frac{a\tau+b}{c\tau+d}.$$

For any subset $G \subseteq \mathrm{SL}_2(\mathbb{Z})$ we denote the image under this map by

$$\overline{G} := \{\overline{\rho} : \rho \in G\}.$$

We note that if G is a subgroup of $\mathrm{SL}_2(\mathbb{Z})$, then $\overline{G}$ is a subgroup of $\overline{\mathrm{SL}_2(\mathbb{Z})} \cong \mathrm{SL}_2(\mathbb{Z})/\{\pm I\}$.

Collecting all these ingredients one can prove as a specialization of Thm. 13.1:

Theorem 13.2 ("valence formula"). *Let Γ be a congruence subgroup and $\mathrm{SL}_2(\mathbb{Z}) = \Gamma\gamma_1 \dot\cup \ldots \dot\cup \Gamma\gamma_m$ a disjoint coset decomposition. Then for any $g \in M(\Gamma)$:*

$$\sum_{\tau \in H(\mathrm{SL}_2(\mathbb{Z}))} \sum_{[\gamma_j\tau]_\Gamma \in S_\Gamma(\tau)} \frac{|\Gamma\gamma_j/\,\mathrm{Stab}(\tau)|}{w(\Gamma)|\overline{\mathrm{Stab}(\tau)}|} \mathrm{ord}_{\gamma_j\tau}(g) + \sum_{\substack{[a/c]_\Gamma \\ \text{cusp of } X(\Gamma)}} \mathrm{ord}^\Gamma_{a/c}(g) = 0, \tag{77}$$

where $H(\mathrm{SL}_2(\mathbb{Z}))$ is a complete set of representatives of the orbits $[\tau]_{\mathrm{SL}_2(\mathbb{Z})}$ with $\tau \in \mathbb{H}$, $\mathrm{ord}_{\gamma_j\tau}(g)$ *is the usual order as in Def. 6.3, and*

$$w(\Gamma) := \begin{cases} 1, \textit{if } -I = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \in \Gamma, \\ 2, \textit{otherwise.} \end{cases}$$

It is well-known that the only points giving rise to non-trivial stabilizers are the elements in the orbits $[i]_{\mathrm{SL}_2(\mathbb{Z})}$ and $[\omega]_{\mathrm{SL}_2(\mathbb{Z})}$, where $\omega := e^{2\pi i/3}$. Indeed one has, for example,

$$\mathrm{Stab}(i) = \{I, -I, T, -T\}, \ \mathrm{Stab}(\omega) = \{I, -I, TS, -TS, (TS)^2, -(TS)^2\}; \tag{78}$$

A detailed analysis of fixed points of modular transformations is given in [34, Ch.I.3].

As examples we consider specializations of the "valence formula" (77) for three choices of Γ: $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, $\Gamma = \Gamma_0(2)$, and $\Gamma = \Gamma_1(5)$.

Example 14. $\Gamma = \mathrm{SL}_2(\mathbb{Z})$: as coset decomposition we have $\mathrm{SL}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})\gamma_1$ with $\gamma_1 = I$; $S_\Gamma(\tau) = \{[\tau]_{\mathrm{SL}_2(\mathbb{Z})}\}$; $\Gamma/\,\mathrm{Stab}(\tau) = \{\Gamma\}$; $w(\Gamma) = 1$ since $-I \in \mathrm{SL}_2(\mathbb{Z})$. Finally, $X(\mathrm{SL}_2(\mathbb{Z}))$ has only one cusp $[\infty]_{\mathrm{SL}_2(\mathbb{Z})}$, hence (77) becomes

$$\sum_{\tau \in H(\mathrm{SL}_2(\mathbb{Z}))} \frac{1}{|\overline{\mathrm{Stab}(\tau)}|} \mathrm{ord}_\tau(g) + \mathrm{ord}^\Gamma_\infty(g) = 0. \tag{79}$$

Because of (78), the "valence formula" (79) turns into the version (43) of Cor. 7.2.

Example 15. $\Gamma = \Gamma_0(2)$: $\mathrm{SL}_2(\mathbb{Z}) = \Gamma\gamma_1 \dot\cup \Gamma\gamma_2 \dot\cup \Gamma\gamma_3$ is the coset decomposition with $\gamma_1 = I$, $\gamma_2 = T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and $\gamma_2 = TS = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$; $S_\Gamma(i) = \{[i]_\Gamma, [TSi]_\Gamma\}$, $S_\Gamma(\omega) = \{[\omega]_\Gamma\}$; $\Gamma/\operatorname{Stab}(i) = \{\Gamma, \Gamma T\}, \Gamma TS/\operatorname{Stab}(i) = \{\Gamma TS\}; \Gamma/\operatorname{Stab}(\omega) = \{\Gamma, \Gamma TS, \Gamma(TS)^2\}; w(\Gamma) = 1$ since $-I \in \Gamma_0(2)$. Finally, Γ has two cusps, $[\infty]_\Gamma$ and $[0]_\Gamma$. Hence (77) turns into the version (47) of Cor. 7.4.

Example 16. $\Gamma = \Gamma_1(5)$: To specify the elements γ_j of the coset decomposition $\mathrm{SL}_2(\mathbb{Z}) = \dot\bigcup_{j=1}^{24} \Gamma\gamma_j$ we use (a,b,c,d) instead of matrix notation $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$:

$$\begin{aligned}
&\gamma_1 := (1,0,0,1), \gamma_2 := (0,-1,1,0), \gamma_3 := (0,-1,1,1), \gamma_4 := (0,-1,1,2),\\
&\gamma_5 := (0,-1,1,3), \gamma_6 := (0,-1,1,4), \gamma_7 := (2,-1,5,-2), \gamma_8 := (-1,-2,-2,-5),\\
&\gamma_9 := (-1,-3,-2,-7), \gamma_{10} := (-1,-4,-2,-9), \gamma_{11} := (-1,-5,-2,-11),\\
&\gamma_{12} := (-1,-6,-2,-13), \gamma_{13} := (3,1,5,2), \gamma_{14} := (1,-3,2,-5),\\
&\gamma_{15} := (1,-2,2,-3), \gamma_{16} := (1,-1,2,-1), \gamma_{17} := (1,0,2,1), \gamma_{18} := (1,1,2,3),\\
&\gamma_{19} := (4,-1,5,-1), \gamma_{20} := (-1,-4,-1,-5), \gamma_{21} := (-1,-5,-1,-6),\\
&\gamma_{22} := (-1,-6,-1,-7), \gamma_{23} := (-1,-7,-1,-8), \gamma_{24} := (-1,-8,-1,-9).
\end{aligned}$$

The action of $\operatorname{Stab}(\omega)$ on the set $C := \{\Gamma\gamma_j : j = 1, \ldots, 24\}$ of cosets results in the disjoint orbit decomposition

$$C = \Gamma\gamma_1/\operatorname{Stab}(\omega) \dot\cup \Gamma\gamma_4/\operatorname{Stab}(\omega) \dot\cup \Gamma\gamma_5/\operatorname{Stab}(\omega) \dot\cup \Gamma\gamma_7/\operatorname{Stab}(\omega). \tag{80}$$

For each $j = 1,4,5,7$ one has $|\Gamma\gamma_j/\operatorname{Stab}(\omega)| = 6$; for instance,

$$\Gamma\gamma_1/\operatorname{Stab}(\omega) = \{\Gamma\gamma_1, \Gamma\gamma_2, \Gamma\gamma_3, \Gamma\gamma_{19}, \Gamma\gamma_{20}, \Gamma\gamma_{21}\}.$$

Hence each of the six elements of $\operatorname{Stab}(\omega)$ gives rise to a different element of $\Gamma\gamma_j/\operatorname{Stab}(\omega)$. This is due to the fact that $-I = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \notin \Gamma$; for example,

$$\Gamma\gamma_{19} = \Gamma \begin{pmatrix} 4 & -1 \\ 5 & -1 \end{pmatrix} = \Gamma \begin{pmatrix} -4 & 1 \\ -5 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \Gamma \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

As another consequence of $-I \notin \Gamma = \Gamma_1(5)$, in the "valence formula" (77) we have to set $w(\Gamma) := 2$.

Finally, owing to the bijection ϕ from (76) we know that the orbit $[\omega]_{\mathrm{SL}_2(\mathbb{Z})}$ splits into four different Γ-orbits with the γ_j as in (80); i.e.,

$$S_\Gamma(\omega) = \{[\gamma_1\omega]_\Gamma, [\gamma_4\omega]_\Gamma, [\gamma_5\omega]_\Gamma, [\gamma_7\omega]_\Gamma\}.$$

Proceeding along these lines one can establish the following "valence formula" for $\Gamma = \Gamma_1(5)$ as a consequence of Thm. 13.2:

Corollary 13.3 ("valence formula" for $\Gamma_1(5)$). *Let $g \in M(\Gamma)$. If $\Gamma = \Gamma_1(5)$ then*

$$\sum_{j\in\{1,4,5,7\}} \frac{6}{2\times 3} \operatorname{ord}_{\gamma_j\omega}(g) + \sum_{j\in\{1,3,4,5,7,9\}} \frac{4}{2\times 2} \operatorname{ord}_{\gamma_j i}(g) + \sum_{\substack{[a/c]_\Gamma \\ \text{cusp of } X(\Gamma)}} \operatorname{ord}^\Gamma_{a/c}(g)$$
$$+ \sum_{\substack{\tau\in H(\mathrm{SL}_2(\mathbb{Z})) \\ [\tau]\neq[i],[\tau]\neq[\omega]}} \sum_{j=1}^{24} \operatorname{ord}_{\gamma_j\tau}(g) = 0, \tag{81}$$

where $H(\mathrm{SL}_2(\mathbb{Z})) \subseteq \mathbb{H}$ *is a complete set of representatives of the orbits* $[\tau]_{\mathrm{SL}_2(\mathbb{Z})}$ *with* $\tau \in \mathbb{H}$, *and where* $\omega := e^{2\pi i/3}$.

14 Conclusion

The Rogers-Ramanujan functions are embedded in a rich web of beautiful mathematics. So there are much more stories to tell. For example, as discussed in [14], one can ask for which evaluations the Rogers-Ramanujan continued fraction $r(\tau)$ gives an algebraic number and if so, in which situations such values can be expressed in terms of radicals over $\mathbb{Q}$. Finally we mention the fact that the Rogers-Ramanujan continued fraction is playing a prominent role in Ramanujan's "Lost" Notebook; see the first five chapters of [6].

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References

1. S.A. Abramov, P. Paule, and M. Petkovsek, *q-Hypergeometric solutions of q-difference equations*, Discrete Math. **180** (1998), 3–22 (english).
2. S.A. Abramov and M. Petkovsek, *Finding all q-hypergeometric solutions of q-difference equations*, Proc. FPSAC95 (Noisy-le-Grand, 1995) (B. Leclerc and J.-Y. Thibon, eds.), Univ. de Marne-la-Vallée, 1995, pp. 1–10 (english).
3. George E. Andrews, *q-Series: Their Development and Application in Analysis, Number Theory, Combinatorics, Physics, and Computer Algebra*, CBMS Reg. Conf. Ser. Math., Vol. 66, AMS, 1986.
4. George E. Andrews, Richard Askey, and Ranjan Roy, *Special Functions*, Cambridge University Press, 1999.
5. George E. Andrews and Bruce C. Berndt, *Ramanujan's Lost Notebook Part I*, Springer, New York, 2005.
6. ______, *Ramanujan's Lost Notebook Part III*, Springer, 2012.
7. Bruce C. Berndt, *Generalized Dedekind eta-functions and generalized Dedekind sums*, Trans. Amer. Math. Soc. **178** (1973), 495–508.
8. Bruce C. Berndt, Heng Huat Chan, Sen-Shan Huang, Soon-Yi Kang, Jaebum Sohn, and Seung Hwan Son, *The Rogers–Ramanujan continued fraction*, J. Comput. Appl. Math. **105** (1999), no. 1, 9–24.
9. Douglas Bowman and James Mc Laughlin, *On the divergence of the Rogers-Ramanujan continued fraction on the unit circle*, Trans. Amer. Math. Soc. **356** (2003), 3325–3345.
10. Fred Diamond and Jerry Shurman, *A First Course in Modular Forms*, Springer, 2005.
11. Ulrich Dieter, *Das Verhalten der Kleinschen Funktionen* $\log_{g,h}(\omega_1,\omega_2)$ *gegenüber Modultransformationen und verallgemeinerte Dedekindsche Summen*, J. Reine Angew. Math. **201** (1959), 37–70.
12. *NIST Digital Library of Mathematical Functions*, http://dlmf.nist.gov/, Release 1.0.14 of 2016-12-21, F.W.J. Olver, A.B. Olde Daalhuis, D.W. Lozier, B.I. Schneider, R.F. Boisvert, C.W. Clark, B.R. Miller, and B.V. Saunders, editors.
13. William Duke, *Continued fractions and modular functions*, Bull. Amer. Math. Soc. **42** (2005), 137–162 (english).
14. Hershel M. Farkas and Irwin Kra, *Theta Constants, Riemann Surfaces and the Modular Group*, Grad. Stud. Math., vol. 37, AMS, 2001.
15. Frank Garvan, *A q-product tutorial for a q-series MAPLE package*, Sém. Lothar. Combin. **42** (1999), Art. B42d, 27 pages (electronic) (english).

16. Ralf Hemmecke, *Dancing samba with Ramanujan partition congruences*, J. Symbolic Comput. (14 pages, 2016. To appear.).
17. Manuel Kauers and Christoph Koutschan, *A Mathematica package for q-holonomic sequences and power series*, Ramanujan J. **19** (2009), no. 2, 137–150 (english).
18. Manuel Kauers and Peter Paule, *The concrete tetrahedron*, Texts Monogr. Symbol. Comput., Springer, 2011. MR 2768529
19. Felix Klein, *Lectures on the Icosahedral Equation and the Solutions of Equations of the Fifth Degree*, Dover, 2nd edition, 1956.
20. Joseph Lewittes, *Analytic continuation of Eisenstein series*, Trans. Amer. Math. Soc. **177** (1972), 469–490.
21. Martin Mazur and Bogdan V. Petrenko, *Representations of analytic functions as infinite products and their application to numerical computations*, Ramanujan J. **34** (2014), 129–141.
22. Henry McKean and Victor Moll, *Elliptic Curves: Function Theory, Geometry, Arithmetic*, Cambridge University Press, 1999.
23. Curt Meyer, *Über einige Anwendungen Dedekindscher Summen*, J. Reine Angew. Math. **198** (1957), 143–203.
24. Rick Miranda, *Algebraic Curves and Riemann Surfaces*, Grad. Stud. Math., vol. 5, AMS, 1995.
25. Ken Ono, *The Web of Modularity: Arithmetic of the Coefficients of Modular Forms and q-Series*, CBMS Reg. Conf. Ser. Math., Vol. 102, AMS, 2004.
26. P. Paule, *Short and easy computer proofs of the Rogers-Ramanujan identities and of identities of similar type*, Electron. J. Combin. **1** (1994), 1–9 (english).
27. P. Paule and A. Riese, *A Mathematica q-analogue of Zeilberger's algorithm based on an algebraically motivated approach to q-hypergeometric telescoping*, Special Functions, q-Series and Related Topics (M.E.H. Ismail and M. Rahman, eds.), Fields Inst. Commun., vol. 14, AMS, 1997, pp. 179–210 (english).
28. Peter Paule and Cristian-Silviu Radu, *Partition analysis, modular functions, and computer algebra*, Recent Trends in Combinatorics, IMA Vol. Math. Appl., Vol. 159 (A. Beveridge, J.R. Griggs, L. Hogben, G. Musiker, and P. Tetali, eds.), Springer, 2016, pp. 511–544.
29. ———, *A new witness identity for* $11|p(11n+6)$, Springer Proc. Math. Stat. (K. Alladi, G.E. Andrews, and F. Garvan, eds.), Springer, 2017, To appear (english).
30. Cristian-Silviu Radu, *An algorithmic approach to Ramanujan-Kolberg identities*, J. Symbolic Comput. **68** (2014), 1–33.
31. Sinai Robins, *Generalized Dedekind products*, Ser. Contemp. Math. **166** (1994), 119–128.
32. Bruno Schoeneberg, *Elliptic Modular Functions*, Springer, 1974.
33. William Stein, *Modular Forms, a Computational Approach*, AMS, 2007.
34. W. A. Stein et al., *Sage Mathematics Software (Version 8.0)*, The Sage Development Team, 2017, `http://www.sagemath.org`.
35. Gabor Toth, *Glimpses of Algebra and Geometry*, Springer, 2002.
36. Herbert S. Wilf and Doron Zeilberger, *An algorithmic proof theory for hypergeometric (ordinary and "q") multisum/integral identities*, Inventiones Math. **108** (1992), 575–633.
37. Yifan Yang, *Transformation formulas for generalized Dedekind eta functions*, Bull. London Math. Soc. **36** (2004), 671–682.
38. Don Zagier, *Ramanujan an Hardy: Vom ersten bis zum letzten Brief*, Mitt. Dtsch. Math.-Ver. **18** (2010), 21–28.