A Symbolic Decision Procedure for Relations Arising among Taylor Coefficients of Classical Jacobi Theta Functions

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Abstract

The class of objects we consider are algebraic relations between the four kinds of classical Jacobi theta functions $\theta_j(z|\tau)$, $j = 1, \ldots, 4$, and their derivatives. We present an algorithm to prove such relations automatically where the function argument z is zero, but where the parameter τ in the upper half complex plane is arbitrary.

Key words: Jacobi theta functions, modular forms, algorithmic zero-recognition, computer algebra, automatic proving of special function identities

1. Introduction

The overall objective of this paper is to provide tools for the computer-assisted treatment of identities among Jacobi theta functions. In the first step of development, this amounts to zero-recognition of Taylor coefficients of the respective series expansions of theta functions. To introduce the general idea and application domain of the method presented in this paper, consider the following lemma that has been used in numerous papers like Berndt et al. (1995), Hirschhorn et al. (1993) and Garvan (2010) to prove relations between Jacobi theta series.

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Lemma 1.1. (Atkin and Swinnerton-Dyer (1954)) Given a non-zero meromorphic function f on $\mathbb{C} \setminus \{0\}$ and $f(wx) \equiv$ [1] $cx^n f(x)$ for some integer n and non-zero complex constants c and w with $0 < |w| < 1$, then

$$\#\ \text{poles}(f) = \#\ \text{zeros}(f) + n$$

in $|w| < |x| \leq 1$.

To do zero recognition of such $f(x) = f(x, q)$, where q is a parameter, the lemma classically is applied as follows: one cleverly chooses sufficiently many zeros $x_1, \ldots, x_m$ in the domain $|w| < |x| \leq 1$. According to the lemma the number m of such zeros needs to be greater than the number of poles of f minus n, in order to show that f is identically zero. By their clever choice of $x_1, \ldots, x_m$, each $f(x_i, q)$ is a modular form when viewed as a function of q. And, zero-recognition of modular forms is algorithmical owing to methods using Sturm bounds or valence formula, e.g., Lemma 4.9 and Proposition 5.13.

Our approach is different and streamlines the idea above by choosing *only one* evaluation point, namely $x_i = 1$ for all i, and by verifying that $f^{(j)}(1, q) = 0$ for $j \in \{0, \ldots, m-1\}$. In this way we prove that there is a zero of multiplicity at least m, which again implies that $f(x) \equiv 0$.

For $j \geq 1$, the Taylor coefficients are not in general modular forms anymore. A crucial point is that, nevertheless, the task of proving relations like $f^{(j)}(1, q) = 0$ can again be carried out algorithmically for a large class of problems specified below. The functions that are the building blocks of this class are the Jacobi theta functions $\theta_j(z|\tau)$ ($j = 1, \ldots, 4$) and their derivatives evaluated at $z = 0$. The $\theta_j(z|\tau)$ are defined as follows.

Definition 1.2. DLMF (2016) Let $\tau \in \mathbb{H} := \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$ and $q = e^{\pi i \tau}$, then

$$\begin{aligned}
\theta_1(z|\tau) = \theta_1(z, q) &:= 2 \sum_{n=0}^{\infty} (-1)^n q^{(n+\frac{1}{2})^2} \sin((2n+1)z), \\
\theta_2(z|\tau) = \theta_2(z, q) &:= 2 \sum_{n=0}^{\infty} q^{(n+\frac{1}{2})^2} \cos((2n+1)z), \\
\theta_3(z|\tau) = \theta_3(z, q) &:= 1 + 2 \sum_{n=1}^{\infty} q^{n^2} \cos(2nz), \\
\theta_4(z|\tau) = \theta_4(z, q) &:= 1 + 2 \sum_{n=1}^{\infty} (-1)^n q^{n^2} \cos(2nz).
\end{aligned}$$

To exemplify our method of using Lemma 1.1, we consider the following classical example.

Example 1.3. DLMF (2016) For $q \in \mathbb{C}$ with $0 < |q| < 1$, prove

$$\theta_3(0, q)^2 \theta_3(z, q)^2 - \theta_4(0, q)^2 \theta_4(z, q)^2 - \theta_2(0, q)^2 \theta_2(z, q)^2 \equiv 0. \tag{1}$$

[1] We use the notation $f_1(z_1, z_2, \ldots) \equiv f_2(z_1, z_2, \ldots)$ if we want to emphasize that the equality between the functions holds for all possible choices of the arguments z_j.

Proof. Let $f_j(x) := \theta_j(z, q)$ with $x(z) = e^{2iz}$. Then using the series expansions in Definition 1.2 one can verify directly that $f_j^2(q^2x) = q^{-2}x^{-2}f_j^2(x)$. Define

$$g(x) := \theta_3(0, q)^2 f_3(x)^2 - \theta_4(0, q)^2 f_4(x)^2 - \theta_2(0, q)^2 f_2(x)^2.$$

Observing that $g(q^2x) = q^{-2}x^{-2}g(x)$, to prove the identity, by Lemma 1.1 it is sufficient to show that $g(x)$ has at least three more zeros than poles in $|q^2| < |x| \leq 1$. By Definition 1.2, $g(x)$ has no pole in $\mathbb{C}$. The Taylor expansion of $g(x)$ around $x = 1$ is

$$g(x) = g(1) + g'(1)(x-1) + \frac{g''(1)}{2}(x-1)^2 + \frac{g^{(3)}(1)}{6}(x-1)^3 + O((x-1)^4).$$

We need to show

$$g(1) = 0, \quad g'(1) = 0 \quad \text{and} \quad g''(1) = 0. \tag{2}$$

Let $h(z) :=$ LHS of (1). Because $h(z) = g(e^{2iz}) = g(x)$, $h'(z) = 2ixg'(x)$ and $h''(z) = -4xg'(x) - 4x^2g''(x)$, to show (2), it is sufficient to show

$$h(0) = \theta_3(0, q)^4 - \theta_2(0, q)^4 - \theta_4(0, q)^4 \equiv 0, \tag{3}$$

$$h'(0) = 2\theta_3(0, q)^3\theta_3'(0, q) - 2\theta_2(0, q)^3\theta_2'(0, q) - 2\theta_4(0, q)^3\theta_4'(0, q) \equiv 0, \tag{4}$$

$$\begin{aligned} \text{and} \quad h''(0) = &\theta_3(0, q)^2\theta_3'(0, q)^2 - \theta_2(0, q)^2\theta_2'(0, q)^2 - \theta_4(0, q)^2\theta_4'(0, q)^2 \\ &+ \theta_3(0, q)^3\theta_3''(0, q) - \theta_2(0, q)^3\theta_2''(0, q) - \theta_4(0, q)^3\theta_4''(0, q) \equiv 0. \end{aligned} \tag{5}$$

Note that identity (4) is trivial because $\theta_2'(0, q) \equiv \theta_3'(0, q) \equiv \theta_4'(0, q) \equiv 0$. The other two equalities will be treated below. In general, proving such identities can be done in a purely algorithmic fashion which will be explained in this paper.

2. Problem specification

As pointed out in the Introduction, this article deals with algorithmic zero recognition of relations arising among Taylor coefficients of theta functions. Throughout the paper $\mathbb{N} := \{0, 1, 2, \dots\}$, $\mathbb{H} := \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$ and $\mathbb{K} \subseteq \mathbb{C}$ is a field. We assume that $\mathbb{K}$ contains all the complex constants we need (i.e., i, $e^{\pi i/4}$, etc.). In algorithmic contexts, $\mathbb{K}$ is an effectively computable field. Throughout the paper for $z = ce^{i\varphi}$ ($c > 0$, $0 \leq \varphi < 2\pi$) we define $z^r := c^r e^{ir\varphi}$ for $r \in \frac{1}{2}\mathbb{Z}$.

For fixed $\tau \in \mathbb{H}$, Definition 1.2 implies that the $\theta_j(z|\tau)$ ($j = 1, \dots, 4$) are analytic functions on the whole complex plane with respect to z. For fixed $z \in \mathbb{C}$, the $\theta_j(z|\tau)$ ($j = 1, \dots, 4$) are analytic functions of τ for all $\tau \in \mathbb{H}$, and correspondingly, analytic functions of q for $|q| < 1$. When $z = 0$, we often denote

$$\theta_j^{(k)}(\tau) := \frac{\partial^k \theta_j}{\partial z^k}(z|\tau)\bigg|_{z=0} \left(= \frac{\partial^k \theta_j}{\partial z^k}(z, q)\bigg|_{z=0} \right), \; k \in \mathbb{N}.$$

Definition 1.2 implies that $\theta_1^{(k_1)}(\tau) \equiv 0$ when $k_1 \in 2\mathbb{N}$, and $\theta_m^{(k_2)}(\tau) \equiv 0$ ($m = 2, 3, 4$) when $k_2 \in 2\mathbb{N} + 1$. Hence in the following setting we omit these cases.

Let $\{x_{j,k}\}_{k \in \mathbb{N}, j=1,\dots,4}$ be a set of indeterminates. For convenience, we denote $x_j^{(k)} := x_{j,k}$. Sometimes we write x_j for $x_j^{(0)}$ and x_j' for $x_j^{(1)}$. Define $R_\Theta := \mathbb{K}[\Theta]$ where

$$\Theta := \left\{\theta_1^{(2k+1)} : k \in \mathbb{N}\right\} \cup \left\{\theta_j^{(2k)} : k \in \mathbb{N} \text{ and } j = 2, 3, 4\right\},$$

and $R_X := \mathbb{K}[X]$ where

$$X := \left\{x_1^{(2k+1)} : k \in \mathbb{N}\right\} \cup \left\{x_j^{(2k)} : k \in \mathbb{N} \text{ and } j = 2, 3, 4\right\}.$$

By homomorphic extension we define the $\mathbb{K}$-algebra homomorphism [2]

$$\begin{aligned} \phi : \quad & R_X \to R_\Theta, \\ & x_j^{(k)} \mapsto \theta_j^{(k)}. \end{aligned}$$

In this paper, we solve the following membership problem algorithmically:

Problem: Given $p \in R_X$, decide whether $p \in \ker\phi$.

To solve this problem, we need to extend the $\mathbb{K}$-algebras and the map ϕ as follows:

$$\begin{aligned} \phi^* : \quad & R_X[s^{\frac{1}{2}}] \to R_\Theta[\delta^{\frac{1}{2}}], \\ & x_j^{(k)} \mapsto \theta_j^{(k)}, \\ & s^{\frac{1}{2}} \mapsto \delta^{\frac{1}{2}}, \end{aligned}$$

where for all $\tau \in \mathbb{H}$ and $r \in \frac{1}{2}\mathbb{N}$, $\delta^r(\tau) := \tau^r$. Since ϕ and ϕ^* are surjective, we have $R_X/\ker\phi \cong R_\Theta$ and $R_X[s^{\frac{1}{2}}]/\ker\phi^* \cong R_\Theta[\delta^{\frac{1}{2}}]$. Here we consider $s^{\frac{1}{2}}$ as a symbol for an indeterminate. We prefer to use $s^{\frac{1}{2}}$ instead of choosing a standard indeterminate like x or y as usual for polynomial rings.

The paper is structured as follows. In Section 3, we introduce a notion of degree in the $\mathbb{K}$-algebra R_X, and based on this we state a way to decompose any $p \in R_X$ into homogeneous polynomials in R_X. We prove that to show $p \in \ker\phi$ is equivalent to showing that the corresponding homogeneous polynomials are in $\ker\phi$. In Section 4 we develop a recursive algorithm to determine for a given homogeneous $g \in R_X$ whether $g \in \ker\phi$ or $g \notin \ker\phi$. In Section 5 we obtain a refined non-recursive algorithm which is more convenient to implement and with linear computational complexity in the length of g.

3. Decomposition of $p \in R_X$

Lemma 3.1. (Serre, 1973, p. 78, Thm. 2) Let

$$\mathrm{SL}_2(\mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc = 1 \text{ and } a, b, c, d \in \mathbb{Z} \right\}.$$

Then $\mathrm{SL}_2(\mathbb{Z})$ is generated by $S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

[2] Here a $\mathbb{K}$-algebra homomorphism is a ring homomorphism and a $\mathbb{K}$-vector space homomorphism.

Definition 3.2. For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, $k \in \mathbb{Z}$ and $f : \mathbb{H} \to \mathbb{C}$, we define $f|_k\gamma : \mathbb{H} \to \mathbb{C}$ by

$$(f|_k\gamma)(\tau) := (c\tau + d)^{-k} f\left(\frac{a\tau + b}{c\tau + d}\right).$$

For instance, for the generators S and T, we have

$$(f|_k S)(\tau) \equiv \tau^{-k} f(-1/\tau) \text{ and } (f|_k T)(\tau) \equiv f(\tau + 1).$$

Note. This action $f|_k\gamma$ of γ on f (for fixed k) is a group action. Hence knowing the action of generators (here S and T acting on the function space) gives the full action.

Let us consider the actions on theta functions when $\tau \mapsto -1/\tau$ and $\tau \mapsto \tau + 1$. We first look at the case $\tau \mapsto -1/\tau$.

Lemma 3.3. (Whittaker and Watson, 1965, p. 475) For the action of S on $\theta_j(z|\tau)$ $(j = 1, \ldots, 4)$ we have

$$\theta_1\left(z\Big| -\frac{1}{\tau}\right) \equiv -i(-i\tau)^{\frac{1}{2}} e^{\frac{i\tau z^2}{\pi}} \theta_1(z\tau|\tau); \quad \theta_2\left(z\Big| -\frac{1}{\tau}\right) \equiv (-i\tau)^{\frac{1}{2}} e^{\frac{i\tau z^2}{\pi}} \theta_4(z\tau|\tau);$$
$$\theta_3\left(z\Big| -\frac{1}{\tau}\right) \equiv (-i\tau)^{\frac{1}{2}} e^{\frac{i\tau z^2}{\pi}} \theta_3(z\tau|\tau); \quad \theta_4\left(z\Big| -\frac{1}{\tau}\right) \equiv (-i\tau)^{\frac{1}{2}} e^{\frac{i\tau z^2}{\pi}} \theta_2(z\tau|\tau).$$

We extend Lemma 3.3 to derivatives.

Proposition 3.4. Define $A := (-i\tau)^{\frac{1}{2}}$ and $E := e^{\frac{i\tau z^2}{\pi}}$. For $(u, v) \in \{(1,1), (2,4), (3,3), (4,2)\}$ and $k \in \mathbb{N}$ define

$$g_u(k) := (E\,A)^{-1} \frac{\partial^k \theta_v}{\partial z^k}\left(z\Big| -\frac{1}{\tau}\right).$$

Then $g_u(k)$ can be written as

$$g_u(k) = p_{k,0}(z)\theta_u(z\tau|\tau) + p_{k,1}(z)\theta_u'(z\tau|\tau) + \cdots + p_{k,k}(z)\theta_u^{(k)}(z\tau|\tau) \tag{$*$}$$

with $p_{k,j}(z) = \frac{k!}{j!}\left(\frac{i}{\pi}\right)^{\frac{k-j}{2}} \tau^{\frac{k+j}{2}} B_{k,j}(z)$ and

$$B_{k,j}(z) = \begin{cases} a_0(k,j) + a_2(k,j)z^2 + a_4(k,j)z^4 + \cdots + a_{k-j}(k,j)z^{k-j}, & k-j \text{ even;} \\ a_1(k,j)z + a_3(k,j)z^3 + a_5(k,j)z^5 + \cdots + a_{k-j}(k,j)z^{k-j}, & k-j \text{ odd,} \end{cases}$$

where for $\ell \in \{0, \ldots, k-j\}$ when $(u, v) = (1, 1)$,

$$a_\ell(k,j) = -i\left(\frac{i\tau}{\pi}\right)^{\frac{\ell}{2}} \frac{2^\ell}{\ell!(\frac{k-j-\ell}{2})!}; \tag{$**$}$$

and when $(u, v) \in \{(2,4), (3,3), (4,2)\}$,

$$a_\ell(k,j) = \left(\frac{i\tau}{\pi}\right)^{\frac{\ell}{2}} \frac{2^\ell}{\ell!(\frac{k-j-\ell}{2})!}.$$

Proof. We prove the statement for $(u, v) = (1, 1)$. The other three cases are analogous. We first prove by complete induction on k that for $k \in \mathbb{N}$ the relation $(*)$ holds where the $p_{k,j}(z)$ $(0 \le j \le k)$ are polynomials in z. Then we prove that $B_{k,j}(z)$ has the desired form.

For $k = 0$ we have $p_{0,0}(z) = -i$ by Lemma 1.1. Assume that $(*)$ holds for $k = n$ where the $p_{n,j}(z)$ $(0 \le j \le n)$ are polynomials in z.

Let $k = n + 1$. We have

$$g_1(n+1) = (E\,A)^{-1}\frac{\partial^{n+1}\theta_1}{\partial z^{n+1}}\left(z\Big| -\frac{1}{\tau}\right) = \frac{2i\tau z}{\pi}g_1(n) + \frac{\partial g_1(n)}{\partial z}.$$

Since $\frac{\partial g_1(n)}{\partial z} = p'_{n,0}(z)\theta_1(z\tau|\tau) + (\tau p_{n,0}(z) + p'_{n,1}(z))\theta'_1(z\tau|\tau) + \cdots + \tau p_{n,n}\theta_1^{(n+1)}(z\tau|\tau)$, we obtain $g_1(n+1) = p_{n+1,0}(z)\theta_1(z\tau|\tau) + p_{n+1,1}(z)\theta'_1(z\tau|\tau) + \cdots + p_{n+1,n}(z)\theta_1^{(n+1)}(z\tau|\tau)$, where the $p_{n+1,j}(z)$ $(j = 0, \ldots, n+1)$ are polynomials in z.

Using the fact just proven we can exploit a recursive relation for $g_1(k)$ in the following way. On one hand, by

$$E\,A\,g_1(k+1) = \frac{\partial^{k+1}\theta_1}{\partial z^{k+1}}\left(z\Big| -\frac{1}{\tau}\right) = \frac{\partial(E\,A\,g_1(k))}{\partial z},$$

we obtain

$$\begin{aligned} g_1(k+1) =& \frac{2iz\tau}{\pi}g_1(k) + \frac{\partial g_1(k)}{\partial z} \\ =& \frac{2iz\tau}{\pi}\sum_{j=0}^{k} p_{k,j}(z)\theta_1^{(j)}(z\tau|\tau) + \sum_{j=1}^{k}\left(\frac{\partial p_{k,j}(z)}{\partial z} + \tau p_{k,j-1}(z)\right)\theta_1^{(j)}(z\tau|\tau) \\ &+ \tau p_{k,k}(z)\theta_1^{(k+1)}(z\tau|\tau) + \frac{\partial p_{k,0}(z)}{\partial z}\theta_1(z\tau|\tau). \end{aligned}$$

On the other hand,

$$g_1(k+1) = \sum_{j=0}^{k+1} p_{k+1,j}(z)\theta_1^{(j)}(z\tau|\tau),$$

and by coefficient comparison, and defining $p_{k,-1}(z) := 0$ and $p_{k,k+1}(z) := 0$ we obtain,

$$p_{k+1,j}(z) = \frac{2iz\tau}{\pi}p_{k,j}(z) + \frac{\partial p_{k,j}(z)}{\partial z} + \tau p_{k,j-1}(z), \quad 0 \le j \le k. \tag{6}$$

Now we can prove that $B_{k,j}(z)$ has the desired form by induction on $k \in \mathbb{N}$. By definition we know $B_{0,0}(z) = -i$. Assume for $k = n$, $B_{k,j}(z)$ has the desired form. Let $k = n + 1$. Applying (6) we have for $j = 0, \ldots, n+1$,

$$(n+1)B_{n+1,j}(z) = 2z\left(\frac{i\tau}{\pi}\right)^{\frac{1}{2}} B_{n,j}(z) + \left(\frac{i\tau}{\pi}\right)^{-\frac{1}{2}}\frac{\partial B_{n,j}(z)}{\partial z} + jB_{n,j-1}(z).$$

Case 1: $n - j$ is even. Then

$$\begin{aligned} (n+1)B_{n+1,j}(z) =& 2\left(\frac{i\tau}{\pi}\right)^{\frac{1}{2}}(a_0(n,j)z + a_2(n,j)z^3 + \cdots + a_{n-j}(n,j)z^{n-j+1}) \\ &+ \left(\frac{i\tau}{\pi}\right)^{-\frac{1}{2}}(2a_2(n,j)z + 4a_4(n,j)z^3 + \cdots + (n-j)a_{n-j}z^{n-j-1}) \\ &+ j(a_1(n,j-1)z + a_3(n,j-1)z^3 + \cdots + a_{n-j+1}(n,j-1)z^{n-j+1}). \end{aligned}$$

We compare this to

$B_{n+1,j}(z) = a_1(n+1,j)z + a_3(n+1,j)z^3 + a_5(n+1,j)z^5 + \cdots + a_{n-j+1}(n+1,j)z^{n-j+1}$,

and obtain for $1 \le 2m+1 \le n-j-1$,

$$\begin{aligned}
a_{2m+1}(n+1,j) =& \frac{1}{n+1}\left(2\left(\frac{i\tau}{\pi}\right)^{\frac{1}{2}} a_{2m}(n,j) + \left(\frac{i\tau}{\pi}\right)^{-\frac{1}{2}} (2m+2)a_{2m+2}(n,j) + ja_{2m+1}(n,j-1)\right)\\
=& \frac{-i}{n+1}\left(\left(\frac{i\tau}{\pi}\right)^{m+\frac{1}{2}} \frac{2^{2m+1}}{(2m)!(\frac{n-j}{2}-m)!} + \left(\frac{i\tau}{\pi}\right)^{m+\frac{1}{2}} \frac{2^{2m+2}}{(2m+1)!(\frac{n-j}{2}-m-1)!}\right.\\
&\left. + j\left(\frac{i\tau}{\pi}\right)^{m+\frac{1}{2}} \frac{2^{2m+1}}{(2m+1)!(\frac{n-j}{2}-m)!}\right)\\
=&(-i)\left(\frac{i\tau}{\pi}\right)^{m+\frac{1}{2}} \frac{2^{2m+1}}{(2m+1)!(\frac{n-j}{2}-m)!};
\end{aligned}$$

and for $2m+1 = n-j+1$,

$$\begin{aligned}
a_{2m}(n+1,j) =& \frac{1}{n+1}\left(2\left(\frac{i\tau}{\pi}\right)^{\frac{1}{2}} a_{2m}(n,j) + ja_{2m+1}(n,j-1)\right)\\
=& \frac{-i}{n+1}\left(\frac{i\tau}{\pi}\right)^{m+\frac{1}{2}} \frac{2^{2m+1}}{(2m)!(\frac{n-j}{2}-m)!}\left(1+\frac{j}{2m+1}\right)\\
=&(-i)\left(\frac{i\tau}{\pi}\right)^{m+\frac{1}{2}} \frac{2^{2m+1}}{(2m+1)!(\frac{n-j}{2}-m)!}.
\end{aligned}$$

Thus for $1 \le 2m+1 \le n-j+1$, $a_{2m+1}(n+1,j)$ satisfies $(**)$.

Case 2: $n-t$ is odd. This case can be treated in the same way as Case 1, and the computation shows that for $0 \le 2m \le n-j+1$, $a_{2m}(n+1,j)$ satisfies $(**)$. Thus we have proven that $B_{n+1,j}(z)$ has the desired form. □

Applying Proposition 1 with $z = 0$, we have:

Corollary 3.5. For $k \in 2\mathbb{N}+1$,

$$\theta_1^{(k)}\left(-\frac{1}{\tau}\right) = (-i)^{\frac{3}{2}} \sum_{\substack{j=1\\ j\in 2\mathbb{N}+1}}^{k} \left(\frac{i}{\pi}\right)^{\frac{k-j}{2}} \frac{k!}{j!(\frac{k-j}{2})!}\tau^{\frac{k+j+1}{2}}\theta_1^{(j)}(\tau);$$

for $k \in 2\mathbb{N}$ and $(u,v) \in \{(2,4),(3,3),(4,2)\}$,

$$\theta_u^{(k)}\left(-\frac{1}{\tau}\right) = (-i)^{\frac{1}{2}} \sum_{\substack{j=0\\ j\in 2\mathbb{N}}}^{k} \left(\frac{i}{\pi}\right)^{\frac{k-j}{2}} \frac{k!}{j!(\frac{k-j}{2})!}\tau^{\frac{k+j+1}{2}}\theta_v^{(j)}(\tau).$$

We carry these analytic relations over to the symbolic algebra.

Definition 3.6. We define two $\mathbb{K}$-algebra homomorphisms:

$$S_0 : R_\Theta \to R_\Theta[\delta^{\frac{1}{2}}]$$

by

$$(S_0 f)(\tau) :\equiv (f|_0 S)(\tau) \quad \left(\equiv f\left(-\frac{1}{\tau}\right)\right);$$

and

$$S_X : \quad R_X \longrightarrow R_X[s^{\frac{1}{2}}]$$

by the homomorphic extension of

$$S_X(x_1^{(k)}) := (-i)^{\frac{3}{2}} \sum_{\substack{j=1\\ j\in 2\mathbb{N}+1}}^{k} \left(\frac{i}{\pi}\right)^{\frac{k-j}{2}} \frac{k!}{j!(\frac{k-j}{2})!} s^{\frac{k+j+1}{2}} x_1^{(j)},$$

if $k \in 2\mathbb{N}+1$; and of

$$S_X(x_u^{(k)}) := (-i)^{\frac{1}{2}} \sum_{\substack{j=0\\ j\in 2\mathbb{N}}}^{k} \left(\frac{i}{\pi}\right)^{\frac{k-j}{2}} \frac{k!}{j!(\frac{k-j}{2})!} s^{\frac{k+j+1}{2}} x_{\sigma(u)}^{(j)},$$

if $k \in 2\mathbb{N}$ and $u \in \{2,3,4\}$, where σ is the permutation on $\{1,2,3,4\}$ that transposes 2 and 4.

Lemma 3.7. The following diagram commutes:

$$\begin{array}{ccc} R_X & \xrightarrow{S_X} & R_X[s^{\frac{1}{2}}] \\ \phi\downarrow & & \downarrow\phi^* \\ R_\Theta & \underset{S_0}{\longrightarrow} & R_\Theta[\delta^{\frac{1}{2}}] \end{array}$$

Proof. The way S_X was introduced in Definition 3.6 as a homomorphic extension satisfies exactly the required property. $\square$

By Definition 3.6 we know the explicit form of $S_X(p)$ for any $p \in R_X$, and can set up the following convention.

Convention. Whenever for a non-zero $p \in R_X$ we write

$$S_X(p) = \sum_{j=1}^{n} s^{c_j} p_j,$$

we assume that

$$p_j \in R_X \setminus \{0\} \quad \text{and} \quad c_1 < \cdots < c_n \text{ with } c_j \in \frac{1}{2}\mathbb{N}.$$

For $c \in \frac{1}{2}\mathbb{N}$ the notation $\langle s^c \rangle q$ refers to the coefficient of s^c in $q \in R_X[s^{\frac{1}{2}}]$.

Example 3.8. Let $p = x_2^{(4)} x_4''$. Then

$$S_X(p) = p_4 s^7 + p_3 s^6 + p_2 s^5 + p_1 s^4,$$

where $p_4 := -i x_2'' x^{(4)}$, $p_3 := \frac{2}{\pi} x_2 x^{(4)} + \frac{12}{\pi} x_2'' x_4''$, $p_2 := \frac{12i}{\pi^2} x_4 x_2'' + \frac{24i}{\pi^2} x_2 x_4''$ and $p_1 := \frac{24}{\pi^3} x_2 x_4$.

Now we consider the action when $\tau \mapsto \tau + 1$. One can derive from Definition 1.2 the following lemma.

Lemma 3.9. For the action of T on $\theta_j^{(k)}(\tau)$ $(j = 1, \ldots, 4)$ we have for $k \in \mathbb{N}$,

$$\theta_1^{(k)}(\tau+1) \equiv e^{\frac{\pi i}{4}} \theta_1^{(k)}(\tau); \quad \theta_2^{(k)}(\tau+1) \equiv e^{\frac{\pi i}{4}} \theta_2^{(k)}(\tau);$$
$$\theta_3^{(k)}(\tau+1) \equiv \theta_4^{(k)}(\tau); \quad \theta_4^{(k)}(\tau+1) \equiv \theta_3^{(k)}(\tau).$$

Again, we carry these relations over to the algebraic side.

Definition 3.10. For $k \in \mathbb{N}$ we define two $\mathbb{K}$-algebra homomorphisms

$$T_0 : R_\Theta \to R_\Theta$$

by

$$(T_0 f)(\tau) :\equiv (f|_0 T)(\tau) \quad (\equiv f(\tau+1));$$

and

$$T_X : \quad R_X \longrightarrow R_X,$$

by the homomorphic extension of

$$T_X(x_1^{(2k+1)}) := e^{\frac{\pi i}{4}} x_1^{(2k+1)}, \quad T_X(x_2^{(2k)}) := e^{\frac{\pi i}{4}} x_2^{(2k)},$$
$$T_X(x_3^{(2k)}) := x_4^{(2k)} \quad \text{and} \quad T_X(x_4^{(2k)}) := x_3^{(2k)}.$$

Analogous to Lemma 3.7 we have:

Lemma 3.11. The following diagram commutes:

$$\begin{array}{ccc} R_X & \xrightarrow{T_X} & R_X \\ \phi\downarrow & & \downarrow\phi \\ R_\Theta & \xrightarrow[T_0]{} & R_\Theta \end{array}$$

Proof. By Lemma 3.9 and Definition 3.10 we have
$\phi(T_X(x_1^{(k)}))(\tau) \equiv \phi(e^{\frac{\pi i}{4}} x_1^{(k)})(\tau) \equiv e^{\frac{\pi i}{4}} \theta_1^{(k)}(\tau) \equiv \theta_1^{(k)}(\tau+1) \equiv \phi(x_1^{(k)})(\tau+1) \equiv (T_0 \phi(x_1^{(k)}))(\tau).$
Analogously we have $\phi(T_X(x_j^{(k)}))(\tau) \equiv \phi(x_j^{(k)})(\tau+1)$ for $j = 2, 3, 4$. The rest follows from the fact that T_X is defined by homomorphic extension. □

Example 3.12. Let $p = x_2^{(4)} x_4''$. Then

$$T_X(p) = e^{\frac{\pi i}{4}} x_2^{(4)} x_3''.$$

Note. Obviously, $T_X^8 = id$.

A non-trivial monomial in R_X is a finite product of elements in $\{x_j^{(k)} : k \in \mathbb{N}, j = 1, \ldots, 4\}$. The empty product gives $1 \in R_X$; it is considered to be the trivial monomial. Hence a polynomial in R_X is a $\mathbb{K}$-linear combination of monomials in R_X.

Definition 3.13. We define the degree of a non-trivial monomial $x_{j_1}^{(k_1)}x_{j_2}^{(k_2)}\cdots x_{j_n}^{(k_n)} \in R_X$ where $k_i \in \mathbb{N}$ and $j_i \in \{1,\dots,4\}$ by

$$\mathrm{Deg}(x_{j_1}^{(k_1)}x_{j_2}^{(k_2)}\cdots x_{j_n}^{(k_n)}) := \frac{n}{2} + \sum_{i=1}^{n} k_i,$$

and define the degree of the trivial monomial by $\mathrm{Deg}(1) := 0$. For every polynomial $p \in R_X$, define $\mathrm{Deg}(p)$:= highest degree of the monomials in its $\mathbb{K}$-linear representation. If all these monomials have the same degree, we say this polynomial is a homogeneous polynomial.

Example 3.14. $\mathrm{Deg}(-x_1^{(3)}) = \frac{7}{2}$, $\mathrm{Deg}(2x_1^{(3)}x_4) = 4$, and $2x_1^{(3)}x_4 - 3x_4^{(2)}x_1'$ is a homogeneous polynomial.

Note. This definition is related to the weight of modular forms. See Definition 4.5 and Lemma 4.6.

According to Definition 3.13, we can write a polynomial $p \in R_X$ into a sum of homogeneous polynomials with pairwise different degrees. As mentioned in the Introduction, we are going to show that $p \in R_X$ is in $\ker\phi$ if and only if these homogeneous parts are all in $\ker\phi$. The key tool we use here is the S_X operation. We shall start by studying the S_X patterns on monomials of R_X.

Lemma 3.15. Let $p \in R_X$ be a non-trivial monomial and $S_X(p) = \sum_{t=1}^{n} s^{c_t}p_t$. Then the p_t are homogeneous and

$$\mathrm{Deg}\left(\langle s^{c_t}\rangle S_X(p)\right) = \mathrm{Deg}(p_t) = 2c_t - \mathrm{Deg}(p), \quad 1 \le t \le n.$$

Moreover, we have

$$c_n = \mathrm{Deg}(p)$$

and, if $p = x_{i_1}^{(k_1)}x_{i_2}^{(k_2)}\cdots x_{i_m}^{(k_m)}$:

$$\langle s^{c_n}\rangle S_X(p) = \langle s^{\mathrm{Deg}(p)}\rangle S_X(p) = (-i)^{\frac{m}{2}+c}x_{\sigma(i_1)}^{(k_1)}x_{\sigma(i_2)}^{(k_2)}\cdots x_{\sigma(i_m)}^{(k_m)}$$

where c is the number of 1s in $(i_1, i_2, \dots, i_m)$ and σ is the permutation on $\{1,2,3,4\}$ that transposes 2 and 4.

Proof. Suppose $p = x_{i_1}^{(k_1)}x_{i_2}^{(k_2)}\cdots x_{i_m}^{(k_m)}$ with $x_{i_1} = x_{i_2} = \cdots = x_{i_c} = x_1$ and $x_{i_j} \neq x_1$ for $c+1 \le j \le m$. Then, writing $\square$ for coefficients in $\mathbb{K}$ whose exact values are irrelevant for the proof, we have

$$\begin{aligned}
S_X(p) =& S_X(x_{i_1}^{(k_1)})S_X(x_{i_2}^{(k_2)})\cdots S_X(x_{i_m}^{(k_m)})\\
=& \left((-i)^{\frac{3}{2}}x_1^{(k_1)}s^{k_1+\frac{1}{2}} + \square x_1^{(k_1-2)}s^{k_1-\frac{1}{2}} + \cdots + \square x_1' s^{\frac{k_1}{2}+1}\right)\\
&\cdots\\
&\left((-i)^{\frac{1}{2}}x_{\sigma(i_{c+1})}^{(k_{c+1})}s^{k_{c+1}+\frac{1}{2}} + \square x_{\sigma(i_{c+1})}^{(k_{c+1}-2)}s^{k_{c+1}-\frac{1}{2}} + \cdots + \square x_{\sigma(i_{c+1})}s^{\frac{k_{c+1}}{2}+\frac{1}{2}}\right)\\
&\cdots\\
&\left((-i)^{\frac{1}{2}}x_{\sigma(i_m)}^{(k_m)}s^{k_m+\frac{1}{2}} + \square x_{\sigma(i_m)}^{(k_m-2)}s^{k_m-\frac{1}{2}} + \cdots + \square x_{\sigma(i_m)}s^{\frac{k_m}{2}+\frac{1}{2}}\right).
\end{aligned}$$

Hence

$$\langle s^{\mathrm{Deg}(p)}\rangle S_X(p) = \langle s^{c_n}\rangle S_X(p) = (-i)^{\frac{m}{2}+c} x^{(k_1)}_{\sigma(i_1)} x^{(k_2)}_{\sigma(i_2)} \cdots x^{(k_m)}_{\sigma(i_m)}$$

and

$$c_n = \sum_{j=1}^{m}(k_j + \frac{1}{2}) = \frac{m}{2} + \sum_{j=1}^{m} k_j = \mathrm{Deg}(p).$$

In the expansion of $S_X(p)$ each monomial has the form

$$\begin{aligned}\prod_{j=1}^{m} x^{(k_j-2a_j)}_{\sigma(i_j)} s^{k_j+\frac{1}{2}-a_j} =& s^{\sum\limits_{j=1}^{m} k_j + \frac{m}{2} - \sum\limits_{j=1}^{m} a_j} \prod_{j=1}^{m} x^{(k_j-2a_j)}_{\sigma(i_j)} \\ =& s^{\mathrm{Deg}(p)-a} \prod_{j=1}^{m} x^{(k_j-2a_j)}_{\sigma(i_j)},\end{aligned}$$

where the a_j are integers with $0 \le a_j \le \frac{k_j-1}{2}$ for $1 \le j \le c$ and $0 \le a_j \le \frac{k_j}{2}$ for $c+1 \le j \le m$ and $a := \sum\limits_{j=1}^{m} a_j$. Thus

$$\mathrm{Deg}\left(\langle s^{\mathrm{Deg}(p)-a}\rangle S_X(p)\right) = \mathrm{Deg}\left(\prod_{j=1}^{m} x^{(k_j-2a_j)}_{i_j}\right) = \frac{m}{2} + \sum_{j=1}^{m}(k_j - 2a_j) = \mathrm{Deg}(p) - 2a.$$

Substituting $\mathrm{Deg}(p) - a$ by c_t we obtain

$$\mathrm{Deg}\left(\langle s^{c_t}\rangle S_X(p)\right) = 2c_t - \mathrm{Deg}(p), \quad 1 \le t \le n.$$

□

For convenience we have:

Definition 3.16. For monomials $p = x^{(k_1)}_{i_1} \dots x^{(k_m)}_{i_m} \in R_X$ we define

$$\begin{aligned}\mu(p) :&= m;\\ \nu_1(p) :&= \text{number of 1s in } (i_1, \dots, i_m),\\ \nu_2(p) :&= \text{number of 2s in } (i_1, \dots, i_m), \text{ and}\\ \sigma(p) :&= x^{(k_1)}_{\sigma(i_1)} \dots x^{(k_m)}_{\sigma(i_m)},\end{aligned}$$

where $\sigma(i_j)$ is defined to be the permutation on $\{1, 2, 3, 4\}$ that transposes 2 and 4..

Now we study the S_X operator on homogeneous polynomials.

Corollary 3.17. Let $p \in R_X$ be homogeneous. Then $S_X(p) = 0$ if and only if $p = 0$.

Proof. " $\Longleftarrow$ " is obvious. So we prove " $\Longrightarrow$ ". Assume $0 \neq p = a_1p_1 + \dots + a_np_n$ with the $p_j \in R_X$ linearly independent monomials over $\mathbb{K} \setminus \{0\}$ with the same degree and the $a_j \in \mathbb{K} \setminus \{0\}$. Then the $\sigma(p_j)$ are also linearly independent monomials over $\mathbb{K} \setminus \{0\}$

because the involution σ is an automorphism on R_X, and

$$\begin{aligned}\langle s^{\mathrm{Deg}(p)}\rangle S_X(p) &= \langle s^{\mathrm{Deg}(p)}\rangle (a_1 S_X(p_1) + \cdots + a_n S_X(p_n)) \\ &= a_1 \langle s^{\mathrm{Deg}(p)}\rangle S_X(p_1) + \cdots + a_n \langle s^{\mathrm{Deg}(p)}\rangle S_X(p_n) \\ &= a_1 (-i)^{\nu_1(p_1) + \frac{\mu(p_1)}{2}} \sigma(p_1) + \cdots + a_n (-i)^{\nu_1(p_n) + \frac{\mu(p_n)}{2}} \sigma(p_n).\end{aligned}$$

Since the $(-i)^{\nu_1(p_j) + \frac{\mu(p_j)}{2}}$ are non-zero, we obtain $\langle s^{\mathrm{Deg}(p)}\rangle S_X(p) \neq 0$. Therefore $S_X(p) \neq 0$. $\square$

Lemma 3.18. Given $p \in R_X$ homogeneous, and $S_X(p) = \sum_{t=1}^{n} s^{c_t} p_t$ with $p_t \in R_X$ and $c_t \in \frac{1}{2}\mathbb{N}$ such that $c_1 < \cdots < c_n$. Then
(i) $\mathrm{Deg}(p_n) = \mathrm{Deg}(p) = c_n$;
(ii) for $t \in \{1, \ldots, n\}$ the p_t are homogeneous ;
(iii) for $i, j \in \{1, \ldots, n\}$ with $i < j$ we have $\mathrm{Deg}(p_i) < \mathrm{Deg}(p_j)$.

Proof. Suppose

$$p = r_1 h_1 + \cdots + r_q h_q$$

with $r_\ell \in \mathbb{K} \setminus \{0\}$ and pairwise different monomials $h_\ell \in R_X$. By assumption on p we have $\mathrm{Deg}(h_\ell) = \mathrm{Deg}(p) =: d$ for all $\ell \in \{1, \ldots, q\}$. Suppose for $1 \leq \ell \leq q$:

$$S_X(h_\ell) = \sum_{t=1}^{n_\ell} s^{c_{\ell,t}} p_{\ell,t}$$

where the $c_{\ell,t}$ and $p_{\ell,t}$ are as in Lemma 3.15.

Then

$$S_X(p) = \sum_{\ell=1}^{q} r_\ell S_X(h_\ell) = \sum_{j=1}^{n} s^{c_j} \sum_{(\ell,t) \in C_j} r_\ell p_{\ell,t}$$

where $\{c_1, \ldots, c_n\} = \{c_{\ell,t} : 1 \leq \ell \leq q,\ 1 \leq t \leq n_\ell\}$ with the ordering $c_1 < c_2 < \cdots < c_n$, and

$$C_j := \{(\ell, t) \in \{1, \ldots, q\} \times \{1, \ldots, n_\ell\} : c_{\ell,t} = c_j\}.$$

Now the statements follow from observing that for $(\ell, t) \in C_j$ by Lemma 3.15:

$$\mathrm{Deg}(p_{\ell,t}) = 2c_{\ell,t} - d = 2c_j - d,$$

and for $(\ell, t) \in C_n$ (i.e., $t = n_\ell$) again by Lemma 3.15:

$$\mathrm{Deg}(p_{\ell,t}) = \mathrm{Deg}(p_{\ell,n_\ell}) = c_{\ell,n_\ell} = \mathrm{Deg}(h_\ell) = d = c_n.$$

$\square$

Remark. Note that Lemma 3.18 actually justifies the definition of Deg and also the Convention we introduced after Definition 3.6. Namely, the highest power of s in $S_X(p)$ is $\mathrm{Deg}(p)$.

Definition 3.19. For each $q \in R_X[s^{\frac{1}{2}}]$ with $q = \sum_{t=1}^{n} s^{c_t} p_t$, using the Convention, we call p_n the leading coefficient of q, denoted by $\mathrm{lc}(q)$. We define $\mathrm{lc}(0) := 0$.

Definition 3.20. Let $R_X^d := \{p \in R_X : p \text{ homogeneous with } \mathrm{Deg}(p) = d\} \cup \{0\}$. We define the map

$$\widetilde{S}: \quad R_X^d \longrightarrow R_X^d,$$

by $\widetilde{S}(0) := 0$ and if $p \neq 0$:

$$\widetilde{S}(p) := \mathrm{lc}(S_X(p)).$$

Example 3.21. $\widetilde{S}(x_1^{(3)} x_4 - x_4^{(2)} x_2') = -x_1^{(3)} x_2 + i\, x_2^{(2)} x_4'$ by Lemmas 3.15 and 3.18.

Proposition 3.22. The map $\widetilde{S}$ is $\mathbb{K}$-linear and $\widetilde{S}^8 = id$.

Proof. The linearity of $\widetilde{S}$ is obvious by Lemma 3.18. Let $p \in R_X^d$ be such that $p = \sum_{\ell=1}^{q} r_\ell h_\ell$ with $r_\ell \in \mathbb{K} \setminus \{0\}$ and with monomials $h_\ell \in R_X^d$. Then, by Lemma 3.15, for $\sigma = (2,4)$:

$$\begin{aligned}\widetilde{S}^8(p) =& r_1 \widetilde{S}^8(h_1) + \cdots + r_q \widetilde{S}^8(h_q) \\ =& r_1 (-i)^{8(\frac{\mu(h_1)}{2} + \nu_1(h_1))} \sigma^8(h_1) + \cdots + r_q (-i)^{8(\frac{\mu(h_q)}{2} + \nu_1(h_q))} \sigma^8(h_q) \\ =& p\end{aligned}$$

□

According to Lemma 3.18, for any homogeneous $p \in R_X$, $S_X(p)$ has a presentation of the form

$$S_X(p) = \sum_{i=1}^{n} s^{c_i} p_i \in R_X[s^{\frac{1}{2}}] \tag{7}$$

with homogeneous $p_i \in R_X$ and where

$$c_1 < \cdots < c_n \quad \text{and} \quad \mathrm{Deg}(p_1) < \cdots < \mathrm{Deg}(p_n);$$

moreover,

$$c_n = \mathrm{Deg}(p_n) = \mathrm{Deg}(p).$$

Definition 3.23. A sum presentation of $S_X(p)$ as in (7) is called S-form presentation. We also say that $S_X(p)$ written as in (7) is in S-form.

Lemma 3.24. Suppose $p \in R_X$ with $p = \sum_{t=1}^{n} p_t$, where the p_t are homogeneous and $\mathrm{Deg}(p_i) < \mathrm{Deg}(p_j)$ if $i < j$. If $S_X(p) = \sum_{t=1}^{m} s^{c_t} q_t$ is in S-form then $\widetilde{S}(p_n) = q_m$.

Proof. First, by Lemma 3.18, we observe that

$$\mathrm{Deg}(p_i) = \text{ highest power of } s \text{ in } S_X(p_i). \tag{8}$$

One has

$$q_m = \mathrm{lc}(S_X(p)) = \mathrm{lc}(S_X(p_1) + \cdots + S_X(p_n)) = \mathrm{lc}(S_X(p_n)) = \widetilde{S}(p_n),$$

where we used (8) together with $\mathrm{Deg}(p_i) < \mathrm{Deg}(p_n)$ for $i \in \{1, \ldots, n-1\}$. □

For our context a special case of the slash operator, introduced in Definition 2, is of special importance.

Recall S_0 from Definition 3.6.

Lemma 3.25. Given $F(\tau) \in R_\Theta$, let $(S_0F)(\tau) \equiv \sum_{t=1}^{n} \tau^{c_t} f_t(\tau)$ ($c_t \in \frac{1}{2}\mathbb{N}$) with $f_t(\tau) \in R_\Theta$ and $c_1 < c_2 < \cdots < c_n$. Then $F(\tau) \equiv 0$ if and only if $f_t(\tau) \equiv 0$ for all $t \in \{1, \ldots, n\}$.

Proof. " $\Longleftarrow$ " is immediate.
" $\Longrightarrow$ ". If $F(\tau) \equiv 0$ then $(S_0F)(\tau) \equiv 0$. Since $f_t(\tau) \equiv f_t(\tau + 8)$, the rest can be done by using the same method as used to prove Lemma 1.1 in Radu (2015). □

Applying Lemma 3.7, we carry Lemma 3.25 over to the symbolic world R_X.

Lemma 3.26. Let $p \in R_X$ and $S_X(p) = \sum_{t=1}^{n} s^{c_t} p_t$ in S-form. Then $p \in \ker\phi$ if and only if $p_t \in \ker\phi$ for all $t \in \{1, \ldots, n\}$.

Proof. The definitions of ϕ and ϕ^* imply that $\phi^*\big|_{R_X} = \phi$. Hence for $\tau \in \mathbb{H}$,

$$\phi^*(S_X(p))(\tau) \equiv \sum_{t=1}^{n} \phi^*(s^{c_t} p_t)(\tau) \equiv \sum_{t=1}^{n} \tau^{c_t} \phi(p_t)(\tau) \equiv S_0(\phi(p))(\tau),$$

where the last equality is by Lemma 3.7. Using also Lemma 3.25, we have the following chain of equivalences:

$$p \in \ker\phi \iff \phi(p) = 0 \iff S_0(\phi(p)) = 0 \iff \forall t: \phi(p_t) = 0,$$

which completes the proof. □

Theorem 3.27. Let $p \in R_X$ with $p = \sum_{t=1}^{n} p_t$, where the $p_t \in R_X$ are homogeneous and $\mathrm{Deg}(p_i) < \mathrm{Deg}(p_j)$ if $i < j$. Then $p \in \ker\phi$ if and only if $p_t \in \ker\phi$ for all $t \in \{1, \ldots, n\}$.

Proof. " $\Longleftarrow$ " is immediate.
" $\Longrightarrow$ ". Suppose $p \in \ker\phi$ with $S_X(p) = \sum_{t=1}^{n_1} s^{c_{1,t}} p_{1,t}$ in S-form. By Lemma 3.24, $\widetilde{S}(p_n) = p_{1,n_1}$, and by Lemma 3.26, $p_{1,n_1} \in \ker\phi$. Next, if $S_X(p_{1,n_1}) = \sum_{t=1}^{n_2} s^{c_{2,t}} p_{2,t}$ in S-form, then $\widetilde{S}(p_{1,n_1}) = p_{2,n_2}$ and $p_{2,n_2} \in \ker\phi$. Iterating this process after k steps gives $\widetilde{S}^k(p_n) = p_{k,n_k}$ with $p_{k,n_k} \in \ker\phi$. For $k = 8$, Proposition 2 gives $p_n = \widetilde{S}^8(p_n) = p_{8,n_8} \in \ker\phi$. Because of $p \in \ker\phi$ we conclude that $\sum_{t=1}^{n-1} p_t \in \ker\phi$. Applying the same procedure to this element we obtain $p_{n-1} \in \ker\phi$. Iterating we eventually obtain $p_t \in \ker\phi$ for all $t \in \{1, 2, \ldots, n\}$. □

Note. Theorem 1 is fundamental for our kernel membership test.

4. Membership recognition for homogeneous $p \in R_X$

Definition 4.1. Given $p \in R_X$ homogeneous, define:

$$\mathrm{LT}(p) := \{\widetilde{S}^{k_1} T_X^{k_2} \widetilde{S}^{k_3} T_X^{k_4} \cdots (p) : \ k_i \in \mathbb{N}\}.$$

We call $\mathrm{LT}(p)$ the leading term orbit of p.

Proposition 4.2. For homogeneous $p \in R_X$, one has $|\mathrm{LT}(p)| \leq 2^7 \cdot 3$ and the bound is sharp.

Proof. Since $p \in R_X$, p is a polynomial in infinitely many variables, that is $p = f(x_1, \dots, x_4, x_1^{(1)}, \dots, x_4^{(1)}, \dots)$. Assume $q \in \mathrm{LT}(p)$, then $q = \hat{\sigma} f(x_1, \dots, x_4, x_1^{(1)}, \dots, x_4^{(1)}, \dots)$ where $\hat{\sigma} = \widetilde{S}^{k_1} T_X^{k_2} \widetilde{S}^{k_3} T_X^{k_4} \cdots \widetilde{S}^{k_{n-1}} T_X^{k_n}$. One can verify that

$$\hat{\sigma} f(x_1, \dots, x_4, x_1^{(1)}, \dots, x_4^{(1)}, \dots) = f(\hat{\sigma} x_1, \dots, \hat{\sigma} x_4, \hat{\sigma} x_1^{(1)}, \dots, \hat{\sigma} x_4^{(1)}, \dots).$$

Therefore the number of possible $\hat{\sigma} f$ is bounded by the number of possible infinite vectors of the form $(\hat{\sigma} x_1, \dots, \hat{\sigma} x_4, \hat{\sigma} x_1^{(1)}, \dots, \hat{\sigma} x_4^{(1)}, \dots)$. Such a vector is uniquely determined by the first four entries. We checked by computer that there are 384 possible values for the first four entries. Therefore there are at most $384 = 2^7 \cdot 3$ different $\hat{\sigma} f$. $\square$

Note. In fact, in view of $T_X^8 = id = \widetilde{S}^8$, $\mathrm{LT}(p)$ is the p-orbit of a corresponding group action. For instance,

$$\text{if } p_1 \in \mathrm{LT}(p) \text{ then } \mathrm{LT}(p_1) = \mathrm{LT}(p).$$

Lemma 4.3. Suppose $p \in R_X$. If $p \in \ker\phi$ then $T_X(p) \in \ker\phi$.

Proof. If $p \in \ker\phi$ then $\phi(p) = 0$. Hence $\phi(T_X(p))(\tau) \equiv \phi(p)(\tau + 1) \equiv 0$ by Lemma 3.11. Therefore $T_X(p) \in \ker\phi$. $\square$

Lemma 4.4. Suppose $p \in R_X$ and $g \in \mathrm{LT}(p)$. Then $p \in \ker\phi$ if and only if $g \in \ker\phi$.

Proof. "$\Longrightarrow$". Suppose $S_X(p) = \sum_{t=1}^{n} s^{c_t} p_t$ in S-form. From Lemma 3.26 we know that if $p \in \ker\phi$, then $\widetilde{S}(p) = p_n \in \ker\phi$. By Lemma 4.3, $T_X(p) \in \ker\phi$. According to Definition 8, for each $g \in \mathrm{LT}(p)$, there exist $k_j \in \mathbb{N}$ such that $g = \widetilde{S}^{k_1} T_X^{k_2} \widetilde{S}^{k_3} T_X^{k_4} \cdots \widetilde{S}^{k_{n-1}} T_X^{k_n}(p)$. Thus if $p \in \ker\phi$ then $g \in \ker\phi$. "$\Longleftarrow$". Noting that $p \in \mathrm{LT}(p) = \mathrm{LT}(g)$ we can apply "$\Longrightarrow$". $\square$

Definition 4.5. (Freitag and Busam, 2005, p.317) Let $q = e^{\pi i \tau}$, $\tau \in \mathbb{H}$. Given $k \in \mathbb{N}$, a modular form of weight k is an analytic function f on $\mathbb{H}$ such that

$$f\left(\frac{a\tau + b}{c\tau + d}\right) \equiv (c\tau + d)^k f(\tau) \quad \text{for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}), \tag{9}$$

and where $f(\tau)$ can be written as a Taylor series in powers of q with complex coefficients; i.e.,

$$f(\tau) \equiv \sum_{n=0}^{\infty} a_n e^{\pi i \tau n} \equiv \sum_{n=0}^{\infty} a_n q^n. \tag{10}$$

Note. Substituting $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ in (9) we obtain $f(\tau+1) \equiv f(\tau)$. Therefore $f(\tau+1) \equiv \sum_{n=0}^{\infty} a_n(-q)^n \equiv \sum_{n=0}^{\infty} a_n q^n \equiv f(\tau)$, which implies by comparison of coefficients that $a_n = -a_n$ for all odd $n \in \mathbb{N}$. Consequently,

$$f(\tau) \equiv \sum_{n=0}^{\infty} a_{2n} q^{2n}.$$

Lemma 4.6. Given $p \in R_X$ homogeneous, if $\phi(p) \in M(k) \setminus \{0\}$ then $\mathrm{Deg}(p) = k$.

Proof. By Lemma 3.18, the highest power of s in $S_X(p)$ is Deg(p). Thus by Lemma 3.7 we know that the highest power of τ in $(S_0\phi(p))(\tau)$ is Deg(p). If $\phi(p) \in M(k) \setminus \{0\}$, then $(S_0\phi(p))(\tau) = \phi(p)(-1/\tau) = \tau^k \phi(p)(\tau)$. Therefore $\mathrm{Deg}(p) = k$. $\square$

Example 4.7. Let $p = -\frac{1}{9}(x_2^4 + x_3^4)(x_3^4 + x_4^4)(x_2^4 - x_4^4)$. One can easily verify that p is homogeneous and $\mathrm{Deg}(p) = 6$. On the other hand, $\phi(p) = e_1 e_2 e_3$ where $e_1 := \frac{1}{3}(\theta_3(0,q)^4 + \theta_4(0,q)^4)$, $e_2 := -\frac{1}{3}(\theta_2(0,q)^4 + \theta_3(0,q)^4)$ and $e_3 := \frac{1}{3}(\theta_2(0,q)^4 - \theta_4(0,q)^4)$. One also verifies that the product $e_1 e_2 e_3$ is a modular form of weight 6.

Definition 4.8. The $\mathbb{K}$-vector space of modular forms of weight $k \in \mathbb{N}$ is denoted by $M(k)$.

By the valence formula (Freitag and Busam, 2005, Th. VI.2.3), one deduces the following lemma.

Lemma 4.9. Given $f \in R_\Theta \cap M(k)$:

$$\text{if } f(\tau) \equiv \sum_{t > \frac{k}{6}} a_t q^t, \text{ then } f = 0.$$

According to Lemma 4.9, to prove that $f \in R_\Theta$ is identically zero we follow two steps: first check if f is a modular form, then check if the first few coefficients of the q-expansion of f are zero.

But usually the $f \in R_\Theta$ given in out context is not a modular form in the sense of Definition 4.5. To be able to apply Lemma 4.9, instead of directly dealing with $f = \phi(p)$ (with homogeneous $p \in R_X$) we deal with $\prod_{u \in \mathrm{LT}(p)} \phi(u)$. We first check if this product is a modular form, and then we check whether the first few coefficients of the q-expansion of this product are zero. We will also show that if this product is zero then each single $\phi(u)$ is zero. This will imply $f = \phi(p) = 0$ because $p \in \mathrm{LT}(p)$.

Lemma 4.10. Let $p \in R_X$ be homogeneous and $\mathrm{LT}(p) = \{p_1, \ldots, p_m\}$ with $S_X(p_j) = \sum_{t=1}^{n_j} s^{c_{j,t}} p_{j,t}$ in S-form. If $p_{j,1}, p_{j,2}, \ldots, p_{j,n_j-1} \in \ker\phi$ for all $j \in \{1, \ldots, m\}$ then

$$\prod_{j=1}^{m} \phi(p_j)(\tau) \in M(m\,\mathrm{Deg}(p)).$$

Proof. By Lemma 3.7 we have for $j \in \{1, \ldots, m\}$,

$$(\phi(p_j)|_0 S)(\tau) \equiv \phi^*(S_X(p_j))(\tau) \equiv \phi^*\left(\sum_{t=1}^{n_j} s^{c_{j,t}} p_{j,t}\right) \equiv \sum_{t=1}^{n_j} \tau^{c_{j,t}} \phi(p_{j,t})(\tau).$$

Let $d = \mathrm{Deg}(p)$. Applying Lemma 3.18 we have $c_{1,n_1} = c_{2,n_2} \cdots = c_{m,n_m} = d$. Suppose $j \in \{1, \ldots, m\}$ is fixed. If $p_{j,1}, p_{j,2}, \ldots, p_{j,n_j-1} \in \ker\phi$ then

$$(\phi(p_j)|_0 S)(\tau) \equiv \tau^d \phi(p_{j,n_j}).$$

Note that $p_{j,n_j} \in \mathrm{LT}(p)$ by Definitions 3.20 and 4.4. Thus

$$(\phi(p_j)|_0 S)(\tau) \equiv \tau^d \phi(p_i)$$

for some $p_i \in \mathrm{LT}(p)$. Moreover, by Definition 4.4 we have $T_X(p_j) = p_t$ for some $p_t \in \mathrm{LT}(p)$ and thus, by Lemma 3.11,

$$(\phi(p_j)|_0 T)(\tau) \equiv \phi^*(T_X(p_j))(\tau) \equiv \phi(p_t)(\tau).$$

Therefore

$$(\phi(p_j)|_d S)(\tau) \equiv \phi(p_i)(\tau) \quad \text{and} \quad (\phi(p_j)|_d T)(\tau) \equiv \phi(p_t)(\tau).$$

Thus for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$:

$$\left(\prod_{j=1}^{m} \phi(p_j)\bigg|_{dm} \gamma\right)(\tau) \equiv \prod_{j=1}^{m}\left(\phi(p_j)\bigg|_{d} \gamma\right)(\tau) \equiv \prod_{j=1}^{m} \phi(p_j)(\tau). \tag{11}$$

In fact the functions $\theta_i^{(k)}$ are analytic on $\mathbb{H}$, which can be seen from their q-expansion. Therefore the above product is analytic on $\mathbb{H}$. Again by Definition 1.2, each of the functions $\theta_i^{(k)}$ is a Taylor series in powers of $q^{1/4}$, therefore also the above product is a Taylor series in powers of $q^{1/4}$. Setting $\gamma = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ in (11) implies that the above product is invariant under the mapping $\tau \mapsto \tau + 2$. It is known that analytic functions with this property may be written as a Laurent series in q; by the uniqueness of Laurent series we must have that the product is a Taylor series in q as required from the definition of modular form. $\square$

Theorem 4.11. Let $p \in R_X$ be homogeneous, $\mathrm{LT}(p) = \{p_1, \ldots, p_m\}$ with $S_X(p_j) = \sum_{t=1}^{n_j} s^{c_{j,t}} p_{j,t}$ in S-form. If for all $j \in \{1, \ldots, m\}$,

$$p_{j,1}, p_{j,2}, \ldots, p_{j,n_j-1} \in \ker\phi \quad \text{and} \quad \mathrm{ord}\left(\prod_{j=1}^{m} \phi(p_j)\right) > \frac{m\mathrm{Deg}(p)}{6},$$

where ord is the order of a power series in q in the usual sense, then $p \in \ker\phi$.

Proof. If for all $j \in \{1, \dots, m\}$, $p_{j,1}, p_{j,2}, \dots, p_{j,n_j-1} \in \ker\phi$, then by Lemma 4.10 we have $\prod_{j=1}^{m} \phi(p_j) \in M(m\,\mathrm{Deg}(p))$. This together with $\mathrm{ord}\left(\prod_{j=1}^{m} \phi(p_j)\right) > \frac{m\mathrm{Deg}(p)}{6}$, by Lemma 4.9, we obtain $\phi\left(\prod_{j=1}^{m} p_j\right) = \prod_{j=1}^{m} \phi(p_j) = 0$. Thus for some j, $p_j \in \ker\phi$, which by Lemma 4.4 implies that for any $h \in \mathrm{LT}(p_j) = \mathrm{LT}(p)$, $h \in \ker\phi$. Therefore $p \in \ker\phi$. □

Algorithm 4.12. Let p, $\mathrm{LT}(p)$ and $S_X(p_j)$ be the same as in Theorem 4.11, and $d := \mathrm{Deg}(p)$. Define

$$\mathrm{Prove}(p) := \begin{cases} \text{True}, & \text{if } p \in \ker\phi; \\ \text{False}, & \text{if } p \notin \ker\phi. \end{cases}$$

We have the following algorithm to prove or disprove $p \in \ker\phi$.

Input: homogeneous $p \in R_X$; output: True or False.

If $d = 0$ then Prove(p) =True if $p = 0$; else Prove(p)= False.

If $d > 0$ then

$$\mathrm{Prove}(p) = \text{True if } \mathrm{ord}\left(\prod_{j=1}^{m} \phi(p_j)\right) > \frac{d\,m}{6}$$

and Prove($p_{j,1}$) = True and ... and Prove(p_{j,n_j-1}) = True;
else Prove(p) = False.

Theorem 4.13. Algorithm 4.12 is correct.

Proof. Suppose $p \in \ker\phi$. Using $\mathrm{LT}(p) = \mathrm{LT}(p_j)$ and Lemma 4.4 we have the equivalences

$$\begin{aligned} p \in \ker\phi &\iff p_j \in \ker\phi \text{ for all } j \in \{1, \dots, m\} \\ &\iff p_j \in \ker\phi \text{ for some } j \in \{1, \dots, m\} \\ &\iff \prod_{j=1}^{m} \phi(p_j)(\tau) \equiv 0. \end{aligned}$$

According to Theorem 4.11, this together with

(1) True=Prove($p_{j,1}$) $= \cdots =$Prove(p_{j,n_j-1}), $j = 1, \dots, m$, gives Prove(p) =True; i.e., $p \in \ker\phi$. We note that owing to Lemma 3.18 the procedure terminates; namely

$$\mathrm{Deg}(p_{j,1}) < \cdots < \mathrm{Deg}(p_{j,n_j-1}) < \mathrm{Deg}(p_{j,n_j}) = d.$$

Suppose $p \notin \ker\phi$. This is equivalent to

(2) $p_j \notin \ker\phi$ for all $j \in \{1, \ldots, m\}$. In case (1) holds, then by Lemma 4.10,

$$f(\tau) := \prod_{j=1}^{m} \phi(p_j)(\tau) \in M(dm).$$

Because of (2) we know that $f(\tau) \not\equiv 0$; thus $\mathrm{ord}(f) \leq \frac{dm}{6}$ and Algorithm 4.12 returns Prove(p) =False. If at least one of the $p_{j,1}, \ldots, p_{j,n_j-1}$ ($j = 1, \ldots, m$) is not in $\ker\phi$, the algorithm detects this in a base case (i.e., $p \in \mathbb{K} \setminus \{0\}$) when applying its steps recursively. $\square$

Example 4.14. Let us return to the task to do zero-recognition for (5) from Example 1.3. Since $\theta_2'(0, q) \equiv \theta_3'(0, q) \equiv \theta_4'(0, q) \equiv 0$, we needed to prove the following identity.

$$\theta_2(\tau)^3\theta_2''(\tau) - \theta_3(\tau)^3\theta_3''(\tau) + \theta_4(\tau)^3\theta_4''(\tau) \equiv 0.$$

Proof. For $p := x_2^3x_2^{(2)} - x_3^3x_3^{(2)} + x_4^3x_4^{(2)} \in R_X^4$ we want to prove $p \in \ker\phi$. We compute

$$\mathrm{LT}(p) = \{p_1, p_2\} = \{x_2^3x_2^{(2)} - x_3^3x_3^{(2)} + x_4^3x_4^{(2)}, -(x_2^3x_2^{(2)} - x_3^3x_3^{(2)} + x_4^3x_4^{(2)})\}.$$

Since $\mathrm{Deg}(p) = 4$ and $|\mathrm{LT}(p)| = 2$, we need to show that $\phi(p_1p_2)(\tau)$ has the form $\sum_{t>\frac{8}{6}} a_t q^t$, which holds because

$$\begin{aligned}\phi(p_1p_2)(\tau) \equiv &(\theta_2(\tau)^3\theta_2''(\tau) - \theta_3(\tau)^3\theta_3''(\tau) + \theta_4(\tau)^3\theta_4''(\tau))\\ &(-\theta_2(\tau)^3\theta_2''(\tau) + \theta_3(\tau)^3\theta_3''(\tau) - \theta_4(\tau)^3\theta_4''(\tau))\\ \equiv &\square q^2 + \square q^3 + \ldots.\end{aligned}$$

Moreover we have

$$S_X(p_1) = (-x_2^3x_2^{(2)} + x_3^3x_3^{(2)} - x_4^3x_4^{(2)})s^4 + \frac{2i}{\pi}(-x_2^4 + x_3^4 - x_4^4)s^3 = p_2s^4 + \frac{2i}{\pi}p_{1,2}s^3$$

and

$$S_X(p_2) = (x_2^3x_2^{(2)} - x_3^3x_3^{(2)} + x_4^3x_4^{(2)})s^4 + \frac{2i}{\pi}(x_2^4 - x_3^4 + x_4^4)s^3 = p_1s^4 + \frac{2i}{\pi}p_{2,2}s^3.$$

According to Theorem 2, it is now left to show that $p_{1,2}, p_{2,2} \in \ker\phi$. We compute

$$\mathrm{LT}(p_{1,2}) = \mathrm{LT}(p_{2,2}) = \{-x_2^4 + x_3^4 - x_4^4, x_2^4 - x_3^4 + x_4^4\} = \{p_{1,2}, p_{2,2}\}.$$

Since $\mathrm{Deg}(p_{1,2}) = 2$ and $|\mathrm{LT}(p_{1,2})| = 2$, we need to show $\phi(p_{1,2}p_{2,2})(\tau)$ has the form $\sum_{t>\frac{4}{6}} a_t q^t$, which holds because

$$\phi(p_{1,2}p_{2,2})(\tau) \equiv (\theta_2(\tau)^4 - \theta_3(\tau)^4 + \theta_4(\tau)^4)(-\theta_2(\tau)^4 + \theta_3(\tau)^4 - \theta_4(\tau)^4) \equiv \square q + \square q^2 + \ldots.$$

We also have

$$S_X(p_{1,2}) = (x_2^4 - x_3^4 + x_4^4)s^2 = p_1s^2 \quad \text{and} \quad S_X(p_{2,2}) = (-x_2^4 + x_3^4 - x_4^4)s^2 = p_2s^2.$$

Thus $p_{1,2}, p_{2,2} \in \ker\phi$. Consequently we obtain $p \in \ker\phi$. $\square$

Example 4.15. As another example we present an identity from the famous book by Rademacher, (93.22) in Rademacher (1973), which was used to derive the formula for the number of presentations of a natural number as a sum of 10 squares:

$$\theta_3^{(4)}(\tau)\theta_3(\tau) - 3(\theta_3''(\tau))^2 - 2\theta_3(\tau)^2\theta_2(\tau)^4\theta_4(\tau)^4 \equiv 0.$$

The algorithmic effort to prove this identity is as simple as in Example 4.14; in contrast, Rademacher used three pages of clever arguments.

5. A Refined Algorithm

Definition 5.1. For any $\bar{k} = (k_1, \ldots, k_m) \in \mathbb{N}^m$ and $t \in \mathbb{N}$ we define

$$D(\bar{k}, t) := \left\{ (b_1, \ldots, b_m) \in \mathbb{N}^m : \sum_{i=1}^{m} b_i = \sum_{i=1}^{m} k_i - 2t,\ b_i \le k_i \text{ and } b_i \equiv k_i \pmod 2 \right\}.$$

Lemma 5.2. Let $p = x_{i_1}^{(k_1)} \cdots x_{i_m}^{(k_m)} \in R_X$. Then there exists an $r \in \mathbb{N}$ such that

$$S_X(p) = s^{\mathrm{Deg}(p)} p_0 + s^{\mathrm{Deg}(p)-1} p_1 + \cdots + s^{\mathrm{Deg}(p)-r} p_r$$

is in S-form. Furthermore, $r = \frac{\mathrm{Deg}(p)}{2} - \frac{m}{4} - \frac{\nu_1(p)}{2}$ and for $0 \le t \le r$,

$$p_t = (-i)^{\nu_1(p) + \frac{m}{2}} \left(\frac{i}{\pi} \right)^t \sum_{\bar{b} \in D(\bar{k}, t)} \prod_{v=1}^{m} \frac{k_v!}{b_v! (\frac{k_v - b_v}{2})!} x_{\sigma(i_1)}^{(b_1)} \cdots x_{\sigma(i_m)}^{(b_m)},$$

where σ is the permutation on $\{1, 2, 3, 4\}$ that transposes 2 and 4.

Proof. Suppose $S_X(p) = s^{c_n} g_n + \cdots + s^{c_1} g_1$ with $g_j \in R_X \setminus \{0\}$, and $x_{i_1} = x_{i_2} = \cdots = x_{i_{\nu_1(p)}} = x_1$ and $x_{i_j} \ne x_1$ for $\nu_1(p) + 1 \le j \le m$. By Definition 3.6, for every $x_\ell^{(k)}$, regardless that ℓ is even or odd, in the S-form $S_X(x_\ell^{(k)})$ the power of s decreases by 1 when the j in Definition 3.6 increases by 2. By Lemma 3.18 we have $c_n = \mathrm{Deg}(p)$. Thus there exists $r \in \mathbb{N}$ such that

$$S_X(p) = s^{\mathrm{Deg}(p)} p_0 + s^{\mathrm{Deg}(p)-1} p_1 + \cdots + s^{\mathrm{Deg}(p)-r} p_r,$$

where $p_t \in R_X$ for $t \in \{0, \ldots, r\}$ and $p_r \ne 0$. By the proof of Lemma 3.15 we find

$$\begin{aligned} c_n - c_1 &= \left(\sum_{j=1}^{\nu_1(p)} \left(k_j + \frac{1}{2} \right) + \sum_{j=\nu_1(p)+1}^{m} \left(k_j + \frac{1}{2} \right) \right) - \left(\sum_{j=1}^{\nu_1(p)} \left(\frac{k_j}{2} + 1 \right) + \sum_{j=\nu_1(p)+1}^{m} \left(\frac{k_j}{2} + \frac{1}{2} \right) \right) \\ &= \sum_{j=1}^{\nu_1(p)} \left(\frac{k_j}{2} - \frac{1}{2} \right) + \sum_{j=\nu_1(p)+1}^{m} \frac{k_j}{2} \\ &= \frac{\mathrm{Deg}(p)}{2} - \frac{m}{4} - \frac{\nu_1(p)}{2}. \end{aligned}$$

Since $k_j \in 2\mathbb{N}$ when $j \in \{1, \ldots, \nu_1(p)\}$ and $k_j \in 2\mathbb{N} + 1$ when $j \in \{\nu_1(p) + 1, \ldots, m\}$, we confirm that $r = c_n - c_1 = \frac{\mathrm{Deg}(p)}{2} - \frac{m}{4} - \frac{\nu_1(p)}{2} \in \mathbb{N}$.

Now we show that $p_t \ne 0$ for all $t \in \{0, \ldots, r\}$. By fully invoking Definition 3.6, for

$0 \le t \le r$ we derive

$$\begin{aligned}\langle s^{d-t}\rangle S_X(p) =&\langle s^{d-t}\rangle S_X(x_{i_1}^{(k_1)})S_X(x_{i_2}^{(k_2)})\cdots S_X(x_{i_m}^{(k_m)})\\ =&(-i)^{\nu_1(p)+\frac{m}{2}}\sum_{\bar{b}\in D(\bar{k},t)}\left(\frac{i}{\pi}\right)^{\sum_{i=1}^{m}\frac{k_i-b_i}{2}}\prod_{v=1}^{m}\frac{k_v!}{b_v!(\frac{k_v-b_v}{2})!}x_{\sigma(i_1)}^{(b_1)}\cdots x_{\sigma(i_m)}^{(b_m)}\\ =&(-i)^{\nu_1(p)+\frac{m}{2}}\left(\frac{i}{\pi}\right)^{t}\sum_{\bar{b}\in D(\bar{k},t)}\prod_{v=1}^{m}\frac{k_v!}{b_v!(\frac{k_v-b_v}{2})!}x_{\sigma(i_1)}^{(b_1)}\cdots x_{\sigma(i_m)}^{(b_m)},\end{aligned}$$

where $\bar{b} = (b_1, \dots, b_m)$ and $\bar{k} = (k_1, \dots, k_m)$. Since $k_v \ge b_v \ge 0$, we have $\prod_{v=1}^{m}\frac{k_v!}{b_v!(\frac{k_v-b_v}{2})!} > 0$, which implies $\langle s^{d-t}\rangle S_X(p) \ne 0$. Therefore the expression of $S_X(p)$ in the statement is in S-form. $\square$

We shall see that the following refined sets of compositions of numbers play a crucial role. Throughout $\bar{b} \in \mathbb{N}^m$ has to be interpreted as $\bar{b} = (b_1, \dots, b_m)$.

Definition 5.3. Given $\bar{d} \in \mathbb{N}^m$, $\bar{k} \in \mathbb{N}^m$, and $j, t \in \mathbb{N}$:

$$B(\bar{d},\bar{k},t,j) := \left\{\bar{b}\in D(k,t) : \sum_{i=1}^{m} b_i = \sum_{i=1}^{m} d_i + 2j, d_i \le b_i \text{ and } d_i \equiv b_i \pmod 2\right\}$$

Lemma 5.4. Given $j, t \in \mathbb{N}$ and $\bar{d} \in \mathbb{N}^m$ and $\bar{k} \in \mathbb{N}^m$, then

$$j\sum_{\bar{b}\in B(\bar{d},\bar{k},t,j)}\prod_{v=1}^{m}\alpha(k_v,b_v)\beta(b_v,d_v) = (t+1)\sum_{\bar{e}\in B(\bar{d},\bar{k},t+1,j-1)}\prod_{v=1}^{m}\alpha(k_v,e_v)\beta(e_v,d_v), \quad (12)$$

where

$$\alpha(k_v,e_v) := \frac{k_v!}{(\frac{k_v-e_v}{2})!} \text{ and } \beta(b_v,c_v) := \frac{1}{c_v!(\frac{b_v-c_v}{2})!}.$$

Proof. Let

$$M_1 := \{(\bar{b},\bar{b}-2z_i) : \bar{b}\in B(\bar{d},\bar{k},t,j),\ 1\le i\le m \text{ and } b_i \ge d_i+2\}$$

and

$$M_2 := \{(\bar{e}+2z_i,\bar{e}) : \bar{e}\in B(\bar{d},\bar{k},t+1,j-1),\ 1\le i\le m \text{ and } e_i \le k_i-2\},$$

where $z_i = (a_1, \dots, a_m)$ with $a_i = 1$ and $a_j = 0$ $(j \ne i)$. Then

$$\begin{aligned}\text{LHS of (12)} &= \sum_{\bar{b}\in B(\bar{d},\bar{k},t,j)}\left(\sum_{i=1}^{m}\frac{b_i-d_i}{2}\prod_{v=1}^{m}\alpha(k_v,b_v)\beta(b_v,d_v)\right)\\ &= \sum_{\bar{b}\in B(\bar{d},\bar{k},t,j)}\prod_{v=1}^{m}\alpha(k_v,b_v)\sum_{i=1}^{m}\beta(b_i-2,d_i)\prod_{\substack{v=1\\v\ne i}}^{m}\beta(b_v,d_v)\\ &= \sum_{(\bar{b},\bar{e})\in M_1}\prod_{v=1}^{m}\alpha(k_v,b_v)\beta(e_v,d_v)\end{aligned}$$

and

$$\text{RHS of (12)} = \sum_{\bar{e}\in B(\bar{d},\bar{k},t+1,j-1)} \left(\sum_{i=1}^{m} \frac{k_i - e_i}{2} \prod_{v=1}^{m} \alpha(k_v, e_v)\beta(e_v, d_v) \right)$$

$$= \sum_{\bar{e}\in B(\bar{d},\bar{k},t+1,j-1)} \prod_{v=1}^{m} \beta(e_v, d_v) \sum_{i=1}^{m} \alpha(k_v, e_i + 2) \prod_{\substack{v=1 \\ v\neq i}}^{m} \alpha(k_v, e_v)$$

$$= \sum_{(\bar{b},\bar{e})\in M_2} \prod_{v=1}^{m} \alpha(k_v, b_v)\beta(e_v, d_v),$$

where we define $\beta(b_i - 2, d_i) := 0$ if $b_i = d_i$, and define $\alpha(k_i, e_i + 2) := 0$ if $e_i = k_i$.

To prove the lemma we finally prove that $M_1 = M_2$.

Take $(\bar{b}, \bar{e}) := (\bar{b}, \bar{b} - 2z_i) \in M_1$ for some $i \in \{1, \ldots, m\}$. Then $\bar{b} = \bar{e} + 2z_i$, and we can write $(\bar{b}, \bar{e}) = (\bar{e} + 2z_i, \bar{e})$. Additionally, from the definition of M_1 we have $\bar{b} \in B(\bar{d}, \bar{k}, t, j)$ and $d_i + 2 \leq b_i \leq k_i$, which implies $\bar{e} + 2z_i \in B(\bar{d}, \bar{k}, t, j)$ and $d_i + 2 \leq e_i + 2 \leq k_i$. Then $d_i \leq e_i \leq k_i - 2$ and by Definition 5.3 we have

$$\sum_{v=1}^{m} e_v + 2 = \sum_{v=1}^{m} k_v - 2t = \sum_{v=1}^{m} d_v + 2j.$$

Hence

$$\sum_{v=1}^{m} e_v = \sum_{v=1}^{m} k_v - 2(t+1) = \sum_{v=1}^{m} d_v + 2(j-1)$$

and $d_i \leq e_i \leq k_i - 2$, which implies $\bar{e} \in B(\bar{d}, \bar{k}, t+1, j-1)$ and $d_i \leq e_i \leq k_i - 2$. Therefore $(\bar{b}, \bar{e}) = (\bar{e} + 2z_i, \bar{e}) \in M_2$. The other direction goes analogously. $\square$

Theorem 5.5. Given $p = x_{i_1}^{(k_1)} \cdots x_{i_m}^{(k_m)} \in R_X$, according to Lemma 5.2 let $S_X(p) = s^{\mathrm{Deg}(p)} p_0 + \cdots + s^{\mathrm{Deg}(p)-r} p_r$ with non-zero $p_i \in R_X$ and $r = \frac{\mathrm{Deg}(p)}{2} - \frac{m}{4} - \frac{\nu_1(p)}{2}$. We have

(1) $S_X(p_r) = s^{\mathrm{Deg}(p_r)} g$ for some non-zero $g \in R_X$; and

(2) for any neighboring pair (p_t, p_{t+1}), $t \in \{0, \ldots, r-1\}$,

$$S_X(p_{t+1}) = \frac{1}{t+1} \sum_{j=1}^{r-t} s^{\mathrm{Deg}(p_t)-j-1} j q_{t,j}$$

and

$$S_X(p_t) = \sum_{j=0}^{r-t} s^{\mathrm{Deg}(p_t)-j} q_{t,j}.$$

with $q_{t,j} \in R_X$.

Proof. (1) According to Lemma 5.2, $p_r \neq 0$. Therefore the statement is implied by Definition 3.6.

(2) We first prove that the low degree of $S_X(p_t)$ with respect to s is $\mathrm{Deg}(p_t) - r + t$, then we prove the coefficient relation

$$\frac{\langle s^{\mathrm{Deg}(p_t)-j} \rangle S_X(p_t)}{\langle s^{\mathrm{Deg}(p_t)-j-1} \rangle S_X(p_{t+1})} = \frac{t+1}{j}$$

is true for $j \in \{0, \ldots, r-t\}$. Suppose $x_{i_1} = x_{i_2} = \cdots = x_{i_a} = x_1$ and $x_{i_j} \neq x_1$ for $a+1 \leq j \leq m$. Let $C(p) := (-i)^{\nu_1(p)+\frac{m}{2}}$. Applying Lemma 5.2 we have

$$\begin{aligned} S_X(p_t) &= S_X\left(C(p)\left(\frac{i}{\pi}\right)^t \sum_{\bar{b}\in D(\bar{k},t)} \prod_{v=1}^{m} \alpha(k_v, b_v) x_{\sigma(i_1)}^{(b_1)} \cdots x_{\sigma(i_m)}^{(b_m)}\right) \\ &= C(p)\left(\frac{i}{\pi}\right)^t \sum_{\bar{b}\in D(\bar{k},t)} \prod_{v=1}^{m} \alpha(k_v, b_v) S_X\left(x_{\sigma(i_1)}^{(b_1)} \cdots x_{\sigma(i_m)}^{(b_m)}\right), \end{aligned} \tag{13}$$

where $\sigma(i_j)$ is the permutation on $\{1, 2, 3, 4\}$ that transposes 2 and 4. Now let $d_t :=$

$\mathrm{Deg}(p_t)$. Concerning (13), for $\bar{b} \in D(\bar{k}, t)$ we apply Lemma 5.2 and obtain

$$\begin{aligned} S_X\left(x_{\sigma(i_1)}^{(b_1)} \cdots x_{\sigma(i_m)}^{(b_m)}\right) =& s^{d_t} C(p) x_{i_1}^{(b_1)} \cdots x_{i_m}^{(b_m)} \\ &+ s^{d_t-1} C(p)\left(\frac{i}{\pi}\right) \sum_{\bar{c}\in D(\bar{b},1)} \prod_{v=1}^{m} \beta(b_v, c_v) x_{i_1}^{(c_1)} \cdots x_{i_m}^{(c_m)} \\ &+ s^{d_t-2} C(p)\left(\frac{i}{\pi}\right)^2 \sum_{\bar{c}\in D(\bar{b},2)} \prod_{v=1}^{m} \beta(b_v, c_v) x_{i_1}^{(c_1)} \cdots x_{i_m}^{(c_m)} \\ &+ \ldots \\ &+ s^{d_t-r_t} C(p)\left(\frac{i}{\pi}\right)^{r_t} \sum_{\bar{c}\in D(\bar{b},r_t)} \prod_{v=1}^{m} \beta(b_v, c_v) x_{i_1}^{(c_1)} \cdots x_{i_m}^{(c_m)}, \end{aligned} \tag{14}$$

where $r_t := \frac{d_t}{2} - \frac{m}{4} - \frac{\nu_1(p)}{2}$ according to Lemma 5.2 and $\nu_1(p) = \nu_1\left(x_{\sigma(i_1)}^{(b_1)} \cdots x_{\sigma(i_m)}^{(b_m)}\right)$. Since $\bar{b} \in D(\bar{k}, t)$, i.e., $\sum_{i=1}^{m} b_i = \sum_{i=1}^{m} k_i - 2t$, we have $\sum_{i=1}^{m} b_i + \frac{m}{2} = \sum_{i=1}^{m} k_i + \frac{m}{2} - 2t$, which means $d_t = \mathrm{Deg}(p) - 2t$. This together with $r = \frac{\mathrm{Deg}(p)}{2} - \frac{m}{4} - \frac{\nu_1(p)}{2}$ implies

$$r_t = \frac{\mathrm{Deg}(p) - 2t}{2} - \frac{m}{4} - \frac{\nu_1(p)}{2} = r - t.$$

Plugging (14) into (13) we get

$$\begin{aligned}
S_X(p_t) =&C(p)^2\left(\frac{i}{\pi}\right)^t\Bigg(s^{d_t}\sum_{\bar b\in D(\bar k,t)}\prod_{v=1}^m\alpha(k_v,b_v)\prod_{v=1}^m\beta(b_v,c_v)x_{i_1}^{(b_1)}\cdots x_{i_m}^{(b_m)}\\
&\quad+s^{d_t-1}\left(\frac{i}{\pi}\right)\sum_{\bar b\in D(\bar k,t)}\prod_{v=1}^m\alpha(k_v,b_v)\sum_{\bar c\in D(\bar b,1)}\prod_{v=1}^m\beta(b_v,c_v)x_{i_1}^{(c_1)}\cdots x_{i_m}^{(c_m)}\\
&\quad+s^{d_t-2}\left(\frac{i}{\pi}\right)^2\sum_{\bar b\in D(\bar k,t)}\prod_{v=1}^m\alpha(k_v,b_v)\sum_{\bar c\in D(\bar b,2)}\prod_{v=1}^m\beta(b_v,c_v)x_{i_1}^{(c_1)}\cdots x_{i_m}^{(c_m)}\\
&\quad+\dots\\
&\quad+s^{d_t-r_t}\left(\frac{i}{\pi}\right)^{r_t}\sum_{\bar b\in D(\bar k,t)}\prod_{v=1}^m\alpha(k_v,b_v)\prod_{v=1}^m\beta(b_v,c_v)(x_1')^a x_{i_{a+1}}\cdots x_{i_m}\Bigg)\\
=&s^{d_t}h_0+s^{d_t-1}h_1+\cdots+s^{d_t-r_t}h_{r_t},
\end{aligned}$$

where for $j\in\{0,\dots,r_t\}$

$$h_j=C(p)^2\left(\frac{i}{\pi}\right)^{t+j}\sum_{\bar b\in D(\bar k,t)}\prod_{v=1}^m\alpha(k_v,b_v)\sum_{\bar c\in D(\bar b,j)}\prod_{v=1}^m\beta(b_v,c_v)x_{i_1}^{(c_1)}\cdots x_{i_m}^{(c_m)}.$$

Analogously we have

$$\begin{aligned}
S_X(p_{t+1}) =&C(p)^2\left(\frac{i}{\pi}\right)^{t+1}\Bigg(s^{d_t-2}\sum_{\bar e\in D(\bar k,t+1)}\prod_{v=1}^m\alpha(k_v,e_v)x_{i_1}^{(e_1)}\cdots x_{i_m}^{(e_m)}\\
&\quad+s^{d_t-3}\left(\frac{i}{\pi}\right)\sum_{\bar e\in D(\bar k,t+1)}\prod_{v=1}^m\alpha(k_v,e_v)\sum_{\bar u\in D(\bar e,1)}\prod_{v=1}^m\beta(e_v,u_v)x_{i_1}^{(u_1)}\cdots x_{i_m}^{(u_m)}\\
&\quad+s^{d_t-4}\left(\frac{i}{\pi}\right)^2\sum_{\bar e\in D(\bar k,t+1)}\prod_{v=1}^m\alpha(k_v,e_v)\sum_{\bar u\in D(\bar e,2)}\prod_{v=1}^m\beta(e_v,u_v)x_{i_1}^{(u_1)}\cdots x_{i_m}^{(u_m)}\\
&\quad+\dots\\
&\quad+s^{d_t-r_t-1}\left(\frac{i}{\pi}\right)^{r_t-1}\sum_{\bar e\in D(\bar k,t+1)}\prod_{v=1}^m\alpha(k_v,e_v)\prod_{v=1}^m\beta(e_v,u_v)(x_1')^a x_{i_{a+1}}\cdots x_{i_m}\Bigg)\\
=&s^{d_t-2}q_1+s^{d_t-3}q_2+\cdots+s^{d_t-r_t-1}q_{r_t},
\end{aligned}$$

where for $j\in\{1,\dots,r_t\}$

$$q_j=C(p)^2\left(\frac{i}{\pi}\right)^{t+j}\sum_{\bar e\in D(\bar k,t+1)}\prod_{v=1}^m\alpha(k_v,e_v)\sum_{\bar u\in D(\bar e,j-1)}\prod_{v=1}^m\beta(e_v,u_v)x_{i_1}^{(u_1)}\cdots x_{i_m}^{(u_m)}.$$

Thus proving the statement to be true is equivalent to proving that

$$\frac{h_j}{q_j}=\frac{t+1}{j}.$$

For any fixed $\bar{c} = (c_1, \dots, c_m) \in \mathbb{N}^m$, the set of all possible $\bar{b}$ contributing to the coefficient of $x_{i_1}^{(c_1)} \cdots x_{i_m}^{(c_m)}$ in h_j is equal to $B(\bar{c}, \bar{k}, t, j)$, and for any fixed $\bar{u} = (u_1, \dots, u_m) \in \mathbb{N}^m$ the set of all possible $\bar{e}$ contributing to the coefficient of $x_{i_1}^{(u_1)} \cdots x_{i_m}^{(u_m)}$ in q_j is equal to $B(\bar{u}, \bar{k}, t+1, j-1)$. Therefore

$$h_j = C(p)^2 \left(\frac{i}{\pi}\right)^{t+j} \sum_{\bar{c}\in\mathbb{N}^m} \left(\sum_{\bar{b}\in B(\bar{c},\bar{k},t,j)} \prod_{v=1}^{m} \alpha(k_v, b_v)\beta(b_v, c_v) \right) x_{i_1}^{(c_1)} \cdots x_{i_m}^{(c_m)}$$

and

$$q_j = C(p)^2 \left(\frac{i}{\pi}\right)^{t+j} \sum_{\bar{u}\in\mathbb{N}^m} \left(\sum_{e_v\in B(\bar{u},\bar{k},t+1,j-1)} \prod_{v=1}^{m} \alpha(k_v, e_v)\beta(e_v, u_v) \right) x_{i_1}^{(u_1)} \cdots x_{i_m}^{(u_m)}.$$

Now fix $(d_1, d_2, \dots, d_m) \in \mathbb{N}^m$. We need to prove that

$$\frac{\langle x_{i_1}^{(d_1)} \cdots x_{i_m}^{(d_m)} \rangle h_j}{\langle x_{i_1}^{(d_1)} \cdots x_{i_m}^{(d_m)} \rangle q_j} = \frac{t+1}{j}.$$

Applying Lemma 5.4 we immediately obtain the correctness of this equality.
□

Corollary 5.6. Let $p \in R_X$ be homogeneous and $S_X(p) = s^{\mathrm{Deg}(p)}p_0 + \cdots + s^{\mathrm{Deg}(p)-\gamma}p_\gamma$ with $\gamma \in \mathbb{N}$, $p_t \in R_X$ and $p_\gamma \neq 0$. Then

(1) $S_X(p_\gamma) = s^{\mathrm{Deg}(p_\gamma)}g$ for some non-zero $g \in R_X$; and

(2) for any neighboring pair (p_t, p_{t+1}), $t \in \{0, \dots, \gamma\}$, defining $p_{\gamma+1} := 0$, there exists $\gamma_t \in \mathbb{N}$ such that

$$S_X(p_{t+1}) = \frac{1}{t+1} \sum_{j=1}^{\gamma_t} s^{\mathrm{Deg}(p_t)-j-1} j p_{t,j}$$

and

$$S_X(p_t) = \sum_{j=0}^{\gamma_t} s^{\mathrm{Deg}(p_t)-j} p_{t,j}$$

with $p_{t,j} \in R_X$.

Proof. We first prove statement (2). Suppose

$$p = a_1 h_1 + \cdots + a_n h_n$$

where the h_j are monomials in R_X with the same degree and the $a_j \in \mathbb{K} \setminus \{0\}$.

Let $d := \mathrm{Deg}(p) = \mathrm{Deg}(h_j)$. By Lemma 5.2, there exists an integer r_j such that

$$S_X(h_j) = s^d h_{j,0} + s^{d-1} h_{j,1} + \cdots + s^{d-r_j} h_{j,r_j}$$

in S-form. Let $r := \max_{j=1,\dots,n} \{r_j\}$. Then

$$\begin{aligned} S_X(p) =& s^d(a_1 h_{1,0} + \cdots + a_n h_{n,0}) + \cdots + s^{d-r}(a_1 h_{1,r} + \cdots + a_n h_{n,r}) \\ =& s^d p_0 + \cdots + s^{d-r} p_r, \end{aligned}$$

where $p_t = a_1 h_{1,t} + \cdots + a_n h_{n,t}$ for $t = 0, \ldots, r$ and $h_{j,t} = 0$ when $t > r_j$.

Since the p_t are homogeneous by Lemma 3.18.2, we can define $d_t := \mathrm{Deg}(p_t)$. Hence we can suppose for $t \in \{0, \ldots, r\}$ that

$$S_X(h_{j,t}) = s^{d_t} q_{j,0} + s^{d_t - 1} q_{j,1} + \cdots + s^{d_t - r_{j,t}} q_{j,r_{j,t}} \tag{15}$$

where the $q_{j,i} \in R_X$ and $q_{j,r_{j,t}} \neq 0$. Therefore by letting $\gamma_t := \max_{j=1,\ldots,m} \{r_{j,t}\}$ we obtain

$$S_X(p_t) = a_1 S_X(h_{1,t}) + \cdots + a_n S_X(h_{n,t}) = s^{d_t} q_0 + s^{d_t - 1} q_1 \cdots + s^{d_t - \gamma_t} q_{\gamma_t}$$

where $q_i = a_1 q_{1,i} + \cdots + a_n q_{n,i}$ and $q_{j,i} = 0$ if $i > \gamma_t$. Furthermore, since the h_j are monomials we immediately obtain from (15) by Theorem 5.5:

$$S_X(h_{j,t+1}) = s^{d_t - 2} \frac{1}{t+1} q_{j,1} + \cdots + s^{d_t - r_{j,t} - 1} \frac{r_{j,t}}{t+1} q_{j,r_{j,t}}.$$

Hence

$$\begin{aligned} S_X(p_{t+1}) =& a_1 S_X(h_{1,t+1}) + \cdots + a_n S_X(h_{n,t+1}) \\ =& a_1 \left(s^{d_t - 2} \frac{1}{t+1} q_{1,1} + \cdots + s^{d_t - r_{1,t} - 1} \frac{r_{1,t}}{t+1} q_{1,r_{1,t}} \right) \\ &+ \ldots \\ &+ a_n \left(s^{d_t - 2} \frac{1}{t+1} q_{n,1} + \cdots + s^{d_t - \gamma_t - 1} \frac{r_{n,t}}{t+1} q_{n,r_{n,t}} \right) \\ =& s^{d_t - 2} \frac{1}{t+1} q_1 + \cdots + s^{d_t - \gamma_t - 1} \frac{\gamma_t}{t+1} q_{\gamma_t}. \end{aligned}$$

It remains to prove statement (1). This follows immediately from statement (2). □

Now we introduce a definition that will serve to increase readability.

Definition 5.7. For half integers $a, b \in \frac{1}{2}\mathbb{Z}$, such that $a \leq b$ and $b - a \in \mathbb{N}$:

$$\{a, \ldots, b\} := \{a, a+1, a+2, \ldots, b\}$$

and

$$\sum_{j=a}^{b} h(j) := h(a) + h(a+1) + \cdots + h(b).$$

Corollary 5.8. Given $p \in R_X$ homogeneous, suppose $S_X(p) = \sum_{j=\gamma}^{\mathrm{Deg}(p)} s^j p_j$ with $p_j \in R_X$ and $p_\gamma \neq 0$. Then the sum is in S-form.

Proof. Assume $p_j \neq 0$ for $j \geq \gamma$. Then $S_X(p_j) \neq 0$ by Corollary 3.17, which by Corollary 5.6. (2) implies $S_X(p_{j+1}) \neq 0$, which again implies $p_{j+1} \neq 0$. □

By Definition 5.7 and Corollary 5.8, for homogeneous $p \in R_X$, the notation of S-form $S_X(p) = \sum_{i=1}^{n} s^{c_i} q_i$ turns into $S_X(p) = \sum_{j=\gamma}^{\mathrm{Deg}(p)} s^j p_j$ where $\gamma \in \frac{1}{2}\mathbb{Z}$ such that $\gamma = c_1$. The next theorem is crucial for refining Algorithm 4.12.

Theorem 5.9. Let $p, g \in R_X$ be homogeneous and assume that both sums

$$S_X(p) = \sum_{j=\gamma_p}^{\mathrm{Deg}(p)} s^j p_j \quad \text{and} \quad S_X(g) = \sum_{j=\gamma_g}^{\mathrm{Deg}(g)} s^j g_j$$

are in S-form. If $g \in \mathrm{LT}(p)$ then $\mathrm{Deg}(p) = \mathrm{Deg}(g)$, $\gamma_p = \gamma_g$, and

$$g_j \in \mathrm{LT}(p_j), \quad j \in \{\gamma_p, \ldots, \mathrm{Deg}(p)\}.$$

Proof. By Definition 4.1, the LT orbit is built up by the powers of $\widetilde{S}$ and T_X. Since $\widetilde{S}$ and T_X both keep the degree, we deduce that if $g \in \mathrm{LT}(p)$ then $\mathrm{Deg}(p) = \mathrm{Deg}(g)$.

The proof of the remaining part proceeds by induction on the length of

$$g = S^{k_1} T^{\ell_1} \cdots S^{k_m} T^{\ell_m}(p).$$

For the induction step, it suffices to prove the statement for two neighboring situations:

$$g = \widetilde{S}(p) \quad \text{and} \quad g = T_X(p).$$

Assume $g = \widetilde{S}(p)$. Let $p = a_1 h_1 + a_2 h_2 + \cdots + a_n h_n$ where the h_t are monomials in R_X with the same degree and the $a_t \in \mathbb{K} \setminus \{0\}$. Suppose $S_X(h_t) = \sum\limits_{j=r_t}^{d} s^j h_{t,j}$ in S-form with $d := \mathrm{Deg}(p)$.

We first prove that $S_X(\sigma(h_t)) = \sigma(S_X(h_t))$. Since σ and S_X are homomorphisms, it suffices to show this is true for the generators, which means we have to prove $S_X(\sigma(x_i^{(k)})) = \sigma(S_X(x_i^{(k)}))$ for any $i \in \{1, \ldots, 4\}$ and $k \in \mathbb{N}$. This is implied immediately by Definitions 3.6 and 3.16.

Let $r := \max\limits_{t=1,\ldots,n} \{r_t\}$ and $h_{t,j} := 0$ when $j < r_t$. Then by Lemma 5.2 we have

$$\begin{aligned} S_X(p) =& a_1 S_X(h_1) + \cdots + a_n S_X(h_n) \\ =& a_1 \sum_{j=r_1}^{d} s^j h_{1,j} + \cdots + a_n \sum_{j=r_n}^{d} s^j h_{n,j} \\ =& s^d (a_1 h_{1,d} + \cdots + a_n h_{n,d}) + \cdots + s^r (a_1 h_{1,r} + \cdots + a_n h_{n,r}). \end{aligned}$$

By Definitions 3.6, 3.16 and the linearity of $\widetilde{S}$ we also have

$$g = \widetilde{S}(p) = a_1 \widetilde{S}(h_1) + \cdots + a_n \widetilde{S}(h_n) = a_1 (-i)^{k_1} \sigma(h_1) + \cdots + a_n (-i)^{k_n} \sigma(h_n),$$

where the $k_t := \nu_1(h_t) + \frac{\mu(h_t)}{2}$. Then

$$\begin{aligned} S_X(g) =& a_1 (-i)^{k_1} S_X(\sigma(h_1)) + \cdots + a_n (-i)^{k_1} S_X(\sigma(h_n)) \\ =& a_1 (-i)^{k_1} \sigma\left(\sum_{j=r_1}^{d} s^j h_{1,j}\right) + \cdots + a_n (-i)^{k_n} \sigma\left(\sum_{j=r_n}^{d} s^j h_{n,j}\right) \\ =& s^d (a_1 (-i)^{k_1} \sigma(h_{1,d}) + \cdots + a_n (-i)^{k_n} \sigma(h_{n,d})) \\ & + \ldots \\ & + s^r (a_1 (-i)^{k_1} \sigma(h_{1,r}) + \cdots + a_n (-i)^{k_n} \sigma(h_{n,r})). \end{aligned}$$

Since for $j \in \{r, r+1, \ldots, d\}$,

$$\begin{aligned}\widetilde{S}(a_1 h_{1,j} + \cdots + a_n h_{n,j}) &= a_1 \widetilde{S}(h_{1,j}) + \cdots + a_n \widetilde{S}(h_{n,j}) \\ &= a_1(-i)^{k_1}\sigma(h_{1,j}) + \cdots + a_n(-i)^{k_n}\sigma(h_{n,j}),\end{aligned}$$

we obtain

$$a_1(-i)^{k_1}\sigma(h_{1,j}) + \cdots + a_n(-i)^{k_n}\sigma(h_{n,j}) \in \mathrm{LT}(a_1 h_{1,j} + \cdots + a_n h_{n,j}).$$

Hence

$$g_j \in \mathrm{LT}(p_j).$$

Then $g_j = 0$ if and only if $p_j = 0$. Therefore $\gamma_p = \gamma_g$.

For $g = T_X(p)$ the proof is analogous. $\square$

Applying Corollary 5.6 and Theorem 5.9 we can simplify Algorithm 4.12 substantially. The essence of the simplification is the following theorem.

Theorem 5.10. Given $p \in R_X$ homogeneous and $S_X(p) = \sum_{j=r}^{\mathrm{Deg}(p)} s^j q_j$ in S-form, then

$$p \in \ker\phi \text{ if and only if } \mathrm{ord}\left(\prod_{g \in \mathrm{LT}(q_j)} \phi(g)\right) > \frac{\mathrm{Deg}(q_j)|\mathrm{LT}(q_j)|}{6} \text{ for all } j \in \{r \ldots, \mathrm{Deg}(p)\}.$$

Proof. Assume $p \in \ker\phi$. By Lemma 3.26, $q_j \in \ker\phi$ for all $j \in \{r, \ldots, \mathrm{Deg}(p)\}$. Therefore, for any $g \in \mathrm{LT}(q_j)$, by Lemma 4.4 we have $g \in \ker\phi$. This implies $\prod_{g \in \mathrm{LT}(q_j)} \phi(g)(\tau) \equiv 0$. And hence

$$\infty = \mathrm{ord}\left(\prod_{g \in \mathrm{LT}(q_j)} \phi(g)\right) > \frac{\mathrm{Deg}(q_j)|\mathrm{LT}(q_j)|}{6}.$$

Assume $p \notin \ker\phi$. According to Lemma 3.26, at least one of the q_j is not in $\ker\phi$. Take $t \in \{r, \ldots, \mathrm{Deg}(p)\}$ such that $q_t \notin \ker\phi$ and $q_i \in \ker\phi$ when $i < t$. We prove that $\prod_{g \in \mathrm{LT}(q_t)} \phi(g)(\tau)$ is a modular form.

Case 1: $t = r$. By Corollary 5.6.1, $S_X(q_t) = S_X(q_r) = s^{r_t} h$ in S-form, where $h \notin \ker\phi$ because $q_t \notin \ker\phi$. Hence for every $g \in \mathrm{LT}(q_t)$, by Theorem 5.9, there exists $q \in R_X$ such that $S_X(g) = s^{r_t} q$ in S-form and $q \notin \ker\phi$. By Lemma 4.10, $\prod_{g \in \mathrm{LT}(q_t)} \phi(g)(\tau) \in M(\mathrm{Deg}(q_t)|\mathrm{LT}(q_t)|)$.

Case 2: $t > r$. Suppose $S_X(q_t) = \sum_{j=r_t}^{\mathrm{Deg}(q_t)} s^j h_j$ in S-form. Since $q_t \notin \ker\phi$, at least one of the h_j is not in $\ker\phi$. By rewriting of Corollary 5.6.2,

$$S_X(q_{t-1}) = \sum_{j=r_t}^{\mathrm{Deg}(q_t)-1} s^{j-1} \frac{\mathrm{Deg}(q_t) - j - 1}{t+1} h_j \text{ in } S\text{-form},$$

where, again by Lemma 3.26, $h_j \in \ker\phi$ for $r_t \le j \le \mathrm{Deg}(q_t) - 1$ because $q_{t-1} \in \ker\phi$. Thus $h_{\mathrm{Deg}(q_t)} \notin \ker\phi$. Hence for $g \in \mathrm{LT}(q_t)$, applying Theorem 5.9 we have $S_X(g) =$

$\sum_{j=r_t}^{\mathrm{Deg}(q_t)} s^j g_j$ in S-form with $g_j \in \mathrm{LT}(h_j)$, which yields $g_j \in \ker\phi$ for $r_t \leq j \leq \mathrm{Deg}(q_t) - 1$ and $g_{\mathrm{Deg}(q_t)} \notin \ker\phi$. Again by Lemma 4.10, $\prod_{g \in \mathrm{LT}(q_t)} \phi(g)(\tau) \in M(\mathrm{Deg}(q_t)|\mathrm{LT}(q_t)|)$.

In addition, $q_t \notin \ker\phi$ implies $\prod_{g \in \mathrm{LT}(q_t)} \phi(g)(\tau) \neq 0$. Therefore by Lemma 4.9 we obtain

$$\mathrm{ord}\left(\prod_{g \in \mathrm{LT}(q_t)} \phi(g)\right) \leq \frac{\mathrm{Deg}(q_t)|\mathrm{LT}(q_t)|}{6}.$$

□

The algorithmic content of Theorem 5.10 is the following:

Algorithm 5.11. Given $p \in R_X$ homogeneous and $S_X(p) = \sum_{j=r}^{\mathrm{Deg}(p)} s^j q_j$ in S-form, we have the following algorithm to prove or disprove $p \in \ker\phi$.

Input: homogeneous $p \in R_X$; output: True or False.

If $\mathrm{Deg}(p) > 0$ set $j := r$. While $j \leq \mathrm{Deg}(p)$ do

if $\mathrm{ord}\left(\prod_{g \in \mathrm{LT}(q_j)} \phi(g)\right) > \frac{\mathrm{Deg}(q_j)|\mathrm{LT}(q_j)|}{6}$

then $j := j + 1$;

else return False;

exit;

end do;

return True.

If $\mathrm{Deg}(p) = 0$ then True if $p = 0$; False if $p \in \mathbb{K} \setminus \{0\}$.

One can connect our method to classical methods using "Sturm bounds". Namely, in Theorem 5.10 one can replace $|\mathrm{LT}(q_j)|$ with 384, owing to Proposition 4.2. Moreover, for every $g \in \mathrm{LT}(q_j)$, the q-expansion of $\phi(g)$ only contains non-negative powers of q. Thus to show $\mathrm{ord}\left(\prod_{g \in \mathrm{LT}(q_j)} \phi(g)\right)$ is greater than a certain number, it suffices to show that $\mathrm{ord}(\phi(q_j))$ is greater than this number. Summarizing, we have the following corollary.

Corollary 5.12. Let $p \in R_X$ be homogeneous and $S_X(p) = \sum_{j=r}^{\mathrm{Deg}(p)} s^j q_j$ in S-form. Then

$$p \in \ker\phi \text{ if and only if } \mathrm{ord}(\phi(q_j)) > 2^6 \cdot \mathrm{Deg}(p) \text{ for all } j \in \{r \ldots, \mathrm{Deg}(p)\}.$$

We also present a modular form version.

Proposition 5.13. Let $p \in R_X$ be homogeneous. If $\phi(p) \in M(k) \setminus \{0\}$ then

$$\mathrm{ord}(\phi(p)) \leq 2^6 \cdot k.$$

Proof. Let $S_X(p) = \sum_{j=r}^{\mathrm{Deg}(p)} s^j q_j$ in S-form. If $\phi(p) \neq 0$, by Corollary 5.12 we have

$$\mathrm{ord}(\phi(q_j)) \leq 2^6 \cdot \mathrm{Deg}(p) \text{ for all } j \in \{r \ldots, \mathrm{Deg}(p)\}. \tag{16}$$

If $\phi(p) \in M(k) \setminus \{0\}$, by Definition 3.6 and Definition 4.5 we have

$$(S_0\phi(p))(\tau) = \phi(p)(-1/\tau) = \tau^k \phi(p)(\tau). \tag{17}$$

This together with Lemma 3.7 implies that $S_X(p) = s^k \widetilde{S}(p)$. Then (16) can be stated as

$$\begin{aligned} \mathrm{ord}(\phi(\widetilde{S}(p))) &\leq 2^6 \cdot \mathrm{Deg}(p) \\ &= 2^6 \cdot k, \end{aligned}$$

where the last equality follows from Lemma 4.6. In the end we show that $\phi(\widetilde{S}(p)) = \phi(p)$. Again by using Lemma 3.7 we have

$$S_0\phi(p) = \phi^*(S_X(p)) = \phi^*(s^k \widetilde{S}(p)) = \tau^k \phi^*(\widetilde{S}(p)) = \tau^k \phi(\widetilde{S}(p)). \tag{18}$$

We plug (17) into (18) and complete the proof. □

Next we do the complexity analysis.

Definition 5.14. For homogeneous $p \in R_X$ define $s(p)$ to be the number of S_X operations required to run Algorithm 4.12 on p.

Definition 5.15. For homogeneous $p \in R_X$ define $o(p)$ to be the number of LT operations required to run Algorithm 4.12 on p. An LT operation is a function that computes the elements of the leading term orbit of a given polynomial in R_X.

Definition 5.16. Let $p \in R_X$ be homogeneous with $S_X(p) = p_1 s^{\mathrm{Deg}(p)} + p_2 s^{\mathrm{Deg}(p)-1} + \cdots + p_r s^{\mathrm{Deg}(p)-r+1}$ in S-form. We define $\ell(p) := r$ to be the length of p.

Lemma 5.17. Let $p \in R_X$ be homogeneous with $S_X(p) = p_1 s^{\mathrm{Deg}(p)} + p_2 s^{\mathrm{Deg}(p)-1} + \cdots + p_r s^{\mathrm{Deg}(p)-r+1}$ in S-form and $|\mathrm{LT}(p_j)| = 384$. Then the number of $o(\tilde{p})$ and $s(\tilde{p})$ applications on any polynomial $\tilde{p}$ appearing when running Algorithm 4.12 on p depends only on the length of $\tilde{p}$.

Proof. Suppose $M = \{\tilde{p}_1, \ldots, \tilde{p}_m\}$ are the polynomials appearing when running Algorithm 4.12 on p. Consider $M = M_1 \cup M_2 \cup \cdots \cup M_n$ where $M_j := \{\tilde{p} \in M : \ell(\tilde{p}) = j\}$. By Corollary 5.6 (1) it is clear that for any $\tilde{p} \in R_X$ with $\ell(\tilde{p}) = 1$ we have $o(\tilde{p}) = 1$. Then by induction on j we prove that for every $f, g \in M_j : o(f) = o(g)$. Assume this is true for $j < k$. We prove that this is also true for $j = k$. Define $\tilde{o} : \{1, \ldots, k-1\} \to \mathbb{N}$ by $\tilde{o}(j) := o(q)$ where $q \in M_j$, which by the induction hypothesis is well-defined. Let $\tilde{p} \in M_k$. Then by Theorem 5.9 we have $\tilde{p} \in \mathrm{LT}(p_j)$ for some $j \in \{1, \ldots, r\}$, hence $|\mathrm{LT}(\tilde{p})| = 384$. Suppose $\mathrm{LT}(\tilde{p}) = \{q_1, \ldots, q_{384}\}$ with $S_X(q_j) = \sum_{t=r_j}^{\mathrm{Deg}(q_j)} s^t q_{j,t}$ in S-form.

Note that $\mathrm{Deg}(q_j) - r_j + 1 = k$. We know from Theorem 5.5 that $\ell(q_{j,r_j}) = 1$ and $\ell(q_{j,i}) = \ell(q_{j,i-1}) + 1$ for all $j \in \{1, \ldots, 384\}$. Therefore running Algorithm 4.12 on $\tilde{p}$ results in one orbit computation on $\tilde{p}$ and triggers a running of Algorithm 4.12 on $q_{j,t}$ for all $j \in \{1, \ldots, 384\}$ and for all $t \in \{1, \ldots, \mathrm{Deg}(q_j) - 1\}$. For the operation count this means,

$$\begin{aligned} o(\tilde{p}) &= 1 + \sum_{j=1}^{384} \sum_{t=r_j}^{\mathrm{Deg}(p_j)-1} o(q_{j,t}) = 1 + \sum_{j=1}^{384} \sum_{t=r_j}^{\mathrm{Deg}(p_j)-1} \tilde{o}(\ell(q_{j,t})) \\ &= 1 + \sum_{j=1}^{384} \sum_{t=r_j}^{\mathrm{Deg}(p_j)-1} \tilde{o}(t - r_j + 1) = 1 + 384 \sum_{t=1}^{\mathrm{Deg}(p_j)-r_j} \tilde{o}(t) \\ &= 1 + 384 \sum_{t=1}^{k-1} \tilde{o}(t). \end{aligned}$$

Since this shows that $o(\tilde{p})$ is only dependent on $k = \ell(\tilde{p})$, this completes the induction proof for the $o(p)$ statement. The $s(p)$ statement is proven analogously. □

Corollary 5.18. Let $N_1(p)$ and $N_2(p)$, respectively, be the total number of LT and S_X operations when running Algorithm 4.12 and Algorithm 5.11 on a given homogeneous $p \in R_X$. Let k be the length of p. Then in the worst case $N_1(p)$ is exponential and $N_2(p)$ is linear in k.

Proof. According to Proposition 4.2, in the worst case $|\mathrm{LT}(\tilde{p})| = 384$ for every polynomial $\tilde{p}$ appearing when running Algorithm 4.12 on p. By Lemma 5.17 we have

$$o(p) = \tilde{o}(k) = 1 + 384 \sum_{t=1}^{k-1} \tilde{o}(t)$$

with $\tilde{o}(1) = 1$. Analogously we define $\tilde{s} : \{1, \ldots, k-1\} \to \mathbb{N}$ by $\tilde{s}(j) := s(q)$ where $q \in M_j$. Then by doing the same induction steps as Lemma 5.17 one can prove that

$$s(p) = \tilde{s}(k) = 384 \sum_{t=1}^{k-1} \tilde{s}(t)$$

with $\tilde{s}(1) = 1$. Thus we obtain $o(p) = 385^{k-1}$ and $s(p) = 385^{k-1} - 385^{k-2}$ for $k \geq 2$. Therefore

$$N_1(p) = o(p) + s(p) = \begin{cases} 2 \cdot 385^{k-1} - 385^{k-2} & \text{if } k \geq 2 \\ 2 & \text{if } k = 1 \end{cases}.$$

For Algorithm 5.11, since only one S_X operation and k LT operations happen, we have $N_2(p) = 1 + k$. □

6. Conclusion

There are several natural extensions and generalizations of our algorithmic approach. First, one could extend from q to powers of q as, for instance,

$$\theta_3(0,q)\theta_3(0,q^3) - \theta_4(0,q)\theta_4(0,q^3) - \theta_2(0,q)\theta_2(0,q^3) \equiv 0$$

from (3.8) Joyce (1998). Another direction deals with the argument z; we are preparing a paper that uses our algorithmic setting not only for $z = 0$ but also for identities in arbitrary z. This concerns identities like those presented in Example 1.3, with possible further extensions to identities involving derivatives $\theta_j^{(k)}(z, q)$. A further generalization concerns identities involving $\theta_j(w + z, q)$ like

$$\theta_4(0,q)^2\theta_4(w+z,q)\theta_4(w-z,q) - \theta_3(w,q)^2\theta_3(z,q)^2 + \theta_2(w,q)^2\theta_2(z,q)^2 \equiv 0$$

from (20.7.9) DLMF (2016).

Another aspect is the computer-assisted discovery of relations among Jacobi theta functions. Based on the homogeneous decomposition described in this paper, we are able to derive all homogeneous polynomials $p \in R_X$ with given degree d which map to identities $\phi(p) = 0$ in R_Θ. Two examples of degree 3 and 4, respectively, are:

$$\theta_4^3\theta_1' + \theta_3\theta_2'' - \theta_2\theta_3'' \equiv 0$$

and

$$\theta_3\theta_4^2\theta_2'' + \theta_2\theta_4^2\theta_3'' + \theta_1'\theta_4'' - \theta_4\theta_1^{(3)} \equiv 0$$

where $\theta_j^{(k)} := \theta_j^{(k)}(0, q)$. In Chapter 6 of Ye (2016) we present the method in detail. In addition, in this context one can derive an upper bound for the dimension of $\ker\phi$ where ϕ is restricted to the $\mathbb{C}$-vector space of the homogeneous polynomials in R_X^d. Details will be presented in a forthcoming paper.

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