Silviu Radu

An Algorithmic Approach to Ramanujan's Congruences and Related Problems

Doctoral Thesis

Research Institute for Symbolic Computation
Johannes Kepler University Linz

Advisor: Univ.-Prof. Dr. Peter Paule

Examiners:
Univ.-Prof. Dr. Peter Paule
Prof. Dr. James Sellers

My PhD Thesis work has been supported by grants
SFB F013, P20162 and W1214-DK6 of the Austrian Science Funds FWF.

Abstract

In this thesis we study Ramanujan's congruences and related problems from an algorithmic point of view. We present an algorithm that proves such congruences. Furthermore, we also describe an algorithm that proves identities involving certain infinite products which are tightly connected to Ramanujan's congruences. The techniques involved are based on modular forms and we spend some time on nailing down the definitions and results involved in this area. In the last part of the thesis we give a proof of a result conjectured by James Sellers about generalized Frobenius partitions, which resulted from joint work with Peter Paule.

Kurzzusammenfassung

In der vorliegenden Arbeit werden Ramanujan Kongruenzen und verwandte Probleme von einem algorithmischen Gesichtspunkt aus behandelt. Wir präsentieren einen neuen Algorithmus, der derartige Kongruenzen behandeln kann. Ausserdem beschreiben wir eine algorithmische Methode, mit der Identitäten über bestimmte unendliche Produkte, die eng mit Ramanujan Kongruenzen verbunden sind, bewiesen werden können. Die verwendeten Techniken basieren auf modularen Formen. Die notwendigen theoretischen Grundlagen für modulare Formen werden ausführlich eingeführt. Den Abschluss dieser Arbeit stellt der Beweis einer Vermutung von James Sellers über verallgemeinerte Frobenius Partitionen, der in Kooperation mit Peter Paule entstanden ist.

Acknowledgments

Δόξα τω Θεώ πάντων ένεκεν

Αγιος Ιωάννης Χρυσόστομος

It is very difficult to give enough credit to all the people that helped me with my thesis and I ask the people that read this section to forgive my poor skills in writing this very important part.

Naturally Professor Peter Paule, my advisor is the first person I want to thank. Already when he took me as a student he knew that he had to spend more time on polishing me than with other students but this did not discourage him. He did not tell me this but I know that he took a lot of his free time to read my bad quality manuscripts. At some point any normal person will lose hope with me but this is not the case of Professor Paule. I feel that the time I spent on my doctorate is the most productive time of my life and I have learned things I could not dream of. I learned to think in an algorithmic fashion which opened completely unexpected paths to studying mathematics. I greatly improved my writing and communication skills. Most important though I learned not to fall into "traps" that lead to waste of time on problems that are tempting but lead nowhere. Professor Paule helped me to keep focused on the right thing at the right time. I really can not thank him enough. All this thesis is the result of very tight collaboration with Professor Paule. My contribution is that I always followed his advice.

I would further like to thank the Austrian Science Funds FWF for the grants SFB F013, P20162 and W1214-DK6 from which I have been employed.

I would further like to thank my wife Luminiţa and my daughter Cristina for following me with their love and support on this path I chose to take. Without them I could never have been here.

I would further like to thank my father Florin for many discussions before and after the thesis. He always gave me the advice to listen to my advisor and woke me up many times when I started to go off track. Also before my doctorate studies he gave me advice and support without which I would not have been here. This is just a small part of his many other contributions.

I would further like to thank my mother Lucia that helped us with our daughter Cristina. Whenever she came to us our research production significantly grew. With this occasion I would also like to thank my grandmother Anica who spend over 1 year in Austria to help us. Again this is just a small part of their contribution.

I would further like to thank Professor Gert Almkvist who advised my diploma thesis. He basically found me this position and personally called Professor Paule. Thanks to his efforts I was accepted as a student.

I would further like to thank Professor Bruno Buchberger who founded this wonderful place RISC, where we have a great chance to do research.

I would further like to thank Professor James Sellers thanks to whom chapter 6 and 7 of this thesis were possible. He also kindly agreed to be an examiner of this work and make all the trip from America.

I would further like to thank my brother Sorin who called me many times during my doctorate. Our discussions helped me very much.

A special thanks goes to my best friend Cătălin Anghel thanks to whom I switched to mathematics. He was there in one of those moments when one realizes important things in ones life.

Another special thanks goes to Eduard Teodorescu who had a contribution in my rediscovery of Electronics. This topic I first learned from my father. Electronics later lead me to Mathematics and Eduard was another person who was there at the right time.

I would further like to thank my father in law Augustin with which I had many philosophical and scientifical discussions that influenced me in the direction of following university studies. Also the other members of my wife's family helped me to go in this direction. Especially her brother Daniel and her mother Camelia.

I would further like to thank my adoptive country Sweden for accepting my family and for giving me the wonderful opportunity to study at the Lunds University of Sweden. This is the place where I met my advisor Professor Gert Almkvist and Professor Arne Meurman who taught me algebra and which I would also like to thank. Many other people in Sweden affected my mathematical evolution such as the professors: Annika Sparr, Lars-Christer Böiers, Victor Ufnarovski, Jörg Schmeling, Pavel Kurasov, Jan Gustavsson, Anders Holst, Gunnar Traustason and many others.

I would further like to thank the colleagues of my group Ralf Hemmecke, Burkhard Zimmermann, Manuel Kauers, Veronika Pillwein, Flavia Stan, Burçin Eröcal, Clemens Raab, Carsten Schneider, Christoph Koutschan and Zafeirakis Zafeirakopoulos. Ralf was the man thanks to whom I got my apartment. Without it I could not have brought my family to Austria. Since Ralf was my temporary advisor during the first year I could not enumerate the many other situations where he helped me. Burkhard shared with me many interesting mathematical ideas that helped me very much in this thesis. Especially he explained me the paper [19]. Burkhard was also involved in finding my first apartment in Hagenberg. Manuel helped me with a huge variety of different tasks. For example he helped me to look for bugs in my own programs and encouraged me to program everything that can be programmed. This helped me to discover many errors. Veronika similarly helped with a huge list of tasks. As an example, I would like to mention that she gave me the latex version of her thesis which saved me a lot of time. Flavia gave me a lot of tips during my first year of studies and also later. Burçin was in the same office with me for one year and we had many interesting mathematical discussions and he has always been kind and helpful to me. Clemens shared his expertise in many topics with me. His patience is formidable. I would like to thank Carsten for always helping me kindly in any possible matter. Cristoph helped me very much with Maple. Finally I thank Zafeirakis for many interesting discussions.

I would further like to thank Adela and Cosmin Mitran for their special interest in the well being of our family and of my doctorate studies. The discussions we had helped me very much.

I would further like to thank Cornelius Nevrinceanu who influenced me enormously under my first year. He is a great friend and we had many mathematical discussions.

I would further like to thank Professor Franz Winkler because of his wonderful Computer Algebra course. I would also like to thank Professor Josef Schicho for answering many questions about Algebraic Geometry and Dr. Günter Landsmann for his lectures on various algebraic topics.

I would further like to thank Professor Heinrich Rolletschek for giving me many rides to RISC and for many interesting mathematical and philosophical discussions.

I would further like to thank Dr. Wolfgang Windsteiger for always being very helpful in many important situations.

I would further like to thank Professor Tudor Jebelean for his TSW course.

I would further like to thank Chau Ngo for many mathematical and philosophical discussions. I extremely appreciate that he never gets angry. I also like to thank Johannes Middeke who taught me a lot of algebra.

I would further like to thank Mădălina Eraşcu, Mădălina Hodorog and Camelia Rosenkranz for many interesting discussions.

I would further like to thank Dan Matei for teaching me how to eat healthy. The eating part showed to be an important factor because the right food helped me to stay focussed. I also got interesting food advice from Muttalip Inekçi and Cornelius Nevrinceanu for which I thank both.

I would further like to thank Tanja Gutenbrunner, Gabriela Hahn and Ramona Pöchinger for being very nice and helpful.

I would further like to thank Mihai Irimia and Dacian Tudor who helped me to think about other things than only mathematics. Our discussions really helped me to relax.

I would further like to thank the family of our priest Sorin Emanuel Bugner for praying for my family and for introducing me to the saints of orthodoxy. Especially I would like to thank Saint Mary through which God gave me a lot of help.

Finally I thank God for his active help and for sending me to all this wonderful people I mentioned and many others that should have been mentioned.

Contents

Chapter 1

Extended Abstract

Our main focus is to find algorithmic solutions to problems related to congruences for infinite products. More precisely, we consider power series generated by products as follows:

$$\sum_{m=0}^{\infty} a_r(m) q^m := \prod_{\delta|N} \prod_{n=1}^{\infty} (1 - q^{\delta n})^{r_\delta}, \tag{1.1}$$

where N is a positive integer and $r = (r_\delta)_{\delta|N}$ is an integer sequence indexed by the positive divisors δ of N; we denote the set of all such sequences by $R(N)$. By a congruence associated to such an infinite product we mean a tuple (m, t, p, r) such that:

$$a_r(mn + t) \equiv 0 \pmod{p}, \quad n \in \mathbb{N}, \tag{1.2}$$

where m is a positive integer, $t \in \{0, \ldots, m-1\}$, p a prime and $r \in R(N)$ for some positive integer N. For example, the generating function for $p(n)$ the number of partitions of n is given by

$$\sum_{m=0}^{\infty} p(m) q^m = \prod_{n=1}^{\infty} (1 - q^n)^{-1}.$$

Ramanujan discovered that:

$$p(5n + 4) \equiv 0 \pmod{5}, \quad n \in \mathbb{N}, \tag{1.3}$$

$$p(7n + 5) \equiv 0 \pmod{7}, \quad n \in \mathbb{N}, \tag{1.4}$$

and

$$p(11n + 6) \equiv 0 \pmod{11}, \quad n \in \mathbb{N}. \tag{1.5}$$

The congruence (1.3) follows from Ramanujan's most beautiful identity (as stated by Hardy):

$$\sum_{m=0}^{\infty} p(5m + 4) q^m = 5 \prod_{n=1}^{\infty} \frac{(1 - q^{5n})^5}{(1 - q^n)^6}. \tag{1.6}$$

We also investigate generalizations of (1.6) because of the tight connection to the congruence (1.3). Our work started when Peter Paule asked for an algorithmic proof of the congruences

$$\Delta_2(10n + 2) \equiv 0 \pmod{2} \tag{1.7}$$

and

$$\Delta_2(25n + 14) \equiv 0 \pmod 5 \tag{1.8}$$

where $\Delta_2(n)$ counts the number of broken 2-diamonds of length n as introduced by George E. Andrews and Peter Paule in [2]. The congruences (1.7)-(1.8) appeared first as conjectures in [2] and became theorems thanks to the work of Michael Hirschhorn and James Sellers [17] who proved (1.7) and Song Heng Chan [8] who proved (1.8). In [2] it was also shown that

$$\sum_{n=0}^{\infty} \Delta_2(n) q^n = \prod_{n=1}^{\infty} \frac{(1-q^{2n})(1-q^{5n})}{(1-q^n)^3(1-q^{10n})}.$$

Consequently, the generating function for $\Delta_2(n)$ is of the desired form (1.1). When looking at the available literature we found an algorithm by Eichhorn and Ono [9] that we could modify to cover (1.7) and (1.8). It should be mentioned that Dennis Eichhorn [11] proposed an algorithm that could deal with any problem of the type (1.2) which is very similar to the one in [9]. The idea of the algorithm is as follows. First we transform the congruence (1.2) into a congruence for the coefficients of some modular form (abbreviated by **MF**). At this point we need to explain what a **MF** is. To make a long story short, a weak modular form (abbreviated by **WMF**) of integer weight k for a subgroup $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$ is a function f defined on some domain, which is invariant under a certain action (depending on k) of the subgroup Γ. The subgroup Γ is required to be of finite index in $\mathrm{SL}_2(\mathbb{Z})$. From now on we denote the set of all **WMF** of weight k for the group Γ by $A_k(\Gamma)$. One can show that to each $f \in A_k(\Gamma)$ there corresponds uniquely an element $(f_1(q), \ldots, f_h(q)) \in \mathbb{C}((q))^h$ (where $\mathbb{C}((q))$ is the ring of formal Laurent series), which we call the expansion of f. Then number $h = h(\Gamma)$ can be computed and depends only on Γ. From now on we identify f with its expansion. In particular, for $f, g \in A_k(\Gamma)$ we mean by $f = g$ that $f_i(q) = g_i(q)$ for all $i \in \{1, \ldots, h\}$. One can also prove that $0 \in A_k(\Gamma)$ and of course $f = 0$ means $f_i(q) = 0$ for all $i \in \{1, \ldots, h\}$. Moreover, we can add **WMF**s by adding their expansions, and we note that if f, $g \in A_k(\Gamma)$, then $f + g \in A_k(\Gamma)$.

The most important observation about $f \neq 0 \in A_k(\Gamma)$ comes from the fact that

$$\sum_{i=1}^{h} \mathrm{ord}(f_i(q)) \leq \frac{k}{12}[\mathrm{SL}_2(\mathbb{Z}) : \Gamma]. \tag{1.9}$$

The formula (1.9) can be used to prove an identity like $f = g$ where f, $g \in A_k(\Gamma)$. The method is as follows. We first rewrite the identity in the form $f - g = 0$. Next we assign to $f - g$ its expansion $(f_1(q) - g_1(q), \ldots, f_h(q) - g_h(q))$ and prove that $\sum_{i=1}^{h} \mathrm{ord}(f_i(q) - g_i(q)) > \frac{k}{12}[\mathrm{SL}_2(\mathbb{Z}) : \Gamma]$. Then necessarily $f - g = 0$, otherwise we get a contradiction to (1.9). In practice one imposes more conditions on $f \in A_k(\Gamma)$, namely that $\mathrm{ord}(f_i(q)) \geq 0$ for all $i \in \{1, \ldots, h\}$. In this case, f is called a modular form (which is abbreviated by **MF**) of weight k for the subgroup Γ, and we denote the set of all such **MF**s by $M_k(\Gamma)$. For $f \neq 0 \in M_k(\Gamma)$ we immediately have by (1.9) that

$$\mathrm{ord}(f_1(q)) \leq \frac{k}{12}[\mathrm{SL}_2(\mathbb{Z}) : \Gamma], \tag{1.10}$$

which is easier to deal with in practice because proving $f = g$ where f, $g \in M_k(\Gamma)$, is equivalent to showing $\mathrm{ord}(f_1(g) - g_1(q)) > \frac{k}{12}[\mathrm{SL}_2(\mathbb{Z}) : \Gamma]$. The connection between **MF**s and congruences comes from the fact that typically for $f \in M_k(\Gamma)$ we have $f_1(q) \in \mathbb{Z}([q])$, the ring of formal power series with integer coefficients. In this case we say that f has integral coefficients. Given f, $g \in M_k(\Gamma)$ with integral coefficients assume we want to prove $f_1(q) \equiv g_1(q) \pmod{p}$. Then Jacob Sturm [40] has discovered an analogue of (1.10). Namely

$$\mathrm{ord}_p(f_1(q)) \leq \frac{k}{12}[\mathrm{SL}_2(\mathbb{Z}) : \Gamma]. \tag{1.11}$$

By $\mathrm{ord}_p(f_1(q))$ we mean taking the usual order of $f_1(q)$ after reducing the coefficients modulo p. At this point we can continue to explain the algorithm of Eichhorn and Ono. Given (1.2) they find some $F \in M_K(\Gamma')$ (for some integer K and group Γ') with integral coefficients and prove that showing (1.2) is equivalent to showing $F_1(q) \equiv 0 \pmod{p}$, which they prove using (1.11). Namely they prove that the first $\nu := \frac{K}{12}[\mathrm{SL}_2(\mathbb{Z}) : \Gamma]$ coefficients of $F_1(q)$ are congruent 0 modulo p. For this approach the main problem is that ν can sometimes gets very big and the computational effort needed can be tremendous. In Chapter 3 we present an algorithm that takes as input the congruence (1.2) and transforms it into an equivalent congruence of the form $f_1(q) \equiv 0 \pmod{p}$ where f is a **MF** with integral coefficients. Our method of obtaining f is based on papers by Rademacher [32], Kolberg [22] and Newman [27], [28] and differs from the method of Eichhorn and Ono. In short our approach leads to significant improvements of the bound ν. Based on the methods we develop in Chapter 3, we are able to prove identities of the type (1.6) in algorithmic fashion, and we show in Chapter 4 how one can obtain such identities by various examples. This thesis can be said to have as main scope the task to take infinite product identities (e.g., (1.6)) resp. congruences (e.g., (1.7)-(1.8)) and to transform them into identities resp. congruences between **MF**s. As seen by (1.9)-(1.11) it is crucial to optimize the weight k and the order $[\mathrm{SL}_2(\mathbb{Z}) : \Gamma]$ of the **MF**s involved. In Chapter 2 we show how one can prove certain identities involving infinite products of the form (1.1). Most part of our description is spent on showing that the identity we wish to prove is equivalent to several identities between certain **MF**s for the Hecke subgroup $\Gamma_0(N)$, $N \in \mathbb{N}^*$, which is the only class of subgroups that will be used throughout this thesis. We conclude the thesis with chapters 5 and 6, which to some part emerged from joint work with Peter Paule, that deal with proving a result conjectured by James Sellers [37]. The only tools involved in these chapters are those introduced in Chapter 1 where we provide the exact definitions, and where we prove the main properties of **WMF**s and **MF**s.

Finally I would like to mention that the material in Chapter 4 is already published [33]. The material of Chapter 6 is submitted. The remaining parts is in preparation.

Chapter 2

Introduction to Modular Forms

In this chapter we will introduce the theory of modular forms that we will need throughout this thesis.

2.1 Main Definitions

Throughout we will use the following conventions: $\mathbb{N} = \{0, 1, \ldots\}$ and $\mathbb{N}^* = \{1, 2, \ldots\}$ denote the nonnegative and positive integers, respectively. The complex upper half plane is denoted by $\mathbb{H} := \{\tau \in \mathbb{C} : \mathrm{Im}(\tau) > 0\}$.

The general linear group

$$\mathrm{GL}_2^+(\mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z} \text{ and } ad - bc > 0 \right\}$$

acts on elements τ of the upper half plane $\mathbb{H}$ as usual; i.e., for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2^+(\mathbb{Z})$:

$$\gamma\tau := \frac{a\tau + b}{c\tau + d}.$$

We recall basic notions related to the modular group

$$\mathrm{SL}_2(\mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z} \text{ and } ad - bc = 1 \right\}.$$

For integers a and c not both zero we let $\gcd(a, c)$ be the greatest positive integer dividing a and c.

Proposition 2.1. *Let* $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2^+(\mathbb{Z})$ *and* x, y *any integers such that*

$$ax + cy = \gcd(a, c).$$

Then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a/\gcd(a,c) & -y \\ c/\gcd(a,c) & x \end{pmatrix} \begin{pmatrix} \gcd(a,c) & bx + dy \\ 0 & (ad - bc)/\gcd(a,c) \end{pmatrix},$$

where the left matrix of the product is in $\mathrm{SL}_2(\mathbb{Z})$ *and the right matrix is in* $\mathrm{GL}_2^+(\mathbb{Z})$.

For $\tau \in \mathbb{H}$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2^+(\mathbb{Z})$ we define as in Rankin [35]:

$$(\gamma : \tau) := c\tau + d.$$

Note that for $\gamma_1, \gamma_2 \in \mathrm{GL}_2^+(\mathbb{Z})$ and $\tau \in \mathbb{H}$ we have the identity:

$$(\gamma_1\gamma_2 : \tau) = (\gamma_1 : \gamma_2\tau)(\gamma_2 : \tau). \tag{2.1}$$

Let $k \in \mathbb{Q}$. For every non-zero $w \in \mathbb{C}$ we define

$$w^k := e^{k \log(w)},$$

where the principal value of the logarithm is used; i.e. the argument is between $-\pi$ and π. For $\gamma_1, \gamma_2 \in \mathrm{GL}_2^+(\mathbb{Z})$ and $k \in \mathbb{Q}$ define

$$\sigma_k(\gamma_1, \gamma_2) := \frac{(\gamma_1 : \gamma_2\tau)^k(\gamma_2 : \tau)^k}{(\gamma_1\gamma_2 : \tau)^k}, \quad \tau \in \mathbb{H}.$$

Because of (2.1) we see that $|\sigma_k(\gamma_1, \gamma_2)| = 1$ for all $\tau \in \mathbb{H}$. One can show that $\sigma_k(\gamma_1, \gamma_2)$ is independent of τ. For $\gamma \in \mathrm{GL}_2^+(\mathbb{Z})$, $f : \mathbb{H} \to \mathbb{C}$ and $k \in \mathbb{Q}$ we define the "stroke operator" $f|_k\gamma : \mathbb{H} \to \mathbb{C}$ by the formula:

$$(f|_k\gamma)(\tau) := \det(\gamma)^{\frac{k}{2}}(\gamma : \tau)^{-k} f(\gamma\tau), \quad \tau \in \mathbb{H}. \tag{2.2}$$

The following formula is easily proven (e.g., Rankin [35, Th. 4.3.9]):

$$f|_k(\gamma_1\gamma_2) = \sigma_k(\gamma_1, \gamma_2)(f|_k\gamma_1)|_k\gamma_2, \quad \gamma_1, \gamma_2 \in \mathrm{GL}_2^+(\mathbb{Z}), k \in \mathbb{Z}. \tag{2.3}$$

Note. We should often write $f|_k\gamma_1\gamma_2$ instead of $f|_k(\gamma_1\gamma_2)$.

Definition 2.2. *We denote by $\mathcal{S}$ the set of all subgroups of $\mathrm{SL}_2(\mathbb{Z})$ of finite index.*

Our next task is to define the notion of weak modular form but before this we recall some facts from complex analysis.

Lemma 2.3. *Let $f : \mathbb{H} \to \mathbb{C}$ be holomorphic and l a positive integer. Let $D^* := \{q \in \mathbb{C} | |q| < 1, q \neq 0\}$ be the punctured unit disc. Assume that f is l-periodic, i.e. $f(\tau + l) = f(\tau)$ for $\tau \in \mathbb{H}$. Then there is a function $\tilde{f} : D^* \to \mathbb{C}$ holomorphic on D^* such that $\tilde{f}(e^{2\pi i\tau/l}) = f(\tau)$ for all $\tau \in \mathbb{H}$. In particular there exists unique $a : \mathbb{Z} \to \mathbb{C}$ with $n \mapsto a(n)$ such that*

$$\tilde{f}(e^{2\pi i\tau/l}) = \sum_{n=-\infty}^{\infty} a(n)e^{2\pi i\tau n/l}, \quad \tau \in \mathbb{H}. \tag{2.4}$$

Proof. Because of $f(\tau + l) = f(\tau)$ we can define a function $\tilde{f} : D^* \to \mathbb{C}$ by the formula $\tilde{f}(e^{2\pi i\tau/l}) := f(\tau)$ which is well defined because for given $q_0 \in \mathbb{C}$ and $\tau_0 \in \mathbb{C}$ such that $q_0 = e^{2\pi i\tau_0/l}$ the set of all solutions τ to the equation $q_0 = e^{2\pi i\tau/l}$ is given by the set $\{\tau_0 + ln | n \in \mathbb{Z}\}$ and $f(\tau_0 + ln) = f(\tau_0)$ by assumption. It remains to prove that $\tilde{f}$ is holomorphic.

Let $q_0 \in D^*$ and τ_0 be such that $q_0 = e^{2\pi i\tau_0/l}$. Let $\log_{\tau_0}$ be the complex logarithm with argument between $2\pi\frac{\mathrm{Re}(\tau_0)}{l} - \pi$ and $2\pi\frac{\mathrm{Re}(\tau_0)}{l} + \pi$. Then

$$\tilde{f}(q) = f\left(\frac{l}{2\pi i}\log_{\tau_0}(q)\right)$$

for $q \in D^*$ with $2\pi\frac{\mathrm{Re}(\tau_0)}{l} - \pi < \arg(q) < 2\pi\frac{\mathrm{Re}(\tau_0)}{l} + \pi$. Since $\frac{l}{2\pi i}\log_{\tau_0}$ is holomorphic in a neighborhood of q_0 and f is holomorphic in a neighbourhood of $\frac{l}{2\pi i}\log_{\tau_0}(q_0)$, their composition $\tilde{f}$ must be holomorphic at q_0.

Finally because $\tilde{f}$ is holomorphic on D^* it admits a Laurent expansion proving (2.4). □

Definition 2.4. *Let l be a positive integer. For $f : \mathbb{H} \to \mathbb{C}$ a holomorphic l-periodic function we define $a_{f,l} : \mathbb{Z} \to \mathbb{C}$ to be the unique sequence of Lemma 2.3 satisfying*

$$f(\tau) = \sum_{n=-\infty}^{\infty} a_{f,l}(n)e^{2\pi i\tau n/l}, \quad \tau \in \mathbb{H}.$$

Definition 2.5. *Let $k \in \mathbb{Z}$ and $f : \mathbb{H} \to \mathbb{C}$ be holomorphic. Then f is called a L_k-function if for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ there exist a positive integer l_γ and a integer m_γ such that $f_\gamma := f|_k\gamma$ is l-periodic and $a_{f_\gamma,l}(n) = 0$ if $n < m_\gamma$; moreover if $m_\gamma \geq 0$ (for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$) then f is called a T_k-function. We define the sets*

$$\mathbb{L}_k := \{f | f \text{ a } L_k\text{-function}\} \quad \text{and} \quad \mathbb{T}_k := \{f | f \text{ a } T_k\text{-function}\}.$$

Definition 2.6. *Let $f : \mathbb{H} \to \mathbb{C}$, $\Gamma \in \mathcal{S}$ and $k \in \mathbb{Z}$. Then f is called a weak modular form of weight k for the group Γ iff*

(i) f is holomorphic on $\mathbb{H}$;

(ii) $f|_k\gamma = f$ for all $\gamma \in \Gamma$;

(iii) $f \in \mathbb{L}_k$.

If in addition $f \in \mathbb{T}_k$ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ then f is called a modular form of weight k. The space of weak modular forms (resp. modular forms) of weight k for Γ will be denoted by $A_k(\Gamma)$ (resp. $M_k(\Gamma)$).

We next develop some tools that enables us to check when a function satisfies conditions *(i)* and *(iii)* of Definition 2.6.

Proposition 2.7. *Let $f : \mathbb{H} \to \mathbb{C}$ be holomorphic, $\gamma \in \mathrm{GL}_2^+(\mathbb{Z})$ and $k \in \mathbb{Z}$. Then the function $f|_k\gamma$ is holomorphic.*

Proof. The functions $g_\gamma : \mathbb{H} \to \mathbb{H}$ and $h_{\gamma,k} : \mathbb{H} \to \mathbb{C}$ defined by $g_\gamma(\tau) := \gamma\tau$ and $h_{\gamma,k}(\tau) = (\gamma : \tau)^{-k}$ for $\tau \in \mathbb{H}$ are holomorphic. From Complex Analysis we know that the property being holomorphic is preserved by composition and taking product of functions so that $h_{\gamma,k} \cdot f \circ g_\gamma$ is holomorphic on $\mathbb{H}$. By (2.2) we have $(f|_k\gamma) = \det(\gamma)^{k/2}h_{\gamma,k} \cdot f \circ g_\gamma$. This proves that $(f|_k\gamma)$ is holomorphic. □

Proposition 2.8. *Let k, $t \in \mathbb{Z}$ and $A \in \mathrm{GL}_2^+(\mathbb{Z})$. Then the following statements are true:*

(i) If f is in $\mathbb{L}_k$ (resp. $\mathbb{T}_k$) then $f|_kA$ is in $\mathbb{L}_k$ (resp. $\mathbb{T}_k$).

(ii) If f, g are in $\mathbb{L}_k$ (resp. $\mathbb{T}_k$) then $f + g$ is in $\mathbb{L}_k$ (resp. $\mathbb{T}_k$).

(iii) If f is in $\mathbb{L}_k$ (resp. $\mathbb{T}_k$) and g is in $\mathbb{L}_t$ (resp. $\mathbb{T}_t$) then fg is in $\mathbb{L}_{k+t}$ (resp. $\mathbb{T}_{k+t}$).

Proof. *(i):* Take $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ arbitrary but fixed. By Proposition 2.1 one can write $A\gamma = \xi T$ where $\xi \in \mathrm{SL}_2(\mathbb{Z})$ and $T = \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \in \mathrm{GL}_2^+(\mathbb{Z})$ is upper triangular. Let $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. Since f is in $\mathbb{L}_k$ (resp. $\mathbb{T}_k$) there exist a integer m (with $m \geq 0$ if $f \in \mathbb{T}_k$) and a positive integer l such that

$$g := (f|_k\xi)(\tau) = \sum_{n=m}^{\infty} a_{g,l}(n) e^{2\pi i \tau n/l}, \quad \tau \in \mathbb{H}.$$

This implies together with (2.3) and $A\gamma = \xi T$ that

$$\begin{aligned}((f|_kA)|_k\gamma)(\tau) &= (f|_kA\gamma)(\tau) = (f|_k\xi T)(\tau) = (f|_k\xi)|_kT \\ &= (ad)^{k/2}d^{-k}(f|_k\xi)\left(\frac{a\tau+b}{d}\right) = (ad)^{k/2}d^{-k}\sum_{n=m}^{\infty} a_{g,l}(n)e^{2\pi i\frac{bn}{ld}}e^{2\pi i\frac{a\tau n}{ld}}\end{aligned}$$

for $\tau \in \mathbb{H}$, showing the desired property for $f|_kA$ in agreement with Definition 2.5.

The proof of *(ii)* and *(iii)* are immediate from Definition 2.5. □

Definition 2.9. *Let $\Gamma \in \mathcal{S}$ and $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. Then we define $\omega_{\Gamma,\gamma}$ to be the least positive integer h such that $\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \in \gamma^{-1}\Gamma\gamma$. When Γ is clear from the context we will write ω_γ instead of $\omega_{\Gamma,\gamma}$.*

We will derive an explicit expression (in Lemma 2.37 below) for $\omega_{\Gamma,\gamma}$ in the special case when Γ is a Hecke subgroup.

Lemma 2.10. *Let $\Gamma \in \mathcal{S}$, k a positive integer, $f \in A_k(\Gamma)$ and $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. Then*

$$(f|_k\gamma)(\tau + \omega_\gamma) = (f|_k\gamma)(\tau), \quad \tau \in \mathbb{H}.$$

Proof. By assumption there exists $\xi \in \Gamma$ such that $\gamma\begin{pmatrix} 1 & \omega_\gamma \\ 0 & 1 \end{pmatrix} = \xi\gamma$. Then by (2.3) (because $\sigma_k \equiv 1$ for $k \in \mathbb{Z}$) and because of $f \in A_k(\Gamma)$ we have

$$f|_k\gamma\begin{pmatrix} 1 & \omega_\gamma \\ 0 & 1 \end{pmatrix} = f|_k\xi\gamma = (f|_k\xi)|_k\gamma = f|_k\gamma.$$

Noting that $\left(f|_k\gamma\begin{pmatrix} 1 & \omega_\gamma \\ 0 & 1 \end{pmatrix}\right)(\tau) = (f|_k\gamma)(\tau+\omega_\gamma)$ finishes the proof. □

Lemma 2.11. *Let $\Gamma \in \mathcal{S}$, k a positive integer, $f \in A_k(\Gamma)$ and $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. Then there exist an integer t and a sequence $b : \mathbb{Z} \to \mathbb{C}$ such that*

$$g(\tau) := (f|_k\gamma)(\tau) = \sum_{n=t}^{\infty} b(n)e^{2\pi i n\tau/\omega_\gamma}, \quad \tau \in \mathbb{H}.$$

This implies in particular that $g(\tau+\omega_\gamma) = g(\tau)$ for $\tau \in \mathbb{H}$ and $b(n) = a_{g,\omega_\gamma}(n)$ for $n \in \mathbb{Z}$ with $n \geq t$. If additionally $f \in M_k(\Gamma)$ then $t \geq 0$.

Proof. By Lemma 2.10 $g(\tau+\omega_\gamma)=g(\tau)$, and since $g\in\mathbb{L}_k$, because of *(iii)* in Definition 2.6, there exist a integer m and a positive integer l (where $m\geq 0$ if $f\in M_k(\Gamma)$) such that

$$g(\tau)=\sum_{n=m}^{\infty}a_{g,l}(n)e^{2\pi in\tau/l}. \tag{2.5}$$

Next change τ by $\tau+\omega_\gamma$ in (2.6) to obtain

$$g(\tau)=\sum_{n=m}^{\infty}a_{g,l}(n)e^{2\pi in\tau/l}=\sum_{n=m}^{\infty}a_{g,l}(n)e^{2\pi in\omega_\gamma/l}e^{2\pi i\tau n/l} \tag{2.6}$$

for $\tau\in\mathbb{H}$. If we substitute $q_l:=e^{2\pi i\tau/l}$ in (2.6) then we obtain an equality of Laurent series for every $q_l\in\mathbb{C}$ with $0<|q_l|<1$. From Complex Analysis, we know that the coefficients of Laurent series are unique and we can make coefficient comparison to obtain

$$a_{g,l}(n)=a_{g,l}(n)e^{2\pi in\omega_\gamma/l},\quad n\in\mathbb{Z},\ n\geq m,$$

which implies that $a_{g,l}(n)=0$ if $l\nmid n\omega_\gamma$. Finally, we can write

$$\begin{aligned}\sum_{n=m}^{\infty}a_{g,l}(n)e^{2\pi in\tau/l}&=\sum_{n\geq m,\frac{l}{\gcd(\omega_\gamma,l)}|n}a_{g,l}(n)e^{2\pi in\tau/l}\\&=\sum_{k=\lceil m\gcd(l,\omega_\gamma)/l\rceil}^{\infty}a_{g,l}(kl/\gcd(l,\omega_\gamma))e^{2\pi ik\tau/\gcd(\omega_\gamma,l)}\\&=\sum_{k=\lceil m\gcd(l,\omega_\gamma)/l\rceil}^{\infty}a_{g,l}(kl/\gcd(l,\omega_\gamma))e^{2\pi i\tau\frac{\omega_\gamma k}{\gcd(\omega_\gamma,l)}/\omega_\gamma}\\&=\sum_{j=\frac{\omega_\gamma}{\gcd(l,\omega_\gamma)}\lceil m\gcd(l,\omega_\gamma)/l\rceil}^{\infty}a_{g,l}(jl/\omega_\gamma)e^{2\pi i\tau j/\omega_\gamma}.\end{aligned}$$

Here we assume the convention $a_{g,l}(d)=0$ if $d\notin\mathbb{Z}$. The proof is finished by setting $t:=\frac{\omega_\gamma}{\gcd(l,\omega_\gamma)}\lceil m\gcd(l,\omega_\gamma)/l\rceil$ and

$$b(n):=\begin{cases}0 & \text{if } n<t\\ a_{g,l}(nl/\omega_\gamma) & \text{otherwise.}\end{cases}$$

□

We get the following equivalent characterization of (weak) modular forms.

Corollary 2.12. *Let $\Gamma\in\mathcal{S}$, k a positive integer and $f:\mathbb{H}\to\mathbb{C}$. Then $f\in A_k(\Gamma)$ (resp. $f\in M_k(\Gamma)$) iff* (i) *and* (ii) *as in Definition 2.6 hold, and if for all $\gamma\in\mathrm{SL}_2(\mathbb{Z})$ there exist an integer m (with $m\geq 0$ if $f\in M_k(\Gamma)$) and a sequence $a:\mathbb{Z}\to\mathbb{C}$ with $n\mapsto a(n)$ such that*

$$(f|_k\gamma)(\tau)=\sum_{n=m}^{\infty}a(n)e^{2\pi in\tau/\omega_\gamma},\quad \tau\in\mathbb{H}. \tag{2.7}$$

The following concept plays a very important role in the study of weak modular forms.

Definition 2.13. *Let* $\Gamma \in \mathcal{S}$, $k \in \mathbb{Z}$, $f \neq 0 \in A_k(\Gamma)$ *and* $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. *Then we define* $\mathrm{Ord}_\Gamma(f, \gamma)$ *to be the least integer* m *such that* $a_{g,\omega_{\Gamma,\gamma}}(m) \neq 0$ *where* $g := f|_k\gamma$.

We next define a concept similar to Ord in Definition 2.13 that will be needed later.

Definition 2.14. *Let* $f : \mathbb{H} \to \mathbb{C}$ *and assume there exist* $a : \mathbb{Z} \to \mathbb{C}$ *and* $m \in \mathbb{Z}$ *such that*

$$f(\tau) = \sum_{n=m}^{\infty} a(n)e^{2\pi i\tau n}, \quad \tau \in \mathbb{H},$$

and $a(m) \neq 0$. *Then we define* $\mathrm{ord}(f) := m$.

The connection between Ord and ord is made explicit in the following lemma.

Lemma 2.15. *Let* $\Gamma \in \mathcal{S}$, $k \in \mathbb{Z}$, $f \in A_k(\Gamma)$ *and* $\gamma \in \Gamma$. *Then* $\mathrm{Ord}_\Gamma(f, \gamma) = \mathrm{ord}\left(f|_k\gamma \begin{pmatrix} \omega_{\Gamma,\gamma} & r \\ 0 & 1 \end{pmatrix}\right)$ *for all* $r \in \mathbb{Z}$.

Proof. Let $g := f|_k\gamma$. Then by Definition 2.13 we have

$$g(\tau) = \sum_{n=\mathrm{Ord}_\Gamma(f,\gamma)}^{\infty} a_{g,\omega_{\Gamma,\gamma}}(n)e^{2\pi i n\tau/\omega_{\Gamma,\gamma}}, \quad \tau \in \mathbb{H},$$

and

$$\begin{aligned} \left(f|_k\gamma \begin{pmatrix} \omega_{\Gamma,\gamma} & r \\ 0 & 1 \end{pmatrix}\right) &= \left(g|_k \begin{pmatrix} \omega_{\Gamma,\gamma} & r \\ 0 & 1 \end{pmatrix}\right) \\ &= \omega_{\Gamma,\gamma}^{k/2} \sum_{n=\mathrm{Ord}_\Gamma(f,\gamma)}^{\infty} a_{g,\omega_{\Gamma,\gamma}}(n)e^{2\pi i n(\omega_{\Gamma,\gamma}\tau + r)/\omega_{\Gamma,\gamma}} \\ &= \omega_{\Gamma,\gamma}^{k/2} \sum_{n=\mathrm{Ord}_\Gamma(f,\gamma)}^{\infty} a_{g,\omega_{\Gamma,\gamma}}(n)e^{2\pi i n r/\omega_{\Gamma,\gamma}} e^{2\pi i n\tau}, \quad \tau \in \mathbb{H}, \end{aligned} \tag{2.8}$$

which by Definition 2.14 implies that $\mathrm{Ord}_\Gamma(f, \gamma) = \mathrm{ord}\left(f|_k\gamma \begin{pmatrix} \omega_\gamma & r \\ 0 & 1 \end{pmatrix}\right)$ as desired. □

We see that we need to verify condition *(iii)* of Definition 2.6 for *all* $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ and this is an *infinite* set. We would like to have some conditions that allows us to do this verification for *finitely* many $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. For this purpose we introduce the group

$$SL_2(\mathbb{Z})_\infty := \left\{ \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \mid h \in \mathbb{Z} \right\}.$$

Recall that for subgroups H_1, H_2 of a group G the double coset of $x \in G$ is defined as

$$H_1xH_2 := \{h_1xh_2 | h_1 \in H_1, h_2 \in H_2\}.$$

The set of double cosets is denoted by $H_1\backslash G/H_2$ and they form a partition of G.

Lemma 2.16. *Let $\Gamma \in \mathcal{S}$, $\gamma_1 \in \mathrm{SL}_2(\mathbb{Z})$, $\gamma_2 \in \Gamma\gamma_1\mathrm{SL}_2(\mathbb{Z})_\infty$, $k \in \mathbb{Z}$ and $f : \mathbb{H} \to \mathbb{C}$. Assume that f satisfies* (ii) *of Definition 2.6. Assume that there are $a : \mathbb{Z} \to \mathbb{C}$ with $n \mapsto a(n)$ and $m \in \mathbb{Z}$ such that*

$$(f|_k\gamma_1) = \sum_{n=m}^{\infty} a(n) e^{2\pi i n\tau/\omega_\gamma}, \quad \tau \in \mathbb{H}. \tag{2.9}$$

Then for any h such that $\gamma_2 \in \Gamma\gamma_1 \left(\begin{smallmatrix} 1 & h \\ 0 & 1 \end{smallmatrix}\right)$ we have $(f|_k\gamma_2)(\tau) = (f|_k\gamma_1)(\tau+h)$ or, equivalently,

$$(f|_k\gamma_2) = \sum_{n=m}^{\infty} a(n) e^{2\pi i n h/\omega_\gamma} e^{2\pi i n\tau/\omega_\gamma}, \quad \tau \in \mathbb{H}. \tag{2.10}$$

In particular, if $f \in A_k(\Gamma)$, then $\mathrm{Ord}_\Gamma(f, \gamma_1) = \mathrm{Ord}_\Gamma(f, \gamma_2)$.

Proof. By assumption there exist $\gamma_0 \in \Gamma$ such that $\gamma_2 = \gamma_0\gamma_1 \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}$. By (2.3) and $f|_k\gamma_0 = f$ (because of *(ii)* in Definition 2.6) we obtain

$$\begin{aligned} (f|_k\gamma_2)(\tau) &= \left(f|_k\gamma_0\gamma_1 \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \right)(\tau) \\ &= \left((f|_k\gamma_0)|_k\gamma_1 \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \right)(\tau) \\ &= \left(f|_k\gamma_1 \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \right)(\tau) \\ &= (f|_k\gamma_1)(\tau + h). \end{aligned} \tag{2.11}$$

□

We obtain immediately:

Lemma 2.17. *Let $f : \mathbb{H} \to \mathbb{C}$, $\Gamma \in \mathcal{S}$, $R \subseteq \Gamma$ and $k \in \mathbb{Z}$. Assume that*

$$\bigcup_{r \in R} \Gamma r \mathrm{SL}_2(\mathbb{Z})_\infty = \mathrm{SL}_2(\mathbb{Z}).$$

Then $f \in A_k(\Gamma)$ iff (i) *and* (ii) *of Definition 2.6 hold, and if for all $\gamma \in R$ there exist a integer m and a sequence $a : \mathbb{Z} \to \mathbb{C}$ with $n \mapsto a(n)$ such that*

$$(f|_k\gamma)(\tau) = \sum_{n=m}^{\infty} a(n) e^{2\pi i n\tau/\omega_\gamma}, \quad \tau \in \mathbb{H}.$$

Note that since Γ is of finite index, the set R can be finitely chosen because

$$|\Gamma\backslash\mathrm{SL}_2(\mathbb{Z})/\mathrm{SL}_2(\mathbb{Z})_\infty| \leq \nu$$

where $\nu := [\mathrm{SL}_2(\mathbb{Z}) : \Gamma]$ is the index of Γ in $\mathrm{SL}_2(\mathbb{Z})$. The most important fact about weak modular forms is the following lemma.

Lemma 2.18. *Let $\Gamma \in \mathcal{S}$, $R \subseteq \Gamma$ a complete set of representatives of the double cosets $\Gamma\backslash\mathrm{SL}_2(\mathbb{Z})/\mathrm{SL}_2(\mathbb{Z})_\infty$ and $f \neq 0 \in A_k(\Gamma)$. Then*

$$\sum_{r \in R} \mathrm{Ord}_\Gamma(f, r) \leq \frac{k}{12}[\mathrm{SL}_2(\mathbb{Z}) : \Gamma]. \tag{2.12}$$

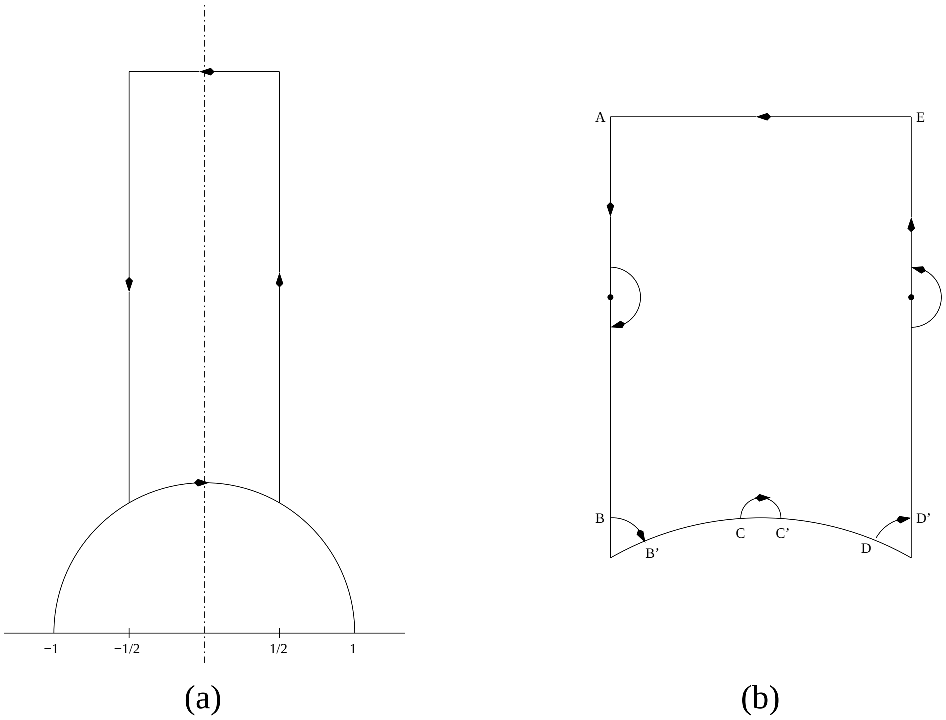

Figure 2.1: Contour of integration

The next lemma is the specialization of Lemma 2.18 to $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ and its proof is based on Complex Analysis. The extension of Lemma 2.19 to Lemma 2.18 involves only algebra; see our proof of Lemma 2.18 after Proposition 2.21.

Lemma 2.19. *Let $k \in \mathbb{Z}$ and $f \neq 0$ be a weak modular form of weight k that is $f \in A_k(\mathrm{SL}_2(\mathbb{Z}))$. Denote by* id *the identity matrix. Then*

$$\mathrm{Ord}_{\mathrm{SL}_2(\mathbb{Z})}(f, \mathrm{id}) \leq \frac{k}{12}. \tag{2.13}$$

Proof. We recall that any holomorphic function f in some neighborhood of $P \in \mathbb{H}$ has locally a representation of the form

$$f(\tau) = \sum_{n=0}^{\infty} a(n)(\tau - P)^N.$$

For f as above we denote by $\nu_P(f)$ the minimal integer m such that $a(m) \neq 0$. We integrate f'/f along the contour of Figure 2.1(a), but modified by taking small arcs around the possible poles of f/f' on the boundary, as on Figure 2.1(b). For simplicity we phrase the proof under the assumption that f has no zero on the boundary of the domain (that is f/f' has no pole on the boundary) other than at $i, e^{2\pi i/3}$ and $e^{4\pi i/3}$. Let $D \subset \mathbb{C}$ be the interior of the domain shown in Figure 2.1. Following e. g. Whittaker and Watson [42, p. 119-120] we see that

$$\frac{1}{2\pi i} \int_{\partial D} \frac{f'}{f} = \sum_{P \in D} v_P(f). \tag{2.14}$$

The integral $\frac{1}{2\pi i}\int_{\partial D}\frac{f'(\tau)}{f(\tau)}d\tau$ can also be computed by integrating over each segment and arc listed in Figure 2.1(b). We start with the segment EA. Let $\tilde{f}$ be defined by the formula $\tilde{f}(e^{2\pi i\tau}) = f(\tau)$ for $\tau \in \mathbb{H}$. Because $g(q) := q^{-\operatorname{Ord}_{\mathrm{SL}_2(\mathbb{Z})}(f,\mathrm{id})}\tilde{f}(q)$ is a Taylor series in q because of (2.7) and $\omega_{\mathrm{id}} = 1$, there exists an ϵ such that $g(q) \neq 0$ for $|q| < \epsilon$ (otherwise g accumulates at 0 and is identically 0). We may assume that E and A are chosen with imaginary part large enough for $e^{-2\pi \mathrm{Im}(E)} < \epsilon$ to hold. Then

$$\frac{1}{2\pi i}\int_E^A \frac{f'(\tau)}{f(\tau)}d\tau = \int_C \frac{\tilde{f}'(q)}{\tilde{f}(q)}dq = -\operatorname{Ord}_{\mathrm{SL}_2(\mathbb{Z})}(f,\mathrm{id}) \tag{2.15}$$

where the contour C is the circle around 0 of radius $e^{-2\pi\mathrm{Im}(E)}$ taken clockwise.

Next we consider the sum of the integrals of the segments AB and $D'E$:

$$\int_A^B \frac{f'(\tau)}{f(\tau)}d\tau + \int_{D'}^E \frac{f'(\tau)}{f(\tau)}d\tau$$

which is 0 because $f(\tau+1) = f(\tau)$. Next we observe that

$$\frac{f'(\tau)}{f(\tau)} = \frac{\nu_{e^{4\pi i/3}}(f)}{\tau - e^{4\pi i/3}} + \text{holomorphic terms.}$$

Consequently the integral

$$\int_B^{B'} \frac{f'(\tau)}{f(\tau)}d\tau = -\frac{1}{6}\nu_{e^{4\pi i/3}}(f) + \chi_1$$

where χ_1 is the integral over the holomorphic terms and goes to zero if the radius of the arc BB' goes to 0. Similarly

$$\int_D^{D'} \frac{f'(\tau)}{f(\tau)}d\tau = -\frac{1}{6}\nu_{e^{2\pi i/3}}(f) + \chi_2 \tag{2.16}$$

with $\chi_2 \to 0$ if the radius of the arc DD' goes to 0. Since $e^{4\pi i/3} + 1 = e^{2\pi i/3}$ and because of $f(\tau+1) = f(\tau)$ we have $\nu_{e^{2\pi i/3}}(f) = \nu_{e^{4\pi i/3}}(f)$ and hence

$$\int_B^{B'} \frac{f'(\tau)}{f(\tau)}d\tau + \int_D^{D'} \frac{f'(\tau)}{f(\tau)}d\tau = -\frac{1}{3}\nu_{e^{2\pi i/3}}(f) + \chi_1 + \chi_2.$$

The same argument is used for the arc CC' around the point i to obtain

$$\int_C^{C'} \frac{f'(\tau)}{f(\tau)}d\tau = -\frac{1}{2}\nu_i(f) + \chi. \tag{2.17}$$

It remains to compute the integrals over the arcs $B'C$ and $C'D$. The map $\tau \mapsto -1/\tau$ transforms the arc $B'C$ to the arc DC'. By property *(ii)* of Definition 2.6 we have

$$f(-1/\tau) = \tau^k f(\tau), \tag{2.18}$$

and

$$\frac{df(-1/\tau)}{d\tau} = \frac{f'(-1/\tau)}{\tau^2} = \tau^k f'(\tau) + k\tau^{k-1}f(\tau). \tag{2.19}$$

Since

$$\int_{C'}^{D} \frac{f'(w)}{f(w)} dw = \int_{C}^{B'} \frac{1}{\tau^2} \frac{f'(-1/\tau)}{f(-1/\tau)} d\tau,$$

and after dividing (2.19) by (2.18) we obtain

$$\frac{1}{\tau^2} \frac{f'(-1/\tau)}{f(-1/\tau)} = \frac{f'(\tau)}{f(\tau)} + \frac{k}{\tau}.$$

We see that the integral over the second arc has one term which cancels the integral over the first arc, plus another term which is

$$\int_{C}^{B'} \frac{k}{\tau} d\tau. \tag{2.20}$$

Finally, we let the radius of the arcs BB' and DD' go to 0. Then (2.20) approaches $k/12$ and we sum (2.15), (2.16), (2.17) and (2.20) and compare it with (2.14). We obtain

$$-\operatorname{Ord}_{\mathrm{SL}_2(\mathbb{Z})}(f, \mathrm{id}) - \frac{1}{2}\nu_i(f) - \frac{1}{3}\nu_{e^{2\pi i/3}}(f) + \frac{k}{12} = \sum_{P \in D} v_P(f). \tag{2.21}$$

Because f is holomorphic on $\mathbb{H}$ by property *(i)* of Definition 2.6 we have $\nu_P(f) \geq 0$ for all $P \in \mathbb{H}$. Using this on (2.21) we obtain (2.13). □

Before we can prove Lemma 2.18 we need the following two simple propositions.

Proposition 2.20. *Let* $\Gamma \in \mathcal{S}$ *and* $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. *Then*

$$\Gamma\gamma \begin{pmatrix} 1 & i_1 \\ 0 & 1 \end{pmatrix} = \Gamma\gamma \begin{pmatrix} 1 & i_2 \\ 0 & 1 \end{pmatrix} \quad \textit{iff} \quad \omega_\gamma | (i_1 - i_2). \tag{2.22}$$

Proof. We see that (2.22) is equivalent to $\begin{pmatrix} 1 & i_2 - i_1 \\ 0 & 1 \end{pmatrix} \in \gamma^{-1}\Gamma\gamma$ iff $\omega_\gamma | (i_2 - i_1)$, which is immediate from Definition 2.9. □

Proposition 2.21. *Let* $\Gamma \in \mathcal{S}$, $R \subseteq \Gamma$ *be a complete set of representatives of* $\Gamma \backslash \mathrm{SL}_2(\mathbb{Z})$, $k \in \mathbb{Z}$ *and* $f \in A_k(\Gamma)$. *Then* $N(f) := \prod_{r \in R}(f|_k r) \in A_\kappa(\mathrm{SL}_2(\mathbb{Z}))$ *where* $\kappa := k \cdot [\mathrm{SL}_2(\mathbb{Z}) : \Gamma]$.

Proof. First we observe that for $k \in \mathbb{Z}$, m a positive integer, $f_1, \ldots, f_m : \mathbb{H} \to \mathbb{C}$ and $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ we have

$$\left(\prod_{i=1}^{m} f_i\right) |_{km}\gamma = \prod_{i=1}^{m}(f_i|_k\gamma). \tag{2.23}$$

Assume that $R = \{\gamma_1, \ldots, \gamma_m\}$ where $m := [\mathrm{SL}_2(\mathbb{Z}) : \Gamma]$. Then there exists $r_1, \ldots, r_m \in \Gamma$ and a bijection $\sigma : \{1, \ldots, m\} \to \{1, \ldots, m\}$ such that $\gamma_i\gamma = r_i\gamma_{\sigma(i)}$ for $i \in \{1, \ldots, m\}$, which together with (2.3), (2.23) and $f \in A_k(\Gamma)$ implies

$$\begin{aligned} N(f) &= \left(\prod_{i=1}^{m}(f|_k\gamma_i)\right) |_{km}\gamma = \prod_{i=1}^{m}\{(f|_k\gamma_i)|_k\gamma\} = \prod_{i=1}^{m}(f|_k\gamma_i\gamma) \\ &= \prod_{i=1}^{m}(f|_k r_i\gamma_{\sigma(i)}) = \prod_{i=1}^{m}\{(f|_k r_i)|_k\gamma_{\sigma(i)}\} = \prod_{i=1}^{m}(f|_k\gamma_{\sigma(i)}) = N(f). \end{aligned}$$

This shows property *(ii)* of Definition 2.6. Furthermore, $N(f)$ is holomorphic by Proposition 2.7 and because of the fact that products of holomorphic functions are again holomorphic. This proves *(i)* of Definition 2.6. By *(i)* and *(iii)* of Proposition 2.8 we have $N(f) \in \mathbb{T}_{km}$ proving *(iii)* of Definition 2.6. □

Now we are ready for the

Proof of Lemma 2.18: Let $R = \{\gamma_1, \ldots, \gamma_n\}$ $(n = |\Gamma\backslash \mathrm{SL}_2(\mathbb{Z})/\mathrm{SL}_2(\mathbb{Z})_\infty|)$. Define

$$S_t := \{\gamma_t \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix} \mid i \in \{0, \ldots, \omega_{\gamma_t} - 1\}\}$$

and $m_t := \mathrm{Ord}_\Gamma(f, \gamma_t)$ for $t \in \{1, \ldots, n\}$. Then there is $a_t : \mathbb{Z} \to \mathbb{C}$ with $a(m_t) \neq 0$ such that

$$g_t(\tau) := \prod_{s \in S_t} (f|_k s)(\tau) = \prod_{j=0}^{\omega_{\gamma_t}-1} (f|_k \gamma_t)(\tau + j) = \sum_{n=m_t}^{\infty} a_t(n) e^{2\pi i \tau n}, \quad \tau \in \mathbb{H},$$

because by Lemma 2.16 and Corollary 2.12 we have

$$(f|_k \gamma_t)(\tau + j) = \sum_{n=m_t}^{\infty} a_{f|_k \gamma_t, \omega_{\gamma_t}}(n) e^{2\pi i j/\omega_{\gamma_t}} e^{2\pi i \tau n/\omega_{\gamma_t}}.$$

Define

$$N(f)(\tau) := \prod_{t=1}^{n} \prod_{s \in S_t} (f|_k s) = \prod_{t=1}^{n} \sum_{j=m_t}^{\infty} a_t(j) e^{2\pi i j \tau}, \quad \tau \in \mathbb{H}. \tag{2.24}$$

By Proposition 2.20 $\cup_{\gamma \in S_k} \Gamma\gamma = \Gamma\gamma_k \mathrm{SL}_2(\mathbb{Z})_\infty$ for all $k \in \{1, \ldots, n\}$. In other words, the set $S := \cup_{k=1}^{n} S_k$ is a complete set of representatives of $\Gamma\backslash\mathrm{SL}_2(\mathbb{Z})$. This implies that $N(f) \in A_\kappa(\mathrm{SL}_2(\mathbb{Z}))$ by Proposition 2.21 where $\kappa := k \cdot [\mathrm{SL}_2(\mathbb{Z}) : \Gamma]$. Furthermore, because of (2.24) we have

$$\mathrm{Ord}_{\mathrm{SL}_2(\mathbb{Z})}(N(f), \mathrm{id}) = \sum_{t=1}^{n} \mathrm{Ord}_\Gamma(f, \gamma_t)$$

which together with Lemma 2.19 implies

$$\sum_{t=1}^{n} \mathrm{Ord}_\Gamma(f, \gamma_t) = \mathrm{Ord}_{\mathrm{SL}_2(\mathbb{Z})}(N(f), \mathrm{id}) \leq \frac{k \cdot [\mathrm{SL}_2(\mathbb{Z}) : \Gamma]}{12}.$$

□

Corollary 2.22. *Let $\Gamma \in \mathcal{S}$, $k \in \mathbb{Z}$ and $f \neq 0 \in M_k(\Gamma)$. Then*

$$\mathrm{Ord}_\Gamma(f, \mathrm{id}) \leq \frac{k \cdot [\mathrm{SL}_2(\mathbb{Z}) : \Gamma]}{12}.$$

Proof. We choose R in Lemma 2.18 such that $\mathrm{id} \in R$. Then by (2.12) we have:

$$\mathrm{Ord}_\Gamma(f, \mathrm{id}) + \sum_{r \in R, r \neq \mathrm{id}} \mathrm{Ord}_\Gamma(f, r) \leq \frac{k}{12}[\mathrm{SL}_2(\mathbb{Z}) : \Gamma]. \tag{2.25}$$

Because of $\mathrm{Ord}_\Gamma(f, r) \geq 0$ for all $r \in R$, by Definition 2.6 we have $\sum_{r \in R, r \neq \mathrm{id}} \mathrm{Ord}_\Gamma(f, r) \geq 0$ and hence the result follows from inequality (2.25). □

Corollary 2.23. *For all* $\Gamma \in \mathcal{S}$ *we have* $M_0(\Gamma) = \mathbb{C}$.

Proof. Let $f \in M_0(\Gamma)$. Then by Definition 2.6 $f = \sum_{n=0}^{\infty} a(n) e^{2\pi i n\tau/l}$ for some $a : \mathbb{Z} \to \mathbb{C}$ and $l \in \mathbb{N}^*$. If $f \notin \mathbb{C}$ then $f - a(0) \neq 0$ and $f - a(0) \in M_0(\Gamma)$. Then $\mathrm{Ord}_\Gamma(f - a(0), \mathrm{id}) > 0$ contradicting Corollary 2.22. □

Remark 2.24. *We make the following observations. For* $\Gamma \in \mathcal{S}$

- $M_k(\Gamma)$ *and* $A_k(\Gamma)$ *are vector spaces over* $\mathbb{C}$ *for* $k \in \mathbb{Z}$*;*
- *if* $f \in A_k(\Gamma)$ *and* $g \in A_l(\Gamma)$ *then* $f \cdot g \in A_{k+l}(\Gamma)$ *for* $k, l \in \mathbb{Z}$*;*
- $A_0(\Gamma)$ *is a ring and* $M_0(\Gamma) = \mathbb{C}$ *(because of Corollary 2.22).*

2.2 Examples of Weak Modular Forms

In the previous section we studied weak modular forms from a general point of view. In this chapter we will introduce a class of weak modular forms that will play an important role throughout.

For $\tau \in \mathbb{H}$ the Dedekind eta function $\eta : \mathbb{H} \to \mathbb{C}$ is defined by

$$\eta(\tau) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n) \text{ where } q := e^{2\pi i \tau}. \tag{2.26}$$

We will also use the short hand notation:

$$\eta_n(\tau) := \eta(n\tau), \quad n \in \mathbb{Z}, \quad \tau \in \mathbb{H}. \tag{2.27}$$

The connection between the eta function and weak modular forms comes from Lemma 2.27 which we will introduce after some definitions.

Definition 2.25. *Let* $a \in \mathbb{Z}$. *For an odd integer* $n > 0$ *we define:*

- *If* $n = 1$ *then:*
$$\left(\frac{a}{n}\right) = \left(\frac{a}{1}\right) := 1.$$
- *If* n *is a prime* p *then:*
$$\left(\frac{a}{n}\right) = \left(\frac{a}{p}\right) := \begin{cases} 0 & \text{if } p \mid a \\ 1 & \text{if } a \text{ is a square modulo } p \\ -1 & \text{otherwise} \end{cases}.$$
- *If* $p_1^{\alpha_1} \cdot \ldots \cdot p_k^{\alpha_k}$ *is the prime factorization of* n *then:*
$$\left(\frac{a}{n}\right) := \left(\frac{a}{p_1}\right)^{\alpha_1} \cdot \ldots \cdot \left(\frac{a}{p_k}\right)^{\alpha_k}.$$

The symbol $\left(\frac{a}{n}\right)$ *is called the Legendre-Jacobi symbol.*

Definition 2.26. *Let c, d be integers with $\gcd(c,d) = 1$ and d odd. Then we define*

$$\left(\frac{c}{d}\right)_* = \begin{cases} \left(\frac{c}{|d|}\right)(-1)^{\frac{\operatorname{sgn}(c)-1}{2}\frac{\operatorname{sgn}(d)-1}{2}}, & \text{if } c \neq 0, \\ \operatorname{sgn}(d), & \text{otherwise,} \end{cases}$$

and

$$\left(\frac{c}{d}\right)^* = \begin{cases} \left(\frac{c}{|d|}\right), & \text{if } c \neq 0, \\ 1, & \text{otherwise,} \end{cases}$$

where $\operatorname{sgn} : \mathbb{R}^* \to \{-1, 1\}$ *is defined by*

$$\operatorname{sgn}(a) := \begin{cases} 1, & \text{if } a > 0, \\ -1, & \text{otherwise.} \end{cases}$$

The next lemma is proven in Knopp's book [20, p. 51]).

Lemma 2.27. *For* $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$*:*

$$\eta\left(\frac{a\tau + b}{c\tau + d}\right) = \upsilon_\eta(a,b,c,d)(c\tau + d)^{1/2}\eta(\tau), \quad \tau \in \mathbb{H}, \tag{2.28}$$

where

$$\upsilon_\eta(a,b,c,d) = \begin{cases} \left(\frac{d}{c}\right)^* e^{2\pi i\{c(a+d)-bd(c^2-1)-3c\}/24} & \text{if } c \text{ is odd,} \\ \left(\frac{c}{d}\right)_* e^{2\pi i\{(a+d)c-bd(c^2-1)+3d-3-3cd\}/24} & \text{if } c \text{ is even.} \end{cases} \tag{2.29}$$

The following lemma gives us another formula for $\upsilon_\eta(a,b,c,d)$ that will turn out to be useful.

Lemma 2.28. *Let* $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ *and* $\upsilon_\eta(a,b,c,d)$ *the unique complex number given by* (2.28)*. Then*

$$\upsilon_\eta(a,b,c,d) = \begin{cases} \left(\frac{a}{-c}\right)^* e^{2\pi i\{c(a+d)-ba(c^2-1)-3c\}/24} & \text{if } c \text{ is odd,} \\ \left(\frac{-c}{a}\right)_* e^{2\pi i\{c(a+d)-ba(c^2-1)-3a+3-3ca\}/24} & \text{if } c \text{ is even.} \end{cases} \tag{2.30}$$

Proof. We know that $A^{-1} = \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$ and by (2.28)

$$\begin{aligned} \eta(\tau) = \eta(AA^{-1}\tau) &= \upsilon_\eta(a,b,c,d)(c(A^{-1}\tau) + d)^{1/2}\upsilon_\eta(d,-b,-c,a)(-c\tau + a)^{1/2}\eta(\tau) \\ &= \upsilon_\eta(a,b,c,d)\upsilon_\eta(d,-b,-c,a)(-c\tau + a)^{-1/2}(-c\tau + a)^{1/2}\eta(\tau) \\ &= \upsilon_\eta(a,b,c,d)\upsilon_\eta(d,-b,-c,a)\eta(\tau), \quad \tau \in \mathbb{H}. \end{aligned} \tag{2.31}$$

Observe that we used above that $\sqrt{z} \cdot \sqrt{\frac{1}{z}} = 1$ for all $z \in \mathbb{C} \setminus \{x \in \mathbb{R} | x \leq 0\}$. After canceling $\eta(\tau)$ from both sides of (2.31) we obtain

$$\upsilon_\eta(a,b,c,d)\upsilon_\eta(d,-b,-c,a) = 1$$

which implies (2.30). □

We get immediately the following simplified formula for $\upsilon_\eta(a,b,c,d)$ if we restrict $c > 0$ and $\gcd(c,6)$, which Newman [27] discovered in slightly different form.

Lemma 2.29. *Let* $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ *with* $c > 0$, $\gcd(c, 6) = 1$ *and* $\upsilon_\eta(a, b, c, d)$ *the unique complex number given by* (2.28)*. Then*

$$\upsilon_\eta(a, b, c, d) = \left(\frac{a}{c}\right) e^{2\pi i \{c(a+d)-3c\}/24}. \tag{2.32}$$

Proof. By using $c^2 - 1 \equiv 0 \pmod{24}$ in (2.30) and Definition 2.26 we immediately obtain (2.32). ☐

If we impose $a > 0$, $c > 0$, $\gcd(a, 6) = 1$, then Newman [28] discovered a more simple expression for $\upsilon_\eta(a, b, c, d)$ that will be more useful than the one in Lemma 2.27. This is given in the next lemma and we prove it directly using Lemma 2.28. But first some properties of the Legendre-Jacobi symbol.

Lemma 2.30. *Let* $n > 0$ *be an odd integer, then the following relations hold:*

- *If* a *and* b *are integers then*
$$\left(\frac{a}{n}\right)\left(\frac{b}{n}\right) = \left(\frac{ab}{n}\right). \tag{2.33}$$
- $$\left(\frac{2}{n}\right) = (-1)^{\frac{n^2-1}{8}}. \tag{2.34}$$
- $$\left(\frac{-1}{n}\right) = (-1)^{\frac{n-1}{2}}. \tag{2.35}$$
- *If* m *is an odd integer then*
$$\left(\frac{m}{n}\right) = \left(\frac{n}{m}\right)(-1)^{\frac{m-1}{2}\frac{n-1}{2}}. \tag{2.36}$$

Proof. We refer to [31]: For (2.36) see Satz 28, p. 71, (9) p. 70 for (2.33), (39) p. 84 for (2.35), and (40) p. 85 for (2.34). ☐

The following notion turns out to be easier to deal with in some applications.

Definition 2.31. *For* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ *we define*

$$\epsilon(a, b, c, d) := e^{\pi i/4} \upsilon_\eta(a, b, c, d)$$

where $\upsilon_\eta(a, b, c, d)$ *is the unique complex number given by* (2.28)*.*

Lemma 2.32. *Let* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ *with* $a > 0$, $c > 0$ *and* $\gcd(a, 6) = 1$*. Then*

$$\epsilon(a, b, c, d) = e^{-\frac{\pi i a}{12}(c-b-3)}. \tag{2.37}$$

Proof. By Definition 2.26 and (2.35):

$$\left(\frac{-c}{a}\right)_* = \left(\frac{-c}{a}\right) = \left(\frac{c}{a}\right)(-1)^{\frac{a-1}{2}} = \left(\frac{c}{a}\right)e^{2\pi i\frac{6(a-1)}{24}}, \tag{2.38}$$

and in case c is odd we have by Definition 2.26 and by (2.33):

$$\left(\frac{a}{-c}\right)^* = \left(\frac{a}{c}\right) = \left(\frac{c}{a}\right)(-1)^{\frac{(c-1)(a-1)}{4}} = \left(\frac{c}{a}\right)e^{2\pi i\frac{-3(c-1)(a-1)}{24}}. \tag{2.39}$$

Note that because $\gcd(a,6)=1$ we have $a^2 \equiv 1 \pmod{24}$ which together with $ad-bc=1$ gives

$$d \equiv abc + a \pmod{24}. \tag{2.40}$$

Next we substitute (2.38)-(2.40) into (2.30) which together with Definition 2.31 will simplify to (2.37). □

The conditions stated in Lemma 2.32 motivate the following definition.

Definition 2.33. *Let $\Gamma \in \mathcal{S}$. Then we define*

$$\Gamma^* := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma | a > 0, c > 0, \gcd(a,6) = 1 \right\}.$$

From Lemma 2.27 we see that $\eta^{24} \in M_{12}(\mathrm{SL}_2(\mathbb{Z}))$ but this is not the only modular form (or weak modular form) that can be constructed using the eta function. Lemma 2.34 below is a mild extension of an extremely useful result stated and exploited first by M. Newman in [27, Th. 1] and [28, Th. 1] which shows us how to construct weak modular forms for the Hecke subgroup

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) | N | c \right\}, \quad N \in \mathbb{N}^*, \tag{2.41}$$

which will play a very important role throughout this work.

Lemma 2.34 ("Newman's Lemma")**.** *Let $r = (r_\delta)_{\delta|N}$ be a finite sequence of integers indexed by the positive divisors δ of $N \in \mathbb{N}^*$. Let $f_r : \mathbb{H} \to \mathbb{C}$ be defined by $f_r(\tau) := \prod_{\delta|N} \eta^{r_\delta}(\delta\tau)$. Then*

$$f_r \in A_k(N) \text{ for } k = \frac{1}{2}\sum_{\delta|N} r_\delta,$$

if the following conditions are satisfied:

(i) $\sum_{\delta|N} \delta r_\delta \equiv 0 \pmod{24}$*;*

(ii) $\sum_{\delta|N} N r_\delta/\delta \equiv 0 \pmod{24}$*;*

(iii) $\prod_{\delta|N} \delta^{r_\delta}$ *is the square of a rational number;*

(iv) $\sum_{\delta|N} r_\delta \equiv 0 \pmod{4}$*.*

If (i)-(iv) *are satisfied, then $f_r \in M_k(N)$ iff*

$$\sum_{\delta|N} \gcd^2(\delta,d) r_\delta/\delta \geq 0$$

for all $d|N$.

The following result due to Newman will be used to prove Lemma 2.34 and will also be used later.

Lemma 2.35. *Let $k \in \mathbb{Z}$, $f : \mathbb{H} \to \mathbb{C}$ and N be a positive integer. Then the following statements are true:*

(i) $\Gamma_0(N)^*$ *generates* $\Gamma_0(N)$*;*

(ii) If $f|_k\gamma = f$ *for all* $\gamma \in \Gamma_0(N)^*$*, then* $f|_k\gamma = f$ *for all* $\gamma \in \Gamma_0(N)$.

Proof. (i): First we show that $\gamma \in \mathrm{SL}_2(\mathbb{Z})_\infty$ may be written as a product of matrices from $\Gamma_0(N)^*$. If $\gamma = \left(\begin{smallmatrix} -1 & h \\ 0 & -1 \end{smallmatrix}\right)$ then

$$\begin{pmatrix} 6N-1 & -1 \\ 6N & -1 \end{pmatrix} \begin{pmatrix} 1 & -(h+1) \\ 6N & -6N(h+1)+1 \end{pmatrix} = \begin{pmatrix} -1 & h \\ 0 & -1 \end{pmatrix} = \gamma. \tag{2.42}$$

If $\gamma = \left(\begin{smallmatrix} 1 & h \\ 0 & 1 \end{smallmatrix}\right)$ then $\gamma = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\left(\begin{smallmatrix} -1 & -h \\ 0 & -1 \end{smallmatrix}\right)$ and each term in this product may be written as a product of matrices from $\Gamma_0(N)^*$. Next let $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$ with $c \neq 0$. Then one easily verifies that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \mathrm{sgn}(c) & t\,\mathrm{sgn}(c) \\ 0 & \mathrm{sgn}(c) \end{pmatrix} \underbrace{\begin{pmatrix} a\,\mathrm{sgn}(c) - |c|t & b\,\mathrm{sgn}(c) - dt\,\mathrm{sgn}(c) \\ |c| & d\,\mathrm{sgn}(c) \end{pmatrix}}_{\in \Gamma_0(N)^*} \tag{2.43}$$

where t is such that $a \cdot \mathrm{sgn}(c) - |c|t > 0, \gcd(a \cdot \mathrm{sgn}(c) - |c|t, 6) = 1$. This proves that $\Gamma_0(N)^*$ generates $\Gamma_0(N)$ because we already showed that $\left(\begin{smallmatrix} \mathrm{sgn}(c) & t\,\mathrm{sgn}(c) \\ 0 & \mathrm{sgn}(c) \end{smallmatrix}\right)$ is a product of matrices in $\Gamma_0(N)^*$ for any $t \in \mathbb{Z}$.

(ii): For $\gamma \in \Gamma_0(N)$ we define the $L(\gamma)$ to be the least positive integer n such that $\gamma = \gamma_1 \cdots \gamma_n$ with $\gamma_1, \ldots, \gamma_n \in \Gamma_0(N)^*$. Note that the existence of the integer n and of $\gamma_1, \ldots, \gamma_n \in \Gamma_0(N)^*$ is guaranteed by *(i)*. By assumption

$$f|_k\xi = f, \quad \xi \in \Gamma_0(N)^*. \tag{2.44}$$

Assume by induction that for all $\xi \in \Gamma_0(N)$ with $L(\xi) < n$ we have

$$f|_k\xi = f. \tag{2.45}$$

Let $\gamma \in \Gamma_0(N)^*$ be such that $L(\gamma) = n$. Then there exists ξ with $L(\xi) < n$ and $\gamma_1 \in \Gamma_0(N)^*$ such that $\gamma_1\xi = \gamma$. By (2.3), (2.44) and (2.45) we have

$$f|_k\gamma = f|_k\gamma_1\xi = (f|_k\gamma_1)|_k\xi = f|_k\xi = f.$$

□

Proof of Lemma 2.34: We proceed by verifying conditions *(i)*-*(iii)* in Definition 2.6. Property *(i)* is clear. In order to prove condition *(ii)* in Definition 2.6, it is sufficient to show that $f_r|_k\gamma = f_r$ for all $\gamma \in \Gamma_0(N)^*$ by *(ii)* of Lemma 2.35.

By Lemma 2.32 the following formula holds for all $\tau \in \mathbb{H}$ and $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})^*$:

$$\eta\left(\frac{A\tau + B}{C\tau + D}\right) = (-i(C\tau + D))^{1/2} \left(\frac{C}{A}\right) e^{-\frac{A\pi i}{12}(C-B-3)}\eta(\tau), \tag{2.46}$$

with (C/A) being the Legendre-Jacobi symbol.

For $\delta | N$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)^*$ this implies:

$$\begin{aligned} \eta\left(\delta\frac{a\tau + b}{c\tau + d}\right) &= \eta\left(\frac{a(\delta\tau) + b\delta}{\frac{c}{\delta}(\delta\tau) + d}\right) \\ &= (-i(c\tau + d))^{1/2} \left(\frac{c/\delta}{a}\right) e^{-\frac{a\pi i}{12}(c/\delta - \delta b - 3)}\eta(\delta\tau). \end{aligned}$$

Consequently we have:

$$\begin{aligned} \prod_{\delta|N} \eta^{r_\delta}\left(\delta\frac{a\tau + b}{c\tau + d}\right) &= (-i(c\tau + d))^{\frac{1}{2}\sum_{\delta|N} r_\delta} \prod_{\delta|N} \left(\frac{c/\delta}{a}\right)^{r_\delta} \\ &\quad \times e^{-\frac{a\pi i}{12}(c\sum_{\delta|N} r_\delta/\delta - b\sum_{\delta|N} r_\delta\delta - 3\sum_{\delta|N} r_\delta)} \prod_{\delta|N} \eta^{r_\delta}(\delta\tau). \end{aligned}$$

Because of properties *(i)* and *(ii)*, and $k = \frac{1}{2}\sum_{\delta|N} r_\delta$ this reduces to:

$$\prod_{\delta|N} \eta^{r_\delta}\left(\delta\frac{a\tau + b}{c\tau + d}\right) = (-i(c\tau + d))^k \prod_{\delta|N} \left(\frac{c/\delta}{a}\right)^{r_\delta} e^{\frac{\pi i k a}{2}} \prod_{\delta|N} \eta^{r_\delta}(\delta\tau).$$

Next we note that

$$\prod_{\delta|N} \left(\frac{c/\delta}{a}\right)^{r_\delta} = \prod_{\delta|N} \left(\frac{c/\delta}{a}\right)^{r_\delta} \left(\frac{\delta^2}{a}\right)^{r_\delta} = \prod_{\delta|N} \left(\frac{\delta c}{a}\right)^{r_\delta} = \prod_{\delta|N} \left(\frac{\delta}{a}\right)^{r_\delta},$$

where we applied *(iv)*. By property *(iii)* this reduces to 1.

Hence we have proven that for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)^*$:

$$(f_r|_k\gamma)(\tau) = (-i)^k e^{\frac{\pi i k a}{2}} \prod_{\delta|N} \eta^{r_\delta}(\delta\tau).$$

Because of $\gcd(a, 6) = 1$ and *(iv)* we have that $(-i)^k e^{\frac{\pi i k a}{2}} = 1$, which proves the desired property. Owing to the fact that the η function is holomorphic on $\mathbb{H}$ it remains to show that condition *(iii)* of Definition 2.6 holds.

For a fixed $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and a fixed positive divisor δ of N, let x_δ, y_δ be integers satisfying $\delta a x_\delta + c y_\delta = \gcd(\delta a, c)$. Observe that $\gcd(\delta a, c) = \gcd(\delta, c)$ because of $\gcd(a, c) = 1$, and set $\lambda := \gcd(\delta, c)$. Set $\gamma_{0,\delta} := \left(\begin{smallmatrix} \delta a/\lambda & -y_\delta \\ c/\lambda & x_\delta \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and $\gamma_{1,\delta} := \left(\begin{smallmatrix} \lambda & \delta b x_\delta + d y_\delta \\ 0 & \delta/\lambda \end{smallmatrix}\right)$, and verify that $\gamma_{0,\delta}\gamma_{1,\delta} = \left(\begin{smallmatrix} \delta a & \delta b \\ c & d \end{smallmatrix}\right)$. Then by (2.28) and because of

$$\frac{c}{\lambda}\gamma_{1,\delta}\tau + x_\delta = \frac{\lambda}{\delta}(c\tau + d)$$

we have:

$$\eta(\gamma_{0,\delta}\gamma_{1,\delta}\tau) = \left(\frac{\lambda}{\delta}(c\tau+d)\right)^{\frac{1}{2}} \upsilon_\eta\left(\delta a/\lambda, -y_\delta, c/\lambda, x_\delta\right)\eta(\gamma_{1,\delta}\tau).$$

Noting that $\delta(\gamma\tau) = (\gamma_{0,\delta}\gamma_{1,\delta})\tau$ one obtains

$$(f_r|_k\gamma)(\tau) = (c\tau+d)^{-k} f_r(\gamma\tau) = C(a,b,c,d)\cdot \prod_{\delta|N} \eta^{r_\delta}(\gamma_{1,\delta}\tau),$$

where

$$C(a,b,c,d) := \prod_{\delta|N} \upsilon_\eta^{r_\delta}(\delta a/\lambda, -y_\delta, c/\lambda, x_\delta)\prod_{\delta|N}\left(\frac{\lambda}{\delta}\right)^{r_\delta/2}.$$

Finally we observe that

$$\begin{aligned}\eta(\gamma_{1,\delta}\tau) &= \eta\left(\frac{\lambda\tau+\delta b x_\delta + d y_\delta}{\delta/\lambda}\right)\\ &= \eta\left(\frac{\lambda^2\tau+(\delta b x_\delta + d y_\delta)\lambda}{\delta}\right)\\ &= q^{\frac{\lambda^2}{24\delta}} e^{\frac{\pi i(\delta b x_\delta + d y_\delta)\lambda}{12\delta}} \prod_{n=1}^{\infty}\left(1-q^n e^{\frac{2\pi i n(\delta b x_\delta + d y_\delta)\lambda}{\delta}}\right).\end{aligned}$$

Consequently, $\prod_{\delta|N}\eta^{r_\delta}(\gamma_{1,\delta}\tau) = q^{\frac{1}{24}\sum_{\delta|N}\frac{r_\delta\lambda^2}{\delta}} h(q)$ where $h(q)$ is a Taylor series with nonzero constant term in powers of $q^{1/\nu}$ for some positive integer ν, and therefore condition *(iii)* of Definition 2.6 is fulfilled. Recalling $\lambda = \gcd(\delta, c)$ and Definition 2.6 we see that $f_r \in M_k(N)$ if and only if

$$\sum_{\delta|N} \frac{r_\delta \gcd^2(\delta,c)}{\delta} \geq 0 \tag{2.47}$$

for all $c\in\mathbb{Z}$. But since $\gcd(\delta,c) = \gcd(\delta,\gcd(c,N))$ whenever $\delta|N$, we see that we need to check (2.47) only for c being a divisor of N. □

Remark 2.36. *Newman's Lemma in its original version in [27] or [28] can be refined to an "if and only if" statement, as remarked-without proof-for instance by Garvan [14, Thm. 4.7]. Being not relevant for the present context, we only mention that an analogous refinement holds also for our modified version as we will see in the end of next chapter.*

2.3 The Hecke Subgroups

The Hecke subgroups $\Gamma_0(N), N\in\mathbb{N}^*$ are very important throughout this thesis and we will derive some of their properties. We start with an explicit formula for $\omega_{\Gamma_0(N),\gamma}$, $\gamma\in\mathrm{SL}_2(\mathbb{Z})$, in Definition 2.9:

Lemma 2.37. *Let* $\gamma = \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in\mathrm{SL}_2(\mathbb{Z})$ *and* N *a positive integer. Then*

$$\omega_{\Gamma_0(N),\gamma} = \frac{N}{\gcd(c^2,N)}.$$

Proof. By Definition 2.9 we need to find the minimal positive integer h such that

$$\gamma \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \gamma^{-1} = \begin{pmatrix} 1-ach & a^2h \\ -c^2h & 1+ach \end{pmatrix} \in \Gamma_0(N). \tag{2.48}$$

We see that (2.48) is equivalent to

$$-c^2h \equiv 0 \pmod N \Leftrightarrow -\frac{c^2}{\gcd(c^2,N)}h \equiv 0 \pmod{\frac{N}{\gcd(c^2,N)}} \Leftrightarrow h \equiv 0 \pmod{\frac{N}{\gcd(c^2,N)}}.$$

□

Definition 2.38. *Let p be a prime, $N \in \mathbb{N}$ and $\alpha \in \{0,\ldots,N\}$. Then we define:*

$$S_{N,\alpha,p} := \begin{cases} \{0,\ldots,p^N-1\} & \text{if } \alpha = 0; \\ \{x \in \{0,\ldots,p^{N-\alpha}-1\} \mid \gcd(p,x)=1\} & \text{if } 1 \le \alpha \le N-1; \\ \{1\} & \text{if } \alpha = N. \end{cases}$$

Lemma 2.39. *Let M, N, Q be positive integers and p a prime such that $p \nmid QM$. For $\alpha \in \{0,\ldots,N\}$ let $x^{(\alpha)}, y^{(\alpha)} : S_{N,\alpha,p} \to \mathbb{Z}$ be such that for $x_\lambda^{(\alpha)} := x^{(\alpha)}(\lambda)$ and $y_\lambda^{(\alpha)} := y^{(\alpha)}(\lambda)$ where $\lambda \in S_{N,\alpha,p}$:*

$$x_\lambda^{(\alpha)} p^\alpha - y_\lambda^{(\alpha)} Q\lambda M = 1.$$

For $\alpha \in \{0,\ldots,N\}$ set

$$R_{N,\alpha,p} := \left\{ \begin{pmatrix} x_\lambda^{(\alpha)} & y_\lambda^{(\alpha)} \\ Q\lambda M & p^\alpha \end{pmatrix} \mid \lambda \in S_{N,\alpha,p} \right\}.$$

Then $\cup_{i=0}^N R_{N,\alpha,p}$ forms a complete set of representatives of $\Gamma_0(p^N M)$ in $\Gamma_0(M)$. In particular

$$\sum_{i=0}^{N} |R_{N,\alpha,p}| = [\Gamma_0(M) : \Gamma_0(p^N M)] = p^N(1+p^{-1}). \tag{2.49}$$

Proof. Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(M)$. We will show that there exists a $\alpha \in \{0,\ldots,N\}$ and $\lambda \in S_{N,\alpha,p}$ such that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} p^\alpha & -y_\lambda^{(\alpha)} \\ -Q\lambda M & x_\lambda^{(\alpha)} \end{pmatrix} \in \Gamma_0(p^N M).$$

Clearly this is the case if $cp^\alpha - Q\lambda M d \equiv_{p^N} 0$. If $d \equiv_{p^N} 0$ then choose $\alpha = N$ and $\lambda = 1$. Otherwise write $d = d_0 p^\beta$ with $\beta < N$ and $\gcd(b_0, p) = 1$ and choose $\alpha = \beta$ and λ such that $\lambda \equiv Q^{-1}M^{-1}d_0^{-1}c \pmod{p^{N-\alpha}}$, $0 \le \lambda \le p^{N-\alpha}$. Next we show that the representatives are not equivalent. This task is equivalent to showing that

$$\begin{pmatrix} x_{\lambda'}^{(\beta)} & y_{\lambda'}^{(\beta)} \\ Q\lambda' M & p^\beta \end{pmatrix} \begin{pmatrix} p^\alpha & -y_\lambda^{(\alpha)} \\ -Q\lambda M & x_\lambda^{(\alpha)} \end{pmatrix} \in \Gamma_0(p^N M)$$

only if $\lambda = \lambda'$ and $\alpha = \beta$, which is equivalent to showing that $Qp^{\alpha}\lambda' M - Qp^{\beta}\lambda M \equiv_{p^N} 0$ which can happen only if $\alpha = \beta$ and $\lambda = \lambda'$. To show (2.49) we note that $|R_{N,\alpha,p}| = |S_{N,\alpha,p}|$ and by Definition 2.38 we obtain immediately

$$|S_{N,\alpha,p}| = \begin{cases} p^N & \text{if } \alpha = 0; \\ p^{N-\alpha} - p^{N-\alpha-1} & \text{if } 1 \leq \alpha \leq N-1; \\ 1 & \text{if } \alpha = N. \end{cases}$$

This implies that

$$\sum_{i=0}^{N} |S_{N,\alpha,p}| = p^N + \sum_{\alpha=1}^{N-1} (p^{N-\alpha} - p^{N-\alpha-1}) + 1 = p^N + p^{N-1} - 1 + 1 = p^N(1 + p^{-1})$$

which proves (2.49). $\square$

Corollary 2.40. *Let N be a positive integer. Then*

$$[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)] = N \prod_{p|N} (1 + p^{-1}).$$

Proof. Let $p_1^{\beta_1} \cdots p_n^{\beta_n}$ be the prime decomposition of N. Because of (2.41) we see that $\Gamma_0(N) \subseteq \Gamma_0(M)$ iff $M|N$. This implies that we have a chain

$$\Gamma_0(N) \subseteq \Gamma_0(p_1^{\beta_1} \cdots p_n^{\beta_n}) \subseteq \Gamma_0(p_1^{\beta_1} \cdots p_{n-1}^{\beta_{n-1}}) \subseteq \ldots \subseteq \Gamma_0(p^{\beta_1}) \subseteq \Gamma_0(1) = \mathrm{SL}_2(\mathbb{Z}).$$

This chain implies that

$$[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)] = [\Gamma_0(1) : \Gamma_0(p_1^{\beta_1})] \prod_{i=1}^{n-1} [\Gamma_0(p_1^{\beta_1} \cdots p_i^{\beta_i}) : \Gamma_0(p_1^{\beta_1} \cdots p_{i+1}^{\beta_{i+1}})],$$

and which by (2.49) equals $\prod_{i=1}^{n} p_i^{\beta_i}(1 + p_i^{-1})$ finishing the proof. $\square$

One can easily modify this proof of Lemma 2.39 to get in addition (in Lemma 2.44 *(i)*) the explicit coset representatives of $\Gamma_0(MN)$ in $\Gamma_0(M)$ even for N not necessarily a prime power and in this way one obtains Corollary 2.40 directly. Moreover, we obtain (in Lemma 2.44 *(ii)* another formula for the index $[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)]$ which might be preferable for some applications. For the sake of completeness we carry out the details.

Lemma 2.41. *Let m, n be positive integers and $\lambda \in \mathbb{Z}$. Then if* $\gcd(\lambda, m, n) = 1$ *there exists an integer k such that* $\gcd(\lambda + km, n) = 1$.

Proof. For $p|n$ define

$$k_p := \begin{cases} 1, & \text{if } p|\lambda; \\ 0, & \text{otherwise.} \end{cases}$$

By chinese remaindering there exists a $k \in \mathbb{Z}$ such that $k \equiv k_p \pmod{p}$ for all $p|n$. We will show that $\gcd(\lambda + km, n) = 1$. Assume by contradiction that $p| \gcd(\lambda + km, n)$. Then $p|n$ and $\lambda + km \equiv \lambda + k_p m \equiv 0 \pmod{p}$. If $p|\lambda$ then $k_p = 1$ meaning $p|\lambda + m$ and consequently $p|m$ contradicting $\gcd(\lambda, m, n) = 1$. If $p \nmid \lambda$ then $k_p = 0$ contradicting $\lambda + k_p m \equiv 0 \pmod{p}$. $\square$

Lemma 2.41 motivates the next definition.

Definition 2.42. *Let m, n be positive integers and $\lambda \in \mathbb{Z}$ such that $\gcd(\lambda, m, n) = 1$. Then we define $\rho_{m,n}(\lambda)$ to be the minimal positive integer such that $\gcd(\rho_{m,n}(\lambda), n) = 1$ and $\rho_{m,n}(\lambda) \equiv \lambda \pmod{m}$.*

We point out that minimality in Definition 2.42 is only required for uniqueness.

Definition 2.43. *Let N be a positive integer. Then for $\delta|N$ we define*

$$S_{N,\delta} := \left\{ \rho_{N/\delta,\delta}(\lambda) | \lambda \in \left\{ 0, \ldots, \frac{N}{\delta} - 1 \right\} | \gcd(\lambda, N/\delta, \delta) = 1 \right\}.$$

Lemma 2.44. *Let $N, Q, M \in \mathbb{N}^*$ such that $\gcd(N, QM) = 1$. For $\delta|N$ let $x^{(\delta)}, y^{(\delta)} : S_{N,\delta} \to \mathbb{Z}$ with $\lambda \mapsto x_\lambda^{(\delta)}, y_\lambda^{(\delta)}$ be such that*

$$x_\lambda^{(\delta)} \delta - y_\lambda^{(\delta)} Q\lambda M = 1.$$

For $\delta|N$ set

$$R_{N,\delta} := \left\{ \begin{pmatrix} x_\lambda^{(\delta)} & y_\lambda^{(\delta)} \\ Q\lambda M & \delta \end{pmatrix} | \lambda \in S_{N,\delta} \right\}.$$

(i) Then $\cup_{\delta|N} R_{N,\delta}$ forms a complete set of coset representatives of $\Gamma_0(NM)$ in $\Gamma_0(M)$;

(ii)

$$\sum_{\delta|N} |R_{N,\delta}| = \sum_{\delta|N} \frac{\delta\varphi(\gcd(\delta, N/\delta))}{\gcd(\delta, N/\delta)} = [\Gamma_0(M) : \Gamma_0(NM)] = N \prod_{p|N} (1 + p^{-1}) \tag{2.50}$$

where $\varphi : \mathbb{N}^* \to \mathbb{N}^*$ is the Euler φ-function with $\varphi(n)$ equal to the number of positive integers less then n and coprime to n.

Proof. (i): Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(M)$. We will show that there exists a $\delta|N$ and $\lambda \in S_{N,\delta}$ such that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \delta & -y_\lambda^{(\delta)} \\ -Q\lambda M & x_\lambda^{(\delta)} \end{pmatrix} \in \Gamma_0(NM).$$

Clearly this is the case if $c\delta - Q\lambda Md \equiv_N 0$. We choose $\delta = \gcd(d, N)$ and λ such that $c \equiv Q\lambda M\frac{d}{\delta} \pmod{\frac{N}{\delta}}$ or equivalently $\lambda \equiv cQ^{-1}M^{-1}(d/\delta)^{-1} \pmod{\frac{N}{\delta}}$. We also need to show that $cQ^{-1}M^{-1}(d/\delta)^{-1}$ is equivalent modulo N/δ with some element in $S_{N,\delta}$ which is true iff $cQ^{-1}M^{-1}(d/\delta)^{-1}$ is a unit modulo $\gcd(\delta, N/\delta)$. Clearly Q^{-1}, M^{-1} and $(d/\delta)^{-1}$ are units modulo $\gcd(\delta, N/\delta)$ because they are units modulo N/δ. Also, because $\delta|d$ we must have $\gcd(c, \delta) = 1$ because of $ad - bc = 1$ implying that c is a unit modulo δ and consequently also modulo $\gcd(\delta, N/\delta)$.

Next we show that the representatives are not equivalent. This task is equivalent to showing that if for $\delta_1, \delta_2|N$, $\lambda_1 \in S_{N,\delta_1}$ and $\lambda_2 \in S_{N,\delta_2}$ the relation

$$\begin{pmatrix} x_{\lambda_1}^{(\delta_1)} & y_{\lambda_1}^{(\delta_1)} \\ Q\lambda_1 M & \delta_1 \end{pmatrix} \begin{pmatrix} \delta_2 & -y_{\lambda_2}^{(\delta_2)} \\ -Q\lambda_2 M & x_{\lambda_2}^{(\delta_2)} \end{pmatrix} \in \Gamma_0(NM), \tag{2.51}$$

holds then $\lambda_1 = \lambda_2$ and $\delta_1 = \delta_2$. We note that (2.51) is equivalent to

$$N|(Q\lambda_1 M\delta_2 - Q\lambda_2 M\delta_1). \tag{2.52}$$

In particular $\delta_1|(Q\lambda_1 M\delta_2 - Q\lambda_2 M\delta_1)$ hence $\delta_1|Q\lambda_1 M\delta_2$ but $\gcd(QM\lambda_1, \delta_1) = 1$ implying $\delta_1|\delta_2$. By symmetry $\delta_2|\delta_1$ and we may set $\delta = \delta_1 = \delta_2$. By (2.52) we have $\frac{N}{\delta}|(Q\lambda_1 M - Q\lambda_2 M)$ so $\lambda_1 \equiv \lambda_2 \pmod{N/\delta}$ implying $\lambda_1 = \lambda_2$ because $S_{N,\delta}$ contain no two elements equivalent modulo N/δ. This finishes the proof of *(i)*.

(ii): Next we compute the number of elements in $S_{N,\delta}$ for $\delta|N$. We see that x with $0 \le x \le N/\delta - 1$ is in $S_{N,\delta}$ iff x is a unit modulo $\gcd(\delta, N/\delta)$. There are totally $\varphi(\gcd(\delta, N/\delta))$ units modulo $\gcd(\delta, N/\delta)$ and the mapping $\phi : \mathbb{Z}_{N/\delta} \to \mathbb{Z}_{\gcd(\delta,N/\delta)}$ defined by $\phi(x) := x \pmod{\gcd(\delta, N/\delta)}$ has the property $|\phi^{-1}(x)| = \frac{N/\delta}{\gcd(\delta,N/\delta)}$ for all $x \in \mathbb{Z}_{\gcd(\delta,N/\delta)}$ which shows that $|S_{N,\delta}| = \frac{N/\delta}{\gcd(\delta,N/\delta)} \cdot \varphi(\gcd(\delta, N/\delta))$. Since $|R_{N,\delta}| = |S_{N,\delta}|$ we showed that

$$\sum_{\delta|N} \frac{(N/\delta)\varphi(\gcd(\delta, N/\delta))}{\gcd(\delta, N/\delta)} = \sum_{\delta|N} \frac{\delta\varphi(\gcd(\delta, N/\delta))}{\gcd(\delta, N/\delta)} = [\Gamma_0(M) : \Gamma_0(NM)]. \tag{2.53}$$

Assume by induction that for $n < N$ (2.50) holds. Next let $p|N$ and write $N = p^\alpha N_0$ where $p \nmid N_0$. Then

$$\begin{aligned}
&= \sum_{\delta|N} \frac{\delta\varphi(\gcd(\delta, N/\delta))}{\gcd(\delta, N/\delta)} \\
&= \sum_{\delta|p^\alpha N_0} \frac{\delta\varphi(\gcd(\delta, N/\delta))}{\gcd(\delta, N/\delta)} \\
&= \sum_{\delta_1|p^\alpha, \delta_2|N_0} \frac{\delta_1\delta_2\varphi(\gcd(\delta_1\delta_2, p^\alpha N_0/(\delta_1\delta_2)))}{\gcd(\delta_1\delta_2, p^\alpha N_0/(\delta_1\delta_2))} \\
&= \sum_{\delta_1|p^\alpha, \delta_2|N_0} \frac{\delta_1\delta_2\varphi(\gcd(\delta_1, p^\alpha/\delta_1)\gcd(\delta_2, N_0/\delta_2))}{\gcd(\delta_1, p^\alpha/\delta_1)\gcd(\delta_2, N_0/\delta_2)} \\
&= \sum_{\delta_1|p^\alpha, \delta_2|N_0} \frac{\delta_1\delta_2\varphi(\gcd(\delta_1, p^\alpha/\delta_1))\varphi(\gcd(\delta_2, N_0/\delta_2))}{\gcd(\delta_1, p^\alpha/\delta_1)\gcd(\delta_2, N_0/\delta_2)} \\
&= \left(\sum_{\delta_1|p^\alpha} \frac{\delta_1\varphi(\gcd(\delta_1, p^\alpha/\delta_1))}{\gcd(\delta_1, p^\alpha/\delta_1)}\right)\left(\sum_{\delta_2|N_0} \frac{\delta_2\varphi(\gcd(\delta_2, N_0/\delta_2))}{\gcd(\delta_2, N_0/\delta_2)}\right) \\
&= [\Gamma_0(M) : \Gamma_0(p^\alpha M)] \cdot [\Gamma_0(M) : \Gamma_0(N_0 M)].
\end{aligned} \tag{2.54}$$

The induction base for (2.50) is $N = p^\alpha$ for p an arbitrary prime and $\alpha \in \mathbb{N}$. By (2.53) we have

$$\begin{aligned}
[\Gamma_0(M) : \Gamma_0(p^\alpha M)] &= \sum_{\delta | p^\alpha} \frac{\delta \varphi(\gcd(\delta, p^\alpha/\delta))}{\gcd(\delta, p^\alpha/\delta)} \\
&= \sum_{i=0}^{\alpha} \frac{p^i \varphi(\gcd(p^i, p^\alpha/p^i))}{\gcd(p^i, p^\alpha/p^i)} \\
&= \sum_{i=0}^{\alpha} \frac{p^i \varphi(p^{\min(i,\alpha-i)})}{p^{\min(i,\alpha-i)}} \qquad (2.55)\\
&= \varphi(1) + \sum_{i=1}^{\alpha-1} \frac{p^i \varphi(p^{\min(i,\alpha-i)})}{p^{\min(i,\alpha-i)}} + p^\alpha \varphi(1) \\
&= 1 + \sum_{i=1}^{\alpha-1} p^i (1-p^{-1}) + p^\alpha = p^\alpha(1+p^{-1}).
\end{aligned}$$

Note that we used that $\varphi(p^\alpha) = p^\alpha(1-p^{-1})$ which is immediately seen by counting the units in $\mathbb{Z}_{p^\alpha}$. □

For the remaining part of this section we will derive some results that allows us to compute a complete set of representatives of the double cosets $\Gamma_0(N)\backslash \mathrm{SL}_2(\mathbb{Z})/\mathrm{SL}_2(\mathbb{Z})_\infty$ for any given positive integer N.

Lemma 2.45. *Let N be a positive integer. For each $\delta \in \mathbb{N}^*$ such that $\delta | N$ let X_δ be any set of complete representatives of $\mathbb{Z}^*_{\gcd(\delta, \frac{N}{\delta})}$ with the property $\gcd(x, \delta) = 1$ for all $x \in X_\delta$. Write $X_\delta = \{a_1^{(\delta)}, \ldots, a_{\varphi(\gcd(\delta, N/\delta))}^{(\delta)}\}$. Let $b^{(\delta)}, d^{(\delta)} : X_\delta \to \mathbb{Z}$ with $\lambda \mapsto b_\lambda^{(\delta)}, d_\lambda^{(\delta)}$ be such that*

$$a_\lambda^{(\delta)} d_\lambda^{(\delta)} - b_\lambda^{(\delta)} \delta = 1, \quad \lambda \in \{1, \ldots, \varphi\left(\gcd(\delta, N/\delta)\right)\}. \qquad (2.56)$$

Define

$$S_{N,\delta} := \left\{ \begin{pmatrix} a_\lambda^{(\delta)} & b_\lambda^{(\delta)} \\ \delta & d_\lambda^{(\delta)} \end{pmatrix} | \lambda \in \{1, \ldots, \varphi(\gcd(\delta, N/\delta))\} \right\}.$$

Then $\cup_{\delta | N} S_{N,\delta}$ is a complete set of representatives of the double cosets $\Gamma_0(N)\backslash \mathrm{SL}_2(\mathbb{Z})/\mathrm{SL}_2(\mathbb{Z})_\infty$. Furthermore

$$\bigcup_{\delta|N} |S_{N,\delta}| = \sum_{\delta | N} \varphi(\gcd(\delta, N/\delta)).$$

Proof. Let $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ be arbitrary but fixed. We will show that there exist $\delta | N$ and $\left(\begin{smallmatrix} a_\lambda^{(\delta)} & b_\lambda^{(\delta)} \\ \delta & d_\lambda^{(\delta)} \end{smallmatrix}\right) \in X_\delta$ such that $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N) \left(\begin{smallmatrix} a_\lambda^{(\delta)} & b_\lambda^{(\delta)} \\ \delta & d_\lambda^{(\delta)} \end{smallmatrix}\right) \mathrm{SL}_2(\mathbb{Z})_\infty$. Set $\delta = \gcd(c, N)$ and choose λ such that

$$a_\lambda^{(\delta)} \equiv d^{-1} \frac{c}{\delta} \pmod{\gcd(\delta, N/\delta)}. \qquad (2.57)$$

Since $\gcd\left(\frac{c}{\delta}, \frac{N}{\delta}\right) = 1$ such a $a_\lambda^{(\delta)} \in X_\delta$ surely exists. Note that because of (2.56) one can write (2.57) equivalently as

$$d_\lambda^{(\delta)} \equiv d \left(\frac{c}{\delta}\right)^{-1} \pmod{\gcd(\delta, N/\delta)}. \qquad (2.58)$$

We will show that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \begin{pmatrix} a_\lambda^{(\delta)} & b_\lambda^{(\delta)} \\ \delta & d_\lambda^{(\delta)} \end{pmatrix} \mathrm{SL}_2(\mathbb{Z})_\infty. \tag{2.59}$$

We observe that (2.59) is equivalent with the existence of $h \in \mathbb{Z}$ such that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \begin{pmatrix} d_\lambda^{(\delta)} & -b_\lambda^{(\delta)} \\ -\delta & a_\lambda^{(\delta)} \end{pmatrix} \in \Gamma_0(N). \tag{2.60}$$

Further (2.60) is equivalent to

$$\begin{aligned} cd_\lambda^{(\delta)} - \delta(ch+d) \equiv 0 \pmod N \Leftrightarrow \frac{c}{\delta} d_\lambda^{(\delta)} - d &\equiv \frac{c}{\delta}\delta h \pmod{N/\delta} \\ \Leftrightarrow d_\lambda^{(\delta)} - d\left(\frac{c}{\delta}\right)^{-1} &\equiv \delta h \pmod{N/\delta}. \end{aligned} \tag{2.61}$$

Because of (2.58) there exists a $k \in \mathbb{Z}$ such that $k\gcd(\delta, N/\delta) = d_\lambda^{(\delta)} - d\left(\frac{c}{\delta}\right)^{-1}$. Because of this we may write (2.61) as

$$k\gcd(\delta, N/\delta) \equiv \delta h \pmod{N/\delta} \Leftrightarrow k \equiv \frac{\delta}{\gcd(\delta, N/\delta)} h \pmod{N/(\delta\gcd(\delta, N/\delta))}. \tag{2.62}$$

Since $\frac{\delta}{\gcd(\delta,N/\delta)}$ is a unit modulo $\frac{N}{\delta\gcd(\delta,N/\delta)}$ we may choose h such that

$$h \equiv k\left(\frac{\delta}{\gcd(\delta, N/\delta)}\right)^{-1} \pmod{N/(\delta\gcd(\delta, N/\delta))}.$$

We next show that given $\begin{pmatrix} a_\lambda^{(\delta_1)} & b_\lambda^{(\delta_1)} \\ \delta_1 & d_\lambda^{(\delta_1)} \end{pmatrix}, \begin{pmatrix} a_\mu^{(\delta_2)} & b_\mu^{(\delta_2)} \\ \delta_2 & d_\mu^{(\delta_2)} \end{pmatrix} \in \cup_{\delta|N} S_{N,\delta}$ such that

$$\begin{pmatrix} a_\lambda^{(\delta_1)} & b_\lambda^{(\delta_1)} \\ \delta_1 & d_\lambda^{(\delta_1)} \end{pmatrix} \in \Gamma_0(N) \begin{pmatrix} a_\mu^{(\delta_2)} & b_\mu^{(\delta_2)} \\ \delta_2 & d_\mu^{(\delta_2)} \end{pmatrix} \mathrm{SL}_2(\mathbb{Z})_\infty \tag{2.63}$$

implies $\delta_1 = \delta_2$ and $\lambda = \mu$. As before we see that (2.63) is equivalent to the existence of $h \in \mathbb{Z}$ such that

$$\begin{pmatrix} a_\lambda^{(\delta_1)} & b_\lambda^{(\delta_1)} \\ \delta_1 & d_\lambda^{(\delta_1)} \end{pmatrix} \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \begin{pmatrix} d_\mu^{(\delta_2)} & -b_\mu^{(\delta_2)} \\ -\delta_2 & a_\mu^{(\delta_2)} \end{pmatrix} \in \Gamma_0(N)$$

which is equivalent to

$$\delta_1 d_\mu^{(\delta_2)} - \delta_2(\delta_1 h + d_\lambda^{(\delta_1)}) \equiv 0 \pmod N. \tag{2.64}$$

By (2.64) we see that $\delta_1|\delta_2(\delta_1 h + d_\lambda^{(\delta_1)})$ implying $\delta_1|\delta_2$ because of $\gcd(\delta_1, d_\lambda^{(\delta_1)}) = 1$ by (2.56). Again by (2.64) we see that $\delta_2|\delta_1 d_\mu^{(\delta_2)}$ implying $\delta_2|\delta_1$ because of $\gcd(\delta_2, d_\mu^{(\delta_2)}) = 1$. This shows $\delta_2 = \delta_1$. By using this we see that (2.64) is equivalent to

$$d_\mu^{(\delta)} - (\delta h + d_\lambda^{(\delta)}) \equiv 0 \pmod{N/\delta}. \tag{2.65}$$

This implies that $d_\mu^{(\delta)} - d_\lambda^{(\delta)} \equiv 0 \pmod{\gcd(\delta, N/\delta)}$ which together with (2.56) implies $a_\mu^{(\delta)} - a_\lambda^{(\delta)} \equiv 0 \pmod{\gcd(\delta, N/\delta)}$ showing $\lambda = \mu$. □

Chapter 3

Automatic Proofs of Identities Related to Modular Forms

Fix a positive integer N and let $R(N)$ be the set of integer sequences $r = (r_\delta)_{\delta|N}$ indexed by the positive divisors δ of N. Let $r^{(1)}, \ldots, r^{(m)} \in R(N)$ and $v_1, \ldots, v_m \in \bigcup_{k=-\infty}^{\infty} M_{2k}(N)$ where $M_{2k}(N) := M_{2k}(\Gamma_0(N))$. In this chapter we will present an algorithm for proving identities of the form:

$$v_1 \prod_{\delta|N} \eta_\delta^{r_\delta^{(1)}} + \cdots + v_m \prod_{\delta|N} \eta_\delta^{r_\delta^{(m)}} = 0. \tag{3.1}$$

We call an identity of type (3.1) a *general eta identity*. We define an equivalence relation $\sim$ on $R(N) \times \mathbb{Z}$ in the following way. For $(r, k_r), (s, k_s) \in R(N) \times \mathbb{Z}$ we say that $(r, k_r) \sim (s, k_s)$ iff:

- $\sum_{\delta|N} \delta(r_\delta - s_\delta) \equiv 0 \pmod{24}$;
- $\sum_{\delta|N} (N/\delta)(r_\delta - s_\delta) \equiv 0 \pmod{24}$;
- $\sum_{\delta|N} (r_\delta - s_\delta) + 2k_r - 2k_s = 0$;
- $\prod_{\delta|N} \delta^{r_\delta - s_\delta}$ is a rational square.

For $f \in M_k(N)$ of weight k denote the weight of f by $w(f)$; i.e. $w(f) = k$. Our strategy for proving (3.1) is as follows. We write the set

$$\{(r^{(1)}, w(v_1)), \ldots, (r^{(m)}, w(v_m))\}$$

as a partition $S_1 \cup S_2 \cup \ldots \cup S_l$ into equivalence classes under the relation $\sim$. Fix an index j and write the equivalence class S_j as $\{(r^{(I_j(1))}, w(v_{I_j(1)})), \ldots, (r^{(I_j(|S_j|))}, w(v_{I_j(|S_j|)}))\}$ where $I_j : S_j \to \{1, \ldots, |S_j|\}$ is a function that indexes the elements of S_j. Then we will show (Theorem 3.5) that if (3.1) is true then for all $j \in \{1, \ldots, l\}$:

$$v_{I_j(1)} \prod_{\delta|N} \eta_\delta^{r_\delta^{(I_j(1))}} + \cdots + v_{I_j(|S_j|)} \prod_{\delta|N} \eta_\delta^{r_\delta^{(I_j(|S_j|))}} = 0. \tag{3.2}$$

We call an identity of type (3.2) a *fundamental eta identity*. This means that to each S_j corresponds a fundamental eta identity and it is sufficient to prove these l (smaller) fundamental eta identities because they sum up to the general eta identity (3.1). Summarizing we prove that a general eta identity (3.1) is true by proving that these l smaller identities are true. So it is sufficient to restrict ourselves to proving fundamental eta identities (3.2). We divide both sides of (3.2) by $\prod_{\delta|N} \eta_\delta^{r_\delta^{(I_j(1))}}$ and obtain

$$v_{I_j(1)} + v_{I_j(2)} \prod_{\delta|N} \eta_\delta^{r_\delta^{(I_j(2))} - r_\delta^{(I_j(1))}} + \cdots + v_{I_j(|S_j|)} \prod_{\delta|N} \eta_\delta^{r_\delta^{(I_j(|S_j|))} - r_\delta^{(I_j(1))}} = 0. \tag{3.3}$$

A fundamental eta identity in the form (3.3) we call *reduced fundamental eta identity.* For the rest of the proof we apply a method very similar to the one proposed in Chapter 20 of [7]. Denote by F the left hand side of (3.3). We will show in Section 3.2 that for some κ_1, $\kappa_2 \in \mathbb{Z}$ there exists $f \in M_{\kappa_1}(N)$ such that $Ff \in M_{\kappa_2}(N)$. Next we show by computing the q-series of Ff that $\mathrm{Ord}_{\Gamma_0(N)}(Ff, \mathrm{id}) > \frac{\kappa_2 \cdot [\mathrm{SL}_2(\mathbb{Z}):\Gamma_0(N)]}{12}$. This contradicts Corollary (2.22) unless $Ff = 0$, implying $F = 0$ and (3.3) is proven. Note that (2.50) is useful in computing the index $[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)] = N \prod_{p|N}(1 + p^{-1})$.

3.1 Reduction of a General Eta Identity to Fundamental Eta Identities

The purpose of this section is to prove Theorem 3.5. In order to state it we need the following definitions. We recall that $r \in \mathbb{Q}$ is called a rational square if $r = q^2$ for some $q \in \mathbb{Q}$.

Definition 3.1. *The equivalence relation $\approx$ on the positive integers is defined by*

$$z_1 \approx z_2 \text{ iff } z_1/z_2 \text{ is a rational square.}$$

Definition 3.2. *For N a positive integer with prime divisors $p_1, \ldots, p_n$ we define*

$$L(N) := \{p_1^{\alpha_1} \cdots p_n^{\alpha_n} | (\alpha_1, \ldots, \alpha_n) \in Z_2^n\},$$

where $Z_2 := \{0, 1\}$.

Next we introduce some short hand notation to save space.

Definition 3.3. *For $n, m \in \mathbb{Z}$ let $[n, m] := \{x \in \mathbb{Z} | n \leq x \leq m\}$.*

Definition 3.4. *Let N be a positive integer. For $(i, j, k, l) \in [0, 23]^2 \times \mathbb{Z} \times L(N)$ we define*

$$\begin{aligned}
A_N^{(1)}(i) &:= \{r \in R(N) \mid \sum_{\delta|N} \delta r_\delta \equiv i \pmod{24}\};\\
A_N^{(2)}(j) &:= \{r \in R(N) \mid \sum_{\delta|N} (N/\delta) r_\delta \equiv j \pmod{24}\};\\
A_N^{(3)}(k) &:= \{(r, z) \in R(N) \times \mathbb{Z} \mid \sum_{\delta|N} r_\delta + 2z = k\};\\
A_N^{(4)}(l) &:= \{r \in R(N) \mid \prod_{\delta|N} \delta^{r_\delta} \approx l\};\\
A_N^{(i,j,k,l)} &:= A_N^{(3)}(k) \cap \{(A_N^{(1)}(i) \cap A_N^{(2)}(j) \cap A_N^{(4)}(l)) \times \mathbb{Z}\}.
\end{aligned}$$

Theorem 3.5. *Let N be a positive integer, let $p_1, \ldots, p_n$ be the primes dividing N, $\{v_1, \ldots, v_m\} \subseteq \cup_{k=-\infty}^{\infty} A_{2k}(N)$ (where $A_{2k}(N) := A_{2k}(\Gamma_0(N))$ is as in Definition 2.6), $\{R^{(1)}, \ldots, R^{(m)}\} \subseteq R(N)$. Assume that the following identity holds:*

$$v_1(\tau) \prod_{\delta|N} \eta_\delta^{R_\delta^{(1)}}(\tau) + \cdots + v_m(\tau) \prod_{\delta|N} \eta_\delta^{R_\delta^{(m)}}(\tau) = 0, \quad \tau \in \mathbb{H}. \tag{3.4}$$

Then for all $(i, j, k, l) \in [0, \ldots, 23]^2 \times \mathbb{Z} \times L(N)$ we have:

$$\sum_{\substack{x \in [1,m] \\ (w(v_x), R^{(x)}) \in A_N^{(i,j,k,l)}}} v_x(\tau) \prod_{\delta|N} \eta_\delta^{R_\delta^{(x)}}(\tau) = 0, \quad \tau \in \mathbb{H}. \tag{3.5}$$

We split the proof of this theorem into four parts. In the first part we show that if a general eta identity (3.4) is true then the following identities hold:

$$\sum_{\substack{x \in [1,m] \\ (w(v_x), R^{(x)}) \in A_N^{(3)}(k)}} v_x(\tau) \prod_{\delta|N} \eta_\delta^{R_\delta^{(x)}}(\tau) = 0, \quad \tau \in \mathbb{H}, \quad k \in \mathbb{Z}. \tag{3.6}$$

In part two we prove that if an identity of type (3.6) holds for some fixed $k \in \mathbb{Z}$ then the following identities hold:

$$\sum_{\substack{x \in [1,m] \\ (w(v_x), R^{(x)}) \in A_N^{(3)}(k) \\ R^{(x)} \in A_N^{(1)}(i)}} v_x(\tau) \prod_{\delta|N} \eta_\delta^{R_\delta^{(x)}}(\tau) = 0, \quad \tau \in \mathbb{H}, \quad i \in [0, 23]. \tag{3.7}$$

In part three we prove that if an identity of type (3.7) holds for some fixed $k \in \mathbb{Z}$ and some $i \in [0, 23]$ then the following identities hold:

$$\sum_{\substack{x \in [1,m] \\ (w(v_x), R^{(x)}) \in S_N^{(k,i,j)}}} v_x(\tau) \prod_{\delta|N} \eta^{R_\delta^{(x)}}(\delta\tau) = 0, \quad \tau \in \mathbb{H}, \quad j \in [0, 23], \tag{3.8}$$

where $S(k, i, j) \subseteq R(N) \times \mathbb{Z}$ is defined by:

$$S_N^{(k,i,j)} := A_N^{(3)}(k) \cap \{(A_N^{(1)}(i) \cap A_N^{(2)}(j)) \times \mathbb{Z}\}.$$

Finally in the last part of the proof we prove that if an identity of type (3.8) holds for some fixed $k \in \mathbb{Z}$ and some $i, j \in [0, 23]$ then the following fundamental eta identities hold:

$$\sum_{\substack{(w(v_x), R^{(x)}) \in S_N^{(k,i,j)}, \\ \prod_{\delta|N} \delta^{|R_\delta^{(x)}|} \approx p_1^{\alpha_1} \cdots p_n^{\alpha_n}}} v_x(\tau) \prod_{\delta|N} \eta^{R_\delta^{(x)}}(\delta\tau) = 0, \quad \tau \in \mathbb{H}, \quad (\alpha_1, \ldots, \alpha_n) \in Z_2^n, \tag{3.9}$$

where $Z_2 := \{0, 1\}$ and $p_1, \ldots, p_n$ are the primes dividing N. In particular the validity of (3.9) finishes the proof because for $(i, j, k, l := p_1^{\alpha_1} \cdots p_n^{\alpha_n}) \in [0, 23]^2 \times \mathbb{Z} \times L(N)$ we have

$$\begin{aligned} &\{(d, R^{(x)}) \in A_N^{(i,j,k,l)}, x \in [1, m]\} \\ =&\{(d, R^{(x)}) \in S_N^{(k,i,j)} | \prod_{\delta|N} \delta^{|R_\delta^{(x)}|} \approx p_1^{\alpha_1} \cdots p_n^{\alpha_n}, x \in [1, m]\}. \end{aligned}$$

We will need the following obvious proposition in order to make the flow of arguments smooth.

Proposition 3.6. *Let N be a positive integer and $f \in \cup_{k=-\infty}^{\infty} A_{2k}(N)$ then*

$$f(\tau+1) = f(\tau), \quad \tau \in \mathbb{H}. \tag{3.10}$$

3.1.1 Reduction of (3.4) to (3.6)

In order to achieve the goal of this subsection we need Lemma 3.8 below. In order to prove Lemma 3.8 we first need Lemma 3.7.

Lemma 3.7. *Let $m, n \in \mathbb{N}^*$. Let $f_1, \ldots, f_n : \mathbb{H} \to \mathbb{C}$ be such that for $j \in [1, n]$ we have $f_j(\tau + m) = f_j(\tau)$, $\tau \in \mathbb{H}$. Assume that there exist nonzero $p_1(\tau), \ldots, p_n(\tau) \in \mathbb{C}[\tau]$ such that*

$$\deg(p_1(\tau)) < \ldots < \deg(p_n(\tau)) \tag{3.11}$$

and

$$f_1(\tau)p_1(\tau) + \cdots + f_n(\tau)p_n(\tau) = 0, \quad \tau \in \mathbb{H}. \tag{3.12}$$

Then $f_1 = \cdots = f_n = 0$.

Proof. Clearly the lemma is true for $n = 1$. Let $N > 1$ be an integer and assume by induction that the lemma is true for $n < N$. Then we prove that it is also true for $n = N$. For $g(\tau) \in \mathbb{C}[\tau]$ denote

$$g^{(0)}(\tau) := g(\tau)$$

and

$$g^{(k)}(\tau) = g^{(k-1)}(\tau + m) - g^{(k-1)}(\tau), \quad k \in \mathbb{N}^*.$$

If $g^{(k)}(\tau) \neq 0$ then it is immediate that $\deg(g^{(k)}(\tau)) = \deg(g^{(k-1)}(\tau)) - 1$. This implies that for $k \in \mathbb{N}$ and $g(\tau) \neq 0$:

$$\deg(g^{(k)}(\tau)) = \deg(g(\tau)) - k, \ g^{(k)}(\tau) \neq 0 \tag{3.13}$$

if $\deg(g(\tau)) \geq k$, and

$$g^{(k)}(\tau) = 0 \tag{3.14}$$

if $k > \deg(g(\tau))$. Let $F : \mathbb{H} \to \mathbb{C}$. Then $\Delta_m(F) : \mathbb{H} \to \mathbb{C}$ is defined by

$$\Delta_m(F)(\tau) := F(\tau + m) - F(\tau), \quad \tau \in \mathbb{H}.$$

Applying Δ_m to both sides of (3.12) one obtains

$$f_1(\tau)p_1^{(1)}(\tau) + \cdots + f_N(\tau)p_N^{(1)}(\tau) = 0, \quad \tau \in \mathbb{H}.$$

And by repeated application of Δ_m to (3.12) we obtain by induction that

$$f_1(\tau)p_1^{(k)}(\tau) + \cdots + f_N(\tau)p_N^{(k)}(\tau) = 0, \quad \tau \in \mathbb{H}.$$

In particular if $\deg(p_1(\tau)) = d$ then

$$f_2(\tau)p_2^{(d+1)}(\tau) + \cdots + f_N(\tau)p_N^{(d+1)}(\tau) = 0, \quad \tau \in \mathbb{H},$$

because of (3.14). Because of (3.11) and (3.13) we have that $p_j^{(d+1)}(\tau) \neq 0$ for $j \in [2, N]$ and

$$\deg(p_2^{(d+1)}(\tau)) < \ldots < \deg(p_N^{(d+1)}(\tau)).$$

Hence by the induction hypothesis $f_2 = \ldots = f_N = 0$ and by (3.12) also $f_1 = 0$. □

Lemma 3.8. *Let $m, n \in \mathbb{N}^*$. Let $f_1, \ldots, f_n : \mathbb{H} \to \mathbb{C}$ be such that for $j \in [1, n]$ we have $f_j(\tau + m) = f_j(\tau)$, $\tau \in \mathbb{H}$. Let $a_1, \ldots, a_n \in \frac{1}{2}\mathbb{Z}$ be pairwise distinct and $p(\tau) \in \mathbb{C}[\tau] \setminus \{0\}$. Assume that*

$$f_1(\tau)p(\tau)^{a_1} + \cdots + f_n(\tau)p(\tau)^{a_n} = 0, \quad \tau \in \mathbb{H}. \tag{3.15}$$

Then $f_1 = \cdots = f_n = 0$.

Proof. Assume w.l.o.g. that $a_1 < \ldots < a_n$, then (3.15) is equivalent to proving

$$f_1(\tau) + f_2(\tau)p(\tau)^{b_2} + \cdots + f_n(\tau)p(\tau)^{b_n} = 0, \quad \tau \in \mathbb{H}, \tag{3.16}$$

where $b_k = a_k - a_1$ for $k = 1, \ldots, n$. Write (3.16) as

$$f_1(\tau) + f_{i_1}(\tau)p(\tau)^{b_{i_1}} + \cdots + f_{i_r}(\tau)p(\tau)^{b_{i_r}} = -f_{j_1}(\tau)p(\tau)^{b_{j_1}} \ldots - f_{j_s}(q)p(\tau)^{b_{j_s}}, \tag{3.17}$$

where $b_{i_1}, \ldots, b_{i_r} \in \mathbb{Z}$ and $b_{j_1}, \ldots, b_{j_s} \in \mathbb{Z} + 1/2$. Taking the square on both sides of (3.17) and moving the right hand-side to the left hand-side we obtain:

$$f_1^2(\tau) + \sum_{k=1}^{\infty} h_k(\tau)p^k(\tau) = 0, \quad \tau \in \mathbb{H}, \tag{3.18}$$

where $h_k : \mathbb{H} \to \mathbb{C}$ is such that $h_k(\tau + m) = h_k(\tau)$ and $h_k = 0$ for sufficiently large k. Hence by Lemma 3.7 we obtain $f_1^2 = 0$. By induction on n the result follows. □

To simplify notation we also introduce the following definition.

Definition 3.9. *Let N be a positive integer and $r \in R(N)$. Then we define $\Phi_r : \mathbb{H} \to \mathbb{C}$ by*

$$\Phi_r(\tau) := \prod_{\delta|N} \eta^{r_\delta}(\delta\tau), \quad \tau \in \mathbb{H}.$$

In order to apply Lemma 3.8 to (3.4) we need the following lemma.

Lemma 3.10. *Let N be a positive integer, $\gamma := \begin{pmatrix} 1 + 24^2N & 24 \\ 24N & 1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ and $r \in R(N)$, then*

$$\Phi_r(\gamma\tau) = (24N\tau + 1)^{\frac{1}{2}\sum_{\delta|N} r_\delta}\Phi_r(\tau), \quad \tau \in \mathbb{H}.$$

Proof. Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$ and δ a divisor of N. Then

$$\eta\left(\delta\frac{a\tau + b}{c\tau + d}\right) = \eta\left(\frac{a(\delta\tau) + \delta b}{(c/\delta)(\delta\tau) + d}\right) = (c\tau + d)^{1/2}\upsilon_\eta(a, \delta b, c/\delta, d)\eta(\delta\tau), \tag{3.19}$$

for $\tau \in \mathbb{H}$, because of (2.28). In particular (3.19) implies that

$$\Phi_r\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^{\frac{1}{2}\sum_{\delta|N} r_\delta} \prod_{\delta|N} \upsilon_\eta^{r_\delta}(a, b\delta, c/\delta, d)\Phi_r(\tau), \quad \tau \in \mathbb{H}. \tag{3.20}$$

Next we set $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \gamma$ in (3.20). This finishes the proof together with

$$\upsilon_\eta(1 + 24^2N, 24\delta, 24N/\delta, 1) = 1$$

for all $\delta|N$ by (2.30). □

We are ready to prove the goal of this subsection. First we rewrite (3.4) as

$$v_1(\tau)\Phi_{R^{(1)}}(\tau) + \cdots + v_m(\tau)\Phi_{R^{(m)}}(\tau) = 0, \quad \tau \in \mathbb{H}. \tag{3.21}$$

Next we apply the transformation γ in Lemma 3.10 to (3.21) and obtain

$$\begin{aligned} &v_1(\tau)(24N\tau+1)^{\frac{1}{2}\sum_{\delta|N} R_\delta^{(1)} + w(v_1)}\Phi_{R^{(1)}}(\tau) \\ &\quad + \cdots + v_m(\tau)(24N\tau+1)^{\frac{1}{2}\sum_{\delta|N} R_\delta^{(m)} + w(v_m)}\Phi_{R^{(m)}}(\tau) = 0, \quad \tau \in \mathbb{H}. \end{aligned} \tag{3.22}$$

We can rewrite (3.22) as

$$\sum_{k=-\infty}^{\infty} (24N\tau+1)^{k/2} g_k(\tau) = 0, \quad \tau \in \mathbb{H}, \tag{3.23}$$

where for $k \in \mathbb{Z}$

$$g_k(\tau) := \sum_{\substack{x\in[1,m] \\ (w(v_x),R^{(x)})\in A_N^{(3)}(k)}} v_x(\tau)\Phi_{R^{(x)}}(\tau), \quad \tau \in \mathbb{H}.$$

Note that the sum (3.23) is in fact a finite sum and that $g_k(\tau+24) = g_k(\tau)$ because by Lemma 2.27 we have $\eta(\tau+24) = \eta(\tau)$ and because of Proposition 3.6 $v_j(\tau+24) = v_j(\tau)$ for $j \in [1, m]$. So we can apply Lemma 3.8 to (3.23) and obtain that $g_k = 0$ for $k \in \mathbb{Z}$. This accomplishes the goal of this subsection.

3.1.2 Reduction of (3.6) to (3.7)

In order to obtain the desired result we will apply the next lemma.

Lemma 3.11. *Let m be a positive integer and $h_0, \ldots, h_{m-1} : \mathbb{H} \to \mathbb{C}$. Assume that*

$$h_k(\tau+1) = e^{2\pi i k/m} h_k(\tau), \quad \tau \in \mathbb{H}, \tag{3.24}$$

for $k \in [0, m-1]$ and that

$$\sum_{k=0}^{m-1} h_k(\tau) = 0, \quad \tau \in \mathbb{H}. \tag{3.25}$$

Then $h_k = 0$ for $k \in [0, m-1]$.

Proof. Let $\omega = e^{2\pi i/m}$. Then after applying the transformation $\tau \mapsto \tau + l$ to (3.25) for $l \in [0, m-1]$ and using (3.24) to rewrite we obtain m equations

$$\sum_{k=0}^{m-1} \omega^{lk} h_k(\tau) = 0, \quad \tau \in \mathbb{H},$$

which in matrix form reads

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & \omega & \cdots & \omega^{m-1} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & \omega^{m-1} & \ldots & \omega^{(m-1)^2} \end{pmatrix} \begin{pmatrix} h_0(\tau) \\ h_1(\tau) \\ \vdots \\ h_{m-1}(\tau) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \tau \in \mathbb{H}. \tag{3.26}$$

The left matrix in (3.26) is just the Vandermonde matrix and its determinant is nonzero. This implies $h_0(\tau) = \ldots = h_{m-1}(\tau) = 0$. □

In order to apply Lemma 3.11 we need the following lemma.

Lemma 3.12. *Let N be a positive integer and $r \in R(N)$. Define*

$$\mu := \sum_{\delta|N} \delta r_\delta.$$

Then

$$\Phi_r(\tau+1) = e^{2\pi i\mu/24}\Phi_r(\tau), \quad \tau \in \mathbb{H}.$$

Proof. This lemma follows from the relation $\eta(\tau+1) = e^{\pi i/12}\eta(\tau)$ because of Lemma 2.27. □

We are now ready to prove that proving (3.6) reduces to proving (3.7). We see that (3.6) is equivalent to

$$\sum_{i=0}^{23} h_i(\tau) = 0, \quad \tau \in \mathbb{H}, \tag{3.27}$$

where

$$h_i(\tau) = \sum_{\substack{x\in[1,m] \\ (w(v_x),R^{(x)})\in A_N^{(3)}(k) \\ R^{(x)}\in A_N^{(1)}(i)}} v_x(\tau)\Phi_{R^{(x)}}(\tau).$$

By Lemma 3.12 and Proposition 3.6 we have for $k \in [0, 23]$ that

$$h_k(\tau+1) = e^{2\pi ik/24}h_k(\tau), \quad \tau \in \mathbb{H}.$$

Lemma 3.11 gives $h_k = 0$ for $k \in [0, 23]$.

3.1.3 Reduction of (3.7) to (3.8)

We will prove the reduction step of this section by applying the transformation $f|_k W_N$ and then Lemma 3.11. Here $W_N := \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$ is the Atkin-Lehner involution studied in [5]. Also recall that for $f \in A_k(N)$ we defined $w(f) = k$.

Lemma 3.13. *Let N be a positive integer, $r \in R(N)$, $f \in \cup_{t=-\infty}^{\infty} A_{2t}(N)$ and $k := \sum_{\delta|N} r_\delta + 2w(f)$. Then*

$$\left(f \cdot \prod_{\delta|N} \eta_\delta^{r_\delta}\right) |_{\frac{k}{2}} W_N = (f|_{w(f)}W_N)N^{\frac{k-2w(f)}{4}}\upsilon_\eta(0,-1,1,0)^{\sum_{\delta|N} r_\delta}\prod_{\delta|N}\delta^{-\frac{r_\delta}{2}}\prod_{\delta|N}\eta_{N/\delta}^{r_\delta}.$$

Proof. Let $\delta|N$. Then by (2.28) we obtain

$$\eta^{r_\delta}(\delta W_N\tau) = \eta^{r_\delta}(-1/(N\tau/\delta)) = (N\tau/\delta)^{r_\delta/2}\upsilon_\eta^{r_\delta}(0,-1,1,0)\eta^{r_\delta}(N\tau/\delta).$$

The result follows by taking the product over all divisors $\delta|N$ on both sides of the above relation and then applying (2.2). □

Lemma 3.14. *Let N be a positive integer and $f \in \cup_{t=-\infty}^{\infty} A_{2t}(N)$. Then*

$$(f|_{w(f)}W_N)(\tau+1) = (f|_{w(f)}W_N)(\tau), \quad \tau \in \mathbb{H}.$$

Proof. In [5] it is proven that

$$W_N\Gamma_0(N)W_N^{-1} = \Gamma_0(N). \tag{3.28}$$

This implies that $f|_{w(f)}W_N \in A_{w(f)}(N)$ if $f \in A_{w(f)}(N)$. This is proven as follows. Assume that $\gamma \in \Gamma_0(N)$ then by (3.28) there exists $\gamma' \in \Gamma_0(N)$ such that $W_N\gamma = \gamma' W_N$. This implies that

$$f|_{w(f)}W_N\gamma = f|_{w(f)}\gamma' W_N = f|_{w(f)}W_N.$$

Above we used (2.3), Definition 2.6 and that $\sigma_j(\gamma_1,\gamma_2) = 1$ for all $\gamma_1,\gamma_2 \in \mathrm{GL}_2^+(\mathbb{Z})$ if $j \in \mathbb{Z}$. We have shown that $f|_{w(f)}W_N \in A_{w(f)}(N)$, the result follows by Proposition 3.6. □

We see that (3.7) is equivalent to

$$\sum_{j=0}^{23} \sum_{\substack{x\in[1,m] \\ (w(v_x),R^{(x)})\in S_N^{(k,i,j)}}} v_x(\tau)\prod_{\delta|N}\eta^{R_\delta^{(x)}}(\delta\tau) = 0, \quad \tau \in \mathbb{H}. \tag{3.29}$$

Next we apply the stroke operator $|_{\frac{k}{2}}W_N$ to both sides of (3.29) and obtain by Lemma 3.13:

$$\sum_{j=0}^{23} h_j = 0 \tag{3.30}$$

where

$$\begin{aligned} h_j :=& \sum_{\substack{x\in[1,m] \\ (w(v_x),R^{(x)})\in S_N^{(k,i,j)}}} (v_x|_{w(v_x)}W_N)N^{\frac{k-2w(v_x)}{4}} \\ &\qquad \times v_\eta(0,-1,1,0)^{\sum_{\delta|N}R_\delta^{(x)}}\prod_{\delta|N}\delta^{-\frac{R_\delta^{(x)}}{2}}\prod_{\delta|N}\eta_{N/\delta}^{R_\delta^{(x)}} \\ =& \left(\sum_{\substack{x\in[1,m] \\ (w(v_x),R^{(x)})\in S_N^{(k,i,j)}}} v_x\prod_{\delta|N}\eta_\delta^{R_\delta^{(x)}}\right)\Big|_{\frac{k}{2}}W_N\,. \end{aligned} \tag{3.31}$$

By Lemma 3.12 and Lemma 3.14 we obtain $h_j(\tau+1) = e^{\pi ij/12}h_j(\tau)$ and consequently by Lemma 3.11 we obtain $h_j = 0$ for $j \in [0,23]$. By (3.31) and (2.3) we obtain

$$(h_j|W_N)(\tau) = \sigma_{k/2}^{-1}(W_N,W_N) \sum_{\substack{x\in[1,m] \\ R^{(x)}\in S_N^{(k,i,j)}}} v_x\prod_{\delta|N}\eta^{R_\delta^{(x)}}(\delta\tau) = 0.$$

Canceling the term $\sigma_{k/2}^{-1}(W_N,W_N)$ (which is a root of unity) gives (3.8).

3.1.4 Reduction of (3.8) to (3.9)

The main goal of this subsection is to prove Lemma 3.17 from which this reduction step will follow immediately by induction as we will see. In order to prove Lemma 3.17 we need Lemma 3.16 which depends on Lemma 3.15. A proof of the next lemma is given in Chapter 6 of Serre's book [38].

Lemma 3.15 (Dirichlet)**.** *For any two coprime integers a, d there are infinitely many primes p such that $a \equiv p \pmod{d}$.*

Lemma 3.16. *Let $\{q_1, \ldots, q_m, v\}$ be a set of (pairwise distinct) primes such that $\{q_1, \ldots, q_m\}$ are odd primes (and v is an arbitrary prime). Then there are infinitely many primes p such that $\left(\frac{q_k}{p}\right) = 1, k \in [1, m]$ and $\left(\frac{v}{p}\right) = -1$.*

Proof. We split the proof in two cases depending on if $v = 2$ or not.

Case $v = 2$. By the Chinese remainder theorem there is an $a \in \mathbb{Z}$ such that

$$a \equiv 5 \pmod{8} \tag{3.32}$$

and

$$a \equiv 1 \pmod{q_k}, \quad k \in [1, m]. \tag{3.33}$$

Because of $\gcd(a, 8q_1 \cdots q_m) = 1$ there exists by Lemma 3.15 a prime p such that $p \equiv a \pmod{8q_1 \cdots q_m}$. Then the congruences (3.32)-(3.33) are still valid if a is replaced by p. Because $p \equiv 5 \pmod{8}$ and because of (2.34):

$$\left(\frac{2}{p}\right) = \left(\frac{v}{p}\right) = (-1)^{\frac{p^2-1}{8}} = -1.$$

Because of (3.33) and (2.36):

$$\left(\frac{q_k}{p}\right) = (-1)^{\frac{p-1}{2}\frac{q_k-1}{2}} \left(\frac{p}{q_k}\right) = \left(\frac{p}{q_k}\right) = 1 \tag{3.34}$$

for $k \in [1, m]$. This finishes the proof for the case $v = 2$.

Case $v \neq 2$. Again by the Chinese remainder theorem there exists a solution a to (3.33) that also satisfies

$$a \equiv r \pmod{v} \quad \text{and} \quad a \equiv 1 \pmod{4} \tag{3.35}$$

where r is an integer such that $\left(\frac{r}{v}\right) = -1$. Again by Dirichlet's Lemma 3.15 there are infinitely many primes such that

$$a \equiv p \pmod{4q_1 \cdots q_m v}.$$

As in the previous case (because of $p \equiv 1 \pmod{4}$) we verify that p satisfies (3.34). Furthermore, by (2.36) and because of (3.35)

$$\left(\frac{v}{p}\right) = (-1)^{\frac{v-1}{2}\frac{p-1}{2}} \left(\frac{p}{v}\right) = \left(\frac{r}{v}\right) = -1.$$

□

Lemma 3.17. *Let $i, j \in [0, 23]$ and $k \in \mathbb{Z}$. Let N be a positive integer and $p_1, \ldots, p_n$ the prime divisors of N. Let $t \in [0, n-1]$ and define $V_t := \{p_1^{\beta_1} \cdots p_t^{\beta_t} | (\beta_1, \ldots, \beta_t) \in Z_2^t\}$ if $t > 0$ and $V_0 := \{1\}$. Assume that for all $\nu \in V_t$*

$$\sum_{(\alpha_{t+1},\ldots,\alpha_n)\in Z_2^{n-t}} \sum_{\substack{x\in[1,m],\\ (w(v_x),R^{(x)})\in S_N^{(k,i,j)},\\ \prod_{\delta|N}\delta^{|R_\delta^{(x)}|}\approx \nu p_{t+1}^{\alpha_{t+1}}\cdots p_n^{\alpha_n}}} v_x \Phi_{R^{(x)}} = 0. \tag{3.36}$$

Then for all $\nu \in V_{t+1}$ we have: If $t \le n-2$

$$\sum_{(\alpha_{t+2},\ldots,\alpha_n)\in Z_2^{n-t-1}} \sum_{\substack{x\in[1,m],\\ (w(v_x),R^{(x)})\in S_N^{(k,i,j)},\\ \prod_{\delta|N}\delta^{|R_\delta^{(x)}|}\approx \nu p_{t+2}^{\alpha_{t+2}}\cdots p_n^{\alpha_n}}} v_x \Phi_{R^{(x)}} = 0; \tag{3.37}$$

otherwise, if $t = n-1$

$$\sum_{\substack{x\in[1,m],\\ (w(v_x),R^{(x)})\in S_N^{(k,i,j)},\\ \prod_{\delta|N}\delta^{|R_\delta^{(x)}|}\approx \nu}} v_x \Phi_{R^{(x)}} = 0. \tag{3.38}$$

Proof. By Lemma 3.16 there exists infinitely many primes P such that

$$\left(\frac{p_{t+1}}{P}\right) = -1 \tag{3.39}$$

and

$$\left(\frac{p_s}{P}\right) = 1 \quad \text{for } s \in [t+2, n]. \tag{3.40}$$

Especially there must exist at least one such prime P which also satisfies $\gcd(P, 24^2N) = 1$. Let $Y, X \in \mathbb{Z}$ be such that $YP - 24^2NX = 1$. Then $\gamma = \begin{pmatrix} Y & 24X \\ 24N & P \end{pmatrix} \in \Gamma_0(N)$. By Lemma 2.27 we have for $\delta|N$

$$v_\eta(Y, 24X\delta, 24N/\delta, P) = \left(\frac{24N/\delta}{P}\right) e^{\pi i(P-1)/4} = \left(\frac{24N\delta}{P}\right) e^{\pi i(P-1)/4}. \tag{3.41}$$

By Lemma 2.27 together with (3.41) we obtain

$$\begin{aligned} \eta(\delta\gamma\tau) &= \eta\left(\frac{Y(\delta\tau) + 24X\delta}{(24N/\delta)\delta\tau + P}\right) \\ &= v_\eta(Y, 24X\delta, 24N/\delta, P)(24N\tau + P)^{1/2}\eta(\delta\tau) \\ &= \left(\frac{24N}{P}\right)\left(\frac{\delta}{P}\right) e^{\pi i(P-1)/4}(24N\tau + P)^{1/2}\eta(\delta\tau). \end{aligned} \tag{3.42}$$

From (3.42) together with $k = \sum_{\delta|N} R_\delta^{(x)} + 2w(v_x)$ we obtain

$$\begin{aligned} &v_x(\gamma\tau) \quad \Phi_{R^{(x)}}(\gamma\tau) \\ &= e^{\pi i k(P-1)/4}\left(\frac{24N}{P}\right)^k \left(\frac{\prod_{\delta|N}\delta^{R_\delta^{(x)}}}{P}\right)(24N\tau + P)^{\frac{k}{2}+w(v_x)} v_x(\tau)\Phi_{R^{(x)}}(\tau). \end{aligned} \tag{3.43}$$

Define

$$\lambda(k) := e^{\pi ik(P-1)/4}\left(\frac{24N}{P}\right)^k.$$

Then from (3.43) we obtain the following formula for the action of the stroke operator $|_{\frac{k}{2}}\gamma$ on $v_x\Phi_{R^{(x)}}$.

$$v_x\Phi_{R^{(x)}}|_{\frac{k}{2}}\gamma = \lambda(k)\left(\frac{\prod_{\delta|N}\delta^{R_\delta^{(x)}}}{P}\right)v_x\Phi_{R^{(x)}}. \tag{3.44}$$

Next we apply the stroke operator $|_{\frac{k}{2}}\gamma$ to (3.36) and after canceling the nonzero term $\lambda(k)$ we obtain:

$$\sum_{(\alpha_{t+1},\ldots,\alpha_n)\in Z_2^{n-t}}\ \sum_{\substack{x\in[1,m],\\ (w(v_x),R^{(x)})\in S_N^{(k,i,j)},\\ \prod_{\delta|N}\delta^{|R_\delta^{(x)}|}\approx\nu p_{t+1}^{\alpha_{t+1}}\cdots p_n^{\alpha_n}}}\left(\frac{\prod_{\delta|N}\delta^{|R_\delta^{(x)}|}}{P}\right)v_x\Phi_{R^{(x)}} = 0. \tag{3.45}$$

Next we use that $a\approx b \Rightarrow \left(\frac{a}{P}\right)=\left(\frac{b}{P}\right)$, in particular

$$\prod_{\delta|N}\delta^{|R_\delta^{(x)}|}\approx\nu p_{t+1}^{\alpha_{t+1}}\cdots p_n^{\alpha_n}\quad\text{implies}\quad\left(\frac{\prod_{\delta|N}\delta^{|R_\delta^{(x)}|}}{P}\right)=\left(\frac{\nu p_{t+1}^{\alpha_{t+1}}\cdots p_n^{\alpha_n}}{P}\right) \tag{3.46}$$

and because of (3.39) and (3.40) we have

$$\left(\frac{\nu p_{t+1}^{\alpha_{t+1}}\cdots p_n^{\alpha_n}}{P}\right)=\left(\frac{\nu p_{t+1}^{\alpha_{k+1}}}{P}\right)=\left(\frac{\nu}{P}\right)\left(\frac{p_{t+1}^{\alpha_{k+1}}}{P}\right). \tag{3.47}$$

Next applying (3.46) and (3.47) to (3.45) we obtain

$$\left(\frac{\nu}{P}\right)\sum_{(\alpha_{t+1},\ldots,\alpha_n)\in Z_2^{n-t}}\ \sum_{\substack{x\in[1,m],\\ (w(v_x),R^{(x)})\in S_N^{(k,i,j)},\\ \prod_{\delta|N}\delta^{|R_\delta^{(x)}|}\approx\nu p_{t+1}^{\alpha_{t+1}}\cdots p_n^{\alpha_n}}}\left(\frac{p_{t+1}^{\alpha_{t+1}}}{P}\right)v_x\Phi_{R^{(x)}} = 0. \tag{3.48}$$

this is equivalent to

$$\begin{aligned}&\left(\frac{\nu}{P}\right)\sum_{(\alpha_{t+2}\ldots,\alpha_n)\in Z_2^{n-t-1}}\ \sum_{\substack{x\in[1,m],\\ (w(v_x),R^{(x)})\in S_N^{(k,i,j)},\\ \prod_{\delta|N}\delta^{|R_\delta^{(x)}|}\approx\nu p_{t+1}^{0}p_{t+2}^{\alpha_{t+2}}\cdots p_n^{\alpha_n}}}v_x\Phi_{R^{(x)}}\\ &-\left(\frac{\nu}{P}\right)\sum_{(\alpha_{t+2},\ldots,\alpha_n)\in Z_2^{n-t-1}}\ \sum_{\substack{x\in[1,m],\\ (w(v_x),R^{(x)})\in S_N^{(k,i,j)},\\ \prod_{\delta|N}\delta^{|R_\delta^{(x)}|}\approx\nu p_{t+1}^{1}p_{t+2}^{\alpha_{t+2}}\cdots p_n^{\alpha_n}}}v_x\Phi_{R^{(x)}} = 0.\end{aligned} \tag{3.49}$$

Similarly we obtain from (3.36) after multiplying by $\left(\frac{\nu}{P}\right)$ that:

$$\begin{aligned} &\left(\frac{\nu}{P}\right) \sum_{(\alpha_{t+2},\dots,\alpha_n)\in Z_2^{n-t-1}} \sum_{\substack{x\in[1,m],\\ (w(v_x),R^{(x)})\in S_N^{(k,i,j)},\\ \prod_{\delta|N}\delta^{|R_\delta^{(x)}|}\approx \nu p_{t+1}^0 p_{t+2}^{\alpha_{t+2}}\cdots p_n^{\alpha_n}}} v_x\Phi_{R^{(x)}} \\ +&\left(\frac{\nu}{P}\right) \sum_{(\alpha_{t+2},\dots,\alpha_n)\in Z_2^{n-t-1}} \sum_{\substack{x\in[1,m],\\ (w(v_x),R^{(x)})\in S_N^{(k,i,j)},\\ \prod_{\delta|N}\delta^{|R_\delta^{(x)}|}\approx \nu p_{t+1}^1 p_{t+2}^{\alpha_{t+2}}\cdots p_n^{\alpha_n}}} v_x\Phi_{R^{(x)}} = 0 \end{aligned} \tag{3.50}$$

Adding (3.50) to (3.49) and dividing out $2\left(\frac{\nu}{P}\right)\neq 0$ (because of $\gcd(P,N)=1$ by assumption) we obtain:

$$\sum_{(\alpha_{t+2},\dots,\alpha_n)\in Z_2^{n-t-1}} \sum_{\substack{x\in[1,m],\\ (w(v_x),R^{(x)})\in S_N^{(k,i,j)},\\ \prod_{\delta|N}\delta^{|R_\delta^{(x)}|}\approx \nu p_{t+1}^0 p_{t+2}^{\alpha_{k+2}}\cdots p_n^{\alpha_n}}} v_x\Phi_{R^{(x)}} = 0 \tag{3.51}$$

Similarly taking the difference between (3.50) and (3.49) and dividing out $2\left(\frac{\nu}{P}\right)$ we obtain:

$$\sum_{(\alpha_{t+2},\dots,\alpha_n)\in Z_2^{n-t-1}} \sum_{\substack{x\in[1,m],\\ (w(v_x),R^{(x)})\in S_N^{(k,i,j)},\\ \prod_{\delta|N}\delta^{|R_\delta^{(x)}|}\approx \nu p_{t+1}^1 p_{t+2}^{\alpha_{t+2}}\cdots p_n^{\alpha_n}}} v_x\Phi_{R^{(x)}} = 0 \tag{3.52}$$

In particular (3.51) and (3.52) implies (3.37) because $p_{t+1}V_t\cup V_t=V_{t+1}$ and hence one needs to show (3.37) for $v\in p_{t+1}V_t$ which is (3.52) and for $v\in V_t$ which is (3.51). □

In particular (3.8) is just (3.36) for $t=0$. Using induction on k we find that (3.37) holds for $t=n-1$ which is precisely (3.9).

3.2 An Algorithm to Prove Reduced Fundamental Eta Identities

In this very short subsection we recollect the results derived in the form of an informal algorithm description. Recall that $M_k(N)$ is the space of modular forms of weight k for the group $\Gamma_0(N)$. Recall that $M_k(N)$ is a $\mathbb{C}$-vector space.

We have shown in the previous section that any general eta identity (3.1) can be split into smaller fundamental eta identities (3.2). As we saw in the introduction to Chapter 3 any fundamental eta identity (3.2) can be rewritten immediately as a reduced fundamental eta identity (3.3).

Let N be a positive integer and $(d_1,\dots,d_m)$ with $1=d_1<d_2<\dots<d_m=N$ the positive divisors of N. We write a sequence $r\in R(N)$ in the form $r=(r_{d_1},\dots,r_{d_m})$. Let $R^{(1)},\dots,R^{(n)}\in R(N)$, $v_0,v_1,\dots,v_n\in\bigcup_{j=-\infty}^{\infty}M_{2j}(N)$ and

$$v_0+v_1\Phi_{R^{(1)}}+\cdots+v_n\Phi_{R^{(n)}}=0 \tag{3.53}$$

a reduced fundamental eta identity (3.3). Then in particular the $R^{(1)}, \ldots, R^{(n)}$ satisfy the conditions *(i)-(iv)* of Lemma 2.34 and $\sum_{\delta|N} R_\delta^{(j)} + 2w(v_j) = 2w(v_0)$ for $j \in [1, n]$.

This shows that each term in a reduced fundamental eta identity belongs to $A_{w(v_0)}(N)$.

Note that if $r_1, r_2 \in R(N)$ satisfy *(i)-(iv)* then also $r_1 + r_2$ does. It is straightforward to check that the sequence $A^{(k)} := (24k, 0, \ldots, 0) \in R(N)$ satisfies *(i)-(iv)* for $k \in \mathbb{Z}$. It is also easy to see that for any $r = (r_{d_1}, \ldots, r_{d_m})$ satisfying *(i)-(iv)* there exists a minimal $k_r \in \mathbb{Z}$ such that for all $t \geq k_r$ the sequence $r + A^{(t)}$ satisfy *(i)-(v)* and consequently $\Phi_{r+A^{(t)}} \in M_v(N)$ for $v = \frac{1}{2}\sum_{\delta|M} r_\delta + 12t$. Let $\kappa := \max(k_{R^{(1)}}, \ldots, k_{R^{(n)}})$. Then by Lemma 2.34 we have that

$$v_0 \Phi_{A^{(\kappa)}} + v_1 \Phi_{R^{(1)}+A^{(\kappa)}} + \cdots + v_n \Phi_{R^{(n)}+A^{(\kappa)}} \tag{3.54}$$

is in $M_{12\kappa+w(v_0)}(N)$ because for each $j \in [1, n]$ we have $v_j \Phi_{A^{(\kappa)}+R^{(j)}} \in M_{12\kappa+w(v_0)}(N)$. Next we can compute the q-expansion of (3.54) using (2.26). Then by Corollary 2.22 and Corollary 2.40 we obtain that (3.54) is identically 0 if the first

$$1 + \left(\kappa + \frac{w(v_0)}{12}\right) N \prod_{p|N} \left(1 + \frac{1}{p}\right)$$

coefficients are.

Example 3.18. We apply the algorithm to the celebrated identity of Jacobi [42, p. 470]:

$$\prod_{n=1}^{\infty}(1 - q^{2n-1})^8 + 16q \prod_{n=1}^{\infty}(1 + q^{2n})^8 = \prod_{n=1}^{\infty}(1 + q^{2n-1})^8, \tag{3.55}$$

which we rewrite in terms of eta products:

$$\frac{\eta^8}{\eta_2^8} + 16\frac{\eta_4^8}{\eta_2^8} - \frac{\eta_2^{16}}{\eta^8\eta_4^8} = 0. \tag{3.56}$$

It is easily seen that (3.56) is a fundamental eta identity. By dividing both sides of (3.56) by $\frac{\eta^8}{\eta_2^8}$ we obtain the following reduced fundamental eta identity:

$$1 + 16\frac{\eta_4^8}{\eta^8} - \frac{\eta_2^{24}}{\eta^{16}\eta_4^8} = 0. \tag{3.57}$$

Next we multiply both sides of (3.57) by η^{24} so that each term is in $M_{12}(4)$:

$$\eta^{24} + 16\eta_4^8\eta^{16} - \frac{\eta_2^{24}\eta^8}{\eta_4^8} = 0. \tag{3.58}$$

Using (2.26) we obtain

$$\begin{aligned}
\eta^{24} &= q - 24q^2 + 252q^3 - 1472q^4 + 4830q^5 - 6048q^6 \cdots \\
\eta_4^8\eta^{16} &= q^2 - 16q^3 + 104q^4 - 320q^5 + 252q^6 + \cdots \\
\frac{\eta_2^{24}\eta^8}{\eta_4^8} &= q - 8q^2 - 4q^3 + 192q^4 - 290q^5 - 2016q^6 + \cdots .
\end{aligned}$$

Using these expansions we find that the first 7 coefficients of the q-expansion of the left hand side of (3.58) are 0, which by Corollary 2.22 and Corollary 2.40 implies that $\eta^{24} + 16\eta_4^8\eta^{16} - \frac{\eta_2^{24}\eta^8}{\eta_4^8}$ is identically 0, proving (3.58). Since (3.55)-(3.58) are equivalent, we have proven the Jacobi identity (3.55).

Remark 3.19. *We point out that in the algorithm described we multiply a reduced fundamental eta identity by η^{24k} for some integer k. This does not always lead to optimality in the sense that the number of coefficients needed to be checked in the q-expansion of the resulting identity is minimal. For example we can multiply both sides of (3.57) by $\frac{\eta^8\eta_4^8}{\eta_2^8}$. Then each term in the new identity is in $M_4(4)$ and by Corollary 2.22 and Corollary 2.40 we only need to show that the first 3 coefficients of the resulting q-expansion are 0.*

In the next chapter we give examples of other fundamental eta identities. The relations in Section 6.6 of Chapter 6 are of the same type as the ones in the next section and are proven in an analogous way.

3.3 Ramanujan's Most Beautiful Identities and Newman's Lemma Revisited

There is a huge variety of examples of fundamental/general eta identities. Moreover, in this section we restrict to prove the Ramanujan identities where (3.59) according to Hardy [13, p. xxxv] is Ramanujan's most beautiful identity:

$$\sum_{n=0}^{\infty} p(5n+4)q^n = 5\prod_{n=1}^{\infty}\frac{(1-q^{5n})^5}{(1-q^n)^6}, \tag{3.59}$$

and

$$\sum_{n=0}^{\infty} p(7n+5)q^n = 7\prod_{n=1}^{\infty}\frac{(1-q^{7n})^3}{(1-q^n)^4} + 49q\prod_{n=1}^{\infty}\frac{(1-q^{7n})^7}{(1-q^n)^8}. \tag{3.60}$$

The proof will use a variant of the method described above, again based on Newman's lemma. To this end we introduce the U-operator.

Definition 3.20. *Let $f:\mathbb{H}\to\mathbb{C}$ be holomorphic on $\mathbb{H}$. Then for m a positive integer we define*

$$(f|U_m)(\tau) := \frac{1}{m}\sum_{\lambda=0}^{m-1} f\left(\frac{\tau+\lambda}{m}\right).$$

Obviously U_m is linear (on $\mathbb{C}$); in addition, $U_{mn} = U_m \circ U_n = U_n \circ U_m$ $(m,n\in\mathbb{N}^*)$. Next we prove some further important properties of the U-operator.

Lemma 3.21. *Let $f:\mathbb{H}\to\mathbb{C}$ be holomorphic on $\mathbb{H}$, m a positive integer and $R := \{r_0,\ldots,r_{m-1}\}$ a complete set of representatives of the residue classes modulo m. Then if $f(\tau+1)=f(\tau)$ for all $\tau\in\mathbb{H}$ we have*

$$(f|U_m)(\tau) = \frac{1}{m}\sum_{\lambda=0}^{m-1} f\left(\frac{\tau+r_\lambda}{m}\right)$$

for all $\tau\in\mathbb{H}$.

Proof. Because R is a complete set of representatives of the residue classes modulo m there is a permutation $\sigma:\{0,\ldots,m-1\}\to\{0,\ldots,m-1\}$ and a mapping $k:\{0,\ldots,m-1\}\to\mathbb{Z}$ such that $r_{\sigma(\lambda)} = \lambda + mk(\lambda)$ for $\lambda\in\{0,\ldots,m-1\}$. This means that for all $\lambda\in\{0,\ldots,m-1\}$

$$f\left(\frac{\tau+r_{\sigma(\lambda)}}{m}\right) = f\left(\frac{\tau+\lambda+mk(\lambda)}{m}\right) = f\left(\frac{\tau+\lambda}{m}+k(\lambda)\right) = f\left(\frac{\tau+\lambda}{m}\right) \tag{3.61}$$

because of $f(\tau+1) = f(\tau)$. By (3.61) and Definition 3.20 we have

$$\frac{1}{m}\sum_{\lambda=0}^{m-1} f\left(\frac{\tau+r_\lambda}{m}\right) = \frac{1}{m}\sum_{\lambda=0}^{m-1} f\left(\frac{\tau+r_{\sigma(\lambda)}}{m}\right) = \frac{1}{m}\sum_{\lambda=0}^{m-1} f\left(\frac{\tau+\lambda}{m}\right) = (f|U_m)(\tau).$$

□

Lemma 3.22. *Let $f:\mathbb{H}\to\mathbb{C}$ be holomorphic on $\mathbb{H}$. Assume that*

$$f(\tau) = \sum_{n=-\infty}^{\infty} a(n)q^n, \quad (q = e^{2\pi i\tau}), \quad \tau\in\mathbb{H}.$$

Then for m a positive integer

$$(f|U_m)(\tau) = \sum_{n=-\infty}^{\infty} a(mn)q^n, \quad (q = e^{2\pi i\tau}), \quad \tau\in\mathbb{H}.$$

Proof. By Definition 3.20 we have

$$(f|U_m)(\tau) = \frac{1}{m}\sum_{\lambda=0}^{m-1}\sum_{n=-\infty}^{\infty} a(n)e^{2\pi i n\frac{\tau+\lambda}{m}} = \sum_{n=-\infty}^{\infty} a(n)e^{\frac{2\pi i\tau n}{m}}\sum_{\lambda=0}^{m-1} e^{\frac{2\pi i\lambda n}{m}} = \sum_{n=-\infty}^{\infty} a(mn)q^n.$$

□

Here we have used that $\sum_{\lambda=0}^{m-1} e^{\frac{2\pi i\lambda n}{m}} = \begin{cases} 0 & \text{if } m\nmid n; \\ m & \text{otherwise.}\end{cases}$.

Lemma 3.23. *Let $f,g:\mathbb{H}\to\mathbb{C}$ be holomorphic functions on $\mathbb{H}$. Let $f_m:\mathbb{H}\to\mathbb{C}$ be defined by $f_m(\tau) := f(m\tau)$ and assume that $f(\tau+1) = f(\tau)$ for all $\tau\in\mathbb{H}$. Then*

$$(f_m g|U_m) = f(g|U_m).$$

Proof. By Definition 3.20 we obtain

$$(f_m g|U_m)(\tau) = \frac{1}{m}\sum_{\lambda=0}^{m-1} f\left(m\frac{\tau+\lambda}{m}\right) g\left(\frac{\tau+\lambda}{m}\right) = f(\tau)(g|U_m)(\tau), \quad \tau\in\mathbb{H}.$$

□

Lemma 3.24. *Let $k\in\mathbb{Z}$, m, $N\in\mathbb{N}^*$ and $f\in A_k(Nm^2)$ (resp. $f\in M_k(Nm^2)$). Then $f|U_m\in A_k(Nm)$ (resp. $f|U_m\in M_k(Nm)$).*

Proof. For $\lambda\in\mathbb{Z}$ define $T_\lambda := \left(\begin{smallmatrix}1&\lambda\\0&m\end{smallmatrix}\right)$. Then for $\lambda\in\mathbb{Z}$ and $\left(\begin{smallmatrix}a&b\\mc&d\end{smallmatrix}\right)\in\Gamma_0(Nm)$ one has:

$$T_\lambda\begin{pmatrix} a & b \\ mc & d\end{pmatrix} = \begin{pmatrix} a+\lambda mc & -c(b+\lambda d)^2 \\ m^2c & d-mdc(b+\lambda d)\end{pmatrix} T_{\lambda d^2+bd}. \tag{3.62}$$

We also observe by Definition 3.20 that

$$f|U_m = m^{k/2}\sum_{\lambda=0}^{m-1} f|_k T_\lambda. \tag{3.63}$$

By *(i)* and *(ii)* of Proposition 2.8 together with (3.63) we see that $f|U_m \in \mathbb{L}_k$ (resp. $f|U_m \in \mathbb{T}_k$ if $f \in M_k(Nm^2)$). This shows that condition *(iii)* of Definition 2.6 holds. By Proposition 2.7 and because the sum of holomorphic functions is again holomorphic we see that $f|U_m$ is holomorphic proving *(i)* of Definition 2.6.

Using (3.62), (3.63), Lemma 3.21 and $f \in M_k(Nm^2)$ we find

$$\begin{aligned}(f|U_m)|_k \begin{pmatrix} a & b \\ mc & d \end{pmatrix} &= m^{k/2} \sum_{\lambda=0}^{m-1} f|_k \begin{pmatrix} a+\lambda mc & -c(b+\lambda d)^2 \\ m^2 c & d - mdc(b+\lambda d) \end{pmatrix} T_{\lambda d^2+bd} \\ &= m^{k/2} \sum_{\lambda=0}^{m-1} f|_k T_{\lambda d^2+bd} = f|U_m.\end{aligned}$$

This verifies condition *(ii)* of Definition 2.6. □

Definition 3.25. *For $l \in \mathbb{N}^*$ write $\varphi_l(\tau) := \prod_{n=1}^{\infty}(1-q^n(l\tau))$ where $q : \mathbb{H} \to \mathbb{C}$ satisfies $q(\tau) := e^{2\pi i \tau}$ for all $\tau \in \mathbb{H}$.*

Proof of (3.59)*:* By Lemma 3.23 and Lemma 3.22 we have

$$\begin{aligned}\left(q^6 \frac{\varphi_5^4\varphi_{25}^5}{\varphi_1}|U_5\right)(\tau) &= \varphi_5^5(\tau)\varphi^4(\tau)(q^6/\varphi_1|U_5)(\tau) \\ &= q^2(\tau)\prod_{n=1}^{\infty}(1-q^{5n}(\tau))^5(1-q^n(\tau))^4 \sum_{n=0}^{\infty} p(5n+4)q^n(\tau), \quad \tau \in \mathbb{H}.\end{aligned} \tag{3.64}$$

By (2.26) we have that $q^6\frac{\varphi_5^4\varphi_{25}^5}{\varphi_1} = \frac{\eta_5^4\eta_{25}^5}{\eta}$ and $q\prod_{n=1}^{\infty}(1-q^{5n})^5(1-q^n)^4 = q^{-1/6}\eta^4\eta_5^5$ which together with (3.64) implies

$$\left(\frac{\eta_5^4\eta_{25}^5}{\eta}|U_5\right) = q^2\prod_{n=1}^{\infty}(1-q^{5n})^5(1-q^n)^4\sum_{n=0}^{\infty}p(5n+4)q^n. \tag{3.65}$$

Next we multiply both sides of (3.59) by $q^2\prod_{n=1}^{\infty}(1-q^{5n})^5(1-q^n)^4$ and use (3.65) to rewrite the result as

$$\left(\frac{\eta_5^4\eta_{25}^5}{\eta}|U_5\right) = 5q^2\prod_{n=1}^{\infty}\frac{(1-q^{5n})^{10}}{(1-q^n)^2}. \tag{3.66}$$

By (2.26) we have $5q^2\prod_{n=1}^{\infty}\frac{(1-q^{5n})^{10}}{(1-q^n)^2} = \frac{\eta_5^{10}}{\eta^2}$. This allows us to rewrite (3.66) as

$$\left(\frac{\eta_5^4\eta_{25}^5}{\eta}|U_5\right) - 5\frac{\eta_5^{10}}{\eta^2} = 0. \tag{3.67}$$

By Lemma 2.34 $\frac{\eta_5^4\eta_{25}^5}{\eta} \in M_4(25)$ and by Lemma 3.24 we have $\left(\frac{\eta_5^4\eta_{25}^5}{\eta}|U_5\right) \in M_4(5)$. Note that (3.67) can be viewed as a reduced fundamental eta identity because it may be rewritten as

$$v_1 + v_2\frac{\eta_5^{10}}{\eta^2} = 0,$$

where $v_1 := \frac{\eta_5^4\eta_{25}^5}{\eta}|U_5 \in M_4(5)$ and $v_2 := -5 \in M_0(5)$. By using Lemma 2.34 again we obtain $\frac{\eta_5^{10}}{\eta^2} \in M_4(5)$. Putting all this together we obtain

$$f := \left(\frac{\eta_5^4\eta_{25}^5}{\eta}|U_5\right) - 5\frac{\eta_5^{10}}{\eta^2} \in M_4(5). \tag{3.68}$$

By (2.49) we have $[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(5)] = 6$ and by Corollary 2.22 we have that if $f \in M_4(5)$ and $\mathrm{Ord}_{\Gamma_0(5)}(f, \mathrm{id}) > 2$ then $f = 0$. By this together with (3.68) we see that in order to prove (3.66) we need to show

$$\mathrm{Ord}_{\Gamma_0(5)}(f, \mathrm{id}) > 2$$

which is by (3.65) and (3.66) equivalent with showing that the coefficients of q^0, q and q^2 of the series

$$q^2\prod_{n=1}^{\infty}(1-q^{5n})^5(1-q^n)^4\sum_{n=0}^{\infty}p(5n+4)q^n - 5q^2\prod_{n=1}^{\infty}\frac{(1-q^{5n})^{10}}{(1-q^n)^2}$$

are 0. Because $p(4) = 5$ this is obviously the case.

Proof of (3.60)*:* By Lemma 3.23, Lemma 3.22 and (2.26) we have

$$\left(q^{16}\frac{\varphi_{49}^7\varphi_7^6}{\varphi}|U_7\right) = \left(\frac{\eta_{49}^7\eta_7^6}{\eta}|U_7\right) = q^3\varphi_7^7\varphi^6\sum_{n=0}^{\infty}p(7n+5)q^n. \tag{3.69}$$

Multiplying (3.60) by $q^3\varphi_7^7\varphi^6$ we obtain by (3.69) and (2.26)

$$\left(\frac{\eta_{49}^7\eta_7^6}{\eta}|U_7\right) = q^3\varphi_7^{10}\varphi^2 + q^4\frac{\varphi_7^{14}}{\varphi^2} = \eta_7^{10}\eta^2 + \frac{\eta_7^{14}}{\eta^2}. \tag{3.70}$$

By Lemma 2.34 we have $\eta_7^{10}\eta^2, \frac{\eta_7^{14}}{\eta^2} \in M_6(7)$ and $\frac{\eta_{49}^7\eta_7^6}{\eta} \in M_6(49)$. By Lemma 3.24 we have $\left(\frac{\eta_{49}^7\eta_7^6}{\eta}|U_7\right) \in M_6(7)$ showing

$$f := \left(\frac{\eta_{49}^7\eta_7^6}{\eta}|U_7\right) - \eta_7^{10}\eta^2 - \frac{\eta_7^{14}}{\eta^2} \in M_6(7). \tag{3.71}$$

By (2.49) we have $[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(7)] = 8$ and by Corollary 2.22 we have that if $f \in M_4(7)$ and $\mathrm{Ord}_{\Gamma_0(7)}(f, \mathrm{id}) > 4$ then $f = 0$. By this together with (3.71) we see that in order to prove (3.70) we need to show

$$\mathrm{Ord}_{\Gamma_0(7)}(f, \mathrm{id}) > 4$$

which is by (3.69) and (3.70) equivalent with showing that the coefficients of q^0, q, q^2, q^3 and q^4 of the series

$$q^3\varphi_7^7\varphi^6\sum_{n=0}^{\infty}p(7n+5)q^n - q^3\varphi_7^{10}\varphi^2 - q^4\frac{\varphi_7^{14}}{\varphi^2}$$

are 0 which is easily verified.

We conclude this section with a "if and only if version" of Lemma 2.34, which we already announced in Remark 2.36:

Lemma 3.26. *Let N be a positive integer, $r \in R(N)$ and $k \in 2\mathbb{Z}$. Then $\prod_{\delta|N}\eta_\delta^{r_\delta} \in A_k(N)$ iff* *(i)* $k = \frac{1}{2} \sum_{\delta|N} r_\delta$;

(ii) $\sum_{\delta|N} \delta r_\delta \equiv 0 \pmod{24}$;

(iii) $\sum_{\delta|N} \frac{N}{\delta} r_\delta \equiv 0 \pmod{24}$;

(iv) $\prod_{\delta|N} \delta^{r_\delta}$ *is a rational square.*

Furthermore $\prod_{\delta|N} \eta_\delta^{r_\delta} \in M_k(N)$ *iff* (i)-(iv) *and*

$$\sum_{\delta|N} r_\delta \frac{\gcd^2(\delta, d)}{\delta} \geq 0 \tag{3.72}$$

for all $d|N$.

Proof. If *(i)-(iv)* hold then $f_r \in A_k(N)$ by Lemma 2.34. Again by Lemma 2.34 we obtain $f_r \in M_k(N)$ if (3.72). For the converse assume that $\prod_{\delta|N} \eta_\delta^{r_\delta} \in A_k(N)$. Then there exists $f \in A_k(N)$ such that $f - \prod_{\delta|N} \eta_\delta^{r_\delta} = 0$ and by Theorem 3.5 we see that $(0, k) \sim (r, 0)$ which is equivalent with r and k verifying *(i)-(iv)*. Again by Lemma 2.34, because $f \in M_k(N)$ and because of the fact that f satisfies *(i)-(iv)*, then necessarily (3.72) is satisfied by r. □

Chapter 4

An Algorithmic Approach to Ramanujan's Congruences

Introduction

Throughout this chapter M denotes a positive integer, and $r = (r_\delta)$ denotes a sequence of integers r_δ indexed by all positive integer divisors δ of M.

In this chapter we present an algorithm that takes as input a generating function of the form $\prod_{\delta|M}\prod_{n=1}^{\infty}(1-q^{\delta n})^{r_\delta} = \sum_{n=0}^{\infty} a(n)q^n$ together with three positive integers m, t, p; the algorithm returns true if $a(mn+t) \equiv 0 \pmod{p}, n \geq 0$, or false otherwise. A similar algorithm for generating functions of the form $\prod_{n=1}^{\infty}(1-q^n)^{r_1}$ (i.e. the case $M = 1$) has already been given in [9]. Our original plan was to implement that algorithm in order to prove some congruences from [2]. The algorithm we present here and the one in [9] both have in common that at the end one has to check that the congruence is true for the first coefficients up to a bound ν that the algorithm returns, and then to use the theorem of Sturm [40] to conclude that it is true for all coefficients. However we noticed that for our purpose the bound ν given in [9] was extremely high for some inputs. Encouraged by comments of Peter Paule we examined the problem in more detail. Finally our study resulted in a significant improvement of estimating the bound ν a priori. Our main tools to derive a better bound ν are a combination of results by Rademacher [32] and Newman [28]; Kolberg [21] was another major source of inspiration.

The organization of this chapter is as follows: In section 1 we present the basic terminology. In section 2 we prepare some results needed to apply the theorem of Sturm. The main result, Theorem 4.22, can be viewed as a generalization of a theorem of R. Lewis [25]. In section 3 we estimate functions at different points; this is needed in order to prove they are indeed modular forms. In section 4 we show how to apply the theorem of Sturm in order to prove our desired congruence. In section 5 we conclude by giving some examples. A published version of this chapter can be found in [33].

4.1 Basic Terminology and Formulas

Let a be an integer relatively prime to 6, i.e. $\gcd(a, 6) = 1$. For such a one can easily show that $a^2 - 1 \equiv_{24} 0$. Similarly if $\gcd(a, 3) = 1$ then $a^2 - 1 \equiv_3 0$, and finally, if $\gcd(a, 2) = 1$ then $a^2 - 1 \equiv_8 0$. These facts will be used throughout the text.

Definition 4.1. *Let* $D^* := \{q \in \mathbb{C} \| q| < 1, q \neq 0\}$ *be the punctured disc. We define* $q : \mathbb{H} \to D^*$ *by* $q = q(\tau) := e^{2\pi i \tau}$ *for* $\tau \in \mathbb{H}$.

Definition 4.2. *Given a positive integer* m *let* $\varphi : [0, m-1] \times \mathbb{H} \to \mathbb{H}$ *be a function with expansion* $\varphi(t, \tau) = q^{-t} \sum_{n=0}^{\infty} a(n) q^n$*, where* $t \in \{0, \dots, m-1\}$*. Let* S_m *be a complete set of non-equivalent representatives of the residue classes modulo* m*. For* $\kappa \in \mathbb{N}$ *with* $\gcd(m, \kappa) = 1$ *we define:*

$$M_{m,\kappa}(\varphi(t, \tau)) := \sum_{\lambda \in S_m} \varphi\left(t, \frac{\tau + \kappa\lambda}{m}\right). \tag{4.1}$$

In this paper we are always choosing

$$\kappa := \gcd(1 - m^2, 24). \tag{4.2}$$

With this choice clearly $\gcd(\kappa, m) = 1$.
Another property needed later is as follows:

Lemma 4.3. *Let* κ *be as defined in* (4.2)*, then* $6 | \kappa m$.

Proof. One can proceed by case distinction. For instance, if $2 \nmid m$ and $3 | m$, then $m^2 - 1 \equiv_8 0$ because $\gcd(m, 2) = 1$. Hence by (4.2) we have $8 | \kappa$, thus $6 | \kappa m$. The other cases are similar. □

Lemma 4.4. *Given positive integers* m *and* κ*, let* $\varphi(t, \tau)$ *be as in Definition 4.2. Then we have:*

$$M_{m,\kappa}(\varphi(t, \tau)) = m \sum_{n=0}^{\infty} a(mn + t) q^n. \tag{4.3}$$

Proof. By Definition 4.2 we have

$$\begin{aligned} M_{m,\kappa}(\varphi(t, \tau)) &= \sum_{\lambda \in S_m} e^{-2\pi i t \frac{\tau + \kappa\lambda}{m}} \sum_{n=0}^{\infty} a(n) e^{2\pi i n \frac{\tau + \kappa\lambda}{m}} \\ &= \sum_{n=0}^{\infty} a(n) e^{-\frac{2\pi i \tau t}{m}} e^{\frac{2\pi i n \tau}{m}} \sum_{\lambda \in S_m} e^{2\pi i \lambda \frac{-\kappa t + \kappa n}{m}} \\ &= \sum_{\substack{n \geq 0 \\ n \equiv_m t}} m \cdot a(n) e^{-\frac{2\pi i \tau t}{m}} e^{\frac{2\pi i n \tau}{m}} \\ &= m \sum_{n=0}^{\infty} a(mn + t) q^{\frac{-t}{m}} q^{\frac{t}{m}} q^n \\ &= m \sum_{n=0}^{\infty} a(mn + t) q^n. \end{aligned}$$

Note that the sum $\sum_{\lambda \in S_m} e^{2\pi i \lambda \frac{-\kappa t + \kappa n}{m}}$ equals m if $-\kappa t + \kappa n \equiv_m 0$. This is exactly the case when $n \equiv_m t$. For $n \not\equiv_m t$ the sum is 0. □

Definition 4.5. *Let* $M \in \mathbb{N}^*$*. By* $R(M)$ *we denote the set of all integer sequences* (r_δ) *indexed by all positive divisors* δ *of* M*.*

Definition 4.6. *For $m, M \in \mathbb{N}^*$, $t \in \mathbb{N}$ such that $0 \le t \le m-1$ and $r = (r_\delta) \in R(M)$, we define:*

$$f(\tau, r) := \prod_{\delta|M} \prod_{n=1}^{\infty} (1 - q^{\delta n})^{r_\delta} = \sum_{n=0}^{\infty} a(n) q^n, \tag{4.4}$$

and

$$g_{m,t}(\tau, r) := q^{\frac{24t + \sum_{\delta|M} \delta r_\delta}{24m}} \sum_{n=0}^{\infty} a(mn+t) q^n.$$

Lemma 4.7. *For $m, M \in \mathbb{N}^*$, $t \in \mathbb{N}$ such that $0 \le t \le m-1$ and $r = (r_\delta) \in R(M)$ we obtain the following representation:*

$$g_{m,t}(\tau, r) = \frac{1}{m} \sum_{\lambda=0}^{m-1} e^{\frac{2\pi i \kappa \lambda (-24t - \sum_{\delta|M} \delta r_\delta)}{24m}} \prod_{\delta|M} \eta^{r_\delta} \left(\frac{\delta(\tau + \kappa\lambda)}{m} \right). \tag{4.5}$$

Proof. Using (2.26) we see that $f(\tau, r) = q^{-\frac{\sum_{\delta|M} \delta r_\delta}{24}} \prod_{\delta|M} \eta^{r_\delta}(\delta\tau)$. Next applying $M_{m,\kappa}$ to $\varphi(t,\tau) := q^{-t} f(\tau, r)$, by Definition 4.2 we see that:

$$M_{m,\kappa}(\varphi(t,\tau)) = \sum_{\lambda=0}^{m-1} e^{2\pi i (\frac{\tau+\kappa\lambda}{m})(-t - \frac{\sum_{\delta|M} \delta r_\delta}{24})} \prod_{\delta|M} \eta^{r_\delta} \left(\frac{\delta(\tau+\kappa\lambda)}{m} \right)$$

$$= q^{\frac{-24t - \sum_{\delta|M} \delta r_\delta}{24m}} \sum_{\lambda=0}^{m-1} e^{\frac{2\pi i \kappa \lambda (-24t - \sum_{\delta|M} \delta r_\delta)}{24m}} \prod_{\delta|M} \eta^{r_\delta} \left(\frac{\delta(\tau+\kappa\lambda)}{m} \right).$$

Alternatively by Lemma 4.4 we obtain:

$$M_{m,\kappa}(\varphi(t,\tau)) = m \sum_{n=0}^{\infty} a(mn+t) q^n = m q^{-\frac{24t + \sum_{\delta|M} \delta r_\delta}{24m}} g_{m,t}(\tau, r).$$

Comparing the two expressions for $M_{m,\kappa}(\varphi(t,\tau))$ we obtain our assertion. □

The following lemma will be used on several occasions:

Lemma 4.8. *Given a real number k and maps $f : \mathbb{H} \to \mathbb{C}$ and $g : \Gamma \times \mathbb{H} \to \mathbb{C}$. Suppose for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ and for all $\tau \in \mathbb{H}$:*

$$(c\tau + d)^{-k} f(\gamma\tau) = g(\gamma, \tau).$$

Then for all $\xi = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in M_2(\mathbb{Z})^*$ and for all $\tau \in \mathbb{H}$:

$$\left(\frac{\gcd(A,C)}{AD - BC} (C\tau + D) \right)^{-k} f(\xi\tau) = g\left(\begin{pmatrix} \frac{A}{\gcd(A,C)} & -y \\ \frac{C}{\gcd(A,C)} & x \end{pmatrix}, \frac{\gcd(A,C)\tau + Bx + Dy}{\frac{AD-BC}{\gcd(A,C)}} \right)$$

where the integers x and y are chosen such that $Ax + Cy = \gcd(A, C)$.

Proof. Define

$$\gamma := \begin{pmatrix} \frac{A}{\gcd(A,C)} & -y \\ \frac{C}{\gcd(A,C)} & x \end{pmatrix} \text{ and } \gamma' := \begin{pmatrix} \gcd(A,C) & Bx+Dy \\ 0 & \frac{AD-BC}{\gcd(A,C)} \end{pmatrix}.$$

Then the statement follows from the relation $\xi = \gamma\gamma'$ and by

$$f(\xi\tau) = f(\gamma(\gamma'\tau)) = \left(\frac{C}{\gcd(A,C)}(\gamma'\tau) + x\right)^k g(\gamma, \gamma'\tau).$$

□

4.2 The function $g_{m,t}(\tau, r)$ under modular substitutions

Throughout this section we will assume that $\gcd(a,6) = 1, a > 0$ and $c > 0$ so that (2.37) will always apply and $a^2 \equiv_{24} 1$. For this reason recall:

$$\Gamma_0(N)^* = \{\gamma \in \Gamma_0(N) | a > 0, c > 0, \gcd(a,6) = 1\}.$$

Because M and $r = (r_\delta)$ are assumed as fixed we will write $g_{m,t}(\tau) := g_{m,t}(\tau, r)$ and $f(\tau) := f(\tau, r)$ throughout.

We are interested in deriving a formula for $g_{m,t}(\gamma\tau)$ with $\gamma \in \Gamma_0(N)^*$ where N is an integer such that for every prime p with $p|m$ we have also $p|N$, i.e.,

$$p|m \text{ implies } p|N, \tag{4.6}$$

and such that for every $\delta|M$ with $r_\delta \neq 0$ we have $\delta|mN$, i.e.,

$$\delta|M \text{ implies } \delta|mN; \tag{4.7}$$

and some additional properties which we will specify later. For our purpose it is convenient to define the following set:

Definition 4.9. *We define*

$$\Delta := \left\{ (m, M, N, (r_\delta)) \in (\mathbb{N}^*)^3 \times R(M) \mid \begin{array}{l} m, M, N \textit{ and } (r_\delta) \textit{ satisfy} \\ \textit{the conditions } (4.6) \textit{ and } (4.7). \end{array} \right\}.$$

Lemma 4.10. *Let $(m, M, N, (r_\delta)) \in \Delta$, $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)^*$ and λ a nonnegative integer. Then there exist integers x, y, a' such that:*

(i) $(a + \kappa\lambda c)x + mcy = 1$, where $y := y_0(m\kappa c)^3$ for some integer y_0.

(ii) $a'a \equiv_{24c} 1$.

Moreover, setting $\mu := \lambda dx + \frac{bx - ba'm^2}{\kappa}$, *(iii) For ϵ as in Definition 2.31, $\tau \in \mathbb{H}$ and $\delta | M$ with $r_\delta \neq 0$ we have*

$$\eta\left(\frac{\delta(\gamma\tau+\kappa\lambda)}{m}\right) = (-i(c\tau+d))^{\frac{1}{2}}\epsilon(a+\kappa\lambda c, -\delta y, mc/\delta, x)\eta\left(\frac{\delta(\tau+\kappa\mu)}{m}\right)e^{\frac{2\pi i abm\delta}{24}}, \tag{4.8}$$

and

$$\epsilon(a+\kappa\lambda c, -\delta y, mc/\delta, x) = \left(\frac{mc\delta}{a+\kappa\lambda c}\right)e^{-\frac{(a+\kappa\lambda c)\pi i}{12}(mc/\delta-3)}. \tag{4.9}$$

(iv) The value μ is an integer, and if λ runs through a complete set of representatives of residue classes modulo m then so does μ; i.e., $\lambda \mapsto \mu$ is a bijection of $\mathbb{Z}/m\mathbb{Z}$.

(v) We have

$$\lambda \equiv_c \mu a^2 - ab\frac{1-m^2}{\kappa}. \tag{4.10}$$

Proof. We prove each part of Lemma 4.10 separately.

Proof of (i). We know that the equation

$$(a+\kappa\lambda c)x + mcy_0(m\kappa c)^3 = 1 \tag{4.11}$$

has integer solutions x and y_0 iff

$$\gcd(a+\kappa\lambda c, mc(m\kappa c)^3) = 1. \tag{4.12}$$

To prove (4.12) it suffices to prove $\gcd(a+\kappa\lambda c, m) = 1$ and $\gcd(a+\kappa\lambda c, \kappa c) = 1$. We have that

$$\gcd(a+\kappa\lambda c, \kappa c) = \gcd(a, \kappa c).$$

But $\gcd(a, c) = 1$ because $ad - bc = 1$, and $\gcd(a, \kappa) = 1$ because $\gcd(a, 6) = 1$ by assumption and κ being a divisor of 24 from (4.2), so $\gcd(a, \kappa c) = 1$. Next we see that $\gcd(a+\kappa\lambda c, c) = 1$ implies $\gcd(a+\kappa\lambda c, N) = 1$ because $N|c$. But $\gcd(a+\kappa\lambda c, N) = 1$ implies $\gcd(a+\kappa\lambda c, m) = 1$ by (4.6). This proves (4.12).

Note: Because of $y = y_0(m\kappa c)^3$ Lemma 4.3 gives

$$y \equiv_{24} 0. \tag{4.13}$$

Proof of (ii). The assumptions $\gcd(a, 6) = 1$ and $\gcd(a, c) = 1$ imply that $\gcd(a, 24c) = 1$, which is equivalent to the existence of an integer a' such that $a'a \equiv_{24c} 1$.

Proof of (iii). To prove (4.8) we let ϵ be as in Definition 2.31 let

$$K := (-i(c\tau+d))^{\frac{1}{2}}\,\epsilon(a+\kappa\lambda c, -\delta y, \frac{mc}{\delta}, x). \tag{4.14}$$

We apply Lemma 4.8 with

$$\xi = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \delta(a+\kappa\lambda c) & \delta(b+\kappa\lambda d) \\ mc & md \end{pmatrix},$$

$$k = 1/2, f(\tau) = \eta(\tau), g(\gamma, \tau) = (-i)^{1/2}\epsilon(a, b, c, d)\eta(\tau),$$

and we get

$$\eta\left(\frac{\delta((a+\kappa\lambda c)\tau+b+\kappa\lambda d)}{mc\tau+md}\right)=K\eta\left(\frac{\delta\tau+\delta(b+\kappa\lambda d)x+md\delta y}{m}\right) \tag{4.15}$$

which is valid under the assumption that $(a+\kappa\lambda c)x+\delta y\frac{mc}{\delta}=1$ which implies that

$$\delta(a+\kappa\lambda c)x+\delta ymc=\delta=\gcd(\delta(a+\kappa\lambda c),mc),$$

as required in Lemma 4.8.

Note that $\frac{mc}{\delta}$ is a positive integer ($N|c$ and, by (4.7), $\delta|mN$), and that $(a+\kappa\lambda c)x+\delta y\frac{mc}{\delta}=1$ because of *(i)*. Also recall that $\gcd(a+\kappa\lambda c,m)=1$ because of (4.6).

We will also need that for all integers j we have as a trivial consequence of (2.26):

$$\eta(\tau+j)=\eta(\tau)e^{\frac{2\pi ij}{24}}. \tag{4.16}$$

Consequently,

$$\begin{aligned}
\eta\left(\frac{\delta(\gamma\tau+\kappa\lambda)}{m}\right) &=\eta\left(\frac{\delta((a+\kappa\lambda c)\tau+b+\kappa\lambda d)}{mc\tau+md}\right) && \text{(by substituting for } \gamma)\\
&=K\eta\left(\frac{\delta\tau+\delta(b+\kappa\lambda d)x+md\delta y}{m}\right) && \text{(by (4.15))}\\
&=K\eta\left(\frac{\delta\tau+\delta(b+\kappa\lambda d)x}{m}\right) && \text{(by (4.16) and (4.13))}\\
&=K\eta\left(\frac{\delta\tau+\delta(b+\kappa\lambda d)x-\delta ba'm^2}{m}+\delta ba'm\right) &&\\
&=K\eta\left(\frac{\delta\tau+\delta(b+\kappa\lambda d)x-\delta ba'm^2}{m}\right)e^{\frac{2\pi i\delta a'bm}{24}} && \text{(by (4.16))}\\
&=K\eta\left(\frac{\delta(\tau+\kappa\mu)}{m}\right)e^{\frac{2\pi i\delta a'bm}{24}} && \text{(by the def. of } \mu)\\
&=K\eta\left(\frac{\delta(\tau+\kappa\mu)}{m}\right)e^{\frac{2\pi i\delta abm}{24}} && \text{(because of } a'\equiv_{24}a).
\end{aligned}$$

In the last line we used fact *(ii)*, namely $aa'\equiv_{24c}1$. This together with $a^2\equiv_{24}1$ implies that $a\equiv_{24}a'$ because of uniqueness of the inverse modulo 24.

To prove (4.9) we first note that

$$\gcd(a+\kappa\lambda c,6)=1, \tag{4.17}$$

because of $\kappa c\equiv_6 0$ by Lemma 4.3 and (4.6) together with $N|c$.

We have that

$$
\begin{aligned}
&\epsilon(a+\kappa\lambda c, -\delta y, \frac{mc}{\delta}, x) \\
&= \left(\frac{mc/\delta}{a+\kappa\lambda c}\right) e^{-\frac{(a+\kappa\lambda c)\pi i}{12}(mc/\delta+\delta y-3)} && \text{(by (2.37) and (4.17))} \\
&= \left(\frac{mc/\delta}{a+\kappa\lambda c}\right) e^{-\frac{(a+\kappa\lambda c)\pi i}{12}(mc/\delta-3)} && \text{(by (4.13))} \\
&= \left(\frac{mc/\delta}{a+\kappa\lambda c}\right)\left(\frac{\delta^2}{a+\kappa\lambda c}\right) e^{-\frac{(a+\kappa\lambda c)\pi i}{12}(mc/\delta-3)} && \text{(see below)} \\
&= \left(\frac{mc\delta}{a+\kappa\lambda c}\right) e^{-\frac{(a+\kappa\lambda c)\pi i}{12}(mc/\delta-3)} && \text{(by (2.33)).}
\end{aligned}
$$

The third equality is shown as follows. If $\gcd(a+\kappa\lambda c, \delta) = 1$ then Definition 2.25 implies that $\left(\frac{\delta^2}{a+\kappa\lambda c}\right) = 1$. To prove relative primeness we see by (4.6) and (4.7) that each prime p dividing δ also divides N and consequently also c. So $\gcd(a+\kappa\lambda c, p) = \gcd(a, p)$. But since $p|c$ and $\gcd(a, c) = 1$ by $ad - bc = 1$, we conclude that $\gcd(a + \kappa\lambda c, \delta) = 1$.

Proof of (iv). In order to prove that μ is an integer we need to show that $bx - ba'm^2 \equiv_\kappa 0$. By (4.11) we obtain $ax \equiv_\kappa 1$. We also know by *(ii)* that $aa' \equiv_{24c} 1$. Because of $\kappa|24$ by (4.2), we have that $aa' \equiv_\kappa 1$. From this it follows that $x \equiv_\kappa a'$ by uniqueness of inverses mod κ. Consequently,

$$bx - ba'm^2 \equiv_\kappa bx - bxm^2 \equiv_\kappa bx(1 - m^2) \equiv_\kappa 0,$$

using $\kappa|(1 - m^2)$ from (4.2).

Next we show that the mapping $\lambda \mapsto \mu$ is a bijection of $\mathbb{Z}/m\mathbb{Z}$ by providing an inverse using the observation that:

$$\mu - \frac{bx - ba'm^2}{\kappa} \equiv_m \lambda dx \text{ implies } \lambda \equiv_m (xd)^{-1}(\mu - \frac{bx - ba'm^2}{\kappa}).$$

The only non-trivial step is to show that d and x are indeed invertible modulo m. First of all, x is invertible modulo m because of (4.11). Because of $ad - bc = 1$ we have that $\gcd(c, d) = 1$, and since $N|c$ we have that $\gcd(N, d) = 1$. By (4.6) we get that $\gcd(m, d) = 1$ which shows that also d is invertible modulo m.

Proof of (v). By (4.11) we have that $ax \equiv_{\kappa c} 1$. From $aa' \equiv_{24c} 1$ and $\kappa|24$ we conclude that $aa' \equiv_{\kappa c} 1$ which implies $x \equiv_{\kappa c} a'$ by uniqueness of inverses mod κc.

Because of the relation $ad - bc = 1$ we have that $ad \equiv_c 1$. From $ax \equiv_{\kappa c} 1$ it follows that $ax \equiv_c 1$ which implies $d \equiv_c x$ by uniqueness of inverses mod c.

Next we will show the validity of

$$\mu \equiv_c \lambda d^2 + bd\frac{1 - m^2}{\kappa} \tag{4.18}$$

by the following chain of arguments starting with the definition of μ:

$$\kappa\mu \equiv_{\kappa c} \kappa\lambda dx + bx - ba'm^2 \equiv_{\kappa c} \kappa\lambda dx + bx - bxm^2 \equiv_{\kappa c} \kappa(\lambda dx + bx\frac{1 - m^2}{\kappa})$$

which implies that

$$\mu \equiv_c \lambda dx + bx\frac{1-m^2}{\kappa} \equiv_c \lambda d^2 + bd\frac{1-m^2}{\kappa}.$$

We thus have proven (4.18). By multiplying the last congruence with a^2, we obtain:

$$\mu a^2 - ba\frac{1-m^2}{\kappa} \equiv_c \lambda.$$

We have again used that the inverse of d is a modulo c.

□

In order to arrive at our main result, Theorem 4.22, we need to introduce some additional assertions, Lemmas 4.11 to 4.19.

Lemma 4.11. *Let l, j be integers and C, a, s non-negative integers such that:*

1. *the relation $p|l$ implies $p|C$ for any prime p;*
2. $\gcd(a, l) = 1$*;*
3. $l = 2^s j$ *where j is odd;*
4. *a is odd and C is even.*

Then for any non-negative integer λ:

$$\left(\frac{l}{a+\lambda C}\right) = \left(\frac{l}{a}\right)(-1)^{\frac{\lambda C(j-1)}{4}}(-1)^{\frac{s(2a\lambda C+\lambda^2 C^2)}{8}}.$$

Proof. By a similar reasoning as in the proof of (4.9) we see that $\gcd(a+\lambda C, l) = 1$ for all integers λ and thus $\gcd(a+\lambda C, j) = 1$.

Next we can write $j = j_1 j_2$ where j_1 is squarefree and j_2 is a square. Clearly $j_1|C$ by assumption. Then:

$$\begin{aligned}
\left(\frac{j}{a+\lambda C}\right) &= \left(\frac{j_1}{a+\lambda C}\right)\left(\frac{j_2}{a+\lambda C}\right) && \text{(by (2.33))}\\
&= \left(\frac{j_1}{a+\lambda C}\right) && \text{(because of } \gcd(a+\lambda C, j) = 1)\\
&= (-1)^{\frac{a+\lambda C-1}{2}\frac{j_1-1}{2}}\left(\frac{a+\lambda C}{j_1}\right) && \text{(by (2.36))}\\
&= (-1)^{\frac{a+\lambda C-1}{2}\frac{j_1 j_2-1}{2}}\left(\frac{a+\lambda C}{j_1}\right) && \text{(because of } j_2 \equiv_4 1)\\
&= (-1)^{\frac{\lambda C(j-1)}{4}}(-1)^{\frac{a-1}{2}\frac{j_1-1}{2}}\left(\frac{a}{j_1}\right) && \text{(because of } a+\lambda C \equiv_{j_1} a)\\
&= (-1)^{\frac{\lambda C(j-1)}{4}}\left(\frac{j_1}{a}\right) && \text{(by (2.36))}\\
&= (-1)^{\frac{\lambda C(j-1)}{4}}\left(\frac{j}{a}\right) && \text{(by (2.33) and because of } \left(\frac{j_2}{a}\right) = 1).
\end{aligned}$$

Summarizing, we have proven:

$$\left(\frac{j}{a+\lambda C}\right) = (-1)^{\frac{\lambda C(j-1)}{4}} \left(\frac{j}{a}\right). \tag{4.19}$$

Next,

$$\left(\frac{2}{a+\lambda C}\right) = (-1)^{\frac{2a\lambda C+\lambda^2 C^2}{8}} \left(\frac{2}{a}\right) \tag{4.20}$$

is easily seen by

$$\begin{aligned}\left(\frac{2}{a+\lambda C}\right) &= (-1)^{\frac{(a+\lambda C)^2-1}{8}} && \text{(by (2.34))}\\ &= (-1)^{\frac{2a\lambda C+\lambda^2 C^2}{8}} \left(\frac{2}{a}\right) && \text{(by (2.34)).}\end{aligned}$$

The following derivation concludes the proof:

$$\begin{aligned}\left(\frac{l}{a+\lambda C}\right) &= \left(\frac{2}{a+\lambda C}\right)^s \left(\frac{j}{a+\lambda C}\right) && \text{(by (2.33))}\\ &= \left(\frac{2}{a}\right)^s \left(\frac{j}{a}\right) (-1)^{\frac{\lambda C(j-1)}{4}} (-1)^{\frac{s(2a\lambda C+\lambda^2 C^2)}{8}} && \text{(by (4.19) and (4.20))}\\ &= \left(\frac{l}{a}\right) (-1)^{\frac{\lambda C(j-1)}{4}} (-1)^{\frac{s(2a\lambda C+\lambda^2 C^2)}{8}} && \text{(by (2.33)).}\end{aligned}$$

□

In order to make the next lemmas more readable we need to introduce some helpful definitions:

Definition 4.12. *A tuple $(m, M, N, (r_\delta)) \in \Delta$ is said to be κ-proper, if*

$$\kappa N \sum_{\delta|M} r_\delta \frac{mN}{\delta} \equiv_{24} 0, \tag{4.21}$$

and

$$\kappa N \sum_{\delta|M} r_\delta \equiv_8 0, \tag{4.22}$$

where as usual $\kappa = \gcd(1-m^2, 24)$.

Definition 4.13. *For $(m, M, N, (r_\delta)) \in \Delta$, $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)^*$ and λ a non-negative integer we define:*

$$\beta(\gamma, \lambda) := e^{\sum_{\delta|M} \frac{2\pi i r_\delta \delta amb}{24}} \prod_{\delta|M} \left(\frac{mc\delta}{a+\kappa\lambda c}\right)^{|r_\delta|} e^{-\frac{(a+\kappa\lambda c)\pi i}{12} \sum_{\delta|M} r_\delta (mc/\delta - 3)}, \tag{4.23}$$

where $\left(\frac{\cdot}{\cdot}\right)$ is the Jacobi symbol.

Remark 4.14. *It follows by Definition 4.13 that* $(\beta(\gamma,\lambda))^{24} = 1$ *for all* $\lambda \in \mathbb{Z}$.

Definition 4.15. *For* M *a positive integer and* $(r_\delta) \in R(M)$ *let* $\pi(M,(r_\delta)) = (s,j)$ *where* s *is a non-negative integer and* j *an odd integer uniquely determined by* $\prod_{\delta|M} \delta^{|r_\delta|} = 2^s j$.

Lemma 4.16. *Let* $(m,M,N,(r_\delta)) \in \Delta$ *be* κ*-proper,* $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)^*$, $(s,j) := \pi(M,(r_\delta))$. *Then for* λ *a non-negative integer the following relations hold:*

$$\beta(\gamma,\lambda) = \prod_{\delta|M} \left(\frac{mc\delta}{a+\kappa\lambda c}\right)^{|r_\delta|} e^{-\frac{\pi i a}{12}\left(\sum_{\delta|M} \frac{mc}{\delta} r_\delta - \sum_{\delta|M} r_\delta \delta m b - 3\sum_{\delta|M} r_\delta\right)}, \tag{4.24}$$

and

$$\beta(\gamma,\lambda) = \begin{cases} \beta(\gamma,0) & \text{if } \kappa c \equiv_8 0 \\ \beta(\gamma,0)(-1)^{\frac{\kappa\lambda c(j-1)}{4}}(-1)^{s\frac{2a\kappa\lambda c+\kappa^2\lambda^2c^2}{8}} & \text{if } \sum_{\delta|M} r_\delta \equiv_2 0 \end{cases}. \tag{4.25}$$

Proof. To prove (4.24) we see from its definition that $\beta(\gamma,\lambda)$ can be rewritten as

$$= \prod_{\delta|M} \left(\frac{mc\delta}{a+\kappa\lambda c}\right)^{|r_\delta|} e^{-\frac{\pi i a}{12}\left(\sum_{\delta|M} r_\delta \frac{mc}{\delta} - \sum_{\delta|M} mb\delta r_\delta - 3\sum_{\delta|M} r_\delta\right)}$$
$$\cdot e^{-\frac{\pi i\kappa\lambda c}{12}\left(\sum_{\delta|M} r_\delta \frac{mc}{\delta} - 3\sum_{\delta|M} r_\delta\right)}.$$

Because of $N|c$, (4.21) and (4.22) we can conclude that $\sum_{\delta|M} r_\delta c\kappa \frac{mc}{\delta} \equiv_{24} 0$ and $\kappa c \sum_{\delta|M} r_\delta \equiv_8 0$. Hence

$$e^{-\frac{\pi i\kappa\lambda c}{12}\left(\sum_{\delta|M} r_\delta \frac{mc}{\delta} - 3\sum_{\delta|M} r_\delta\right)} = 1.$$

To prove (4.25) we see that condition (4.22) implies that either $\sum_{\delta|M} r_\delta \equiv_2 0$ or $\kappa N \equiv_8 0$. From (4.24) and Lemma 4.11 we see that if $\kappa c \equiv_8 0$ then $\beta(\gamma,\lambda) = \beta(\gamma,0), \lambda \geq 0$.

If $\sum_{\delta|M} r_\delta \equiv_2 0$ then

$$\prod_{\delta|M} \left(\frac{mc}{a+\kappa\lambda c}\right)^{|r_\delta|} = \left(\frac{mc}{a+\kappa\lambda c}\right)^{\sum_{\delta|M}|r_\delta|} = \left(\frac{mc}{a+\kappa\lambda c}\right)^{\sum_{\delta|M} r_\delta} = 1,$$

and we have

$$\begin{aligned} \prod_{\delta|M} \left(\frac{\delta mc}{a+\kappa\lambda c}\right)^{|r_\delta|} &= \left(\frac{\prod_{\delta|M} \delta^{|r_\delta|}}{a+\kappa\lambda c}\right) && \text{(by (2.33))} \\ &= \left(\frac{\prod_{\delta|M} \delta^{|r_\delta|}}{a}\right)(-1)^{\frac{\kappa\lambda c(j-1)}{4}}(-1)^{\frac{s(2a\kappa\lambda c+\kappa^2\lambda^2c^2)}{8}} && \text{(by Lemma 4.11)} \\ &= \prod_{\delta|M} \left(\frac{\delta mc}{a}\right)^{|r_\delta|}(-1)^{\frac{\kappa\lambda c(j-1)}{4}}(-1)^{\frac{s(2a\kappa\lambda c+\kappa^2\lambda^2c^2)}{8}} && \text{(by (2.33)).} \end{aligned}$$

In view of (4.24) this implies that

$$\beta(\gamma,\lambda) = \beta(\gamma,0)(-1)^{\frac{\kappa\lambda c(j-1)}{4}}(-1)^{\frac{s(2a\kappa\lambda c+\kappa^2\lambda^2c^2)}{8}}. \tag{4.26}$$

Note that in order to apply Lemma 4.11 above we need to verify that $p|\prod_{\delta|M}\delta^{|r_\delta|}$ implies $p|\kappa c$ and that $\gcd(a, \prod_{\delta|M}\delta^{|r_\delta|}) = 1$. This follows from (4.6) and (4.7) together with $\gcd(a, c) = 1$ because of $ad - bc = 1$.

□

Lemma 4.17. *Let* $(m, M, N, (r_\delta)) \in \Delta$, $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)^*$ *and* t *an integer with* $0 \leq t \leq m-1$ *such that the relation*

$$\frac{24m}{\gcd(\kappa(-24t - \sum_{\delta|M}\delta r_\delta), 24m)} \mid N \tag{4.27}$$

holds, then for $\tau \in \mathbb{H}$ *we have that*

$$\begin{aligned} g_{m,t}(\gamma\tau) = (-i(c\tau+d))^{\frac{\sum_{\delta|M} r_\delta}{2}} e^{\frac{2\pi i ab(1-m^2)(24t+\sum_{\delta|M}\delta r_\delta)}{24m}} \\ \cdot \frac{1}{m}\sum_{\lambda=0}^{m-1}\beta(\gamma,\lambda) e^{\frac{2\pi i\kappa\mu a^2(-24t-\sum_{\delta|M}\delta r_\delta)}{24m}} \prod_{\delta|M}\eta^{r_\delta}\left(\frac{\delta(\tau+\kappa\mu)}{m}\right), \end{aligned} \tag{4.28}$$

where μ *is defined as in Lemma 4.10.*

Proof. Given two integers λ, λ' such that $\lambda \equiv_c \lambda'$, relation (4.27) implies

$$\lambda \equiv_{\frac{24m}{\gcd(24m, \kappa(-24t-\sum_{\delta|M}\delta r_\delta))}} \lambda',$$

consequently

$$e^{\frac{2\pi i\lambda\kappa(-24t-\sum_{\delta|M}\delta r_\delta)}{24m}} = e^{\frac{2\pi i\lambda'\kappa(-24t-\sum_{\delta|M}\delta r_\delta)}{24m}}.$$

Therefore by *(v)* in Lemma 4.10 we conclude that:

$$e^{\frac{2\pi i\lambda\kappa(-24t-\sum_{\delta|M}\delta r_\delta)}{24m}} = e^{\frac{2\pi i\kappa(\mu a^2 - \frac{ab(1-m^2)}{\kappa})(-24t-\sum_{\delta|M}\delta r_\delta)}{24m}}. \tag{4.29}$$

Hence,

$$g_{m,t}(\gamma\tau) = \frac{1}{m}\sum_{\lambda=0}^{m-1} e^{\frac{2\pi i\kappa\lambda(-24t-\sum_{\delta|M}\delta r_\delta)}{24m}} \prod_{\delta|M}\eta^{r_\delta}\left(\frac{\delta(\gamma\tau+\kappa\lambda)}{m}\right)$$

(by (4.5))

$$\begin{aligned} &= (-i(c\tau+d))^{\frac{\sum_{\delta|M} r_\delta}{2}} \\ &\quad \cdot \frac{1}{m}\sum_{\lambda=0}^{m-1}\beta(\gamma,\lambda) e^{\frac{2\pi i\kappa\lambda(-24t-\sum_{\delta|M}\delta r_\delta)}{24m}} \prod_{\delta|M}\eta^{r_\delta}\left(\frac{\delta(\tau+\kappa\mu)}{m}\right) \end{aligned}$$

(by (4.8), (4.9) and (4.23))

$$\begin{aligned} &= (-i(c\tau+d))^{\frac{\sum_{\delta|M} r_\delta}{2}} e^{\frac{2\pi i ab(1-m^2)(24t+\sum_{\delta|M}\delta r_\delta)}{24m}} \\ &\quad \cdot \frac{1}{m}\sum_{\lambda=0}^{m-1}\beta(\gamma,\lambda) e^{\frac{2\pi i\kappa\mu a^2(-24t-\sum_{\delta|M}\delta r_\delta)}{24m}} \prod_{\delta|M}\eta^{r_\delta}\left(\frac{\delta(\tau+\kappa\mu)}{m}\right) \end{aligned}$$

(by (4.29)).

□

Lemma 4.18. *Let $(m, M, N, (r_\delta)) \in \Delta$ be κ-proper, $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)^*$, and t an integer with $0 \le t \le m-1$ such that (4.27) holds. Let t' be the unique integer satisfying $0 \le t' \le m-1$ and $t' \equiv_m ta^2 + \frac{a^2-1}{24}\sum_{\delta|M}\delta r_\delta$. Assume that $\kappa N \equiv_8 0$, then for $\tau \in \mathbb{H}$ we have that*

$$g_{m,t}(\gamma\tau) = \beta(\gamma,0)\,(-i(c\tau+d))^{\frac{\sum_{\delta|M} r_\delta}{2}}\, e^{\frac{2\pi i ab(1-m^2)(24t+\sum_{\delta|M}\delta r_\delta)}{24m}} g_{m,t'}(\tau).$$

Proof.

$$\begin{aligned} g_{m,t}(\gamma\tau) = \beta(\gamma,0)\,(-i(c\tau+d))^{\frac{\sum_{\delta|M} r_\delta}{2}}\, e^{\frac{2\pi i ab(1-m^2)(24t+\sum_{\delta|M}\delta r_\delta)}{24m}} \cdot \\ \frac{1}{m}\sum_{\lambda=0}^{m-1} e^{\frac{2\pi i\kappa\mu(-24t'-\sum_{\delta|M}\delta r_\delta)}{24m}} \prod_{\delta|M}\eta^{r_\delta}\left(\frac{\delta(\tau+\kappa\mu)}{m}\right) \end{aligned}$$

(by (4.28) and because $\beta(\gamma,0)=\beta(\gamma,\lambda), \lambda\in\mathbb{Z}$ by (4.25))

$$=\beta(\gamma,0)\,(-i(c\tau+d))^{\frac{\sum_{\delta|M} r_\delta}{2}}\, e^{\frac{2\pi i ab(1-m^2)(24t+\sum_{\delta|M}\delta r_\delta)}{24m}} g_{m,t'}(\tau)$$

(by (4.5) and *(iv)* in Lemma 4.10).

□

Lemma 4.19. *Let $(m, M, N, (r_\delta)) \in \Delta$ be κ-proper, $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)^*$, $(s,j) = \pi(M,(r_\delta))$ and t an integer with $0 \le t \le m-1$ such that (4.27) holds. Assume further that $\sum_{\delta|M} r_\delta \equiv_2 0$ and $2|m$.*

(i) If $s \equiv_2 0$ let t' be the unique integer satisfying $t' \equiv_m ta^2 + \frac{a^2-1}{24}\sum_{\delta|M}\delta r_\delta - \frac{3mca^2(j-1)}{24}$ and $0 \le t' \le m-1$. Then for $\tau\in\mathbb{H}$ we have that

$$\begin{aligned} g_{m,t}(\gamma\tau) = (-1)^{\frac{abc(1-m^2)(j-1)}{4}}\beta(\gamma,0) \\ \cdot(-i(c\tau+d))^{\frac{\sum_{\delta|M} r_\delta}{2}}\, e^{\frac{2\pi i ab(1-m^2)(24t+\sum_{\delta|M}\delta r_\delta)}{24m}} g_{m,t'}(\tau). \end{aligned} \tag{4.30}$$

(ii) If $\kappa c \equiv_4 0$ let t' be the unique integer satisfying $t' \equiv_m -\frac{3mcsa^2}{24} + ta^2 + \frac{a^2-1}{24}\sum_{\delta|M}\delta r_\delta$ and $0 \le t' \le m-1$. Then for $\tau\in\mathbb{H}$ we have that

$$\begin{aligned} g_{m,t}(\gamma\tau) = (-1)^{\frac{sa^2bc(1-m^2)}{4}}\beta(\gamma,0) \\ \cdot(-i(c\tau+d))^{\frac{\sum_{\delta|M} r_\delta}{2}}\, e^{\frac{2\pi i ab(1-m^2)(24t+\sum_{\delta|M}\delta r_\delta)}{24m}} g_{m,t'}(\tau). \end{aligned} \tag{4.31}$$

Proof. ***Proof of (i):***

$$g_{m,t}(\gamma\tau) = (-i(c\tau+d))^{\frac{\sum_{\delta|M} r_\delta}{2}} e^{\frac{2\pi i ab(1-m^2)(24t+\sum_{\delta|M}\delta r_\delta)}{24m}} \beta(\gamma,0) \cdot \frac{1}{m}\sum_{\lambda=0}^{m-1} (-1)^{\frac{\kappa\lambda c(j-1)}{4}} e^{\frac{2\pi i\kappa\mu a^2(-24t-\sum_{\delta|M}\delta r_\delta)}{24m}} \prod_{\delta|M} \eta^{r_\delta}\left(\frac{\delta(\tau+\kappa\mu)}{m}\right)$$

(by (4.28) and (4.26), together with $2|m$ which implies $2|c$ because of (4.6))

$$=(-1)^{\frac{abc(1-m^2)(j-1)}{4}} \beta(\gamma,0)\,(-i(c\tau+d))^{\frac{\sum_{\delta|M} r_\delta}{2}} e^{\frac{2\pi i ab(1-m^2)(24t+\sum_{\delta|M}\delta r_\delta)}{24m}} \cdot \frac{1}{m}\sum_{\lambda=0}^{m-1} (-1)^{\frac{\kappa\mu a^2 c(j-1)}{4}} e^{\frac{2\pi i\kappa\mu a^2(-24t-\sum_{\delta|M}\delta r_\delta)}{24m}} \prod_{\delta|M} \eta^{r_\delta}\left(\frac{\delta(\tau+\kappa\mu)}{m}\right)$$

(by (4.10) and $c \equiv_2 0$)

$$=(-1)^{\frac{abc(1-m^2)(j-1)}{4}} \beta(\gamma,0)\,(-i(c\tau+d))^{\frac{\sum_{\delta|M} r_\delta}{2}} e^{\frac{2\pi i ab(1-m^2)(24t+\sum_{\delta|M}\delta r_\delta)}{24m}} \cdot \frac{1}{m}\sum_{\lambda=0}^{m-1} e^{\frac{2\pi i\kappa\mu a^2(3mc(j-1)-24t-\sum_{\delta|M}\delta r_\delta)}{24m}} \prod_{\delta|M} \eta^{r_\delta}\left(\frac{\delta(\tau+\kappa\mu)}{m}\right)$$

$$=(-1)^{\frac{abc(1-m^2)(j-1)}{4}} \beta(\gamma,0)\,(-i(c\tau+d))^{\frac{\sum_{\delta|M} r_\delta}{2}} e^{\frac{2\pi i ab(1-m^2)(24t+\sum_{\delta|M}\delta r_\delta)}{24m}} \cdot \frac{1}{m}\sum_{\lambda=0}^{m-1} e^{\frac{2\pi i\kappa\mu(-24t'-\sum_{\delta|M}\delta r_\delta)}{24m}} \prod_{\delta|M} \eta^{r_\delta}\left(\frac{\delta(\tau+\kappa\mu)}{m}\right)$$

(by substituting for t')

$$=(-1)^{\frac{abc(1-m^2)(j-1)}{4}} \beta(\gamma,0)\,(-i(c\tau+d))^{\frac{\sum_{\delta|M} r_\delta}{2}} e^{\frac{2\pi i ab(1-m^2)(24t+\sum_{\delta|M}\delta r_\delta)}{24m}} g_{m,t'}(\tau)$$

(by (4.5) and *(iv)* in Lemma 4.10).

Proof of (ii):

$$g_{m,t}(\gamma\tau) = (-i(c\tau+d))^{\frac{\sum_{\delta|M} r_\delta}{2}} e^{\frac{2\pi i ab(1-m^2)(24t+\sum_{\delta|M}\delta r_\delta)}{24m}} \beta(\gamma,0) \cdot \frac{1}{m}\sum_{\lambda=0}^{m-1} (-1)^{\frac{sa\kappa\lambda c}{4}} e^{\frac{2\pi i\kappa\mu a^2(-24t-\sum_{\delta|M}\delta r_\delta)}{24m}} \prod_{\delta|M} \eta^{r_\delta}\left(\frac{\delta(\tau+\kappa\mu)}{m}\right)$$

(by (4.28) and (4.26))

$$=(-1)^{\frac{sa^2bc(1-m^2)}{4}} \beta(\gamma,0)\,(-i(c\tau+d))^{\frac{\sum_{\delta|M} r_\delta}{2}} e^{\frac{2\pi i ab(1-m^2)(24t+\sum_{\delta|M}\delta r_\delta)}{24m}} \cdot \frac{1}{m}\sum_{\lambda=0}^{m-1} (-1)^{\frac{\kappa\mu a^3 cs}{4}} e^{\frac{2\pi i\kappa\mu a^2(-24t-\sum_{\delta|M}\delta r_\delta)}{24m}} \prod_{\delta|M} \eta^{r_\delta}\left(\frac{\delta(\tau+\kappa\mu)}{m}\right)$$

(by (4.10) and $c \equiv_2 0$))

$$=(-1)^{\frac{sa^2bc(1-m^2)}{4}}\beta(\gamma,0)\,(-i(c\tau+d))^{\frac{\sum_{\delta|M} r_\delta}{2}}\,e^{\frac{2\pi i ab(1-m^2)(24t+\sum_{\delta|M}\delta r_\delta)}{24m}}$$
$$\cdot\,\frac{1}{m}\sum_{\lambda=0}^{m-1} e^{\frac{2\pi i\kappa\mu(-24t'-\sum_{\delta|M}\delta r_\delta)}{24m}}\prod_{\delta|M}\eta^{r_\delta}\left(\frac{\delta(\tau+\kappa\mu)}{m}\right)$$

(by substituting for t')

$$=(-1)^{\frac{sa^2bc(1-m^2)}{4}}\beta(\gamma,0)\,(-i(c\tau+d))^{\frac{\sum_{\delta|M} r_\delta}{2}}\,e^{\frac{2\pi i ab(1-m^2)(24t+\sum_{\delta|M}\delta r_\delta)}{24m}}\,g_{m,t'}(\tau)$$

(by (4.5) and *(iv)* in Lemma 4.10).

□

Note that if $2 \nmid m$ then $\kappa N \equiv_8 0$ and Lemma 4.18 applies. If $2|m$ and $\kappa N \not\equiv_8 0$ then the Lemma 4.19 applies.

Let $(m, M, N, (r_\delta)) \in \Delta$ and s, j integers such that $\pi(M, (r_\delta)) = (s, j)$. In the next theorem we will also assume that:

$$\kappa N \equiv_4 0 \text{ and } 8|Ns, \tag{4.32}$$

or

$$s \equiv_2 0 \text{ and } 8|N(1-j). \tag{4.33}$$

Definition 4.20. *We define*

$$\Delta^* := \{ \textit{ all tuples } (m, N, N, t, (r_\delta)) \textit{ with properties as listed in } (4.34) \textit{ below} \} :$$

$$\begin{aligned} &(m, M, N, (r_\delta)) \in \Delta \textit{ is } \kappa\textit{-proper, } t \in \mathbb{N}, 0 \le t \le m-1; \\ &\textit{in addition } (4.27) \textit{ hold and } (4.32) \textit{ or } (4.33). \end{aligned} \tag{4.34}$$

Definition 4.21. *Let $m, M, N \in \mathbb{N}^*$ and $(r_\delta) \in R(M)$. Define the operation $\odot : \Gamma_0(N)^* \times \{0, \dots, m-1\} \to \{0, \dots, m-1\}$, $(\gamma, t) \mapsto \gamma \odot t$, where for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ the image $\gamma \odot t$ is uniquely defined by the relation*

$$\gamma \odot t \equiv_m ta^2 + \frac{a^2-1}{24}\sum_{\delta|M}\delta r_\delta. \tag{4.35}$$

Finally we arrive at the main theorem of this section which can be viewed as a generalization of a theorem of R. Lewis; see Remark 4.23 below.

Theorem 4.22. *Let $(m, M, N, t, (r_\delta) = r) \in \Delta^*$, $g_{m,t}(\tau, r)$ be as in Definition 4.6, $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)^*$, and β as in Definition 4.13. Then for all $\tau \in \mathbb{H}$ we have that*

$$g_{m,t}(\gamma\tau, r) = \beta(\gamma,0)\,(-i(c\tau+d))^{\frac{\sum_{\delta|M} r_\delta}{2}}\,e^{2\pi i\frac{ab(1-m^2)(24t+\sum_{\delta|M}\delta r_\delta)}{24m}}\cdot g_{m,\gamma\odot t}(\tau, r). \tag{4.36}$$

Proof. If $2 \nmid m$ then $\kappa N \equiv_8 0$ and (4.36) follows from Lemma 4.18. If $2|m$ we split the proof in two cases depending on if (4.32) or (4.33) holds. If (4.32) holds we have that $\frac{-3mcsa^2}{24} \equiv_m 0$ and $(-1)^{\frac{sa^2bc(1-m^2)}{4}} = 1$ and by *(ii)* in Lemma 4.19 we obtain (4.36). Similarly when (4.33) holds we have that $\frac{3mca^2(j-1)}{24} \equiv_m 0$ and $(-1)^{\frac{abc(1-m^2)(j-1)}{4}} = 1$ and by *(i)* in Lemma 4.19 we obtain (4.36).

□

Remark 4.23. *Theorem 4.22 extends Theorem 1 in [25] which covers products of the form $\prod_{n=1}^{\infty}(1-q^n)^{r_1}$ where r_1 is a fixed integer.*

4.3 Formulas for $g_{m,t}(\gamma\tau)$ when $\gamma \in \Gamma$

Usually $g_{m,t}(\tau) = g_{m,t}(\tau, r)$ as defined in Definition 4.6 is not a modular form. But if we choose a sequence $(a_\delta) \in R(N)$ properly, we can always make sure that $\left(\prod_{t'\in P(t)} g_{m,t'}(\tau)\right)\left(\prod_{\delta|N} \eta^{a_\delta}(\delta\tau)\right)$ (with $P(t)$ as in (4.46)) is a modular form. To prove this we need some formulas for $\prod_{\delta|N} \eta^{a_\delta}(\delta(\gamma\tau))$ and for $g_{m,t}(\gamma\tau)$ that are valid for all γ in Γ, in order to check condition *(iii)* in Definition 2.6 of a modular form. This is done in the Lemmas 4.25 to 4.31 below.

Recall from (4.2) that $\kappa = \gcd(1-m^2, 24)$.

Definition 4.24. *For m a positive integer and $x \in \mathbb{Z}$ we denote by $[x]_m \in \mathbb{Z}/m\mathbb{Z}$ the residue class of x modulo m.*

Lemma 4.25. *Let $(m, M, N, (r_\delta)) \in \Delta$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. For $\delta|M$ with $\delta > 0$ and λ an integer let $x(\delta,\lambda)$ and $y(\delta,\lambda)$ be any fixed solutions to the equation $\delta(a+\kappa\lambda c)\cdot x(\delta,\lambda) + mc \cdot y(\delta,\lambda) = \gcd(\delta(a+\kappa\lambda c), mc)$. Further define*

$$w(\delta,\lambda,\gamma,\tau) := \frac{\gcd(\delta(a+\kappa\lambda c), mc)\tau + \delta(b+\kappa\lambda d)x(\delta,\lambda) + mdy(\delta,\lambda)}{\frac{\delta m}{\gcd(\delta(a+\kappa\lambda c), mc)}}. \tag{4.37}$$

Then there exists a map $C : \Gamma \to \mathbb{C}$ such that for all $\gamma \in \Gamma$ and $\tau \in \mathbb{H}$ the following relation holds:

$$\prod_{\delta|M} \eta^{r_\delta}\left(\frac{\delta(\gamma\tau+\kappa\lambda)}{m}\right) = C(\gamma)(c\tau+d)^{\frac{1}{2}\sum_{\delta|M} r_\delta} \prod_{\delta|M} \eta^{r_\delta}\left(w(\delta,\lambda,\gamma,\tau)\right). \tag{4.38}$$

In addition, there exist mappings $C' : \Gamma \to \mathbb{C}$ and $\mu : \mathbb{Z} \to \mathbb{Z}$ such that for all $\gamma \in \Gamma_0(N)$ and $\tau \in \mathbb{H}$ the following relation holds:

$$\prod_{\delta|M} \eta^{r_\delta}\left(\frac{\delta(\gamma\tau+\kappa\lambda)}{m}\right) = C'(\gamma)(c\tau+d)^{\frac{1}{2}\sum_{\delta|M} r_\delta} \prod_{\delta|M} \eta^{r_\delta}\left(\frac{\delta(\tau+\kappa\mu(\lambda))}{m}\right), \tag{4.39}$$

where μ is chosen such that $[\lambda]_m \mapsto [\mu(\lambda)]_m$ is a bijection of $\mathbb{Z}/m\mathbb{Z}$.

Proof. To prove (4.38) let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. Apply Lemma 4.8 with k set to $\frac{1}{2}$, $f(\tau)$ to $\eta(\tau)$, $g(\gamma,\tau)$ to $(-i)^{\frac{1}{2}}\epsilon(a,b,c,d)\eta(\tau)$ and ξ to $\begin{pmatrix} \delta(a+\kappa\lambda c) & \delta(b+\kappa\lambda d) \\ mc & md \end{pmatrix}$; then for all $\delta|M$

with $\delta > 0$ the following relation holds:

$$\begin{aligned}&\left(\frac{\gcd(\delta(a+\kappa\lambda c), mc)}{\delta m} m(c\tau+d)\right)^{-\frac{1}{2}} \eta\left(\frac{\delta((a+\kappa\lambda c)\tau+b+\kappa\lambda d)}{m(c\tau+d)}\right)\\ =&(-i)^{\frac{1}{2}}\epsilon\left(\frac{\delta(a+\kappa\lambda c)}{\gcd(\delta(a+\kappa\lambda c), mc)}, -y(\delta,\lambda), \frac{mc}{\gcd(\delta(a+\kappa\lambda c), mc)}, x(\delta,\lambda)\right)\eta(w(\delta,\lambda,\gamma,\tau)).\end{aligned}$$

Taking the product over $\delta|M$ on both sides and using that

$$\eta\left(\frac{\delta(\gamma\tau+\kappa\lambda)}{m}\right) = \eta\left(\frac{\delta((a+\kappa\lambda c)\tau+b+\kappa\lambda d)}{m(c\tau+d)}\right)$$

proves (4.38).

To prove (4.39) we first will prove that $\gcd(\delta(a+\kappa\lambda c), mc) = \delta$ if $N|c$. By (4.7) we see that $\delta|mc$ hence $\gcd(\delta(a+\kappa\lambda c), mc) = \delta\gcd(a+\kappa\lambda c, \frac{mc}{\delta})$. Also since $\gcd(a+\kappa\lambda c, c) = 1$ because of $ad-bc=1$, and $\gcd(a+\kappa\lambda c, m) = 1$ because of (4.6), we can conclude that $\gcd(a+\kappa\lambda c, \frac{mc}{\delta}) = 1$. Next, for $\lambda\in\mathbb{Z}$ let $x_0(\lambda)$ and $y_0(\lambda)$ be any solutions to the equation $(a+\kappa\lambda c)x_0(\lambda)+mcy_0(\lambda) = 1$. Then we can define $x(\delta,\lambda) := x_0(\lambda)$ and $y(\delta,\lambda) := \delta y_0(\lambda)$ because of $\gcd(\delta(a+\kappa\lambda c), mc) = \delta$. Consequently,

$$\eta(w(\delta,\lambda,\gamma,\tau)) = \eta\left(\frac{\delta\tau+\delta(b+\kappa\lambda d)x_0(\lambda)}{m}+\delta d y_0(\lambda)\right). \tag{4.40}$$

Next, let X and Y be integers such that $\kappa X + mY = 1$. Such integers clearly exist by (4.2). Define $\mu(\lambda) := (b+\kappa\lambda d)Xx_0(\lambda)$. Then

$$\begin{aligned}\eta\left(\frac{\delta(\tau+\kappa\mu(\lambda))}{m}\right) &= \eta\left(\frac{\delta(\tau+\kappa(b+\kappa\lambda d)Xx_0(\lambda))}{m}\right)\\ &=\eta\left(\frac{\delta(\tau+(b+\kappa\lambda d)x_0(\lambda))}{m}-\delta Y(b+\kappa\lambda d)x_0(\lambda)\right).\end{aligned} \tag{4.41}$$

This shows that

$$\eta(w(\delta,\lambda,\gamma,\tau)) = \epsilon\eta\left(\frac{\delta(\tau+\kappa\mu(\lambda))}{m}\right)$$

for some 24-th root of unity ϵ because of (4.16) and by (4.40) and (4.41). It only remains to show that μ is a bijection of $\mathbb{Z}/m\mathbb{Z}$. Note that $x_0(\lambda)$ is invertible modulo m because of $(a+\kappa\lambda c)x_0(\lambda)+mcy_0(\lambda) = 1$ implying $(\mu(\lambda)X^{-1}x_0(\lambda)^{-1}-b)\kappa^{-1}d^{-1} \equiv_m \lambda$. Note that d is invertible modulo m because of $\gcd(c,d) = 1$ which by (4.6) implies $\gcd(m,d) = 1$. □

Remark 4.26. *Note that* (4.39) *is very similar to* (4.8) *in Lemma 4.10 but here we lifted the restriction* $\gcd(a,6) = 1$, $a>0$, $c>0$.

Proposition 4.27. *Let M be a positive integer and $r, a, b, c \in R(M)$. Then there exists a positive integer k and a Taylor series $h(q)$ in powers of $q^{1/k}$ such that*

$$\prod_{\delta|M}\eta^{r_\delta}\left(\frac{a_\delta\tau+b_\delta}{c_\delta}\right) = q^{\frac{1}{24}\sum_{\delta|M} r_\delta\frac{a_\delta}{c_\delta}}h(q). \tag{4.42}$$

Proof. The proof follows by substituting (2.26) into (4.42). We omit the details. □

Lemma 4.28. *Let $\gamma_0 \in \Gamma$, $(m, M, N, (r_\delta)) \in \Delta$, $t \in \mathbb{Z}$ with $0 \le t \le m-1$, and define the mappings $p : \Gamma \times [0, \ldots, m-1] \to \mathbb{Q}$ and $p : \Gamma \to \mathbb{Q}$ by*

$$p(\gamma, \lambda) := \frac{1}{24} \sum_{\delta|M} r_\delta \frac{\gcd^2(\delta(a + \kappa\lambda c), mc)}{\delta m}, \tag{4.43}$$

and

$$p(\gamma) := \min_{\lambda \in \{0,\ldots,m-1\}} p(\gamma, \lambda). \tag{4.44}$$

Then for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)\gamma_0\mathrm{SL}_2(\mathbb{Z})$ there exists a positive integer k and a Taylor series $h(q)$ in powers of $q^{\frac{1}{k}}$ such that for $\tau \in \mathbb{H}$ we have

$$(c\tau + d)^{-\frac{1}{2}\sum_{\delta|M} r_\delta} g_{m,t}(\gamma\tau) = h(q) q^{p(\gamma_0)}.$$

Proof. We write $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \gamma_N\gamma_0\gamma_\infty$ where $\gamma_N = \begin{pmatrix} a_N & b_N \\ c_N & d_N \end{pmatrix} \in \Gamma_0(N), \gamma_\infty = \begin{pmatrix} 1 & b_\infty \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ and $\gamma_0 = \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} \in \Gamma$. Then

$$g_{m,t}(\gamma\tau) = \frac{1}{m} \sum_{\lambda=0}^{m-1} C_1(\lambda) \prod_{\delta|M} \eta^{r_\delta} \left(\frac{\delta(\gamma\tau + \kappa\lambda)}{m} \right)$$

(by (4.5)) with suitably chosen $C_1 : \{0, \ldots, m-1\} \to \mathbb{C}$)

$$\begin{aligned} =&(c_N(\gamma_0\gamma_\infty\tau) + d_N)^{\frac{1}{2}\sum_{\delta|M} r_\delta} \\ &\cdot \frac{1}{m} \sum_{\mu(\lambda)=0}^{m-1} C_2(\mu(\lambda)) \prod_{\delta|M} \eta^{r_\delta} \left(\frac{\delta(\gamma_0\gamma_\infty\tau + \kappa\mu(\lambda))}{m} \right) \end{aligned}$$

(by (4.39) with suitably chosen $C_2 : \{0, \ldots, m-1\} \to \mathbb{C}$)

$$\begin{aligned} =&((c_N(\gamma_0\gamma_\infty\tau) + d_N)(c_0(\gamma_\infty\tau) + d_0))^{\frac{1}{2}\sum_{\delta|M} r_\delta} \\ &\cdot \frac{1}{m} \sum_{\mu(\lambda)=0}^{m-1} C_3(\mu(\lambda)) \prod_{\delta|M} \eta^{r_\delta} \left(\frac{\gcd^2(\delta(a_0 + \kappa\mu(\lambda)c_0), mc_0)\tau + C_4(\mu(\lambda), \delta)}{\delta m} \right) \end{aligned}$$

(by (4.38) with suitably chosen $C_3 : \{0, \ldots, m-1\} \to \mathbb{C}$ and $C_4 : \{0, \ldots, m-1\} \times \{\delta \mid \delta \in \mathbb{N}, \delta|M\} \to \mathbb{C}$)

$$\begin{aligned} =&(c\tau + d)^{\frac{1}{2}\sum_{\delta|M} r_\delta} \\ &\cdot \frac{1}{m} \sum_{\mu(\lambda)=0}^{m-1} C_3(\mu(\lambda)) \prod_{\delta|M} \eta^{r_\delta} \left(\frac{\gcd^2(\delta(a_0 + \kappa\mu(\lambda)c_0), mc_0)\tau + C_4(\mu(\lambda), \delta)}{\delta m} \right) \end{aligned}$$

(because of $\begin{pmatrix} a_N & b_N \\ c_N & d_N \end{pmatrix} \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} \begin{pmatrix} 1 & b_\infty \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$)

$$=(c\tau+d)^{\frac{1}{2}\sum_{\delta|M} r_\delta} \sum_{\mu(\lambda)=0}^{m-1} C_3(\mu(\lambda)) q^{p(\gamma_0,\mu(\lambda))} h(\mu(\lambda),q)$$

(where for each $\mu(\lambda)$, $h(\mu(\lambda),q)$ is a Taylor series in powers of $q^{24p(\gamma_0,\mu(\lambda))}$ by (4.42))

$$=(c\tau+d)^{\frac{1}{2}\sum_{\delta|M} r_\delta} q^{p(\gamma_0)} h(q)$$

(with $h(q) := q^{p(\gamma_0)} \sum_{\mu(\lambda)=0}^{m-1} C_3(\mu(\lambda)) q^{p(\gamma_0,\mu(\lambda))-p(\gamma_0)} h(\mu(\lambda),q)$).

□

Lemma 4.29. *Let $N \in \mathbb{N}^*$, $(a_\delta) \in R(N)$, $f(\tau) := \prod_{\delta|N} \eta^{a_\delta}(\delta\tau)$, and define the mapping $p^* : \Gamma \to \mathbb{C}$ by $p^*\left(\begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix}\right) := \frac{1}{24}\sum_{\delta|N} \frac{a_\delta \gcd^2(\delta,c_0)}{\delta}$. Then for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ there exists an integer k and a Taylor series $h^*(q)$ in powers of $q^{\frac{1}{k}}$ such that*

$$(c\tau+d)^{-\frac{1}{2}\sum_{\delta|N} a_\delta} f(\gamma\tau) = h^*(q) q^{p^*(\gamma)}. \tag{4.45}$$

Furthermore, for $\gamma_1 \in \Gamma$ and $\gamma_2 \in \Gamma_0(N)\gamma_1 \mathrm{SL}_2(\mathbb{Z})$ we have $p^*(\gamma_1) = p^*(\gamma_2)$.

Proof. Let $w_\delta := \gcd(\delta a, c)\frac{\gcd(\delta a,c)\tau+\delta b x_\delta + d y_\delta}{\delta}$ where $x_\delta, y_\delta \in \mathbb{Z}$ such that $a\delta x_\delta + c y_\delta = \gcd(a\delta, c)$ for any fixed $\delta|N$ with $\delta > 0$. Then

$$(c\tau+d)^{-\frac{1}{2}\sum_{\delta|N} a_\delta} \prod_{\delta|N} \eta^{a_\delta}(\delta\gamma\tau) =(c\tau+d)^{-\frac{1}{2}\sum_{\delta|N} a_\delta} \prod_{\delta|N} \eta^{a_\delta}\left(\frac{\frac{\delta a}{\gcd(\delta a,c)} w_\delta - y_\delta}{\frac{c}{\gcd(\delta a,c)} w_\delta + x_\delta}\right)$$
$$=C\prod_{\delta|N} \eta^{a_\delta}(w_\delta)$$

(by (2.28)) with suitably chosen $C \in \mathbb{C}$)

$$=Cq^{p^*(\gamma)} \prod_{\delta|N} h^*(\delta,q)$$

(by (2.26) for some Taylor series $h^*(\delta,q)$ where $\delta|N$ (with constant term 1)). This proves (4.45).

To prove the remaining part of Lemma 4.29 let $\gamma_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\gamma_2 = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. Because of $\gamma_2 \in \Gamma_0(N)\gamma_1\mathrm{SL}_2(\mathbb{Z})$ we have that $\gamma_2 = \gamma_N\gamma_1\gamma_\infty$ for some $\gamma_N = \begin{pmatrix} a' & b' \\ c'N & d' \end{pmatrix} \in \Gamma_0(N)$ and $\gamma_\infty = \begin{pmatrix} 1 & b_\infty \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$. This shows that $C = ac'N + d'c$ and clearly $\gcd(d', c'N) = 1$ because of $a'd' - c'Nd' = 1$. For $\delta|N$ this implies that $\gcd(\delta, C) = \gcd(\delta, ac'\delta\frac{N}{\delta} + d'c) = \gcd(\delta, d'c) = \gcd(\delta, c)$. By this we have shown that the sums $p^*(\gamma_1)$ and $p^*(\gamma_2)$ have the same summands which proves that they are identical. □

Theorem 4.30. *Let $(m, M, N, (r_\delta)) \in \Delta$, $t \in \mathbb{Z}$ with $0 \le t \le m-1$, p be as in Lemma 4.28, (a_δ) and p^* be as in Lemma 4.29, and $\gamma_0 \in \Gamma$. Then for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)\gamma_0\mathrm{SL}_2(\mathbb{Z})$ the expression*

$$q^{-(p(\gamma_0)+p^*(\gamma_0))}(c\tau+d)^{-\frac{1}{2}\sum_{\delta|M} r_\delta - \frac{1}{2}\sum_{\delta|N} a_\delta} g_{m,t}(\gamma\tau) \prod_{\delta|N} \eta^{a_\delta}(\delta(\gamma\tau))$$

finds a representation as a Taylor series in powers of $q^{\frac{1}{k}}$ for some positive integer k.

Proof. By Lemmas 4.28 and 4.29, for each $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)\gamma_0\mathrm{SL}_2(\mathbb{Z})$ there exists a positive integer k, and Taylor series $h(q)$ and $h^*(q)$ in powers of $q^{\frac{1}{k}}$ such that

$$(c\tau+d)^{-\frac{1}{2}(\sum_{\delta|M} r_\delta + \sum_{\delta|N} a_\delta)} g_{m,t}(\gamma\tau) \prod_{\delta|N} \eta^{a_\delta}(\delta(\gamma\tau)) = h(q)h^*(q)q^{p(\gamma_0)+p^*(\gamma_0)}.$$

□

Lemma 4.31. *Let $F : \mathbb{H} \to \mathbb{C}$ be a mapping, k an integer, and l a positive integer. Assume that for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ there exists a positive integer n and a Taylor series $h(\gamma, q)$ in powers of $q^{\frac{1}{n}}$ such that for all $\tau \in \mathbb{H}$ the relation $(c\tau+d)^{-k}F(\gamma\tau) = h(\gamma, q)$ holds. Then for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ there exists a positive integer n' and a Taylor series $h^*(\gamma, q)$ in powers of $q^{\frac{1}{n'}}$ such that for all $\tau \in \mathbb{H}$ the relation $(c\tau+d)^{-k}F(l(\gamma\tau)) = h^*(\gamma, q)$ holds.*

Proof. We apply Lemma 4.8 with $f(\tau) = F(\tau)$, $g(\gamma, \tau) = h(\gamma, q)$, $\xi = \begin{pmatrix} al & bl \\ c & d \end{pmatrix}$, $g := \gcd(al, c)$, and x, y some integers such that $alx + cy = g$. As a consequence we have that

$$\left(\frac{g}{l}(c\tau+d)\right)^{-k} f(l(\gamma\tau)) = h\left(\begin{pmatrix} \frac{al}{g} & -y \\ \frac{c}{g} & x \end{pmatrix}, q^{\frac{g^2}{l}} e^{\frac{2\pi i g}{l}(blx+dy)}\right).$$

Choosing $n' = \frac{g^2}{l}n$ and

$$h^*(\gamma, q) = (g/l)^k h\left(\begin{pmatrix} \frac{al}{g} & -y \\ \frac{c}{g} & x \end{pmatrix}, q^{\frac{g^2}{l}} e^{\frac{2\pi i g}{l}(blx+dy)}\right)$$

concludes the proof. □

Definition 4.32. *We define*

$$\mathbb{Z}_n^* := \{[x]_n \in \mathbb{Z}_n | \gcd(x,n) = 1\},$$

and

$$\mathbb{S}_n := \{y^2 | y \in \mathbb{Z}_n^*\}.$$

Lemma 4.33. *For all integers $w \geq 2$ we have $24\sum_{s\in\mathbb{S}_w} s = [0]_w$. If $\gcd(w,6) = 1$ then $\sum_{s\in\mathbb{S}_w} s = [0]_w$.*

Proof. If $\gcd(w,6) = 1$ then $[2^2]_w \in \mathbb{S}_w$, which implies that $[2^2]_w \sum_{s\in\mathbb{S}_w} s = \sum_{s\in\mathbb{S}_w} s$. This is because multiplication by an element of $\mathbb{S}_w$ just permutes the summands. Consequently $[2^2-1]_w \sum_{s\in\mathbb{S}_w} s = [0]_w$, but 2^2-1 is invertible modulo w and we can conclude that $\sum_{s\in\mathbb{S}_w} s = [0]_w$. If we assume that $w = 2^s3^t$ then $[5^2-1]_w \sum_{s\in\mathbb{S}_w} s = [0]_w$. Next consider a general $w = 2^s3^tu, \gcd(u,6) = 1$. We have a ring isomorphism $\phi : \mathbb{Z}_{2^s3^tu} \to \mathbb{Z}_{2^s3^t} \times \mathbb{Z}_u$ given by $\phi([x]_{2^s3^tu}) = ([x]_{2^s3^t}, [x]_u)$. Obviously,

$$\phi\left([24]_w \sum_{s\in\mathbb{S}_w} s\right) = \phi([24]_w) \sum_{\substack{s \in \mathbb{S}_{2^s3^t},\\ s' \in \mathbb{S}_u}} (s,s')$$

$$=\phi([24]_w)\left([|\mathbb{S}_u|]_{2^s3^t} \sum_{s\in\mathbb{S}_{2^s3^t}} s, [|\mathbb{S}_{2^s3^t}|]_u \sum_{s\in\mathbb{S}_u} s\right)$$

$$=\left([24|\mathbb{S}_u|]_{2^s3^t} \sum_{s\in\mathbb{S}_{2^s3^t}} s, [24|\mathbb{S}_{2^s3^t}|]_u \sum_{s\in\mathbb{S}_u} s\right) = ([0]_{2^s3^t}, [0]_u).$$

Since ϕ is an isomorphism its kernel is $\{0\}$, which proves the lemma. □

Definition 4.34. *For $m, M \in \mathbb{N}^*$, $(r_\delta) \in R(M)$ and $t \in \mathbb{N}$ with $0 \leq t \leq m-1$ we define the map $\overline{\odot} : \mathbb{S}_{24m} \times \{0,\dots,m-1\} \to \{0,\dots,m-1\}$ where the image $[s]_{24m}\overline{\odot}t$ is uniquely determined by the relation $[s]_{24m}\overline{\odot}t \equiv_m ts + \frac{s-1}{24}\sum_{\delta|M}\delta r_\delta$. We also define*

$$P(t) := \{[s]_{24m}\overline{\odot}t | [s]_{24m} \in \mathbb{S}_{24m}\}. \tag{4.46}$$

Lemma 4.35. *Let m, t, M, N be positive integers with $0 \leq t \leq m-1$ such that (4.6) holds. Let $(r_\delta) \in R(M)$, $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)^*$, and $\odot$ as in (4.35). Moreover, define*

$$w := \frac{24m}{\gcd(\kappa(24t + \sum_{\delta|M}\delta r_\delta), 24m)}.$$

Then the following statements hold:

(i) $\gamma \odot t = [a^2]_{24m}\overline{\odot}t$.

(ii) $[x]_{24m}\overline{\odot}t = [y]_{24m}\overline{\odot}t$ *iff* $x \equiv_w y$ *for all* $x, y \in \mathbb{Z}$.

(iii) $P(t) = \{\gamma \odot t | \gamma \in \Gamma_0(N)^*\}$.

(iv) For $[s]_{24m} \in \mathbb{S}_{24m}$ *we have*

$$P(t) = \{[s]_{24m}\overline{\odot}t' | t' \in P(t)\}.$$

(v) $\chi := \prod_{t' \in P(t)} e^{2\pi i \frac{ab(1-m^2)(24t'+\sum_{\delta|M}\delta r_\delta)}{24m}}$ *is a 24-th root of unity.*

Proof. ***Proof of (i):*** If $\gamma \in \Gamma_0(N)^*$ then $\gcd(a,6)=1$. By (4.6) and because of $\gcd(a,N)=1$ we also have that $\gcd(a,m)=1$, hence $\gcd(a,24m)=1$. This means that $[a^2]_{24m} \in \mathbb{S}_{24m}$.

Proof of (ii): Assume that $[s_1]_{24m}\overline{\odot}t = [s_2]_{24m}\overline{\odot}t$ for $[s_1]_{24m}, [s_2]_{24m} \in \mathbb{S}_{24m}$. Then

$$\kappa(s_1 t + \frac{s_1 - 1}{24}\sum_{\delta|M}\delta r_\delta) \equiv_m \kappa(s_2 t + \frac{s_2-1}{24}\sum_{\delta|M}\delta r_\delta) \tag{4.47}$$

because $\gcd(\kappa, m) = 1$. Consequently (4.47) is equivalent to

$$\kappa(24t + \sum_{\delta|M}\delta r_\delta)(s_1 - s_2) \equiv_{24m} 0 \tag{4.48}$$

and (4.48) is equivalent to

$$s_1 - s_2 \equiv_w 0.$$

Proof of (iii): By *(i)* we have

$$\begin{aligned}&\{\gamma \odot t | \gamma \in \Gamma_0(N)^*\}\\ &=\{[a^2]_{24m}\overline{\odot}t | \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)^*\} \subseteq \{[s]_{24m}\overline{\odot}t | [s]_{24m} \in \mathbb{S}_{24m}\}.\end{aligned}$$

To show the other inclusion let $[s]_{24m} \in \mathbb{S}_{24m}$. By definition there exists an $[a]_{24m} \in \mathbb{Z}^*_{24m}$ such that $[s]_{24m} = [a^2]_{24m}$. Because $\gcd(a,24)=1$ we have $\gcd(a,6)=1$. We want to show that there exists a λ such that $\gcd(a+24\lambda m, N) = 1$ because then there exist integers x and y such that

$$\begin{pmatrix} a + 24\lambda m & -y \\ N & x \end{pmatrix} \odot t = [s]_{24m}\overline{\odot}t$$

and the other inclusion is shown. It is sufficient to show that for each prime p with $p|N$ there exists an integer λ_p s.t. $\gcd(a + 24\lambda_p m, p) = 1$ because then by the Chinese remainder there exists a λ s.t. for all $p|N$ we have that $\lambda \equiv_p \lambda_p$. If p is such that $p|N$ and $p|24m$ then we simply choose $\lambda_p = 0$. If $p|N$ and $p \nmid 24m$ and $p|a$ then choose $\lambda_p = 1$, if $p \nmid a$ choose $\lambda_p = 0$.

Proof of (iv): We have to show that given $[s]_{24m} \in \mathbb{S}_{24m}$, the mapping $[s]_{24m}\overline{\odot}t : P(t) \to P(t)$ is a bijection. This is clear because the inverse is $[s]^{-1}_{24m}\overline{\odot}t$.

Proof of (v): Let S be a subset of $\mathbb{S}_{24m}$ such that for $[r_1]_{24m}, [r_2]_{24m} \in S$ we have $r_1 \not\equiv_w r_2$, and such that for all $[s]_{24m} \in \mathbb{S}_{24m}$ there exists $[r]_{24m} \in S$ with $r \equiv_w s$. Then by *(ii)*:

$$P(t) = \{ts + \frac{s-1}{24}\sum_{\delta|M}\delta r_\delta | [s]_{24m} \in S\}.$$

It is straight-forward to prove that the set S gives a complete set of representatives of $\mathbb{S}_w$. Next note that

$$\begin{aligned}\chi &= \prod_{t'\in P(t)} e^{2\pi i\frac{ab(1-m^2)(24t'+\sum_{\delta|M}\delta r_\delta)}{24m}}\\ &= \prod_{[s]_w\in\mathbb{S}_w} e^{2\pi i\frac{sab(1-m^2)(24t+\sum_{\delta|M}\delta r_\delta)}{24m}}\\ &= e^{\frac{2\pi iab(1-m^2)(24t+\sum_{\delta|M}\delta r_\delta)}{24m}\sum_{[s]_w\in\mathbb{S}_w} s}.\end{aligned}$$

Since $\kappa|(1-m^2)$ and $24\sum_{s\in\mathbb{S}_w} s \equiv_w 0$, by Lemma 4.33 we conclude that χ is a 24-th root of unity.

□

4.4 Proving Congruences by Sturm's Theorem

4.4.1 Proof Strategy

Let M be a positive integer and $r=(r_\delta)\in R(M)$. Let $f(\tau,r)=\prod_{\delta|M}\prod_{n=1}^\infty(1-q^{\delta n})^{r_\delta}=\sum_{n=0}^\infty a(n)q^n$ be as in Definition 4.6. Let m and u be positive integers and t an integer satisfying $0\le t\le m-1$. We want to prove or disprove the conjecture $a(mn+t)\equiv_u 0, n\ge 0$. It is convenient to introduce the following definition:

Definition 4.36. *For u a positive integer and $c(\tau):=\sum_{n=0}^\infty c(n)q^n$ a power series we define* $\mathrm{Ord}_u(c(\tau)):=\inf\{n \mid u\nmid c(n)\}$*; we write $c(\tau)\equiv_u 0$ if* $\mathrm{Ord}_u(c(\tau))=\infty$.

First note that if $c_1(\tau)$ and $c_2(\tau)$ are power series in q and if p is a prime number then the relation $c_1(\tau)c_2(\tau)\equiv_p 0$ implies either $c_1(\tau)\equiv_p 0$ or $c_2(\tau)\equiv_p 0$.

Proposition 4.37. *Let A,u be integers. Assume that for all divisors u' of u and all primes p dividing u/u' we have*

$$u'|A \textit{ implies } (u'p)|A. \tag{4.49}$$

Then $u|A$.

Proof. To prove this statement assume that $u\nmid A$ then there exists a prime q and $\alpha\in\mathbb{N}$ such that $q^\alpha|A$, $q^{\alpha+1}\nmid A$ and $q^{\alpha+1}|u$. Set $u'=p^\alpha$ and $p=q$ then by (4.49) $p^{\alpha+1}|A$ contradicting $q^{\alpha+1}\nmid A$. □

Suppose that we already know that $u'|a(mn+t)$ for all divisors u' of u and all $n\ge 0$. If we can prove that $\frac{a(mn+t)}{u'}\equiv_p 0$ for any prime p dividing u/u', then by Proposition 4.37 we have $a(mn+t)\equiv_u 0, n\ge 0$. In other words, our aim is to prove

$$\frac{1}{u'}\sum_{n=0} a(mn+t)q^n\equiv_p 0,$$

which is equivalent to proving

$$\left(\frac{1}{u'}\sum_{n=0}^\infty a(mn+t)q^n\right)^{24}\equiv_p 0,$$

which in turn is equivalent to proving

$$H(\tau) := \left(\frac{1}{u'}\sum_{n=0}^{\infty} a(mn+t)q^n\right)^{24} h_1(\tau) \equiv_p 0, \tag{4.50}$$

where $h_1(\tau)$ is a power series in q with $h_1(\tau) \not\equiv_p 0$. We will choose $h_1(\tau)$ in such a way that $H(\tau)$ becomes a modular form of weight k for some subgroup G of Γ and some positive integer k. Then by Theorem 4.38 below it is sufficient to show that $\mathrm{Ord}_p(H(\tau)) > \frac{k}{12}[\Gamma : G]$ in order to conclude that $H(\tau) \equiv_p 0$ and hence $a(mn+t)/u' \equiv_p 0, n \geq 0$. In order to derive a bound for $\mathrm{Ord}_p(H(\tau))$ we will use that for given power series $c_1(\tau)$ and $c_2(\tau)$ with $\mathrm{Ord}_p(c_1(\tau)) \geq b_1$ for some $b_1 \in \mathbb{N}$ and $\mathrm{Ord}_p(c_2(\tau)) \geq b_2$ for some $b_2 \in \mathbb{N}$ then $\mathrm{Ord}_p(c_1(\tau)c_2(\tau)) \geq b_1 + b_2$.

We will consider two types of congruences:
Type 1: $a(mn+t) \equiv_u 0, n \geq 0$;
Type 2: $a(mn+t') \equiv_u 0, t' \in P(t), n \geq 0$.

Obviously congruences of Type 2 are special cases of congruences of Type 1 but we have observed that one can be "$|P(t)|$ times faster in practical computations" when considering congruences of Type 2. At the current stage this observation relies on experimental data and is not yet proved; for a comparison see Example 4.42.

Before entering a detailed discussion of how to prove congruences of Type 1 and 2 we recall a theorem of J. Sturm.

Theorem 4.38 (Sturm [40])**.** *Let k be an integer and $c(\tau) = \sum_{n=0}^{\infty} c(n)q^n$ a modular form of weight k for a subgroup G of Γ. Assume that $\mathrm{Ord}_u(c(\tau)) > \frac{k}{12}[\Gamma : G]$ then $c(\tau) \equiv_u 0$.*

For setting up the lemmas in the next two subsections we have collected valuable ideas from [39, p. 134, Cor. 9.1.4], attributed to Buzzard.

Proving Congruences of Type 1

Lemma 4.39. *Let $(m, M, N, t, (r_\delta) = r) \in \Delta^*$, $(a_\delta) \in R(N)$, n be the number of double cosets in $\Gamma_0(N)\backslash\Gamma/\mathrm{SL}_2(\mathbb{Z})$ and $\{\gamma_1, \ldots, \gamma_n\} \subseteq \Gamma$ a complete set of representatives of the double cosets $\Gamma_0(N)\backslash\Gamma/\mathrm{SL}_2(\mathbb{Z})$. Assume that $p^*(\gamma_i) + |P(t)|p(\gamma_i) \geq 0$ for $1 \leq i \leq n$ and with p and p^* as in the lemmas 4.28 and 4.29. Next define:*

$$\begin{aligned}\nu :=& \frac{1}{24}\left(\left(\sum_{\delta|N} a_\delta + |P(t)|\sum_{\delta|M} r_\delta\right)[\Gamma : \Gamma_0(N)] - \sum_{\delta|N}\delta a_\delta\right)\\ &- \frac{1}{m}\sum_{t'\in P(t)} t' - \frac{|P(t)|}{24m}\sum_{\delta|M}\delta r_\delta.\end{aligned}$$

Then for $f(\tau, r) = \sum_{n=0}^{\infty} a(n)q^n$ and $g_{m,t}(\tau, r)$ as in Definition 4.6 the following statements hold:

(i) $\{(\prod_{t'\in P(t)} g_{m,t'}(\tau))(\prod_{\delta|N}\eta^{a_\delta}(\delta\tau))\}^{24}$ is a modular form for the group $\Gamma_0(N)$ of weight $12\sum_{\delta|N} a_\delta + 12|P(t)|\sum_{\delta|M} r_\delta$.

(ii) For any $u \in \mathbb{N}^*$ we have that if $\mathrm{Ord}_u(\sum_{n=0}^{\infty} a(mn+t)q^n) > \nu$ then $\sum_{n=0}^{\infty} a(mn+t)q^n \equiv_u 0$.

Proof. ***Proof of (i):*** Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)^*$ and let χ be as in *(v)* in Lemma 7.6. Then:

$$\left(\prod_{t'\in P(t)} g_{m,t'}(\gamma\tau)\right)^{24} =(\beta(\gamma,0)^{|P(t)|}\chi)^{24}(c\tau+d)^{12|P(t)|\sum_{\delta|M} r_\delta}\left(\prod_{t'\in P(t)} g_{m,[a^2]_{24m}\odot t'}(\tau)\right)^{24}$$

(by (4.36), Remark 4.14 and *(i)* in Lemma 7.6)

$$=(c\tau+d)^{12|P(t)|\sum_{\delta|M} r_\delta}\left(\prod_{t'\in P(t)} g_{m,t'}(\tau)\right)^{24} \tag{4.51}$$

(by *(iv)* and *(v)* in Lemma 7.6).

By (2.28) we get:

$$\left(\prod_{\delta|N}\eta^{a_\delta}\left(\frac{a(\delta\tau)+b\delta}{\frac{c}{\delta}(\delta\tau)+d}\right)\right)^{24} = (c\tau+d)^{12\sum_{\delta|N} a_\delta}\left(\prod_{\delta|N}\eta^{a_\delta}(\delta\tau)\right)^{24}. \tag{4.52}$$

By (4.51) and (4.52) we obtain:

$$\begin{aligned}&\left(\left(\prod_{t'\in P(t)} g_{m,t'}(\gamma\tau)\right)\left(\prod_{\delta|N}\eta^{a_\delta}(\delta(\gamma\tau))\right)\right)^{24}\\ =&(c\tau+d)^{12\sum_{\delta|N} a_\delta+12|P(t)|\sum_{\delta|M} r_\delta}\left(\left(\prod_{t'\in P(t)} g_{m,t'}(\tau)\right)\left(\prod_{\delta|N}\eta^{a_\delta}(\delta\tau)\right)\right)^{24}\end{aligned} \tag{4.53}$$

We want to prove that

$$V(\tau):=\left(\left(\prod_{t'\in P(t)} g_{m,t'}(\tau)\right)\left(\prod_{\delta|N}\eta^{a_\delta}(\delta\tau)\right)\right)^{24} \tag{4.54}$$

is a modular form of weight $12\sum_{\delta|N} a_\delta + 12|P(t)|\sum_{\delta|M} r_\delta$ for the group $\Gamma_0(N)$. Clearly, condition *(i)* of Definition 2.6 is satisfied. Also condition (2) is satisfied because of (4.53) and because of Lemma 2.35. The only assertion left to verify is condition (3). Let $\gamma \in \Gamma_0(N)\gamma_i\mathrm{SL}_2(\mathbb{Z}), i \in \{1,\ldots,n\}$, then by Lemmas 4.29 and 4.28 there exists a positive integer k such that $h_1(q),\ldots,h_{|P(t)|}(q), h^*(q)$ are Taylor series in powers of $q^{\frac{1}{k}}$ such that:

$$(c\tau+d)^{-12\sum_{\delta|N} a_\delta+12|P(t)|\sum_{\delta|M} r_\delta}V(\gamma\tau) = q^{24p^*(\gamma_i)+24|P(t)|p(\gamma_i)}h^*(q)\prod_{j=1}^{|P(t)|} h_j(q).$$

But by assumption $p^*(\gamma_i)+|P(t)|p(\gamma_i)\geq 0$, so also condition *(iii)* of Definition 2.6 is satisfied.

Proof of (ii): Assume that $a(mn+t) \equiv_{u'} 0$ for some integer u' that divides u. Let $l \in \mathbb{N}^*$ be such that

$$h_0(\tau) := \frac{1}{l}\left(\prod_{t'\in P(t), t'\neq t}\left(\sum_{n=0}^{\infty} a(mn+t')q^n\right)\right)^{24}\left(\prod_{n=1}^{\infty}\prod_{\delta|N}(1-q^{\delta n})^{a_\delta}\right)^{24},$$

is a power series with integral coefficients such that for any prime p we have $h_0(\tau) \not\equiv_p 0$. Then $\frac{V(\tau)}{lu'^{24}}$ in (4.54) can be written as:

$$\begin{aligned}\frac{V(\tau)}{lu'^{24}} =& \frac{1}{lu'^{24}} q^{\sum_{t'\in P(t)}(24t'+\sum_{\delta|M}\delta r_\delta)/m}\left(\prod_{t'\in P(t)}\left(\sum_{n=0}^{\infty} a(mn+t')q^n\right)\right)^{24}\\ &\cdot q^{\sum_{\delta|N}\delta a_\delta}\left(\prod_{n=1}^{\infty}\prod_{\delta|N}(1-q^{\delta n})^{a_\delta}\right)^{24}\\ =& \left(\frac{1}{u'}\sum_{n=0}^{\infty} a(mn+t)q^n\right)^{24} q^{\frac{1}{m}\left(24\sum_{t'\in P(t)} t'+|P(t)|\sum_{\delta|M}\delta r_\delta\right)+\sum_{\delta|M}\delta a_\delta} h_0(\tau).\end{aligned}$$

If we choose $h_1(\tau) := q^{\frac{1}{m}\left(24\sum_{t'\in P(t)} t'+|P(t)|\sum_{\delta|M}\delta r_\delta\right)+\sum_{\delta|M}\delta a_\delta} h_0(\tau)$ then in order to prove $\frac{a(mn+t)}{u'} \equiv_p 0, n \geq 0$ for some prime p dividing u/u' we need to prove

$$\frac{V(\tau)}{lu'^{24}} = \left(\frac{1}{u'}\sum_{n=0}^{\infty} a(mn+t)q^n\right)^{24} h_1(\tau) \equiv_p 0, \tag{4.55}$$

which is exactly (4.50) above. From the above derivation we note that

$$\mathrm{Ord}_p(h_1(\tau)) \geq \frac{1}{m}\left(24\sum_{t'\in P(t)} t' + |P(t)|\sum_{\delta|M}\delta r_\delta\right) + \sum_{\delta|M}\delta a_\delta \tag{4.56}$$

which is an integer because $V(\tau+1) = V(\tau)$. Because

$$\mathrm{Ord}_p\left(\left(\frac{1}{u'}\sum_{n=0}^{\infty} a(mn+t)q^n\right)^{24}\right) > 24\nu,$$

by assumption, we have that

$$\mathrm{Ord}_p\left(\frac{1}{lu'^{24}}V(\tau)\right) > \left(\sum_{\delta|N} a_\delta + |P(t)|\sum_{\delta|M} r_\delta\right)[\Gamma : \Gamma_0(N)]$$

because of (4.55), (4.56) and by substituting according to the definition of ν. Theorem 4.38 allows us to conclude that $\frac{1}{lu'^{24}}V(\tau) \equiv_p 0$ which implies that $\frac{a(mn+t)}{u'} \equiv_p 0, n \geq 0$.

□

We display the results of this subsection in the form of an algorithm description. Our input to the algorithm is $m, M, u \in \mathbb{N}^*$, $t \in \{0, \dots, m-1\}$ and $(r_\delta) \in R(M)$. The output of the algorithm is true of false depending on if $a(mn+t) \equiv_u 0$ for all $n \geq 0$ where $\sum_{n=0}^\infty a(n)q^n = \prod_{n=1}^\infty \prod_{\delta|M}(1-q^{\delta n})^{r_\delta}$. The steps are as follows:

- Compute the minimal N such that $(m, M, N, t, (r_\delta)) \in \Delta^*$.
- Compute a complete set of representatives $\gamma_1, \dots, \gamma_d$ for the double cosets $\Gamma_0(N)\backslash\Gamma/\mathrm{SL}_2(\mathbb{Z})$.
- Compute $(a_\delta) \in R(N)$ such that $p^*(\gamma_i) + |P(t)|p(\gamma_i) \geq 0, i \in \{1, \dots, d\}$, p and p^* are as in Lemmas 4.28 and 4.29.
- Let ν be as in Lemma 4.40. If $a(mn+t) \equiv_u 0$ for $n \in \{0, \dots, \nu\}$ then return true otherwise return false.

Proving Congruences of Type 2

Lemma 4.40. *Let u be a positive integer, $(m, M, N, t, (r_\delta)) \in \Delta^*$, $(a_\delta) \in R(N)$, n be the number of double cosets in $\Gamma_0(N)\backslash\Gamma/\mathrm{SL}_2(\mathbb{Z})$ and $\{\gamma_1, \dots, \gamma_n\} \subseteq \Gamma$ a complete set of representatives of the double cosets $\Gamma_0(N)\backslash\Gamma/\mathrm{SL}_2(\mathbb{Z})$. Assume that $p(\gamma_i) + p^*(\gamma_i) \geq 0, i = 1, \dots, n$, with p and p^* as in the Lemmas 4.28 and 4.29. Furthermore, let $l := \frac{24m}{\kappa}$, $t_{\min} := \min_{t' \in P(t)} t'$ and*

$$\nu := \frac{1}{24}\left(\left(\sum_{\delta|N} a_\delta + \sum_{\delta|M} r_\delta\right)[\Gamma : \Gamma_0(N)] - \sum_{\delta|N} \delta a_\delta\right) - \frac{1}{24m}\sum_{\delta|M}\delta r_\delta - \frac{t_{\min}}{m}.$$

Then

(i) $(\prod_{\delta|N}\eta^{a_\delta}(l\delta\tau)\sum_{t'\in P(t)} g_{m,t'}(l\tau))^{24}$ *is a modular form of weight* $12(\sum_{\delta|M} r_\delta + \sum_{\delta|N} a_\delta)$ *for the group* $\Gamma_0(lN)$.

(ii) If $\mathrm{Ord}_u(\sum_{n=0}^\infty a(mn+t')q^n) > \nu$ *for all* $t' \in P(t)$ *then* $\sum_{n=0}^\infty a(mn+t')q^n \equiv_u 0$ *for all* $t' \in P(t)$.

Proof. **Proof of *(i)*:** Clearly condition *(i)* of Definition 2.6 is satisfied.

In order to prove condition *(ii)* we only need to consider $\gamma \in \Gamma_0(lN)^*$ because of Lemma 2.35. Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(lN)^*$ then by Theorem 4.22 the following relation holds:

$$\begin{aligned} g_{m,t}(l(\gamma\tau)) =& \beta\left(\begin{pmatrix} a & lb \\ \frac{c}{l} & d \end{pmatrix}, 0\right)(-i(c\tau+d))^{\frac{\sum_{\delta|M} r_\delta}{2}} \\ & \cdot e^{2\pi i\frac{abl(1-m^2)(24t+\sum_{\delta|M}\delta r_\delta)}{24m}} \cdot g_{m,[a^2]_{24m}\odot t}(l\tau) \\ =& \beta\left(\begin{pmatrix} a & lb \\ \frac{c}{l} & d \end{pmatrix}, 0\right)(-i(c\tau+d))^{\frac{\sum_{\delta|M} r_\delta}{2}} g_{m,[a^2]_{24m}\odot t}(l\tau), \end{aligned} \tag{4.57}$$

because

$$e^{\frac{2\pi iabl(1-m^2)(24t+\sum_{\delta|M}\delta r_\delta)}{24m}} = e^{\frac{2\pi iab(1-m^2)(24t+\sum_{\delta|M}\delta r_\delta)}{\kappa}} = 1.$$

In this derivation we have substituted for l and used that $\kappa|(1-m^2)$.

By (4.57), Remark 4.14 and *(iv)*, *(i)* of Lemma 7.6 we obtain:

$$\left(\sum_{t'\in P(t)} g_{m,t'}(l(\gamma\tau))\right)^{24} = ((c\tau+d))^{12\sum_{\delta|M} r_\delta}\left(\sum_{t'\in P(t)} g_{m,t'}(l\tau)\right)^{24}. \tag{4.58}$$

By (4.58) and (4.52) we obtain:

$$\begin{aligned}&\left(\prod_{\delta|N}\eta^{a_\delta}(\delta l(\gamma\tau))\sum_{t'\in P(t)} g_{m,t'}(l(\gamma\tau))\right)^{24}\\ &\quad= ((c\tau+d))^{12(\sum_{\delta|N}a_\delta+\sum_{\delta|M} r_\delta)}\left(\prod_{\delta|N}\eta^{a_\delta}(\delta l\tau)\sum_{t'\in P(t)} g_{m,t'}(l\tau)\right)^{24};\end{aligned} \tag{4.59}$$

hence condition *(ii)* of Definition 2.6 is satisfied.

In order to prove *(iii)* in Definition 2.6 fix a $t'\in P(t)$, and a $\gamma\in\Gamma_0(N)\gamma_i\mathrm{SL}_2(\mathbb{Z}), i\in\{1,\dots,n\}$. Then by Lemmas 4.28 and 4.29 there exist positive integers k, k' and Taylor series $h(q), h^*(q)$ in powers of $q^{\frac{1}{k}}$ and $q^{\frac{1}{k'}}$, respectively, such that

$$(c\tau+d)^{-\frac{1}{2}(\sum_{\delta|N}a_\delta+\sum_{\delta|M} r_\delta)} g_{m,t'}(\gamma\tau)\prod_{\delta|M}\eta^{a_\delta}(\delta(\gamma\tau)) = h(q)h^*(q)q^{p(\gamma_i)+p^*(\gamma_i)}.$$

Because of the positivity of $p(\gamma_i)+p^*(\gamma_i)$, there exists an positive integer j such that $h(q)h^*(q)q^{p(\gamma_i)+p^*(\gamma_i)}$ is a Taylor series in powers of $q^{\frac{1}{j}}$. Summarizing, we have proven that for all $t'\in P(t)$ and all γ there exists a positive integer k and a Taylor series $h(\gamma,q)$ such that

$$(c\tau+d)^{-\frac{1}{2}(\sum_{\delta|N}a_\delta+\sum_{\delta|M} r_\delta)} g_{m,t'}(\gamma\tau)\prod_{\delta|M}\eta^{a_\delta}(\delta(\gamma\tau)) = h(\gamma,q).$$

Then by Lemma 4.31 there exist positive integers $k'(t'), t'\in P(t)$ and Taylor series $h^*(t',\gamma,q)$, $t'\in P(t)$ in powers of $q^{1/k'(t')}$ such that

$$(c\tau+d)^{-\frac{1}{2}(\sum_{\delta|N}a_\delta+\sum_{\delta|M} r_\delta)} g_{m,t'}(l(\gamma\tau))\prod_{\delta|M}\eta^{a_\delta}(\delta l(\gamma\tau)) = h^*(t',\gamma,q).$$

This proves that

$$\begin{aligned}&(c\tau+d)^{-12(\sum_{\delta|N}a_\delta+\sum_{\delta|M} r_\delta)}\left(\prod_{\delta|N}\eta^{a_\delta}(\delta l(\gamma\tau))\sum_{t'\in P(t)} g_{m,t'}(l(\gamma\tau)\right)^{24}\\ &\qquad= \left(\sum_{t'\in P(t)} h^*(t',\gamma,q)\right)^{24}.\end{aligned}$$

So we have proven condition *(iii)* of Definition 2.6.

Proof of (ii): First we note that given positive integers u', ν' and a power series $c(\tau) := \sum_{n=0}^{\infty} c(n)q^n$ such that $\mathrm{Ord}_{u'}(c(\tau)) > \nu'$ we have that $\mathrm{Ord}_u\left(\sum_{n=0}^{\infty} c(n)q^{an+b}\right) > a\nu' + b$ for any positive integers a and b.

We have proven above that $V_2(\tau) := (\prod_{\delta|N} \eta^{a_\delta}(l\delta\tau) \sum_{t' \in P(t)} g_{m,t'}(l\tau))^{24}$ is a modular form of weight $12(\sum_{\delta|N} a_\delta + \sum_{\delta|M} r_\delta)$ on group $\Gamma_0(lN)$.

Let u' be a divisor of u and p a divisor of u/u'. Assume that $u'|a(mn+t')$ for $n \geq 0$ and $t' \in P(t)$. We have that

$$\frac{V_2(\tau)}{u'^{24}} = q^{\frac{24}{\kappa}\sum_{\delta|M}\delta r_\delta + l\sum_{\delta|N}\delta a_\delta + \frac{24^2}{\kappa}t_{\min}} \left(\sum_{t' \in P(t)} \sum_{n=0}^{\infty} \frac{a(mn+t')}{u'} q^{\frac{24}{\kappa}(mn+t'-t_{\min})}\right)^{24} \cdot \left(\prod_{n=1}^{\infty}\prod_{\delta|N}(1-q^{l\delta n})^{a_\delta}\right)^{24}.$$

For this rewriting we have used the definition of $g_{m,t}(\tau)$, the definition of l and that $\eta(\tau)$ can be written as an infinite product according to (2.26). We observe that

$$\mathrm{Ord}_p(V_2(\tau)/u'^{24}) > \frac{24}{\kappa}\sum_{\delta|M}\delta r_\delta + l\sum_{\delta|N}\delta a_\delta + \frac{24^2}{\kappa}t_{\min} + \frac{24^2}{\kappa}m\nu, \tag{4.60}$$

by looking at the above rewriting of $\frac{V_2(\tau)}{u'^{24}}$ and using the assumption that

$$\mathrm{Ord}_u\left(\sum_{n=0}^{\infty} a(mn+t')q^n\right) > \nu,$$

for $t' \in P(t)$. If we substitute for ν in (4.60) we obtain:

$$\mathrm{Ord}_p(V_2(\tau)/u'^{24}) > \left(\sum_{\delta|N} a_\delta + \sum_{\delta|M} r_\delta\right)[\Gamma : \Gamma_0(N)]l.$$

Next observe that $[\Gamma : \Gamma_0(N)]l = [\Gamma : \Gamma_0(Nl)]$ because there in no prime q such that $q|l$ and $q \nmid N$ together with (2.50). Next apply Theorem 4.38 and we obtain $\frac{V_2(\tau)}{u'^{24}} \equiv_p 0$. This completes the proof.

□

As in the previous subsection we display the results of this subsection in the form of an algorithm description. Our input to the algorithm is $m, M, u \in \mathbb{N}^*$, $t \in \{0, \ldots, m-1\}$ and $(r_\delta) \in R(M)$. The output of the algorithm is true of false depending on if $a(mn+t') \equiv_u 0$ for all $n \geq 0$ and $t' \in P(t)$ where $\sum_{n=0}^{\infty} a(n)q^n = \prod_{n=1}^{\infty}\prod_{\delta|M}(1-q^{\delta n})^{r_\delta}$. The steps are as follows:

- Compute the minimal N such that $(m, M, N, t, (r_\delta)) \in \Delta^*$.
- Compute a complete set of representatives $\gamma_1, \dots, \gamma_d$ for the double cosets $\Gamma_0(N)\backslash\Gamma/\mathrm{SL}_2(\mathbb{Z})$.
- Compute $(a_\delta) \in R(N)$ such that $p^*(\gamma_i) + p(\gamma_i) \geq 0, i \in \{1, \dots, d\}$, p and p^* are as in Lemmas 4.28 and 4.29.
- Let ν be as in Lemma 4.40. If $a(mn + t') \equiv_u 0$ for $n \in \{0, \dots, \nu\}$ and $t' \in P(t)$ then return true otherwise return false.

4.5 Examples

Example 4.41. The generating function for broken 2-diamonds according to Andrews and Paule [2] is given by

$$\prod_{n=1}^{\infty} \frac{(1-q^{2n})(1-q^{5n})}{(1-q^n)^3(1-q^{10n})} = \sum_{n=0}^{\infty} \Delta_2(n) q^n.$$

In this paper they state some conjectures about the congruence properties of this function such as

$$\Delta_2(10n+2) \equiv_2 0, n \geq 0, \tag{4.61}$$

and

$$\Delta_2(25n+14) \equiv_5 0, n \geq 0. \tag{4.62}$$

The first congruence (4.61) has been proven in [17] and the second (4.62) in [8]. Following our approach, alternative proofs can be provided as follows. Since Chan [8] also proved that $\Delta_2(25n+24) \equiv_5 0, n \geq 0$ we can consider this to be a congruence of Type 2; i.e., we will apply Lemma 4.40. We observe that $(25, 10, 10, 14, (r_1, r_2, r_5, r_{10}) = (-3, 1, 1, -1)) \in \Delta^*$. A complete set of representatives of the double cosets $\Gamma_0(10)\backslash\Gamma/\mathrm{SL}_2(\mathbb{Z})$ is given by

$$\gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \gamma_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \gamma_2 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \gamma_3 = \begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix}.$$

Also let $(a_1, a_2, a_5, a_{10}) = (73, -21, -15, 5) \in R(10)$. According to Lemma 4.40 we need to show that $p^*(\gamma_k) + p(\gamma_k) \geq 0, k = 0, 1, 2, 3$ which can be readily verified from the data below.

$$p^*(\gamma_0) = \frac{1}{24}(1^2\frac{73}{1} - 2^2\frac{21}{2} - 5^2\frac{15}{5} + 10^2\frac{5}{10}) = \frac{6}{24},$$

$$p^*(\gamma_1) = \frac{1}{24}(\frac{73}{1} - \frac{21}{2} - \frac{15}{5} + \frac{5}{10}) = \frac{60}{24},$$

$$p^*(\gamma_2) = \frac{1}{24}(1^2\frac{73}{1} - 2^2\frac{21}{2} - 1^2\frac{15}{5} + 2^2\frac{5}{10}) = \frac{30}{24},$$

$$p^*(\gamma_3) = \frac{1}{24}(1^2\frac{73}{1} - 1^2\frac{21}{2} - 5^2\frac{15}{5} + 5^2\frac{5}{10}) = 0,$$

$$p(\gamma_0) = \min_{\lambda \in \{0,\dots,24\}} \frac{1}{24}(-\gcd^2(1 \cdot (1 + 24\lambda \cdot 0), 25 \cdot 0)\frac{3}{1 \cdot 25} + \gcd^2(2 \cdot (1 + 24\lambda \cdot 0), 25 \cdot 0)\frac{1}{2 \cdot 25}$$

$$+\gcd^2(5 \cdot (1 + 24\lambda \cdot 0), 25 \cdot 0)\frac{1}{5 \cdot 25} - \gcd^2(10 \cdot (1 + 24\lambda \cdot 0), 25 \cdot 0)\frac{1}{10 \cdot 25}) = -\frac{1}{100},$$

$$p(\gamma_1) = \min_{\lambda\in\{0,\ldots,24\}} \frac{1}{24}(-\gcd^2(1\cdot(0+24\lambda\cdot 1),25\cdot 1)\frac{3}{1\cdot 25} + \gcd^2(2\cdot(0+24\lambda\cdot 1),25\cdot 1)\frac{1}{2\cdot 25}$$
$$+\gcd^2(5\cdot(0+24\lambda\cdot 1),25\cdot 1)\frac{1}{25\cdot 5} - \gcd^2(10\cdot(0+24\lambda\cdot 1),25\cdot 1)\frac{1}{10\cdot 25}) = -\frac{5}{2},$$
$$p(\gamma_2) = \min_{\lambda\in\{0,\ldots,24\}} \frac{1}{24}(-\gcd^2(1\cdot(1+24\lambda\cdot 2),25\cdot 2)\frac{3}{1\cdot 25} + \gcd^2(2\cdot(1+24\lambda\cdot 2),25\cdot 2)\frac{1}{2\cdot 25}$$
$$+\gcd^2(5\cdot(1+24\lambda\cdot 2),25\cdot 2)\frac{1}{25\cdot 5} - \gcd^2(10\cdot(1+24\lambda\cdot 2),25\cdot 2)\frac{1}{10\cdot 25}) = -\frac{5}{4},$$
$$p(\gamma_3) = \min_{\lambda\in\{0,\ldots,24\}} \frac{1}{24}(-\gcd^2(1\cdot(1+24\lambda\cdot 5),25\cdot 5)\frac{3}{1\cdot 25} + \gcd^2(2\cdot(1+24\lambda\cdot 5),25\cdot 5)\frac{1}{2\cdot 25}$$
$$+\gcd^2(5\cdot(1+24\lambda\cdot 5),25\cdot 5)\frac{1}{5\cdot 25} - \gcd^2(10\cdot(1+24\lambda\cdot 5),25\cdot 5)\frac{1}{10\cdot 25}) = 0.$$

Further we have that $[\Gamma:\Gamma_0(10)] = 18$, $\sum_{\delta|10} a_\delta = 42$, $\sum_{\delta|10} r_\delta = -2$, $\sum_{\delta|10} \delta r_\delta = -6$ and $\sum_{\delta|10} \delta a_\delta = 6$ hence $\nu = \frac{1}{24}(40\cdot 18 - 6) - \frac{1}{24\cdot 25}\cdot(-6) - \frac{14}{25} = 146/5 \approx 30$. Consequently by Lemma 4.40 *(ii)* we have that if $\Delta_2(25n+14) \equiv_5 0$ and $\Delta_2(25n+24) \equiv_5 0$ for $n = 0,\ldots,30$ then $\Delta_2(25n+14) \equiv_5 \Delta_2(25n+24) \equiv_5 0$ for all nonnegative n. Also note that by *(ii)* in Lemma 4.40 we have that

$$q^{144}\left(\sum_{n=0}^{\infty} \Delta_2(25n+14)q^{25n+14} + \Delta_2(25n+24)q^{25n+24}\right)^{24}\left(\prod_{n=1}^{\infty}\prod_{\delta|10}(1-q^{25\delta n})^{a_\delta}\right)^{24},$$

is a modular form of weight 480 for the group $\Gamma_0(250)$.

Hirschhorn and Sellers [17] proved that $\Delta_2(10n+6) \equiv_2 0, n \geq 0$. To prove (4.61) and Hirschhorn and Sellers' result we can again apply Lemma 4.40. This time we have that $(10, 10, 40, 2, (r_1, r_2, r_5, r_{10}) = (-3, 1, 1, -1)) \in \Delta^*$. If we choose $(a_\delta) = (a_1, a_2, a_4, a_5, a_8, a_{10}, a_{20}, a_{40}) = (33, -15, 0, -6, 0, 3, 0, 0)$ then all conditions of Lemma 4.40 apply and we get that $\nu \geq 39$. Consequently, verification of $\Delta_2(10n+2) \equiv_2 0$ and $\Delta_2(10n+6) \equiv_2 0$ for $0 \leq n \leq 39$ implies that (4.61) is true for all $n \geq 0$.

Example 4.42. The generating function

$$\prod_{n=1}^{\infty}\frac{1}{(1-q^{3n})(1-q^n)^3} = \sum_{n=0}^{\infty} a(n)q^n$$

appears in [29]. Here Ono proves that the numbers $a(63n+j), j = 22, 40, 49, n \geq 0$ are divisible by 7.

Ono uses Sturm's criterion and needs to compute 148147 coefficients of a certain generating function.

In order to solve this problem we can again apply Lemma 4.40. We find that $(63, 3, 21, 22, (r_1, r_3) = (-3, -1)) \in \Delta^*$ and see that the Lemma applies with $(a_\delta) = (a_1, a_3, a_7, a_{21}) = (240, -77, -33, 11)$. We find that $\nu \geq 182$ hence we need to verify that $a(63n+22) \equiv_7 a(63n+40) \equiv_7 a(63n+49) \equiv 0$ for $0 \leq n \leq 182$ in order to conclude that this congruence holds for all nonnegative n.

However Ono restates the problem by defining:

$$\sum_{n=0}^{\infty} b(n)q^n = \left(\prod_{n=1}^{\infty}\frac{(1-q^n)^{14}}{(1-q^{7n})^2}\right)\sum_{n=0}^{\infty} a(n)q^n.$$

He observes that $a(63n+j) \equiv_7 0, j = 22, 40, 49$ is equivalent to $b(63n+j) \equiv_7 0, j = 22, 40, 49$. This is clear since $\prod_{n=1}^{\infty} \frac{(1-q^n)^{14}}{(1-q^{7n})^2} \equiv_7 1$.

We can again apply Lemma 4.40 to this reformulated problem. With input $(63, 21, 21, 22, (r_1, r_3, r_7, r_{21}) = (11, -1, -2, 0)) \in \Delta^*$ we see that the lemma applies with $(a_\delta) = (a_1, a_3, a_7, a_{21}) = (5, -1, 0, 0)$. This time we find that $\nu \geq 16$ which is a huge improvement. Because of $a(n) \equiv_7 b(n)$ for all nonnegative n we need to show $a(63n+22) \equiv_7 a(63n+40) \equiv_7 a(63n+49) \equiv 0$ for $0 \leq n \leq 16$ in order to conclude that this congruence holds for all nonnegative n.

We can also prove the congruence $b(63n+22) \equiv_7 0$ with Lemma 4.39 and with the same input $(63, 21, 21, 22, (r_1, r_3, r_7, r_{21}) = (11, -1, -2, 0)) \in \Delta^*$. We see that all conditions of Lemma 4.39 are satisfied if we choose $(a_\delta) = (a_1, a_3, a_7, a_{21}) = (15, -4, 0, 0)$, and we get that $\nu \geq 45$ (approximately 3 times higher in comparison to using Lemma 4.40). Hence we need to verify that $b(63n+22) \equiv_7 0$ for $0 \leq n \leq 45$ in order for the congruence to be true for all nonnegative n. Also *(i)* in Lemma 4.39 gives us that

$$q^{45} \left(\prod_{t' \in \{22,40,49\}} \left(\sum_{n=0}^{\infty} b(63n+t') q^n \right) \right)^{24} \left(\prod_{n=1}^{\infty} \frac{(1-q^n)^{15}}{(1-q^{3n})^4} \right)^{24},$$

is a modular form of weight 420 for the group $\Gamma_0(21)$.

Ex.	gen. funct.	m	t	p	ν	N	$(a_\delta) \in R(N)$
1	$1^{-3}2^{1}5^{1}10^{-1}$	25	14,24	5	30	10	$1^{73}2^{-21}5^{-15}10^{5}$
2	$1^{-3}2^{1}5^{1}10^{-1}$	10	2,6	2	39	40	$1^{33}2^{-15}5^{-6}10^{3}$
3	$3^{-1}1^{-3}$	63	22,40,49	7	182	21	$\frac{1^{240}21^{11}}{3^{77}7^{33}}$
4	$1^{4}3^{-1}7^{-1}$	63	22,40,49	7	8	21	$1^{5}7^{-1}$
5	1^{-1}	5	4	5	1	5	1^{5}
6	1^{-1}	7	5	7	2	7	$1^{8}7^{-1}$
7	1^{-1}	11	6	11	5	11	1^{11}
8	1^{-1}	25	24	25	5	5	$1^{26}5^{-5}$
9	1^{-1}	49	47	49	14	7	$1^{50}7^{-7}$
10	1^{-1}	$11^3 \cdot 13$	$t \in P(237)$	13	103145	143	$\frac{1^{17551}143^{122}}{11^{1595}13^{1342}}$
11	$1^{12}13^{-1}$	$11^3 \cdot 13$	$t \in P(237)$	13	742	143	$1^{104}11^{-9}$
12	1^{-1}	125	74,124	125	26	5	$1^{130}5^{-25}$
13	1^{-2}	5	3	5	2	5	$1^{11}5^{-2}$
14	1^{-2}	25	23	25	10	5	$1^{52}5^{-10}$
15	1^{-8}	11	4	11	37	11	$1^{89}11^{-8}$
16	$1^{3}11^{-1}$	11	4	11	2	11	1^{1}
17	$2^{5}1^{-4}4^{-2}$	625	573	625	1301	20	$\frac{1^{1736}4^{434}10^{217}}{2^{1085}5^{347}20^{86}}$
18	$\frac{1^{-3+5}9^{3}}{3^{1}5^{1}}$	45	22,40	5	7	15	$1^{6}3^{-2}15^{1}$
19	$\frac{1^{-3+7}9^{3}}{3^{1}7^{1}}$	63	49	7	12	21	$1^{5}3^{-1}$
20	$\frac{1^{-3+11}9^{3}}{3^{1}11^{1}}$	99	94	11	22	33	$1^{3}3^{-1}$
21	$\frac{1^{-3+19}9^{3}}{3^{1}19^{1}}$	171	49	19	63	22	1^{2}

Table 4.1: Congruence Table

Example 4.43. In this example we are considering several generating functions and consider their congruence properties. Given a positive integer M we assume a generating function to be of the form $\prod_{\delta|M}\prod_{n=1}^{\infty}(1-q^{\delta n})^{r_\delta}=\sum_{n=0}^{\infty}a(n)q^n$, and we abbreviate such a generating function by $\prod\delta^{r_\delta}$. In the Table 4.1 the second column describes the generating function that we are considering. In columns $3, 4$ and 5 we specify the integers m, t and p for which we wish to prove that $a(mn+t)\equiv_p 0, n\geq 0$. The column labeled by N specifies the integer N as in Lemma 4.40. The last column specifies the (a_δ) in $R(N)$ such that Lemma 4.40 applies; this is also listed in the form $\prod_{\delta|N}\delta^{a_\delta}$. Finally the column ν shows the bound for the "verification proof"; i.e., that, number such that if $a(mn+t)\equiv_p 0$ is true for all $0\leq n\leq \nu$ and all t in column 4, then it is true for all $n\geq 0$.

Remark 4.44. *Note that the examples* $5, 6$ *and* 7 *are the famous Ramanujan congruences. Let* $p(n)$ *denote the number of partitions of* $n\in\mathbb{N}$*; then the entries in example* 5 *show that in order to prove* $p(5n+4)\equiv_5 0$ *for all* $n\geq 0$*, it is sufficient to verify that* $5|p(4)$*. Similarly if* $7|p(5)$ *and* $7|p(12)$ *then* $p(7n+5)\equiv_7 0$ *for all* $n\geq 0$*. Finally in order to prove* $p(11n+6)\equiv_{11} 0$ *for all* $n\geq 0$ *we need to verify that* $p(11n+6)\equiv_{11} 0$ *for* $0\leq n\leq 5$*. Ono [9] obtains bounds twice as big for the same congruences.*

Remark 4.45. *Generally, for some congruences one obtains a much better bound if one multiplies the generating function by* $\prod_{n=1}^{\infty}\frac{(1-q^n)^p}{(1-q^{pn})}\equiv_p 1$ *for some prime* p *when one wants to prove a congruence modulo* p*. This trick has been found by Ono [9]. In the table above examples* 3 *and* 4 *prove the same congruence because their generating functions are equal modulo* p*; the same holds for examples* $10, 11$ *and examples* $15, 16$*; however the bounds* ν *differ.*

Remark 4.46. *Example* 17 *in the table has been studied by Eichhorn and Sellers [10]. The generating function is denoted in their paper by* $\sum_{n=0}^{\infty}c\phi_2(n)q^n$ *and corresponds to 2-colored Frobenius partitions. They conclude that* $c\phi_2(625n+573)\equiv_{625} 0, n\geq 0$ *if and only if* $c\phi_2(625n+573)\equiv_{625} 0, 0\leq n\leq 198745$*. As seen in the table we only require that* $c\phi_2(625n+573)\equiv_{625} 0, 0\leq n\leq 1301$*. This improves the number of coefficients needed to be checked by a factor of approximately* 152*. In the end of the paper they are stating that the computation took 147 hours while with our bound we are decreasing the computation time to less then one hour!*

Remark 4.47. *The congruences in examples* $18, 19, 20$ *and* 21 *are studied by Lovejoy [26]. If we multiply the generating function in examples* $18, 19, 20$ *and* 21 *by* $\prod_{n=1}^{\infty}\frac{1-q^{pn}}{(1-q^n)^p}$ *for* $p=5, 7, 11, 19$ *we then obtain the same generating function* $f(q)$ *(and* $9qf(q)$ *is the generating function for 3-colored Frobenius partitions, e.g., [23]). For examples* $18, 20$ *and* 21*, Lovejoy proves the congruences by checking the first* $181, 505$ *and* 841 *initial values while with the methods developed here we only need to check the first* $7, 22$ *and* 63 *initial values. This gives an improvement by a factor of* $25, 22$ *and* 13*, respectively.*

Remark 4.48. *It should be noted that there is a difference between what Ono and Eichhorn [9] do and the approach here. Let* $f(q)=\sum_{n=0}^{\infty}a(n)q^n$ *and assume that we want to prove that* $\sum_{n=0}^{\infty}a(mn+t)q^n\equiv_p 0$*. Ono multiplies* $f(q)$ *be a suitable* η *product and gets a new generating function* $\sum_{n=0}^{\infty}b(n)q^n$ *which is a modular form. Then he shows that*

$$\sum_{n=0}^{\infty}a(mn+t)q^n\equiv_p 0 \Leftrightarrow \sum_{n=0}^{\infty}b(m'n+t')q^n\equiv_p 0$$

for suitable m' *and* t'. *Finally he uses a lemma which says that if* $\sum_{n=0}^{\infty} b(n)q^n$ *is a modular form for a group* Γ' *then also* $\sum_{n=0}^{\infty} b(m'n+t')q^{m'n+t'}$ *is a modular form for another group for which he applies the theorem of Sturm. We on the other hand are transforming* $\sum_{n=0}^{\infty} a(mn+t)q^n$ *into a modular form by multiplying with a suitable function* $h_1(q)$. *As we have seen, our method which is a generalization of the method in Rademacher [32] in practice gives much better bounds* ν.

Remark 4.49. *We finally mention that the tools of this chapter were used by the author and James Sellers to prove new congruences in [34] involving* t*-core partitions and* k*-broken diamond partitions. For definitions of this partition functions we refer to [2] and [12].*

Chapter 5

Computer-Assisted Discovery of Congruence Identities of Kolberg-Ramanujan Type

This chapter describes an algorithmic approach towards a computer-assisted discovery of identities like Ramanujan's (3.59) and (3.60). For the sake of simplicity we call such identities *Congruence identities of Ramanujan type*. Besides Ramanujan type identities we also consider identities like the one in Corollary 5.10 which we call *Congruence identities of Kolberg type* because they generalize the identities presented in [21]. We will give two examples illustrating our method.

The generating function

$$\sum_{m=0}^{\infty} \Delta_2(m) q^m := \prod_{n=1}^{\infty} \frac{(1-q^{2n})(1-q^{5n})}{(1-q^n)^3(1-q^{10n})}. \tag{5.1}$$

counts the number of broken 2-diamond partitions and was introduced by George Andrews and Peter Paule in [2]. In their paper they present the conjectures:

$$\Delta_2(10n+2) \equiv 0 \pmod{2}, \quad (n \in \mathbb{N}) \tag{5.2}$$

and

$$\Delta_2(25n+14) \equiv 0 \pmod{5}, \quad (n \in \mathbb{N}). \tag{5.3}$$

Conjecture (5.2) has been proven by Michael Hirschhorn and James Sellers [17] and they also proved that

$$\Delta_2(10n+6) \equiv 0 \pmod{2}, \quad (n \in \mathbb{N}). \tag{5.4}$$

Song Heng Chan [8] proved (5.3) together with

$$\Delta_2(25n+24) \equiv 0 \pmod{5}, \quad (n \in \mathbb{N}). \tag{5.5}$$

He also gave proofs for (5.2) and (5.4). At this point we call the congruences (5.2) and (5.4) the *first main congruences* and the congruences (5.3) and (5.5) the *second main congruences*.

We will also give a proof of the first and second main congruences as follows. Define

$$\sum_{m=0}^{\infty} \Delta_{2,5}(m) q^m := \prod_{n=1}^{\infty} \frac{(1-q^{2n})(1-q^n)^2}{(1-q^{10n})}. \tag{5.6}$$

Since $\prod_{n=1}^{\infty} \frac{(1-q^n)^5}{(1-q^{5n})} \equiv 1 \pmod 5$ and

$$\left(\sum_{m=0}^{\infty} \Delta_2(m) q^m\right) \prod_{n=1}^{\infty} \frac{(1-q^n)^5}{(1-q^{5n})} = \left(\sum_{m=0}^{\infty} \Delta_{2,5}(m) q^m\right)$$

we see that

$$\Delta_2(n) \equiv \Delta_{2,5}(n) \pmod 5, \quad (n \in \mathbb{N}). \tag{5.7}$$

Because of (5.7) we can prove the second main congruences (5.3) and (5.5) by replacing Δ_2 with $\Delta_{2,5}$. The first reason we prefer to do this replacement rather then working with the original generating function is because in the proof we will derive some identities similar to (3.59)-(3.60) that imply the second main congruences and by experiment we found that the identity gets considerably smaller if we replace Δ_2 by $\Delta_{2,5}$. This saves us some space (and computation time). The second motivation of the rewritings we will use is to illustrate the standard types of tools than one often uses in manipulations of such problems. For the same reasons we will also rewrite the first main congruences (5.2) and (5.4).

Define

$$\sum_{m=0}^{\infty} a_5(m) q^n := \prod_{n=1}^{\infty} \frac{(1-q^{5n})^5}{(1-q^n)}. \tag{5.8}$$

The sequence $(a_5(n))_{n\geq 0}$ is the generating function for 5-core partitions studied in [12]. We have the following connection with the first main congruences (5.2) and (5.4).

Lemma 5.1. *Let α be a positive integer with $0 \leq \alpha \leq 9$. Then*

$$a_5(10n+\alpha) \equiv 0 \pmod 2, \quad (n \in \mathbb{N})$$

is equivalent

$$\Delta_2(10n+\alpha) \equiv 0 \pmod 2, \quad (n \in \mathbb{N}).$$

In order to prove Lemma 5.1 the following result is useful:

Lemma 5.2. *Let m be a positive integer, $\alpha \in \{0, \ldots, m-1\}$ and $a, b, c : \mathbb{Z} \to \mathbb{C}$ such that*

$$\left(\sum_{n=0}^{\infty} a(n) q^n\right) \left(\sum_{n=0}^{\infty} b(n) q^{mn}\right) = \sum_{n=0}^{\infty} c(n) q^n. \tag{5.9}$$

Then

$$\sum_{n=0}^{\infty} c(mn+\alpha) q^n = \left(\sum_{n=0}^{\infty} a(mn+\alpha) q^n\right) \left(\sum_{n=0}^{\infty} b(n) q^n\right).$$

Proof. Define $b(n/m) := \begin{cases} b(n/m) & \text{if } m|n. \\ 0 & \text{otherwise.} \end{cases}$ Then (5.9) can be rewritten as

$$\left(\sum_{n=0}^{\infty} a(n) q^n\right) \left(\sum_{n=0}^{\infty} b(n/m) q^n\right) = \sum_{n=0}^{\infty} c(n) q^n$$

and $c(n) = \sum_{k=0}^{n} a(k)b\left(\frac{n-k}{m}\right)$. We have

$$\begin{aligned} c(mn+\alpha) &= \sum_{k=0}^{mn+\alpha} a(k)b\left(\frac{mn+\alpha-k}{m}\right) \\ &= \sum_{k=0, m|(\alpha-k)}^{mn+\alpha} a(k)b\left(\frac{mn+\alpha-k}{m}\right) \\ &= \sum_{s=0}^{n} a(ms+\alpha)b(n-s). \end{aligned}$$

The proof is completed after noting that

$$\sum_{n=0}^{\infty}\left(\sum_{s=0}^{n} a(ms+\alpha)b(n-s)\right)q^n = \left(\sum_{n=0}^{\infty} a(mn+\alpha)q^n\right)\left(\sum_{n=0}^{\infty} b(n)q^n\right).$$

□

Proof of Lemma 5.1. Since $\prod_{n=1}^{\infty} \frac{(1-q^n)^2(1-q^{10n})}{(1-q^{2n})(1-q^{5n})^2} \equiv 1 \pmod 2$ we have by (5.1)

$$\begin{aligned} \sum_{n=0}^{\infty} \Delta_2(n)q^n &\equiv \prod_{n=1}^{\infty} \frac{(1-q^n)^2(1-q^{10n})}{(1-q^{2n})(1-q^{5n})^2} \cdot \prod_{n=1}^{\infty} \frac{(1-q^{2n})(1-q^{5n})}{(1-q^n)^3(1-q^{10n})} \pmod 2 \\ &= \prod_{n=1}^{\infty} \frac{1}{(1-q^n)(1-q^{5n})}. \end{aligned} \tag{5.10}$$

Using (5.10), (5.8) and $\prod_{n=0}^{\infty} \frac{(1-q^{10n})^{13}}{(1-q^{5n})^{26}} \equiv 1 \pmod 2$ we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} a_5(n)q^n &\equiv \prod_{n=1}^{\infty}(1-q^{5n})^{26} \sum_{n=0}^{\infty} \Delta_2(n)q^n \\ &\equiv \prod_{n=1}^{\infty}(1-q^{10n})^{13} \sum_{n=0}^{\infty} \Delta_2(n)q^n \pmod 2. \end{aligned} \tag{5.11}$$

Next we apply Lemma 5.2 to (5.11) with $m = 2$,

$$\begin{aligned} \sum_{n=0}^{\infty} b(n)q^n &= \prod_{n=1}^{\infty}(1-q^{10n})^{13}, \\ \sum_{n=0}^{\infty} c(n)q^n &= \sum_{n=0}^{\infty} a_5(n)q^n, \\ \sum_{n=0}^{\infty} a(n)q^n &= \sum_{n=0}^{\infty} \Delta_2(n)q^n \end{aligned}$$

and obtain

$$\sum_{n=0}^{\infty} a_5(10n+\alpha)q^n \equiv \prod_{n=1}^{\infty}(1-q^n)^{13} \sum_{n=0}^{\infty} \Delta_2(10n+\alpha)q^n \pmod 2$$

which implies the desired result. □

Lemma 5.1 shows that the first main congruences are equivalent to proving

$$a_5(10n+2) \equiv a_5(10n+6) \equiv 0 \pmod{2}, \quad (n \in \mathbb{N}). \tag{5.12}$$

In connection with (5.7) we also noted that (5.3) and (5.5) are equivalent to

$$\Delta_{2,5}(25n+14) \equiv \Delta_{2,5}(25n+24) \equiv 0 \pmod{5}, \quad n \in \mathbb{N}. \tag{5.13}$$

We conclude this introduction by describing the strategy we will use to prove (5.12) and (5.13). Take for example (5.12). We start by finding a suitable sequence $a = (a_1, a_2, a_5, a_{10}) \in \mathbb{Z}^4$ and use Theorem 4.22 to show that

$$\prod_{\delta|10} \eta_\delta^{a_\delta} \left(\sum_{n=0}^{\infty} a_5(10n+2)q^n\right) \left(\sum_{n=0}^{\infty} a_5(10n+6)q^n\right) \tag{5.14}$$

is in $A_0(10)$. Next, for another suitable sequence $r = (r_1, r_2, r_5, r_{10})$ such that $g = \prod_{\delta|10} \eta_\delta^{r_\delta} \in A_0(10)$ we will show that (5.14) is a polynomial in g with coefficients in $4\mathbb{Z}$. Since g is a Laurent series in q with integer coefficients (by Lemma 2.34 and (2.26)) this will imply (5.12). This shows how to prove the congruence by proving an identity of the form $A = B$ where $A, B \in A_0(10)$. In order to prove such an identity we will show that $A - B \in M_0(10)$ by showing $\mathrm{Ord}_{\Gamma_0(10)}(A - B, \gamma) \geq 0$ for $\gamma \in R$ and R a complete set of representatives of the double cosets $\Gamma_0(10)\backslash \mathrm{SL}_2(\mathbb{Z})/\mathrm{SL}_2(\mathbb{Z})_\infty$ with $\mathrm{id} \in R$. Then by Corollary 2.23 $M_0(10) = \mathbb{C}$ showing that $A - B = c$ for some $c \in \mathbb{C}$. Next by coefficient comparison we show that $c = 0$ proving $A = B$. We mention at this point (assuming that A is (5.14) and B is polynomial in g) that $a = (a_1, a_2, a_5, a_{10})$ and $r = (r_1, r_2, r_5, r_{10})$ will be chosen such that $\mathrm{Ord}_{\Gamma_0(10)}(A, \gamma), \mathrm{Ord}_{\Gamma_0(10)}(B, \gamma) \geq 0$ for all $\gamma \in R - \{\mathrm{id}\}$. Then in order to prove $\mathrm{Ord}_{\Gamma_0(10)}(A - B, \gamma) \geq 0$ for all $\gamma \in R$ it suffices to prove $\mathrm{Ord}_{\Gamma_0(10)}(A - B, \mathrm{id}) \geq 0$, which is shown by comparing all the coefficients with negative exponent of two q-series. The author has learned this method of proof from [32] and [27]-[28] although we expose it in a different way.

5.1 The Ring $A_0^+(10)$ and its Generators

Definition 5.3. *For $d \in \mathbb{Z}$ let $\gamma_d := \left(\begin{smallmatrix} 1 & 0 \\ d & 1 \end{smallmatrix}\right)$. The ring $A_0^+(10)$ is defined as the subring of $A_0(10)$ consisting of all $f \in A_0(10)$ such that*

$$\mathrm{Ord}_{\Gamma_0(10)}(f, \gamma_d) \geq 0 \tag{5.15}$$

for $d \in \{1, 2, 5\}$.

The ring $A_0^+(10)$ has the following property.

Lemma 5.4. *Let $F \neq 0 \in A_0^+(10)$. Then* $\mathrm{Ord}_{\Gamma_0(10)}(F, \mathrm{id}) \leq 0$.

Proof. Assume by contradiction that $\mathrm{Ord}_{\Gamma_0(10)}(F, \mathrm{id}) > 0$. Then

$$\sum_{s \in R} \mathrm{Ord}_{\Gamma_0(10)}(f, s) > 0,$$

contradicting Lemma 2.18 because $R := \{\gamma_1, \gamma_2, \gamma_5, \mathrm{id}\}$ is a complete set of representatives for the double cosets $\Gamma_0(10)\backslash\mathrm{SL}_2(\mathbb{Z})/\mathrm{SL}_2(\mathbb{Z})_\infty$ by Lemma 2.45.

One can also prove this using Corollary 2.23. Because if $\mathrm{Ord}_{\Gamma_0(10)}(F) > 0$ then $F \in M_0(10)$ hence $F \in \mathbb{C}$ by Corollary 2.23. Since $\mathrm{Ord}_{\Gamma_0(10)}(c) = 0$ for all $c \in \mathbb{C} - \{0\}$ we conclude $F = 0$ because of $\mathrm{Ord}_{\Gamma_0(10)}(F) > 0$. □

We are going to show that $A_0^+(10)$ is generated by

$$f_r := \eta^{r_1}\eta_2^{r_2}\eta_5^{r_5}\eta_{10}^{r_{10}}$$

where r_1, r_2, r_5, r_{10} will be suitably chosen. As we saw, the Ord function plays an important role, and in the present context we have the following simple result.

Lemma 5.5. *Let $r \in R(10)$ and $f_r := \eta^{r_1}\eta_2^{r_2}\eta_5^{r_5}\eta_{10}^{r_{10}}$. If $f_r \in A_0(10)$ then* $\mathrm{Ord}_{\Gamma_0(10)}(f_r, \mathrm{id}) = \frac{1}{24}(r_1 + 2r_2 + 5r_5 + 10r_{10})$.

Proof. By Lemma 2.34 we see that $\frac{1}{24}(r_1 + 2r_2 + 5r_5 + 10r_{10}) \in \mathbb{Z}$ and by (2.26) we have

$$f_r = q^{\frac{1}{24}(r_1+2r_2+5r_5+10r_{10})} \prod_{n=1}^{\infty}(1-q^n)^{r_1}(1-q^{2n})^{r_2}(1-q^{5n})^{r_5}(1-q^{10n})^{r_{10}},$$

proving that $\mathrm{ord}(f_r) = \frac{1}{24}(r_1 + 2r_2 + 5r_5 + 10r_{10})$. By Lemma 2.15 we have $\mathrm{ord}(f_r) = \mathrm{Ord}_{\Gamma_0(10)}(f_r, \mathrm{id})$ because of $\omega_{\Gamma_0(10),\mathrm{id}} = 1$ by Lemma 2.37. □

We will try to find $f_r \in A_0^+(10)$ with $\mathrm{Ord}_{\Gamma_0(10)}(f_r) = -1$, so by Lemma 5.5

$$r_1 + 2r_2 + 5r_5 + 10r_{10} = 24. \tag{5.16}$$

We know by Lemma 2.34 that $f_r \in A_0(10)$ if (5.16) and

$$\begin{aligned} 10r_1 + 5r_2 + 2r_5 + r_{10} &\equiv 0 \pmod{24}, && (5.17)\\ r_1 + r_2 + r_5 + r_{10} &= 0, && (5.18)\\ r_2 + r_{10} &\equiv 0 \pmod 2, && (5.19)\\ r_5 + r_{10} &\equiv 0 \pmod 2; && (5.20) \end{aligned}$$

the conditions (5.19)-(5.20) correspond to *(iii)* in Lemma 2.34. Note that $g_{1,0}(\tau, r) = f_r$ by Definition 4.5, and by Lemma 4.28 (or by Lemma 4.29) we see that (5.15) is satisfied if

$$r_1\mathrm{gcd}^2(1,d) + \frac{r_2\mathrm{gcd}^2(2,d)}{2} + \frac{r_5\mathrm{gcd}^2(5,d)}{5} + \frac{r_{10}\mathrm{gcd}^2(10,d)}{10} \geq 0 \tag{5.21}$$

for $d = 1, 2, 5$. By computer methods we find that the solutions $f_r = g_0$, g_2, g_5 satisfying (5.16)-(5.21) are given by

$$g_0 := \frac{\eta^3\eta_5}{\eta_2\eta_{10}^3}, \quad g_2 := \frac{\eta_2^4\eta_5^2}{\eta^2\eta_{10}^4}, \quad g_5 := \frac{\eta_2\eta_5^5}{\eta\eta_{10}^5},$$

which can be written with (2.26) as

$$g_0 = q^{-1}\prod_{n=1}^{\infty}\frac{(1-q^n)^3(1-q^{5n})}{(1-q^{2n})(1-q^{10n})^3}; \tag{5.22}$$

$$g_2 = q^{-1}\prod_{n=1}^{\infty}\frac{(1-q^{2n})^4(1-q^{5n})^2}{(1-q^{n})^2(1-q^{10n})^4}; \tag{5.23}$$

$$g_5 = q^{-1}\prod_{n=1}^{\infty}\frac{(1-q^{2n})(1-q^{5n})^5}{(1-q^{n})(1-q^{10n})^5}. \tag{5.24}$$

Lemma 5.6. *We have the relations*

$$g_0+5 = g_2, \quad g_0+4 = g_5.$$

Proof. By using (5.22)-(5.24) we find that

$$\begin{aligned} g_0 &= q^{-1}-3+q+2q^2+2q^3-2q^4-q^5+\cdots \\ g_2 &= q^{-1}+2+q+2q^2+2q^3-2q^4-q^5+\cdots \\ g_5 &= q^{-1}+1+q+2q^2+2q^3-2q^4-q^5+\cdots \end{aligned}$$

Using this information we find that $\mathrm{ord}(g_0+5-g_2)\geq 1$ and $\mathrm{ord}(g_0+4-g_5)\geq 1$. By Lemma 2.15 we have $\mathrm{ord}(f)=\mathrm{Ord}_{\Gamma_0(10)}(f,\mathrm{id})$ for all $f\in A_0(10)$, and because $g_0+4-g_5, g_0+5-g_2\in A_0^+(10)$ we obtain by Lemma 5.4 that $g_0+4-g_5=0$ and $g_0+5-g_2=0$. □

Lemma 5.7. *Let $X\in\{g_0,g_2,g_5\}$. Then $A_0^+(10)=\mathbb{C}[X]$.*

Proof. Let $f\in A_0^+(10)$ and consider the set

$$O(f) := \{f+p(X)|p(X)\in\mathbb{C}[X]\}.$$

If $0\in O(f)$ then we are done. Assume that $0\notin O(f)$. Then there exists a integer $n\leq 0$ such that

$$\max_{g\in O(f)} \mathrm{Ord}_{\Gamma_0(10)}(g,\mathrm{id}) = n \tag{5.25}$$

because if for some $G\in A_0^+(10)$ we have $\mathrm{Ord}_{\Gamma_0(10)}(G,\mathrm{id})>0$ then $G=0$ by Lemma 5.4. Let $g\in O(f)$ be such that $\mathrm{Ord}_{\Gamma_0(10)}(g,\mathrm{id})=n$. We observe that $\mathrm{Ord}_{\Gamma_0(10)}(X^{-n},\mathrm{id})=n$ because of $\mathrm{Ord}_{\Gamma_0(10)}(X,\mathrm{id})=-1$ by (5.16) and Lemma 5.5. This implies that there is a $c\in\mathbb{C}$ such that $\mathrm{Ord}_{\Gamma_0(10)}(g-cX^{-n},\mathrm{id})>n$ contradicting (5.25), because $g-cX^{-n}\in O(f)$ because of $n\leq 0$. □

5.2 The First Main Congruences

We prove the first main congruences by proving (5.12).

Recall that $R(10)$ is the set of integer sequences $r=(r_\delta)$ indexed by the positive divisors δ of 10. Let $r',a'\in R(10)$ with $r'=(r'_1,r'_2,r'_5,r'_{10}):=(-1,0,5,0)$ and $a'=(a'_1,a'_2,a'_5,a'_{10})$ where a'_i are unknowns. Then by Definition 4.5 and (5.8) we have

$$\begin{aligned} F(\tau) &:= g_{1,0}(\tau,a')g_{10,2}(\tau,r')g_{10,6}(\tau,r') \\ &= q^{1+\frac{\sum_{\delta|10}\delta a'_\delta}{24}}\prod_{\delta|10}\eta_\delta^{a'_\delta}\left(\sum_{n=0}^{\infty}a_5(10n+2)q^n\right)\left(\sum_{n=0}^{\infty}a_5(10n+6)q^n\right). \end{aligned} \tag{5.26}$$

Note that β in Definition 4.23 depends on the parameters r and m, and we will write $\beta = \beta_{m,r}$. Next we apply Theorem 4.22 on each term of the left hand side of (5.26) with $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_0(10)^*$, $(m, M, N, t, r_\delta) = (1, 10, 10, 0, a')$, $(10, 10, 10, 2, (-1, 0, 5, 0))$, $(10, 10, 10, 6, (-1, 0, 5, 0))$. We obtain

$$F(\gamma\tau) = (\beta_{1,a'}\beta_{10,r'}\beta_{10,r'})(\gamma, 0)(-i(c\tau + d))^{\frac{1}{2}(8+\sum_{\delta|10} a'_\delta)} F(\tau), \tag{5.27}$$

and by Definition 4.23

$$(\beta_{1,a'}\beta_{10,r'}\beta_{10,r'})(\gamma, 0) = \prod_{\delta|10} \left(\frac{C\delta}{A}\right)^{|a'_\delta|} e^{-\frac{\pi i A}{12}(\sum_{\delta|10} \frac{C}{\delta} a'_\delta - \sum_{\delta|10} a'_\delta \delta B - 3\sum_{\delta|10} a'_\delta)}. \tag{5.28}$$

In view of Lemma 2.35, (5.27) and (5.28) we see that $F \in A_0(10)$ if

$$\begin{aligned} a'_1 + 2a'_2 + 5a'_5 + 10a'_{10} &\equiv 0 \pmod{24}, & (5.29)\\ 10a'_1 + 5a'_2 + 2a'_5 + a'_{10} &\equiv 0 \pmod{24}, & (5.30)\\ a'_1 + a'_2 + a'_5 + a'_{10} &= -8, & (5.31)\\ a'_2 + a'_{10} &\equiv 0 \pmod{2}, & (5.32)\\ a'_5 + a'_{10} &\equiv 0 \pmod{2}. & (5.33) \end{aligned}$$

Note that conditions (5.29)-(5.31) ensure that

$$\sum_{\delta|10} \frac{C}{\delta} a'_\delta - \sum_{\delta|10} a'_\delta \delta B - 3\sum_{\delta|10} a'_\delta \equiv 0 \pmod{24},$$

and conditions (5.32) and (5.33) imply $\prod_{\delta|10} \left(\frac{C\delta}{A}\right)^{|a'_\delta|} = 1$ in (5.28).

Next we find conditions on a' such that $F \in A_0^+(10)$. To indicate the dependence on m and r of p in Lemma 4.28 we write $p = p_{m,r}$. Then by Lemma 4.28 we see that for all $d \in \{1, 2, 5\}$ we have

$$(F|_0\gamma_d)(\tau) = H(q) q^{p_{1,a'}(\gamma_d) + 2p_{10,r'}(\gamma_d)} \tag{5.34}$$

for some Taylor series $H(q)$ in powers of $q^{1/l} = e^{2\pi i\tau/l}$ where l is some positive integer. Next note that

$$p_{1,a'}(\gamma_d) = \frac{1}{24} \sum_{\delta|10} a'_\delta \frac{\gcd^2(\delta, d)}{\delta}. \tag{5.35}$$

We also see by the formulas (4.43) and (4.44) that

$$p_{10,r'}(\gamma_d) = \min_{\lambda \in \{0,\dots,9\}} \frac{1}{24 \cdot 10} (25\gcd^2(1 + \kappa\lambda d, 2) - \gcd^2(1 + \kappa\lambda d, 10)) \tag{5.36}$$

and $\kappa = \gcd(1 - 10^2, 24) = 3$. From (5.36) we find

$$p_{10,r'}(\gamma_1) = p_{10,r'}(\gamma_2) = 0 \quad \text{and} \quad p_{10,r'}(\gamma_5) = \frac{1}{10}. \tag{5.37}$$

By (5.34), (5.35) and (5.37) we see that (5.15) is satisfied if

$$\frac{1}{24}\left(a_1' + \frac{a_2'}{2} + \frac{a_5'}{5} + \frac{a_{10}'}{10}\right) \geq 0 \tag{5.38}$$

$$\frac{1}{24}\left(a_1' + 2a_2' + \frac{a_5'}{5} + \frac{2a_{10}'}{5}\right) \geq 0 \tag{5.39}$$

$$\frac{1}{24}\left(a_1' + \frac{a_2'}{2} + 5a_5' + \frac{5a_5'}{2}\right) \geq -\frac{1}{5} \tag{5.40}$$

One verifies immediately that $a' = (a_1', a_2', a_5', a_{10}') = (-2, 4, 10, -20)$ satisfies (5.29)-(5.33) and (5.38)-(5.40) proving that $F_{2,6} \in A_0^+(10)$ with $F_{2,6}$ as in Definition 5.8 below.

Definition 5.8.

$$F_{2,6} := q^{-5} \prod_{n=1}^{\infty} \frac{(1-q^{2n})^4(1-q^{5n})^{10}}{(1-q^n)^2(1-q^{10n})^{20}} \left(\sum_{n=0}^{\infty} a_5(10n+2)q^n\right)\left(\sum_{n=0}^{\infty} a_5(10n+6)q^n\right).$$

Let $X \in \{g_0, g_2, g_5\}$. Then by Lemma 5.7 there exists a $p(X) \in \mathbb{C}[X]$ such that $F_{2,6} = p(X)$. The closed form for $p(X)$ is given in the next lemma.

Lemma 5.9. *We have* $F_{2,6} = 4(3g_0+20)(g_0+4)(g_0+5)^3$.

Proof. By Lemma 5.4 it is sufficient to verify that $\mathrm{Ord}_{\Gamma_0(10)}(F_{2,6} - (3g_0+20)(g_0+4)(g_0+5)^3, \mathrm{id}) > 0$ which is done by the computer. □

Corollary 5.10.

$$\left(\sum_{n=0}^{\infty} a_5(10n+2)q^n\right)\left(\sum_{n=0}^{\infty} a_5(10n+6)q^n\right) = 12\prod_{n=1}^{\infty} \frac{(1-q^{2n})^8(1-q^{5n})^2}{(1-q^n)^2} + 80q\prod_{n=1}^{\infty} \frac{(1-q^{2n})^9(1-q^{5n})(1-q^{10n})^3}{(1-q^n)^5}. \tag{5.41}$$

Proof. By Lemma 5.6 we have $g_0 + 5 = g_2$ and $g_0 + 4 = g_5$ which together with Lemma 5.9 implies

$$q^5\prod_{n=1}^{\infty} \frac{(1-q^n)^2(1-q^{10n})^{20}}{(1-q^{2n})^4(1-q^{5n})^{10}} F_{2,6} = q^5\prod_{n=1}^{\infty} \frac{(1-q^n)^2(1-q^{10n})^{20}}{(1-q^{2n})^4(1-q^{5n})^{10}}(12g_0+80)g_5g_2^3.$$

The result follows by applying Definition 5.8 together with (5.22)-(5.24). □

Corollary 5.11. *For all $n \in \mathbb{N}$ we have*

$$a_5(10n+2) \equiv a_5(10n+6) \equiv 0 \pmod{2}.$$

Proof. Define $A_s(q) := \sum_{n=0}^{\infty} a_5(10n+s)q^n$ for $s \in \{2,6\}$. Because of $a_5(6) = 6$ and $a_5(2) = 2$ we have

$$A_6(q), A_2(q) \not\equiv 0 \pmod{4}. \tag{5.42}$$

By Corollary (5.10) we have $A_2(q)A_6(q) \equiv_4 0$. This implies that either $A_2(q) \equiv 0 \pmod{2}$ or $A_6(q) \equiv 0 \pmod{2}$. Assume without loss of generality that $A_2(q) \equiv 0 \pmod{2}$. Then $\left(\frac{1}{2}A_2(q)\right)A_6(q) \equiv 0 \pmod{2}$ implying $A_6(q) \equiv 0 \pmod{2}$ because $\frac{1}{2}A_2(q) \not\equiv 0 \pmod{2}$ by (5.42). □

5.3 The Second Main Congruences

We prove the second main congruences by proving (5.13). We could proceed as in the previous section and try to find an expression for

$$\left(\sum_{n=0}^{\infty}\Delta_{2,5}(25n+24)q^n\right)\left(\sum_{n=0}^{\infty}\Delta_{2,5}(25n+14)q^n\right)$$

from which (5.13) follow. Indeed this method works also in this instance, but in view of getting additional insight we present a slightly modified approach which has some extra advantages.

Let $r', a' \in R(10)$ with $r' = (r'_1, r'_2, r'_5, r'_{10}) := (2,1,0,-1)$ and $a' = (a'_1, a'_2, a'_5, a'_{10})$. Then by Definition 4.5 and (5.8) we have

$$\begin{aligned} F(\tau) &:= g_{1,0}(\tau, a')g_{5,4}(\tau, r') \\ &= q^{\frac{3}{4}+\frac{\sum_{\delta|10}\delta a'_\delta}{24}}\prod_{\delta|10}\eta_\delta^{a'_\delta}\left(\sum_{n=0}^{\infty}\Delta_{2,5}(5n+4)q^n\right). \end{aligned} \tag{5.43}$$

By Definition 4.23 we see that β depends on the parameters r and m and we will write $\beta = \beta_{m,r}$. Next we apply Theorem 4.22 on each term of the left hand side of (5.26) with $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_0(10)^*$, $(m, M, N, t, r_\delta) = (1, 10, 10, 0, a')$, $(5, 10, 10, 4, (2, 1, 0, -1))$. We obtain

$$F(\gamma\tau) = (\beta_{1,a'}\beta_{5,r'})(\gamma, 0)(-i(c\tau + d))^{\frac{1}{2}(2+\sum_{\delta|10}a'_\delta)}F(\tau), \tag{5.44}$$

and by Definition 4.23

$$(\beta_{1,a'}\beta_{5,r'})(\gamma,0) = \left(\frac{5}{A}\right)e^{-\frac{\pi i A}{12}(30B-6)}\prod_{\delta|10}\left(\frac{C\delta}{A}\right)^{|a'_\delta|}e^{-\frac{\pi i A}{12}(\sum_{\delta|10}\frac{C}{\delta}a'_\delta-\sum_{\delta|10}a'_\delta\delta B-3\sum_{\delta|10}a'_\delta)}. \tag{5.45}$$

In view of Lemma 2.35, (5.44) and (5.45) we see that $F \in A_0(10)$ if

$$\begin{aligned} a'_1 + 2a'_2 + 5a'_5 + 10a'_{10} &\equiv 6 \pmod{24}, &(5.46)\\ 10a'_1 + 5a'_2 + 2a'_5 + a'_{10} &\equiv 0 \pmod{24}, &(5.47)\\ a'_1 + a'_2 + a'_5 + a'_{10} &= -2, &(5.48)\\ a'_2 + a'_{10} &\equiv 0 \pmod{2}, &(5.49)\\ a'_5 + a'_{10} &\equiv 1 \pmod{2}. &(5.50) \end{aligned}$$

Note that conditions (5.46)-(5.48) ensure that

$$\sum_{\delta|10}\frac{C}{\delta}a'_\delta - \sum_{\delta|10}a'_\delta\delta(B-30) - 3(\sum_{\delta|10}a'_\delta + 2) \equiv 0 \pmod{24},$$

and conditions (5.49) and (5.50) imply $\left(\frac{5}{A}\right)\prod_{\delta|10}\left(\frac{C\delta}{A}\right)^{|a'_\delta|} = 1$ in (5.45).

Next we find conditions on a' such that $F \in A_0^+(10)$. To indicate the dependence on m and r of p in Lemma 4.28 we write $p = p_{m,r}$. Then by Lemma 4.28 we see that for all $d \in \{1, 2, 5\}$ we have

$$(F|_0\gamma_d)(\tau) = H(q)q^{p_{1,a'}(\gamma_d)+p_{5,r'}(\gamma_d)} \tag{5.51}$$

for some Taylor series $H(q)$ in powers of $q^{1/l} = e^{2\pi i\tau/l}$ where l is some positive integer. We also see by the formulas (4.43) and (4.44) that

$$p_{5,r'}(\gamma_d) = \min_{\lambda\in\{0,\ldots,4\}} \frac{1}{24\cdot 5}(2\mathrm{gcd}^2(1+24\lambda d, 5d)+\frac{\mathrm{gcd}^2(2(1+24\lambda d), 5d))}{2}-\frac{\mathrm{gcd}^2(10(1+24\lambda d), 5d)}{10})$$

from which we find

$$p_{5,r'}(\gamma_1) = p_{5,r'}(\gamma_5) = 0 \quad \text{and} \quad p_{5,r'}(\gamma_2) = -\frac{1}{20}. \tag{5.52}$$

By (5.51), (5.35) and (5.52) we see that (5.15) is satisfied if

$$\frac{1}{24}\left(a_1' + \frac{a_2'}{2} + \frac{a_5'}{5} + \frac{a_{10}'}{10}\right) \geq 0 \tag{5.53}$$

$$\frac{1}{24}\left(a_1' + 2a_2' + \frac{a_5'}{5} + \frac{2a_{10}'}{5}\right) \geq \frac{1}{20} \tag{5.54}$$

$$\frac{1}{24}\left(a_1' + \frac{a_2'}{2} + 5a_5' + \frac{5a_5'}{2}\right) \geq 0 \tag{5.55}$$

One verifies immediately that $a' = (a_1', a_2', a_5', a_{10}') = (-1, 2, 3, -6)$ satisfies (5.46)-(5.50) and (5.53)-(5.55) proving that $F_4 \in A_0^+(10)$ with F_4 as in Definition 5.12 below.

Definition 5.12.

$$F_4 := q^{-1}\prod_{n=1}^{\infty}\frac{(1-q^{2n})^2(1-q^{5n})^3}{(1-q^n)(1-q^{10n})^6}\left(\sum_{n=0}^{\infty}\Delta_{2,5}(5n+4)q^n\right).$$

Let $X \in \{g_0, g_2, g_5\}$. Then by Lemma 5.7 there exists a $p(X) \in \mathbb{C}[X]$ such that $F_4 = p(X)$. The closed form for $p(X)$ is given in the next lemma.

Lemma 5.13. *We have* $F_4 = g_0$.

Proof. By Lemma 5.4 it is sufficient to verify that $\mathrm{Ord}_{\Gamma_0(10)}(F_4 - g_0, \mathrm{id}) > 0$ which is done by computer. □

Corollary 5.14.

$$\sum_{n=0}^{\infty}\Delta_{2,5}(5n+4)q^n = \prod_{n=1}^{\infty}\frac{(1-q^n)^4(1-q^{10n})^3}{(1-q^{2n})^3(1-q^{5n})^2}. \tag{5.56}$$

Proof. By Lemma 5.13

$$q\prod_{n=1}^{\infty}\frac{(1-q^n)(1-q^{10n})^6}{(1-q^{2n})^2(1-q^{5n})^3}F_4 = q\prod_{n=1}^{\infty}\frac{(1-q^n)(1-q^{10n})^6}{(1-q^{2n})^2(1-q^{5n})^3}g_0.$$

The result follows by applying Definition 5.12 together with (5.22). □

From this point on the proof of the second main congruences follow the same pattern as in Chan's paper [8].

Corollary 5.15. *For all $n \in \mathbb{N}$ we have*

$$\Delta_{2,5}(25n+14) \equiv \Delta_{2,5}(25n+24) \equiv 0 \pmod 5.$$

Proof. By Corollary 5.14 and because of $(1-A)^5 \equiv 1 - A^5 \pmod 5$ we have

$$\begin{aligned}\sum_{n=0}^{\infty} \Delta_{2,5}(5n+4)q^n &= \prod_{n=1}^{\infty} \frac{(1-q^n)^4(1-q^{10n})^3}{(1-q^{2n})^3(1-q^{5n})^2} = \prod_{n=1}^{\infty} \frac{(1-q^n)^4(1-q^{10n})}{(1-q^{2n})^3(1-q^{5n})} \prod_{n=1}^{\infty} \frac{(1-q^{10n})^2}{(1-q^{5n})} \\ &\equiv \prod_{n=1}^{\infty} \frac{(1-q^n)^4(1-q^{2n})^5}{(1-q^{2n})^3(1-q^n)^5} \prod_{n=1}^{\infty} \frac{(1-q^{10n})^2}{(1-q^{5n})} \pmod 5\end{aligned}$$

implying

$$\sum_{n=0}^{\infty} \Delta_{2,5}(5n+4)q^n \equiv \prod_{n=1}^{\infty} \frac{(1-q^{2n})^2}{(1-q^n)} \prod_{n=1}^{\infty} \frac{(1-q^{10n})^2}{(1-q^{5n})} \pmod 5. \tag{5.57}$$

Define $\sum_{n=0} b(n)q^n := \prod_{n=1}^{\infty} \frac{(1-q^{2n})^2}{(1-q^n)}$. Then (5.57) can be rewritten as

$$\left(\sum_{n=0}^{\infty} b(n)q^n\right)\left(\sum_{n=0}^{\infty} b(n)q^{5n}\right) \equiv \sum_{n=0}^{\infty} \Delta_{2,5}(5n+4)q^n \pmod 5.$$

and by Lemma 5.2 we obtain

$$\sum_{n=0}^{\infty} \Delta_{2,5}(5(5n+\alpha)+4)q^n \equiv \left(\sum_{n=0}^{\infty} b(5n+\alpha)q^n\right)\left(\sum_{n=0}^{\infty} b(n)q^n\right)$$

for $\alpha \in \{0, \dots, 4\}$, which implies

$$\Delta_{2,5}(5(5n+\alpha)+4) \equiv 0 \pmod 5 \Leftrightarrow b(5n+\alpha) \equiv 0 \pmod 5$$

so it suffices to prove that

$$b(5n+2) \equiv b(5n+4) \equiv 0 \pmod 5$$

which is clear from the identity $\sum_{n=0}^{\infty} b(n)q^n = \sum_{n=0}^{\infty} q^{\frac{1}{2}n(n+1)}$ by (7.25). □

Chapter 6

Sellers' Conjecture

6.1 Introduction

In his 1984 Memoir [1], George E. Andrews introduced two families of partition functions, $\phi_k(m)$ and $c\phi_k(m)$, which he called generalized Frobenius partition functions. In this chapter we restrict our attention to 2-colored Frobenius partitions. Their generating function reads as follows [1, (5.17)]:

$$\sum_{m=0}^{\infty} c\phi_2(m)q^m = \prod_{n=1}^{\infty} \frac{1-q^{4n-2}}{(1-q^{2n-1})^4(1-q^{4n})}. \tag{6.1}$$

In 1994 James Sellers [37] conjectured that for all integers $n \geq 0$ and $\alpha \geq 1$ one has

$$c\phi_2(5^\alpha n + \lambda_\alpha) \equiv 0 \pmod{5^\alpha},$$

where λ_α is defined to be the smallest positive integer such that

$$12\lambda_\alpha \equiv 1 \pmod{5^\alpha}. \tag{6.2}$$

In his joint paper with Dennis Eichhorn [10] this conjecture was proved for the cases $\alpha = 1, 2, 3, 4$. In this chapter we settle Sellers' conjecture for all α in the spirit of G. N. Watson [41]. Several authors (e.g. [24], [3]) have stated that the method of Watson works well when the modular functions involved live on a Riemann surface of genus 0. The reason for this is that every such modular function can be written as a rational function (in Watson's case polynomial function) in some fixed modular function t. In contrast to this, the modular functions that appear in this chapter belong to a Riemann surface of genus 1. Treatments of this type are very rare in the literature. To the best of our knowledge only the papers by B. Gordon and K. Hughes [15], [16] and [18] apply Watson's method to genus 1 Riemann surfaces. In these papers the authors use a relatively simple trick on the modular equation to make Watson's method work for larger genus then 0. We are applying essentially the same idea in this chapter; see Lemma 6.13 below.

This chapter (a preceding version [30] is submitted) is structured as follows. In Section 6.2 we state the Main Theorem (Theorem 6.8) of this chapter. It describes the action of a class of U-operators on a quotient of eta function products being crucial for the problem Sellers' conjecture then is derived as an immediate consequence (Corollary 6.9). The rest of the chapter deals with proving the Main Theorem. The basic building blocks of our proof are the twenty Fundamental Relations listed in Section 6.6. Despite postponing their proof to Section 6.5, we shall use these relations already in Section 6.3 and Section 6.4. In Section 6.3 a crucial result is proved, the Fundamental Lemma (Lemma 6.13), which has been inspired by work of B. Gordon and K. Hughes as it was mentioned above. The proof of the Main Theorem is presented in Section 6.4. To this end three further lemmas are introduced, all being immediate consequences of the Fundamental Lemma. Finally we mention that in Section 6.5, in order to prove the twenty Fundamental Relations, we utilize a computer-assisted method which is based on a variant of a well-known lemma by M. Newman (Lemma 2.34).

For $x \in \mathbb{R}$ the symbol $\lfloor x \rfloor$ ("floor" of x) as usual denotes the greatest integer less or equal to x. Let $f = \sum_{n\in\mathbb{Z}} a_n q^n$, $f \neq 0$, be such that $a_n = 0$ for almost all $n < 0$. Then the order of f is the smallest integer N such that $a_N \neq 0$; we write $N = \mathrm{ord}(f)$. More generally, let $F = f \circ t = \sum_{n\in\mathbb{Z}} a_n t^n$ with $t = \sum_{n\geq 1} b_n q^n$, then the t-order of F is defined to be the smallest integer N such that $a_N \neq 0$; we write $N = \mathrm{ord}_t(F)$. For example, if $\mathrm{ord}(f) = -1$ and $t = q^2$, then $\mathrm{ord}_t(F) = -1$ but $\mathrm{ord}(F) = -2$.

6.2 The Main Theorem

Let

$$\mathrm{C}\Phi_2(q) := \sum_{m=0}^{\infty} c\phi_2(m) q^m.$$

Lemma 6.1.

$$\mathrm{C}\Phi_2(q) = \prod_{n=1}^{\infty} \frac{(1-q^{2n})^5}{(1-q^n)^4(1-q^{4n})^2}.$$

Proof. From (6.1),

$$\begin{aligned}\mathrm{C}\Phi_2(q) &= \prod_{n=1}^{\infty} \frac{(1-q^{2(2n-1)})(1-q^{2n})^4}{(1-q^n)^4(1-q^{4n})} \\ &= \prod_{n=1}^{\infty} \frac{(1-q^{2n})(1-q^{2n})^4}{(1-q^n)^4(1-q^{4n})^2}.\end{aligned}$$

□

Definition 6.2. *We define*

$$A := \frac{\eta_2^5 \eta_{100}^2}{\eta_{50}^5 \eta_4^2} \tag{6.3}$$

and

$$u := \frac{\eta}{\eta_{25}}. \tag{6.4}$$

The following explicit expressions for λ_α in (6.2) are easily verified.

Lemma 6.3. *For $\beta \in \mathbb{N}^*$:*

$$\lambda_{2\beta-1} = \frac{1 + 7\cdot 5^{2\beta-1}}{12} \quad \textit{and} \quad \lambda_{2\beta} = \frac{1 + 11\cdot 5^{2\beta}}{12}.$$

Definition 6.4. *For $\alpha, m \in \mathbb{N}$ and $f, M : \mathbb{H} \to \mathbb{C}$ we define $L_0^{(m)}(M, f) := f$ and for $\alpha \geq 1$ we define*

$$L_{2\alpha-1}^{(m)}(M, f) := ML_{2\alpha-2}^{(m)}(M, f)|U_m$$

and

$$L_{2\alpha}^{(m)}(M, f) := L_{2\alpha-1}^{(m)}(M, f)|U_m.$$

Here U_m is as in Definition 3.20.

Lemma 6.5. *For $\alpha \in \mathbb{N}^*$ and λ_α as in Lemma 6.3 we have:*

$$L_{2\alpha-1}^{(5)}(Au^{-4}, 1) = q \prod_{n=1}^{\infty} \frac{(1-q^{5n})^4(1-q^{20n})^2}{(1-q^{10n})^5} \sum_{n=0}^{\infty} c\phi_2(5^{2\alpha-1}n + \lambda_{2\alpha-1})q^n \tag{6.5}$$

and

$$L_{2\alpha}^{(5)}(Au^{-4}, 1) = q \prod_{n=1}^{\infty} \frac{(1-q^{n})^4(1-q^{4n})^2}{(1-q^{2n})^5} \sum_{n=0}^{\infty} c\phi_2(5^{2\alpha}n + \lambda_{2\alpha})q^n \tag{6.6}$$

Proof. It is sufficient to prove the following statements:

(i) (6.5) is valid for $\alpha = 1$;

(ii) the truth of (6.5) for $\alpha = N$ with $N \geq 1$ implies the truth of (6.6) for $\alpha = N$;

(iii) the truth of (6.6) for $\alpha = N$ with $N \geq 1$ implies the truth of (6.5) for $\alpha = N + 1$.

Proof of (i)*:*

$$L_1^{(5)}(Au^{-4}, 1) = \frac{\eta_2^5\eta_{100}^2\eta_{25}^4}{\eta_{50}^5\eta_4^2\eta^4}|U_5 = q^2 \prod_{n=1}^{\infty} \frac{(1-q^{2n})^5(1-q^{100n})^2(1-q^{25n})^4}{(1-q^{50n})^5(1-q^{4n})^2(1-q^{n})^4}|U_5$$

(by Definition 6.4, (6.3), (6.4) and (2.26))

$$= \prod_{n=1}^{\infty} \frac{(1-q^{20n})^2(1-q^{5n})^4}{(1-q^{10n})^5} \left(q^2 \prod_{n=1}^{\infty} \frac{(1-q^{2n})^5}{(1-q^{n})^4(1-q^{4n})^2}|U_5 \right)$$

(by Lemma 3.23)

$$= \prod_{n=1}^{\infty} \frac{(1-q^{20n})^2(1-q^{5n})^4}{(1-q^{10n})^5} \sum_{n=1}^{\infty} c\phi_2(5n-2)q^n$$

(by Lemma 6.1 and Lemma 3.22)

$$= q \prod_{n=1}^{\infty} \frac{(1-q^{20n})^2(1-q^{5n})^4}{(1-q^{10n})^5} \sum_{n=0}^{\infty} c\phi_2(5n+3)q^n.$$

Proof of (ii):

$$\begin{aligned}
L_{2N}^{(5)}(Au^{-4},1) &= q\prod_{n=1}^{\infty}\frac{(1-q^{5n})^4(1-q^{20n})^2}{(1-q^{10n})^5}\sum_{n=0}^{\infty}c\phi_2(5^{2N-1}n+\lambda_{2N-1})q^n|U_5 \\
&= \prod_{n=1}^{\infty}\frac{(1-q^{n})^4(1-q^{4n})^2}{(1-q^{2n})^5}\left(\sum_{n=1}^{\infty}c\phi_2(5^{2N-1}(n-1)+\lambda_{2N-1})q^n|U_5\right)
\end{aligned}$$

(by Lemma 3.23)

$$= \prod_{n=1}^{\infty}\frac{(1-q^{n})^4(1-q^{4n})^2}{(1-q^{2n})^5}\sum_{n=1}^{\infty}c\phi_2(5^{2N-1}(5n-1)+\lambda_{2N-1})q^n$$

(by Lemma 3.22)

$$= \prod_{n=1}^{\infty}\frac{(1-q^{n})^4(1-q^{4n})^2}{(1-q^{2n})^5}\sum_{n=0}^{\infty}c\phi_2(5^{2N}n+4\cdot 5^{2N-1}+\lambda_{2N-1})q^n.$$

By Lemma 6.3, $\lambda_{2N}=\lambda_{2N-1}+4\cdot 5^{2N-1}$.

Proof of (iii):

$$\begin{aligned}
L_{2N+1}(1) &= q^3\prod_{n=1}^{\infty}\frac{(1-q^{100n})^2(1-q^{25n})^4}{(1-q^{50n})^5}\sum_{n=0}^{\infty}c\phi_2(5^{2\alpha}n+\lambda_{2\alpha})q^n|U_5 \\
&= \prod_{n=1}^{\infty}\frac{(1-q^{20n})^2(1-q^{5n})^4}{(1-q^{10n})^5}\left(\sum_{n=3}^{\infty}c\phi_2(5^{2N}(n-3)+\lambda_{2N})q^n|U_5\right)
\end{aligned}$$

(by Lemma 3.23)

$$= \prod_{n=1}^{\infty}\frac{(1-q^{20n})^2(1-q^{5n})^4}{(1-q^{10n})^5}\sum_{n=0}^{\infty}c\phi_2(5^{2N+1}n+2\cdot 5^{2N}+\lambda_{2N})q^n$$

(by Lemma 3.22). Again by Lemma 6.3, $\lambda_{2N+1}=\lambda_{2N}+2\cdot 5^{2N}$.

□

Definition 6.6. *Let $t,\rho,\sigma,p_0,$ and p_1 be functions defined on $\mathbb{H}$ as follows:*

$$t:=\frac{\eta_5^6}{\eta^6},\quad \rho:=\frac{\eta_2\eta_{10}^3}{\eta_4^3\eta_{20}},\quad \sigma:=\frac{\eta_2^2\eta_5^4}{\eta^4\eta_{10}^2} \tag{6.7}$$

$$p_0:=\frac{1}{2}(-4t\sigma-25t\rho\sigma^2-2\rho\sigma^2+30t\sigma^2+2\sigma^2+t\rho), \tag{6.8}$$

$$p_1:=\frac{1}{2}(-250t\sigma^2+200t\sigma+20\sigma+\rho-22\sigma^2+5\rho\sigma^2-4\rho\sigma). \tag{6.9}$$

We note that all functions defined in Definition 6.6 have Taylor series expansions in powers of q with coefficients in $\mathbb{Z}$, resp. $\frac{1}{2}\mathbb{Z}$. (In fact, one can show that all the coefficients are in $\mathbb{Z}$ but this is not needed for our purpose.) In particular, $\mathrm{Ord}(\rho) = \mathrm{Ord}(\sigma) = 0$ and $\mathrm{Ord}(t) = 1$, which implies $\mathrm{Ord}(p_0) \geq 1$ and $\mathrm{Ord}(p_1) \geq 1$.

Before stating the Main Theorem of the chapter, we introduce convenient shorthand notation.

Definition 6.7. *A map $a : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ is called discrete array if for each $i \in \mathbb{Z}$ the map $a(i, -) : \mathbb{Z} \to \mathbb{Z}$, $j \mapsto a(i, j)$, has finite support.*

Theorem 6.8 ("Main Theorem")**.** *There exist discrete arrays r, s, u, v such that for $\beta \in \mathbb{N}^*$ and $\tau \in \mathbb{H}$:*

$$L^{(5)}_{2\beta-1}(Au^{-4}, 1) = 5^{2\beta-1} p_0 \sum_{n=0}^{\infty} r(\beta, n) 5^{\lfloor \frac{5n+2}{2} \rfloor} t^n + \sum_{n=1}^{\infty} s(\beta, n) 5^{\lfloor \frac{5n-5}{2} \rfloor} t^n, \tag{6.10}$$

and

$$L^{(5)}_{2\beta}(Au^{-4}, 1) = 5^{2\beta} p_1 \sum_{n=0}^{\infty} u(\beta, n) 5^{\lfloor \frac{5n+1}{2} \rfloor} t^n + \sum_{n=1}^{\infty} v(\beta, n) 5^{\lfloor \frac{5n-4}{2} \rfloor} t^n. \tag{6.11}$$

The remaining sections are devoted to proving the Main Theorem by mathematical induction on β. In Sections 6.3 and 6.4 we describe the algebra underlying the induction step. In Section 6.5 we settle the initial cases, i.e., the correctness of the twenty fundamental relations listed in Section 6.6.

We conclude this section by deriving the truth of Sellers' conjecture as a corollary.

Corollary 6.9. *Sellers' conjecture is true; i.e., for $\alpha \in \mathbb{N}^*$:*

$$c\phi_2(5^\alpha n + \lambda_\alpha) \equiv 0 \pmod{5^\alpha}, \quad n \in \mathbb{N}^*.$$

Proof. The statement is derived immediately by applying the Lemmas 6.5 and 6.3 to (6.10) and (6.11). □

6.3 The Fundamental Lemma

In this section we prove the Fundamental Lemma, Lemma 6.13, which will play a crucial role in the proof of the Main Theorem in Section 6.4.

Definition 6.10. *With $t = t(\tau)$ as in Definition 6.6 we define:*

$$a_0(t) = -t, a_1(t) = -5^3t^2 - 6 \cdot 5t, a_2(t) = -5^6t^3 - 6 \cdot 5^4t^2 - 63 \cdot 5t,$$
$$a_3(t) = -5^9t^4 - 6 \cdot 5^7t^3 - 63 \cdot 5^4t^2 - 52 \cdot 5^2t,$$
$$a_4(t) = -5^{12}t^5 - 6 \cdot 5^{10}t^4 - 63 \cdot 5^7t^3 - 52 \cdot 5^5t^2 - 63 \cdot 5^2t.$$

We define $s : \{0, \dots, 4\} \times \{1, \dots, 5\} \to \mathbb{Z}$ to be the unique function satisfying

$$a_j(t) = \sum_{l=1}^{5} s(j, l) 5^{\lfloor \frac{5l+j-4}{2} \rfloor} t^l. \tag{6.12}$$

Remark 6.11. *Writing $a_j(t)$ as in (6.12) to reveal divisibility by powers of 5 of its coefficients is of help in the proof of Lemma 6.15 and is inspired by [6].*

Lemma 6.12. *For $0 \leq \lambda \leq 4$ let*

$$t_\lambda(\tau) := t\left(\frac{\tau+\lambda}{5}\right), \quad \tau \in \mathbb{H}.$$

Then in the polynomial ring $\mathbb{C}(t)[X]$:

$$X^5 + \sum_{j=0}^{4} a_j(t) X^j = \prod_{\lambda=0}^{4} (X - t_\lambda). \tag{6.13}$$

Proof. First we prove

$$\prod_{\lambda=0}^{4} t_\lambda = -a_0(t) = t. \tag{6.14}$$

With $\omega := e^{48\pi i/5}$ one has for $\tau \in \mathbb{H}$:

$$\begin{aligned}\prod_{\lambda=0}^{4} t_\lambda(\tau) &= \prod_{\lambda=0}^{4} q^{1/5}\omega^\lambda \prod_{n=1}^{\infty}\left(\frac{1-q^n}{1-\omega^{\lambda n}q^{n/5}}\right)^6 = q\prod_{n=1}^{\infty}\prod_{\lambda=0}^{4}\left(\frac{1-q^n}{1-\omega^{\lambda n}q^{n/5}}\right)^6 \\ &= q\prod_{n=1}^{\infty}(1-q^n)^{30}\prod_{n=1}^{\infty}\left(\frac{1-q^{5n}}{1-q^n}\right)^6\prod_{n=1}^{\infty}\left(\frac{1}{1-q^n}\right)^{30} = t(\tau).\end{aligned}$$

Here we used the fact that $\prod_{\lambda=0}^{4}(1-\omega^{\lambda n}z)$ equals $(1-z)^5$ if $5|n$, and $1-z^5$ otherwise.

For the remaining part of the proof we use (6.14) to rewrite (6.13) into the equivalent form

$$X^5 + \sum_{j=0}^{4} a_j(t) X^j = -t\prod_{\lambda=0}^{4}(1 - X t_\lambda^{-1}). \tag{6.15}$$

Hence to complete the proof, in view of $t = \prod_{\lambda=0}^{4} t_\lambda$ it remains to show that

$$a_j(t) = (-1)^{j+1} t e_j(t_0^{-1}, \ldots, t_4^{-1}), \quad 0 \leq j \leq 4, \tag{6.16}$$

where the e_j are the elementary symmetric functions. To this end we utilize the fact that

$$5(t^{-j}|U_5) = \sum_{\lambda=0}^{4} t_\lambda^{-j}, \quad j \in \mathbb{Z}.$$

The first non-trivial case is $j = 1$. Observing

$$e_1(t_0^{-1}, \ldots, t_4^{-1}) = \sum_{\lambda=0}^{4} t_\lambda^{-1} = 5(t^{-1}|U_5),$$

to show (6.16) for $j = 1$ we need to show that

$$5(t^{-1}|U_5) = t^{-1}a_1(t) = -5^3 t - 5 \cdot 6.$$

But this is the second entry of Group III of the twenty fundamental relations from Section 6.6. The next cases $2 \leq j \leq 4$ work analogously with the remaining entries of Group III. For example, if $j = 2$ then Newton's formula, translating elementary symmetric functions into power sums, implies

$$\begin{aligned} e_2(t_0^{-1}, \dots, t_4^{-1}) &= \frac{1}{2}\left(\left(5(t^{-1}|U_5)\right)^2 - 5(t^{-2}|U_5)\right) \\ &= \frac{1}{2}\left((-5^3 t - 5{\cdot}6)^2 - (-5^6 t^2 + 54{\cdot}5)\right) = -t^{-1}a_2(t). \end{aligned}$$

Here we used the third entry of Group III. □

Finally we are ready for the main result of this section.

Lemma 6.13 ("Fundamental Lemma")**.** *For $u : \mathbb{H} \to \mathbb{C}$ and $j \in \mathbb{Z}$:*

$$ut^j|U_5 = -\sum_{l=0}^{4} a_l(t)(ut^{j+l-5}|U_5).$$

Proof. For $\lambda \in \{0, \dots, 4\}$ Lemma 6.12 implies

$$t_\lambda^5 + \sum_{l=0}^{4} a_l(t)t_\lambda^l = 0.$$

Multiplying both sides with $u_\lambda t_\lambda^{j-5}$ where $u_\lambda(\tau) := u((\tau + 24\lambda)/5)$ gives

$$u_\lambda t_\lambda^j + \sum_{l=0}^{4} a_l(t)u_\lambda t_\lambda^{j+l-5} = 0.$$

Summing both sides over all λ from $\{0, \dots, 4\}$ completes the proof of the lemma. □

6.4 Proving the Main Theorem

We need to prepare with some lemmas. Recall that t is as in Definition 6.6.

Lemma 6.14. *Given functions $v_1, v_2, u : \mathbb{H} \to \mathbb{C}$ and $l \in \mathbb{Z}$. Suppose for $l \leq k \leq l+4$ there exist Laurent polynomials $p_k^{(1)}(t)$, $p_k^{(2)}(t) \in \mathbb{Z}[t, t^{-1}]$ such that*

$$ut^k|U_5 = v_1 p_k^{(1)}(t) + v_2 p_k^{(2)}(t) \tag{6.17}$$

and

$$\mathrm{Ord}_t\left(p_k^{(i)}(t)\right) \geq \left\lceil \frac{k + s_i}{5} \right\rceil, \quad i \in \{1, 2\}, \tag{6.18}$$

for some fixed integers s_1 and s_2. Then there exist families of Laurent polynomials $p_k^{(1)}(t), p_k^{(2)}(t) \in \mathbb{Z}[t, t^{-1}]$, $k \in \mathbb{Z}$, such that (6.17) *and* (6.18) *hold for all $k \in \mathbb{Z}$.*

Proof. Let $N > l+4$ be an integer and assume by induction that there are families of Laurent polynomials $p_k^{(i)}(t)$, $i \in \{1,2\}$, such that (6.17) and (6.18) hold for $l \leq k \leq N-1$. Suppose

$$p_k^{(i)}(t) = \sum_{n \geq \left\lceil \frac{k+s_i}{5} \right\rceil} c_i(k,n)t^n, \quad 1 \leq k \leq N-1,$$

with integers $c_i(k,n)$. Applying Lemma 6.13 we obtain:

$$\begin{aligned} ut^N|U_5 &= -\sum_{j=0}^{4} a_j(t)(ut^{N+j-5}|U_5) \\ &= -\sum_{j=0}^{4} a_j(t) \sum_{i=1}^{2} v_i \sum_{n \geq \left\lceil \frac{N+j-5+s_i}{5} \right\rceil} c_i(N+j-5,n)t^n \\ &= -\sum_{i=1}^{2} v_i \sum_{j=0}^{4} a_j(t)t^{-1} \sum_{n \geq \left\lceil \frac{N+j+s_i}{5} \right\rceil} c_i(N+j-5,n-1)t^n. \end{aligned}$$

Recalling the fact that $a_j(t)t^{-1}$ for $0 \leq j \leq 4$ is a polynomial in t, this determines Laurent polynomials $p_N^{(i)}(t)$ with the desired properties. The induction proof for $N < l$ works analogously. □

Lemma 6.15. *Given functions $v_1, v_2, u : \mathbb{H} \to \mathbb{C}$ and $l \in \mathbb{Z}$. Suppose for $l \leq k \leq l+4$ there exist Laurent polynomials $p_k^{(i)} \in \mathbb{Z}[t,t^{-1}]$, $i \in \{1,2\}$, such that*

$$ut^k|U_5 = v_1 p_k^{(1)}(t) + v_2 p_k^{(2)}(t) \tag{6.19}$$

where

$$p_k^{(i)}(t) = \sum_n c_i(k,n) 5^{\left\lfloor \frac{5n-k+\gamma_i}{2} \right\rfloor} t^n \tag{6.20}$$

with integers γ_i and $c_i(k,n)$. Then there exist families of Laurent polynomials $p_k^{(i)}(t) \in \mathbb{Z}[t,t^{-1}]$, $k \in \mathbb{Z}$, of the form (6.20) for which property (6.19) holds for all $k \in \mathbb{Z}$.

Proof. Suppose for an integer $N > l+4$ there are families of Laurent polynomials $p_k^{(i)}(t)$, $i \in \{1,2\}$, of the form (6.20) satisfying property (6.19) for $l \leq k \leq N-1$. We proceed by mathematical induction on N. Applying Lemma 6.13 and using the induction base (6.19) and (6.20) we obtain:

$$ut^N|U_5 = -\sum_{j=0}^{4} a_j(t) \sum_{i=1}^{2} v_i \sum_n c_i(N+j-5,n) 5^{\left\lfloor \frac{5n-(N+j-5)+\gamma_i}{2} \right\rfloor} t^n.$$

Utilizing (6.12) from Definition 6.10 this rewrites into :

$$\begin{aligned} ut^N|U_5 &= -\sum_{j=0}^{4}\sum_{l=1}^{5} s(j,l)5^{\left\lfloor\frac{5l+j-4}{2}\right\rfloor}t^l \\ &\quad\times\sum_{i=1}^{2} v_i \sum_n c_i(N+j-5,n)5^{\left\lfloor\frac{5n-(N+j-5)+\gamma_i}{2}\right\rfloor}t^n \\ &= -\sum_{i=1}^{2} v_i \sum_{j=0}^{4}\sum_{l=1}^{5}\sum_n s(j,l)c_i(N+j-5,n-l) \\ &\quad\times 5^{\left\lfloor\frac{5(n-l)-(N+j-5)+\gamma_i}{2}\right\rfloor+\left\lfloor\frac{5l+j-4}{2}\right\rfloor}t^n. \end{aligned} \tag{6.21}$$

The induction step is completed by simplifying the exponent of 5 as follows:

$$\begin{aligned} &\left\lfloor\frac{5(n-l)-(N+j-5)+\gamma_i}{2}+\left\lfloor\frac{5l+j-4}{2}\right\rfloor\right\rfloor \\ &\geq \left\lfloor\frac{5(n-l)-(N+j-5)+\gamma_i}{2}+\frac{5l+j-5}{2}\right\rfloor \\ &= \left\lfloor\frac{5n-N+\gamma_i}{2}\right\rfloor. \end{aligned}$$

The induction proof for $N<l$ works analogously. □

Before proving the Main Theorem, Theorem 6.8, we need one more lemma.

Lemma 6.16. *Given A as in* (6.3)*, p_0 and p_1 as in* (6.8) *and* (6.9)*, respectively. Then there exist discrete arrays a_i, b_i, c, and d_i, $i\in\{0,1\}$, such that the following relations hold for all $k\in\mathbb{N}$:*

$$Au^{-4}t^k|U_5 = \sum_{n\geq\lceil(k+1)/5\rceil} a_0(k,n)5^{\left\lfloor\frac{5n-k-2}{2}\right\rfloor}t^n + p_0\sum_{n\geq\lceil(k-4)/5\rceil} a_1(k,n)5^{\left\lfloor\frac{5n-k+5}{2}\right\rfloor}t^n, \tag{6.22}$$

$$Au^{-4}p_1t^k|U_5 = \sum_{n\geq\lceil(k+1)/5\rceil} b_0(k,n)5^{\left\lfloor\frac{5n-k-2}{2}\right\rfloor}t^n + p_0\sum_{n\geq\lceil(k-4)/5\rceil} b_1(k,n)5^{\left\lfloor\frac{5n-k+4}{2}\right\rfloor}t^n, \tag{6.23}$$

$$t^k|U_5 = \sum_{n\geq\lceil k/5\rceil} c(k,n)5^{\left\lfloor\frac{5n-k-1}{2}\right\rfloor}t^n, \tag{6.24}$$

$$p_0t^k|U_5 = \sum_{n\geq\lceil(k+1)/5\rceil} d_0(k,n)5^{\left\lfloor\frac{5n-k-2}{2}\right\rfloor}t^n + p_1\sum_{n\geq\lceil k/5\rceil} d_1(k,n)5^{\left\lfloor\frac{5n-k+1}{2}\right\rfloor}t^n. \tag{6.25}$$

Proof. Section 6.6 lists twenty fundamental relations, which are proved in Section 6.5 (Theorem 6.19). The five fundamental relations of Group I fit the pattern of the relation (6.22) for five consecutive values of k. The same observation applies to the relations of the Groups II, III and IV with regard to the relations (6.23), (6.24), and (6.25), respectively. In each of these cases k is less or equal to 0. Hence applying Lemma 6.14 and Lemma 6.15 immediately proves the statement for all $k\geq 0$. □

Now we are ready for the proof of the Main Theorem.

Proof of Theorem 6.8 ("Main Theorem"). We proceed by mathematical induction on β. For $\beta = 1$ the statement is settled by the first fundamental identity $Au^{-4}|U_5 = 5(-t+5p_0)$ of Section 6.6. The induction step will be carried out as follows: In the first step we show that the correctness of (6.10) for $N = 2\beta - 1$, $\beta \in \mathbb{N}^*$, implies (6.11) for $N+1 = 2\beta$, which in the second step is shown to imply the correctness of (6.10) for $N+2 = 2\beta+1$.

For the first step we recall Definition 6.4 and apply the induction hypothesis (6.10) to obtain

$$\begin{aligned} L_{2\beta}^{(5)}(Au^{-4},1) &= L_{2\beta-1}^{(5)}(Au^{-4},1)|U_5 \\ &= 5^{2\beta-1}\left(\sum_{n=0}^{\infty} r(\beta,n)5^{\left\lfloor\frac{5n+2}{2}\right\rfloor}(p_0t^n|U_5) + \sum_{n=1}^{\infty} s(\beta,n)5^{\left\lfloor\frac{5n-5}{2}\right\rfloor}(t^n|U_5)\right). \end{aligned}$$

Utilizing (6.24) and (6.25) of Lemma 6.16 with discrete arrays c and d_i gives

$$\begin{aligned} L_{2\beta}^{(5)}(Au^{-4},1) = 5^{2\beta-1}\Bigg(& p_1 \sum_{m\geq 0}\sum_{n\geq 0} r(\beta,n)d_1(n,m)5^{\left\lfloor\frac{5n+2}{2}\right\rfloor+\left\lfloor\frac{5m-n+1}{2}\right\rfloor}t^m \\ & + \sum_{m\geq 1}\sum_{n\geq 0} r(\beta,n)d_0(n,m)5^{\left\lfloor\frac{5n+2}{2}\right\rfloor+\left\lfloor\frac{5m-n-2}{2}\right\rfloor}t^m \\ & + \sum_{m\geq 1}\sum_{n\geq 1} s(\beta,n)c(n,m)5^{\left\lfloor\frac{5n-5}{2}\right\rfloor+\left\lfloor\frac{5m-n-1}{2}\right\rfloor}t^m \Bigg). \end{aligned} \tag{6.26}$$

Observe that for $m,n \geq 0$:

$$\begin{aligned} \left\lfloor\frac{5n+2}{2}\right\rfloor + \left\lfloor\frac{5m-n+1}{2}\right\rfloor &= \left\lfloor\frac{5m+n+1}{2}\right\rfloor + \left\lfloor\frac{3n+2}{2}\right\rfloor \geq \left\lfloor\frac{5m+1}{2}\right\rfloor + 1, \\ \left\lfloor\frac{5n+2}{2}\right\rfloor + \left\lfloor\frac{5m-n-2}{2}\right\rfloor &= \left\lfloor\frac{5m+n-2}{2}\right\rfloor + \left\lfloor\frac{3n+2}{2}\right\rfloor \geq \left\lfloor\frac{5m-4}{2}\right\rfloor + 1, \end{aligned}$$

and for $m,n \geq 1$:

$$\left\lfloor\frac{5n-5}{2}\right\rfloor + \left\lfloor\frac{5m-n-1}{2}\right\rfloor = \left\lfloor\frac{5m+n-5}{2}\right\rfloor + \left\lfloor\frac{3n-1}{2}\right\rfloor \geq \left\lfloor\frac{5m-4}{2}\right\rfloor + 1.$$

Hence the right hand side of (6.26) is of the desired form (6.11).

For the second step we again recall Definition 6.4 and apply the induction hypothesis (6.11) to obtain

$$\begin{aligned} L_{2\beta+1}^{(5)}(Au^{-4},1) &= Au^{-4}L_{2\beta}^{(5)}(Au^{-4},1)|U_5 \\ &= 5^{2\beta}\left(\sum_{n=0}^{\infty} r(\beta,n)5^{\left\lfloor\frac{5n+1}{2}\right\rfloor}(Au^{-4}p_1t^n|U_5) + \sum_{n=1}^{\infty} s(\beta,n)5^{\left\lfloor\frac{5n-4}{2}\right\rfloor}(Au^{-4}t^n|U_5)\right). \end{aligned}$$

Utilizing (6.22) and (6.23) of Lemma 6.16 with discrete arrays a_i and b_i gives

$$\begin{aligned}L^{(5)}_{2\beta+1}(Au^{-4},1) = 5^{2\beta}&\\ &\times\Bigg(p_0\sum_{m\geq 0}\sum_{n\geq 0} r(\beta,n)b_1(n,m)5^{\lfloor\frac{5n+1}{2}\rfloor+\lfloor\frac{5m-n+4}{2}\rfloor}t^m\\ &+p_0\sum_{m\geq 0}\sum_{n\geq 1} s(\beta,n)a_1(n,m)5^{\lfloor\frac{5n-4}{2}\rfloor+\lfloor\frac{5m-n+5}{2}\rfloor}t^m\\ &+\sum_{m\geq 1}\sum_{n\geq 0} r(\beta,n)b_0(n,m)5^{\lfloor\frac{5n+1}{2}\rfloor+\lfloor\frac{5m-n-2}{2}\rfloor}t^m\\ &+\sum_{m\geq 1}\sum_{n\geq 1} s(\beta,n)a_0(n,m)5^{\lfloor\frac{5n-4}{2}\rfloor+\lfloor\frac{5m-n-2}{2}\rfloor}t^m\Bigg).\end{aligned} \tag{6.27}$$

Similar to above observe that for $m,n\geq 0$:

$$\left\lfloor\frac{5n+1}{2}\right\rfloor+\left\lfloor\frac{5m-n+4}{2}\right\rfloor=\left\lfloor\frac{5m+n+2}{2}\right\rfloor+\left\lfloor\frac{3n+3}{2}\right\rfloor\geq\left\lfloor\frac{5m+2}{2}\right\rfloor+1,$$

for $m\geq 0$ and $n\geq 1$:

$$\left\lfloor\frac{5n-4}{2}\right\rfloor+\left\lfloor\frac{5m-n+5}{2}\right\rfloor=\left\lfloor\frac{5m+n+2}{2}\right\rfloor+\left\lfloor\frac{3n-1}{2}\right\rfloor\geq\left\lfloor\frac{5m+2}{2}\right\rfloor+1,$$

for $m\geq 1$ and $n\geq 0$:

$$\left\lfloor\frac{5n+1}{2}\right\rfloor+\left\lfloor\frac{5m-n-2}{2}\right\rfloor=\left\lfloor\frac{5m+n-4}{2}\right\rfloor+\left\lfloor\frac{3n+3}{2}\right\rfloor\geq\left\lfloor\frac{5m-5}{2}\right\rfloor+1,$$

and for $m,n\geq 1$:

$$\left\lfloor\frac{5n-4}{2}\right\rfloor+\left\lfloor\frac{5m-n-2}{2}\right\rfloor=\left\lfloor\frac{5m+n-6}{2}\right\rfloor+\left\lfloor\frac{3n}{2}\right\rfloor\geq\left\lfloor\frac{5m-5}{2}\right\rfloor+1.$$

Hence the right hand side of (6.27) is of the desired form (6.10) with β replaced by $\beta+1$. This completes the proof of the Main Theorem assuming the validity of the twenty fundamental relation in Section 6.6. Their correctness will be proven in the next section. □

6.5 Proving the Fundamental Relations

6.5.1 A computerized proof of the fundamental relations

At the level of eta products we need the following facts that are immediate from Newman's Lemma 2.34.

Lemma 6.17. *For the functions from Definition 6.6 the following statements are true:*

(i) $\eta_5^{24}\cdot\frac{\eta_{25}^4\eta_{100}^2}{\eta_{50}^5}\cdot\frac{\eta_2^5}{\eta^4\eta_4^2}=\eta_5^{24}Au^{-4}\in M_{12}(100)$*;*

(ii) $t\eta^{24}$, $t\eta_5^{24}\in M_{12}(20)$*;*

(iii) $\sigma\eta^{24}$, $\sigma\eta_5^{24} \in M_{12}(20)$;

(iv) $\rho\eta^{24}$, $\rho\eta_5^{24} \in M_{12}(20)$;

(v) $t^{-j}\eta_5^{24} \in M_{12}(20)$, $0 \le j \le 5$;

(vi) $t^{-6}\eta_5^{48} \in M_{24}(20)$;

(vii) $t^j\eta^{48} \in M_{24}(20)$, $-2 \le j \le 5$;

(viii) $p_1\eta^{72}$, $p_1\eta_5^{72} \in M_{36}(20)$;

(ix) $p_0\eta^{96}$, $p_0\eta_5^{96} \in M_{48}(20)$.

Proof. The statements *(i)-(vii)* are straight-forward verifications invoking Lemma 2.34. In proving *(viii)* and *(ix)* we restrict to showing that $p_1\eta^{72} \in M_{36}(20)$ in *(viii)*, since the other cases are analogous. According to (6.9) we need to show that

$$t\sigma^2\eta^{72}, t\sigma\eta^{72}, \sigma\eta^{72}, \rho\eta^{72}, \sigma^2\eta^{72}, \sigma^2\rho\eta^{72}, \sigma\rho\eta^{72} \in M_{36}(20).$$

By *(ii)* and *(iii)* we have that $t\eta^{24}$ and $\sigma\eta^{24}$ are in $M_{12}(20)$. Consequently

$$\sigma\eta^{24} \cdot \sigma\eta^{24} \cdot t\eta^{24} \in M_{36}(20).$$

Similarly one sees that $t\eta^{24} \cdot \sigma\eta^{24} \cdot \eta^{24} \in M_{36}(20)$ because $\eta^{24} \in M_{12}(20)$. The other monomials are treated analogously. □

Next we connect all the fundamental relations to Newman's lemma 2.34.

Lemma 6.18. *For the functions from Definition 6.6 the following statements are true for any choice of integer coefficients $c(i,j)$ and $d(i,j)$:*

(i) $\eta^{144}\left((Au^{-4}t^{-j}|U_5) - \sum_{i=-1}^{4}(c(i,j)t^i + d(i,j)p_0t^i)\right) \in M_{72}(20)$, $0 \le j \le 4$;

(ii) $\eta^{144}\left((Au^{-4}p_1t^{-j}|U_5) - \sum_{i=-2}^{5}(c(i,j)t^i + d(i,j)p_0t^i)\right) \in M_{72}(20)$, $2 \le j \le 6$;

(iii) $\eta^{144}\left((t^{-j}|U_5) - \sum_{i=0}^{4} c(i,j)t^i\right) \in M_{72}(20)$, $0 \le j \le 4$;

(iv) $\eta^{144}\left((p_0t^{-j}|U_5) - \sum_{i=-2}^{5}(c(i,j)t^i + d(i,j)p_0t^i)\right) \in M_{72}(20)$, $1 \le j \le 5$.

Proof. We only prove *(i)* which corresponds to Group I of the fundamental relations; the other cases are analogous. The statement follows from showing that each term in the sum is in $M_{72}(20)$. We start with the term $\eta^{144}(Au^{-4}t^{-j}|U_5)$ for a fixed $j \in \{0, \dots, 4\}$. By Lemma 3.23,

$$\eta^{144}(Au^{-4}t^{-j}|U_5) = (\eta_5^{144}Au^{-4}t^{-j}|U_5).$$

By (6.3) we have that

$$\eta_5^{24}Au^{-4} = \eta_5^{24}\frac{\eta_{25}^4\eta_{100}^2}{\eta_{50}^5} \cdot \frac{\eta_2^5}{\eta^4\eta_4^2},$$

which is in $M_{12}(100)$ by Lemma 6.17*(i)*. By Lemma 6.17*(v)* we have $t^{-j}\eta_5^{24} \in M_{12}(20) \subseteq M_{12}(100)$, because in general $\Gamma_0(N_1)$ is a subgroup of $\Gamma_0(N_2)$ if $N_2|N_1$. Observing that $\eta_5^{96} \in M_{48}(20) \subseteq M_{48}(100)$, we can conclude that

$$t^{-j}\eta_5^{24}\cdot\eta_5^{24}Au^{-4}\cdot\eta_5^{96} = \eta_5^{144}Au^{-4}t^{-j} \in M_{72}(100).$$

Finally, Lemma 3.24 implies that $(\eta_5^{144}Au^{-4}t^{-j}|U_5) \in M_{72}(20)$. Proving that $\eta^{144}t^i$ and $\eta^{144}p_0t^i$ are in $M_{72}(20)$ for $-1 \le i \le 4$ is done analogously using Lemma 6.17 again. □

Theorem 6.19. *The twenty fundamental relations listed in the Appendix hold true.*

Proof. After multiplication with η^{144} the entries of Group I to IV correspond to elements from $M_k(N)$ with $k = 72$ and $N = 20$. By Corollary 2.40 we have $\mu = [\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(20)] = 36$. Consequently by Lemma 2.22, the proof is completed by verifying equality of the first $1 + \mu k/12 = 217$ coefficients in the Taylor series expansions of both sides of each of the fundamental relations. This task is left to the computer. □

6.6 The Fundamental Relations

Group I:

$$\begin{aligned}
Au^{-4}|U_5 &= -5t + 5^2p_0;\\
Au^{-4}t^{-1}|U_5 &= -1 + p_0t^{-1};\\
Au^{-4}t^{-2}|U_5 &= 5^5t^2 + 11\cdot5^2t + 11 - p_0(5^3 + 2\cdot5t^{-1});\\
Au^{-4}t^{-3}|U_5 &= -5^8t^3 - 34\cdot5^5t^2 - 51\cdot5^3t - 119 + p_0(2\cdot5^6t + 6\cdot5^4 + 21\cdot5t^{-1});\\
Au^{-4}t^{-4}|U_5 &= -5^{11}t^4 + 92\cdot5^6t^2 + 759\cdot5^3t + 253\cdot5 - p_0(8\cdot5^7t + 99\cdot5^4 + 44\cdot5^2t^{-1}).
\end{aligned}$$

Group II:

$$\begin{aligned}
Au^{-4}p_1t^{-2}|U_5 &= -5^5t^2 + 114\cdot5^2t + 59 - p_0(124\cdot5^3 + 59t^{-1});\\
Au^{-4}p_1t^{-3}|U_5 &= 5^8t^3 - 36\cdot5^5t^2 - 103\cdot5^3t - 26 - p_0(5^6t - 9\cdot5^4 + 7\cdot 5t^{-1});\\
Au^{-4}p_1t^{-4}|U_5 &= 5^{11}t^4 + 14\cdot5^9t^3 + 259\cdot5^6t^2 + 1436\cdot5^3t + 38\cdot5\\
&\quad - p_0(5^9t^2 + 122\cdot5^6t + 211\cdot5^4 - 7\cdot5t^{-1});\\
Au^{-4}p_1t^{-5}|U_5 &= -5^{14}t^5 + 12\cdot5^{11}t^4 + 9\cdot5^9t^3 - 1494\cdot 5^6t^2 - 2366\cdot5^4t - 196\cdot5\\
&\quad + p_0(5^{12}t^3 + 8\cdot5^{10}t^2 + 282\cdot5^7t + 409\cdot5^5 - 11\cdot5^2t^{-1});\\
Au^{-4}p_1t^{-6}|U_5 &= -7\cdot5^{15}t^5 - 372\cdot5^{12}t^4 - 917\cdot5^{10}t^3 - 1581\cdot5^7t^2 + 16089\cdot5^4t - 69\cdot5^2\\
&\quad + t^{-1} + p_0(96\cdot5^{12}t^3 + 13\cdot5^{12}t^2 - 404\cdot5^7t - 3152\cdot5^5 + 361\cdot5^2t^{-1} - t^{-2}).
\end{aligned}$$

Group III:

$$\begin{aligned}
1|U_5 &= 1;\\
t^{-1}|U_5 &= -5^2t - 6;\\
t^{-2}|U_5 &= -5^5t^2 + 54;\\
t^{-3}|U_5 &= -5^8t^3 - 102\cdot5;\\
t^{-4}|U_5 &= -5^{11}t^4 + 966\cdot5.
\end{aligned}$$

Group IV:

$$
\begin{aligned}
p_0t^{-1}|U_5 &= 3{\cdot}5^{10}t^4 + 77{\cdot}5^7t^3 + 562{\cdot}5^4t^2 + 41{\cdot}5^3t + 1\\
&\quad - p_1(5^9t^3 + 14{\cdot}5^6t^2 + 44{\cdot}5^3t + 2{\cdot}5);\\
p_0t^{-2}|U_5 &= -5^5t^2 - 14{\cdot}5^2t + 7 - 5p_1;\\
p_0t^{-3}|U_5 &= -5^8t^3 - 14{\cdot}5^5t^2 - 5^4t - 12 - 5^4tp_1;\\
p_0t^{-4}|U_5 &= -5^{11}t^4 - 14{\cdot}5^8t^3 - 5^7t^2 + 12{\cdot}5 - 5^7t^2p_1;\\
p_0t^{-5}|U_5 &= 4{\cdot}5^{14}t^5 + 121{\cdot}5^{11}t^4 + 233{\cdot}5^9t^3 + 738{\cdot}5^6t^2 + 109{\cdot}5^4t - 17{\cdot}5^2\\
&\quad + p_1(4{\cdot}5^{10}t^3 + 14{\cdot}5^8t^2 + 44{\cdot}5^5t + 2{\cdot}5^3 - t^{-1}).
\end{aligned}
$$

Chapter 7

Some Modular Functions Connected to Sellers' Conjecture

In the previous chapter we proved Sellers' conjecture. This was done by proving its equivalence with the congruences

$$L_n^{(5)}(Au^{-4},1) \equiv 0 \pmod{5^n}, \quad n \in \mathbb{N}. \tag{7.1}$$

Here L is as in Definition 6.4. We recall that $L_n^{(5)}(Au^{-4},1)$ in relation (7.1) can be viewed as a Laurent series $\tilde{L}_n(Au^{-4},1)(q)$ (with integer coefficients) in powers of $q = e^{2\pi i\tau}$, and (7.1) means that each coefficient in $\tilde{L}_n^{(5)}(Au^{-4},1)(q)$ is divisible by 5^n.

To prove (7.1) in Theorem 6.8 we found identities of the form

$$L_{2n-1}^{(5)}(Au^{-4},1) = 5^{2n-1}(P_{2n-1}(t) + p_0 Q_{2n-1}(t)) \tag{7.2}$$

and

$$L_{2n}^{(5)}(Au^{-4},1) = 5^{2n}(P_{2n}(t) + p_1 Q_{2n}(t)), \tag{7.3}$$

where $P_n(t), Q_n(t) \in \mathbb{Z}[t]$ and where $p_0, p_1, t \in A_0(20)$ are Laurent series in powers of q with integer coefficients. Let $p(n)$ denote the number of partitions of n, and let μ_α be such that $24\mu_\alpha \equiv 1 \pmod{11^\alpha}$ and $0 \le \mu_\alpha \le 11^\alpha - 1$. We note that Atkin [3] proved the longstanding Ramanujan conjecture

$$p(11^\alpha n + \mu_\alpha) \equiv 0 \pmod{11^\alpha} \tag{7.4}$$

for $\alpha, n \in \mathbb{N}$ in a similar way. More precisely, he proved that (7.4) is implied by the identities

$$\begin{aligned} &L_{2n-1}^{(11)}(\eta_{121}/\eta,1) \\ &\quad = 11^{2n-1}(P_{2n-1}^{(0)}(T) + J_2 P_{2n-1}^{(2)}(T) + J_3 P_{2n-1}^{(3)}(T) + J_4 P_{2n-1}^{(4)}(T) + J_6 P_{2n-1}^{(6)}(T)) \end{aligned} \tag{7.5}$$

and

$$\begin{aligned} &L_{2n}^{(11)}(\eta_{121}/\eta,1) \\ &\quad = 11^{2n}(P_{2n}^{(0)}(T) + J_2 P_{2n}^{(2)}(T) + J_3 P_{2n}^{(3)}(T) + J_4 P_{2n}^{(4)}(T) + J_6 P_{2n}^{(6)}(T)) \end{aligned} \tag{7.6}$$

where $P_n^{(j)} \in \mathbb{Z}[T]$ for $j \in \{0,2,3,4,6\}$, $n \in \mathbb{N}^*$ and $T, J_2, J_3, J_4, J_6 \in A_0(11)$ are Laurent series in powers of q with integral coefficients.

Atkin also proved that the $\mathbb{C}[T,T^{-1}]$-module generated by $1, J_2, J_3, J_4, J_6$ is the whole $A_0(11)$. In our case neither the $\mathbb{C}[t,t^{-1}]$-module generated by $1, p_0$ nor the one generated by $1, p_1$ is the whole $A_0(20)$ (and also not the one generated by $1, p_0, p_1$).

One can easily see by looking at Atkin's proof that for any element $f = P^{(0)}(T) + J_2P^{(2)}(T) + J_3P^{(3)}(T) + J_4P^{(4)}(T) + J_6P^{(6)}(T)$ with $P^{(j)} \in \mathbb{Z}[T,T^{-1}]$ for $j \in \{0,2,3,4,6\}$ and any $\alpha \in \mathbb{N}$ there exists a $n \in \mathbb{N}$ such that $L_n^{(11)}(\eta_{121}/\eta, f) \equiv 0 \pmod{11^\alpha}$. This is the same as saying that the sequence $(L_n^{(11)}(\eta_{121}/\eta, f))_{n\geq 0}$ converges to 0 in the 11-adic metric.

It seems that without this property the identities (7.5) and (7.6) could not have been proven. In fact, all the papers that deal with congruences similar to (7.4) (see for example [4], [15], [16], [24] and [18]) involve identities of the type

$$L_n^{(p)}(g,1) = p^\alpha(J_1P_n^{(1)}(T) + \cdots + J_jP_n^{(j)}(T)), \quad P_n^{(I)}(T) \in \mathbb{Z}[T,T^{-1}], \quad i = 0,\ldots,j,$$

and the $\mathbb{C}[T,T^{-1}]$ module generated by $1, J_1, \ldots, J_j$ is the whole $A_0(pN)$ for some N. Furthermore, the sequence $(L_n^{(p)}(g,f))_{n\geq 0}$ converges to 0 in the p-adic metric for every $f \in A_0(pN)$.

In our problem one can also show that for every given $\alpha \in \mathbb{N}$ and any $F(t), G(t) \in \mathbb{C}[t]$ the sequence $(L_n^{(5)}(Au^{-4}, F(t) + p_1G(t))_{n\geq 0}$ converges to 0 in the 5-adic metric. However we also found examples of functions $f \in A_0(20)$ such that the sequence $(L_n^{(5)}(Au^{-4}, f))_{n\geq 0}$ does not converge to 0 in the 5-adic metric. In our point of view, this is the most important fact that distinguishes the Sellers' problem from the previous ones. Our key observation is that the "Sellers functions" are satisfying special functional relations which we first discovered by computer experiments. In this chapter we will turn our attention to the main properties that characterizes the special $\mathbb{C}[t,t^{-1}]$ modules generated by $1, p_0$ and $1, p_1$. We will show that the functional equations we mentioned earlier together with $A_0(20)$ membership gives rise to submodules of $A_0(20)$ that are generated by $\{1, p_0\}$ and $\{1, p_1\}$, respectively.

7.1 Preparatory Notions and Results

Definition 7.1. *For $f : \mathbb{H} \to \mathbb{C}$ and $\gamma \in \mathrm{GL}_2^+(\mathbb{Z})$ we define $f|\gamma := f|_0\gamma$.*

Definition 7.2. *We define $q : \mathbb{H} \to \mathbb{C}$ by $q(\tau) := e^{2\pi i\tau}$ for $\tau \in \mathbb{H}$.*

Definition 7.3. *For $m, r, t \in \mathbb{Z}$ we define $V_{m,r,t} := \left(\begin{smallmatrix} m & r \\ 0 & t \end{smallmatrix}\right)$, $V_m := V_{m,0,1}$ and $V_{m,r} := V_{m,r,1}$.*

Definition 7.4. *For $m \in \mathbb{Z}$ we introduce* $\gamma_m := \begin{cases} \left(\begin{smallmatrix} 1 & 0 \\ m & 1 \end{smallmatrix}\right), & \text{if } m \neq 0, \\ \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right), & \text{otherwise.} \end{cases}$

Definition 7.5. *For $p \geq 5$ a prime we define*

$$\begin{array}{rclrclrcl} \tilde{F}_\infty(q) & := & \prod_{n=1}^\infty \frac{(1-q^{2n})^5}{(1-q^{4n})^2}, & F_\infty & := & \frac{\eta_2^5}{\eta_4^2}, & f_{\infty,p} & := & \frac{F_\infty}{F_\infty|V_{p^2}}, \\ \tilde{F}_{1/2}(q) & := & \prod_{n=1}^\infty \frac{(1-q^n)^2(1-q^{4n})^2}{(1-q^{2n})}, & F_{1/2} & := & \frac{\eta^2\eta_4^2}{\eta_2}, & f_{1/2,p} & := & \frac{F_{1/2}}{F_{1/2}|V_{p^2}}, \\ \tilde{F}_0(q) & := & \prod_{n=1}^\infty \frac{(1-q^{2n})^5}{(1-q^n)^2}, & F_0 & := & \frac{\eta_2^5}{\eta^2}, & f_{0,p} & := & \frac{F_0}{F_0|V_{p^2}}. \end{array}$$

Definition 7.6. *Let $p \geq 5$ be a prime and $\alpha \in \{1,2\}$. We define $R_1(p^\alpha)$ to be the set of all $f \in A_0(4p^\alpha)$ such that*

$$(F_0|V_p)(f|\gamma_{p^2}V_{4,-1}) = (F_\infty|V_{4p})(f|V_4) + 2(F_{1/2}|V_{4p})(f|\gamma_{2p^2}V_4). \tag{7.7}$$

Definition 7.7. *Let $p \geq 5$ be a prime and $\alpha \in \{1,2\}$. We define $R_2(p^\alpha)$ to be the set of all $f \in A_0(4p^\alpha)$ such that*

$$F_0(f|\gamma_{p^2}V_{4,-1}) = (F_\infty|V_4)(f|V_4) + 2(F_{1/2}|V_4)(f|\gamma_{2p^2}V_4). \tag{7.8}$$

The following proposition is a straight forward consequence of (2.26) and Definitions 7.5, 7.6 and 7.7.

Proposition 7.8. *Let $p \geq 5$ be a prime and $\alpha \in \{1,2\}$. If $f \in R_1(p^\alpha)$ then*

$$\tilde{F}_0(q^p)(f|\gamma_{p^2}V_{4,-1}) = \tilde{F}_\infty(q^{4p})(f|V_4) + 2q^p\tilde{F}_{1/2}(q^{4p})(f|\gamma_{2p^2}V_4), \tag{7.9}$$

and if $f \in R_2(p^\alpha)$ then

$$\tilde{F}_0(q)(f|\gamma_{p^2}V_{4,-1}) = \tilde{F}_\infty(q^4)(f|V_4) + 2q\tilde{F}_{1/2}(q^4)(f|\gamma_{2p^2}V_4). \tag{7.10}$$

For ord as in Definition 2.14 we have the following important property.

Lemma 7.9. *Let $p \geq 5$ be a prime and $\alpha \in \{1,2\}$. If $f \in R_1(p^\alpha)$ then*

$$\begin{aligned} \mathrm{ord}(f|\gamma_{p^2}V_{4,-1}) &\leq 4 \cdot \mathrm{ord}(f), & (7.11)\\ \mathrm{ord}(f|\gamma_{p^2}V_{4,-1}) &\leq 4 \cdot \mathrm{ord}(f|\gamma_{2p^2}) + p, & (7.12) \end{aligned}$$

with equality in either (7.11) or (7.12). If $f \in R_2(p^\alpha)$ then

$$\begin{aligned} \mathrm{ord}(f|\gamma_{p^2}V_{4,-1}) &\leq 4 \cdot \mathrm{ord}(f), & (7.13)\\ \mathrm{ord}(f|\gamma_{p^2}V_{4,-1}) &\leq 4 \cdot \mathrm{ord}(f|\gamma_{2p^2}) + 1, & (7.14) \end{aligned}$$

with equality in either (7.13) or (7.14).

Proof. By Lemma 2.37 we have

$$\omega_{\Gamma_0(4p^\alpha),\gamma_{p^2}} = \frac{4p^\alpha}{\gcd(p^4, 4p^\alpha)} = 4,\ \omega_{\Gamma_0(4p^\alpha),\gamma_{2p^2}} = \frac{4p^\alpha}{\gcd(4p^4, 4p^\alpha)} = 1 \text{ and } \omega_{\Gamma_0(4p^\alpha),\mathrm{id}} = 1.$$

Consequently by Lemma 2.11 there exist $b, b_1, b_2 : \mathbb{Z} \to \mathbb{C}$ and $t, t_1, t_2 \in \mathbb{Z}$ such that $b(t), b_1(t_1), b_2(t_2) \neq 0$ and for $\tau \in \mathbb{H}$:

$$(f|\gamma_{p^2})(\tau) = \sum_{n=t}^{\infty} b(n)e^{2\pi in\tau/4},\ f(\tau) = \sum_{n=t_1}^{\infty} b_1(n)e^{2\pi in\tau} \text{ and } (f|\gamma_{2p^2})(\tau) = \sum_{n=t_2}^{\infty} b_2(n)e^{2\pi in\tau},$$

which implies that

$$(f|\gamma_{p^2}V_{4,-1}) = \sum_{n=t}^{\infty} b(n)e^{-2\pi in/4}e^{2\pi in\tau},$$

which by Definition 2.14 implies that

$$\mathrm{ord}(f|\gamma_{p^2}V_{4,-1}) = t,\ \mathrm{ord}(f) = t_1 \text{ and } \mathrm{ord}(f|\gamma_{2p^2}) = t_2. \tag{7.15}$$

Define $c : \mathbb{Z} \to \mathbb{C}$ by the formula

$$\sum_{n=t}^{\infty} c(n)e^{2\pi i\tau n} := \tilde{F}_0(e^{2\pi ip\tau})(f|\gamma_{p^2}V_{4,-1})(\tau) = \tilde{F}_0(e^{2\pi ip\tau})\sum_{n=t}^{\infty} b(n)e^{-2\pi in/4}e^{2\pi in\tau}, \quad \tau \in \mathbb{H}. \tag{7.16}$$

Clearly $c(t) \neq 0$ because $\tilde{F}_0(e^{2\pi i\tau p})$ is of the form $1 + e^{2\pi i\tau p}\sum_{n=0}^{\infty} d(n)e^{2\pi i\tau pn}$ for some $d : \mathbb{Z} \to \mathbb{C}$. Also define $c_1, c_2 : \mathbb{Z} \to \mathbb{C}$ by the formulas

$$\sum_{n=t_1}^{\infty} c_1(n)e^{8\pi in\tau} := \tilde{F}_\infty(e^{8\pi ip\tau})(f|V_4)(\tau) = \tilde{F}_\infty(e^{8\pi ip\tau})\sum_{n=t}^{\infty} b_1(n)e^{8\pi in\tau} \tag{7.17}$$

and

$$\sum_{n=t_2}^{\infty} c_2(n)e^{8\pi in\tau} := \tilde{F}_{1/2}(e^{8\pi ip\tau})(f|\gamma_{2p^2}V_4)(\tau) = \tilde{F}_{1/2}(e^{8\pi ip\tau})\sum_{n=t}^{\infty} b_2(n)e^{8\pi in\tau}, \quad \tau \in \mathbb{H}. \tag{7.18}$$

Also here $c_1(t_1), c_2(t_2) \neq 0$ by the same reason as $c(t) \neq 0$. By (7.16)-(7.18) and (7.10) we obtain

$$\sum_{n=t}^{\infty} c(n)q^n = \sum_{n=t_1}^{\infty} c_1(n)q^{4n} + q^p\sum_{n=t_2}^{\infty} c_2(n)q^{4n}$$

and by coefficient comparison we observe $t \leq 4t_1$ and $t \leq 4t_2+p$, with equality in one of them, which together with (7.15) gives (7.11)-(7.12). The proof of (7.13)-(7.14) is analogous. □

The most important result of this chapter is that $p_0t^j \in R_1(5)$***,*** $p_1t^j \in R_2(5)$ ***and*** $t^j \in R_1(5) \cap R_2(5)$ ***for all*** $j \in \mathbb{Z}$***.***

Furthermore, all functions in $R_1(5)$ ***may be written as a linear combination of*** p_0t^j ***and*** t^j ***and that all functions in*** $R_2(5)$ ***may be written as a linear combination of*** p_1t^j ***and*** t^j ***for*** $j \in \mathbb{Z}$***.***

7.2 $R_1(p^i)$ and $R_2(p^i)$ are $A_0(p^i)$-modules Containing $A_0(p^i)$

The next lemma shows that if $i \in \{1, 2\}$ then $R_1(p^i)$, $R_2(p^i)$ are $A_0(p^i)$ modules.

Lemma 7.10. *Let* $i \in \{1, 2\}$ *and* $f \in A_0(p^i)$*. If* $f' \in R_1(p^i)$ *(resp.* $f' \in R_2(p^i)$*) then* $ff' \in R_1(p^i)$ *(resp.* $ff' \in R_2(p^i)$*).*

Proof. We have:

$$\begin{aligned}
&(F_0|V_p)(ff'|\gamma_{p^2}V_{4,-1}) - (F_\infty|V_{4p})(ff'|V_4) - 2(F_{1/2}|V_{4p})(ff'|\gamma_{2p^2}V_4)\\
&= (F_0|V_p)(f|\gamma_{p^2}V_{4,-1})(f'|\gamma_{p^2}V_{4,-1}) - (F_\infty|V_{4p})(f|V_4)(f'|V_4)\\
&\quad - 2(F_{1/2}|V_{4p})(f|\gamma_{2p^2}V_4)(f'|\gamma_{2p^2}V_4)\\
&= (F_0|V_p)(f|V_{4,-1})(f'|\gamma_{p^2}V_{4,-1}) - (F_\infty|V_{4p})(f|V_4)(f'|V_4)\\
&\quad - 2(F_{1/2}|V_{4p})(f|V_4)(f'|\gamma_{2p^2}V_4)\\
&= (f|V_4)((F_0|V_p)(f'|\gamma_{p^2}V_{4,-1}) - (F_\infty|V_{4p})(f'|V_4) - 2(F_{1/2}|V_{4p})(f'|\gamma_{2p^2}V_4))\\
&= 0.
\end{aligned}$$

The other case is analogous. □

The rest of this section will be devoted to show that $1 \in R_1(p) \cap R_2(p)$ for $p \geq 5$ a prime. We will need the following lemma (e.g. [42, p. 469]).

Lemma 7.11 (Jacobi Triple Product)**.** *For $q, y \in \mathbb{C}$ with $y \neq 0$ and $|q| < 1$:*

$$\prod_{m=1}^{\infty}(1-q^{2m})(1+q^{2m-1}y^2)(1+q^{2m-1}y^{-2}) = \sum_{n=-\infty}^{\infty} q^{n^2}y^{2n}.$$

Lemma 7.12. *We have the following relation:*

$$\tilde{F}_0(q) = \tilde{F}_\infty(q^4) + 2q\tilde{F}_{1/2}(q^4). \tag{7.19}$$

Proof. We define

$$\sum_{n=0}^{\infty} a(n)q^n := \prod_{m=1}^{\infty} \frac{(1-q^{2m})^5}{(1-q^m)^2(1-q^{4m})^2} \tag{7.20}$$

Because of the relation

$$\prod_{m=1}^{\infty}(1+q^{2m-1}) = \prod_{m=1}^{\infty}\frac{1+q^m}{1+q^{2m}} = \prod_{m=1}^{\infty}\frac{(1-q^{2m})^2}{(1-q^m)(1-q^{4m})}$$

we have

$$\prod_{m=1}^{\infty}(1-q^{2m})(1+q^{2m-1})^2 = \prod_{n=1}^{\infty}\frac{(1-q^{2m})^5}{(1-q^m)^2(1-q^{4m})^2}. \tag{7.21}$$

By Lemma 7.11 (after replacing $y = 1$), (7.21) and (7.20) we have that

$$\sum_{n=0}^{\infty} a(n)q^n = 1 + 2\sum_{n=1}^{\infty} q^{n^2}. \tag{7.22}$$

From (7.22) we see that

$$\sum_{n=0}^{\infty} a(4n)q^{4n} = 1 + 2\sum_{n=0}^{\infty} q^{4n^2} = \prod_{m=1}^{\infty}\frac{(1-q^{8m})^5}{(1-q^{4m})^2(1-q^{16m})^2} \tag{7.23}$$

and

$$\sum_{n=0}^{\infty} a(4n+1)q^{4n+1} = 2q\sum_{n=1}^{\infty} q^{4n(n+1)} \tag{7.24}$$

We again see from Lemma 7.11 (after replacing $y = q^{1/2}$ and making some elementary simplifications) that

$$\sum_{n=0}^{\infty} q^{n(n+1)} = \prod_{m=1}^{\infty}\frac{(1-q^{4m})^2}{(1-q^{2m})} \tag{7.25}$$

which together with (7.24) gives

$$\sum_{n=0}^{\infty} a(4n+1)q^{4n+1} = 2q\prod_{m=1}^{\infty}\frac{(1-q^{16m})^2}{(1-q^{8m})}. \tag{7.26}$$

From (7.22) we see that $a(4n+2) = a(4n+3) = 0$ for all $n \in \mathbb{N}$, showing that

$$\sum_{n=0}^{\infty} a(n)q^n = \sum_{n=0}^{\infty} a(4n)q^n + \sum_{n=0}^{\infty} a(4n+1)q^{4n+1}.$$

This together with (7.20), (7.23) and (7.24) implies

$$\prod_{m=1}^{\infty} \frac{(1-q^{2m})^5}{(1-q^m)^2(1-q^{4m})^2} = \prod_{m=1}^{\infty} \frac{(1-q^{8m})^5}{(1-q^{4m})^2(1-q^{16m})^2} + 2q \prod_{m=1}^{\infty} \frac{(1-q^{16m})^2}{(1-q^{8m})},$$

which by Definition 7.5 gives after multiplication by $\prod_{m\geq 1}(1-q^{4m})^2$ the desired result (7.19). □

We obtain immediately by Lemma 7.12 and Proposition 7.8:

Corollary 7.13. *Let $p \geq 5$ be a prime. Then $1 \in R_1(p) \cap R_2(p)$.*

Corollary 7.14. *Let $i \in \{1,2\}$ and $f \in A_0(p^i)$, then $f \in R_2(p^i) \cap R_1(p^i)$.*

Proof. By Corollary 7.13 we have $1 \in R_2(p) \cap R_1(p) \subseteq R_2(p^2) \cap R_1(p^2)$, and by Lemma 7.10 we have $f \cdot 1 \in R_1(p^i) \cap R_2(p^i)$. □

7.3 A $A_0(p)$-module Isomorphism Between $R_1(p)$ and $R_2(p)$

Definition 7.15. *We define $g_\infty(\tau) := F_\infty(\tau)/F_0(4\tau)$ and $g_{1/2}(\tau) := F_{1/2}(\tau)/F_0(4\tau)$ for $\tau \in \mathbb{H}$.*

Remark 7.16. *Note that $g_\infty, g_{1/2} \in A_0(16)$ by Lemma 2.34.*

Definition 7.17. *For $h \in \mathbb{Z}$ we define $B_h := \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}$.*

We need the following easily verifiable proposition in the proof of the next two lemmas.

Proposition 7.18. *Let $m \in \mathbb{N}^*$ be odd and $h \in \mathbb{Z}$ then:*

(i) if $(m^2+4)h - 1 \equiv 0 \pmod{4m}$ then $\Gamma_0(4m)\gamma_{m^2}\gamma_4 = \Gamma_0(4m)\gamma_0 B_h$;

(ii) if $4h+1 \equiv 0 \pmod{m}$ then $\Gamma_0(4m)\gamma_{2m^2}\gamma_4 = \Gamma_0(4m)\gamma_2 B_h$.

Lemma 7.19. *Let $p \geq 5$ be a prime and $f \in R_2(p)$. Then the following identity holds:*

$$(F_0|V_p)(f|\gamma_0 V_{4p}) = (F_\infty|V_{4p})(f|\gamma_4 V_{4p,x_\infty}) + 2(F_{1/2}|V_{4p})(f|\gamma_2 V_{4p,x_{1/2}}) \qquad (7.27)$$

where $x_\infty, x_{1/2}$ are any integers satisfying $1+4x_\infty \equiv 0 \pmod{p}$, $x_\infty + 1 \equiv 0 \pmod{4}$ and $1 + 2x_{1/2} \equiv 0 \pmod{p}$.

Proof. Set $\xi_1 := \begin{pmatrix} 5 & 1 \\ 16 & 4 \end{pmatrix}$ and $\xi_2 := \begin{pmatrix} 5 & -1 \\ 16 & -3 \end{pmatrix}$. Note that $\xi_2 \in \Gamma_0(16)$. One can easily verify that

$$\xi_1 = \xi_2 V_{1,1,4}. \qquad (7.28)$$

Then

$$\begin{aligned}
0 =& F_0(f|\gamma_{p^2}V_{4,-1}) - F_\infty(f|V_4) - F_{1/2}(f|\gamma_{2p^2}V_4) && \text{by Definition 7.6}\\
=& (f|\gamma_{p^2}V_{4,-1}) - g_\infty(f|V_4) - g_{1/2}(f|\gamma_{2p^2}V_4) && \text{by Def. 7.15 after div. by } F_0\\
=& (f|\gamma_{p^2}) - (g_\infty|V_{1,1,4})f - (g_{1/2}|V_{1,1,4})(f|\gamma_{2p^2}) && \text{by applying } V_{1,1,4}\\
=& (f|\gamma_{p^2}\gamma_4) - (g_\infty|\xi_1)(f|\gamma_4) + 2(g_{1/2}|\xi_1)(f|\gamma_{2p^2}\gamma_4) && \text{by applying } \gamma_4\\
=& (f|\gamma_{p^2}\gamma_4) - (g_\infty|\xi_2V_{1,1,4})(f|\gamma_4)\\
&+ 2(g_{1/2}|\xi_2V_{1,1,4})(f|\gamma_{2p^2}\gamma_4) && \text{because of (7.28)}\\
=& (f|\gamma_{p^2}\gamma_4) - (g_\infty|V_{1,1,4})(f|\gamma_4)\\
&+ 2(g_{1/2}|V_{1,1,4})(f|\gamma_{2p^2}\gamma_4) && \text{because } g_\infty, g_{1/2} \in A_0(16)\\
=& (f|\gamma_0B_{-x_\infty}) - (g_\infty|V_{1,1,4})(f|\gamma_4)\\
&+ 2(g_{1/2}|V_{1,1,4})(f|\gamma_2B_{x_{1/2}-x_\infty}) && \text{by Proposition 7.18}\\
=& (f|\gamma_0V_{4p}) - (g_\infty|B_{\frac{x_\infty+1}{4}}V_p)(f|\gamma_4V_{4p,x_\infty})\\
&+ 2(g_{1/2}|B_{\frac{x_\infty+1}{4}}V_p)(f|\gamma_2V_{4p,x_{1/2}}) && \text{by applying } V_{4p,x_\infty}\\
=& (f|\gamma_0V_{4p}) - (g_\infty|V_p)(f|\gamma_4V_{4p,x_\infty}) && \text{by } g_\infty|B_{\frac{x_\infty+1}{4}} = g_\infty\\
&+ 2(g_{1/2}|V_p)(f|\gamma_2V_{4p,x_{1/2}}) && \text{and } g_{1/2}|B_{\frac{x_\infty+1}{4}} = g_{1/2}\\
=& (F_0|V_p)(f|\gamma_0V_{4p}) - (F_\infty|V_p)(f|\gamma_4V_{4p,x_\infty}) && \text{after multiplying by } F_0|V_p\\
&+ 2(F_{1/2}|V_p)(f|\gamma_2V_{4p,x_{1/2}}) && \text{and using Def. 7.15.}
\end{aligned}$$

□

Lemma 7.20. *Let $p \geq 5$ be a prime and $f \in R_1(p)$. Then the following identity holds:*

$$F_0(f|\gamma_0V_{4p}) = (F_\infty|V_4)(f|\gamma_4V_{4p,x_\infty}) + 2(F_{1/2}|V_4)(f|\gamma_2V_{4p,x_{1/2}}), \tag{7.29}$$

where $x_\infty, x_{1/2}$ are any integers satisfying $1 + 4x_\infty \equiv 0 \pmod p$, $x_\infty + 1 \equiv 0 \pmod 4$ and $1 + 2x_{1/2} \equiv 0 \pmod p$.

Proof. Set $\xi_1 := \left(\begin{smallmatrix} 5p & p \\ 16 & 4 \end{smallmatrix}\right)$ and $\xi_2 := \begin{pmatrix} 5p & \frac{1+5x_\infty}{4} \\ 16 & \frac{1+4x_\infty}{p} \end{pmatrix}$. Note that $\xi_2 \in \Gamma_0(16)$. One can easily verify that

$$\xi_1 = \xi_2V_{1,-x_\infty,4p}. \tag{7.30}$$

Then

$$\begin{aligned}
0 =& (F_0|V_p)(f|\gamma_{p^2}V_{4,-1}) - (F_\infty|V_p)(f|V_4)\\
&- (F_{1/2}|V_p)(f|\gamma_{2p^2}V_4) && \text{by Definition 7.7}\\
=& (f|\gamma_{p^2}V_{4,-1}) - (g_\infty|V_p)(f|V_4) && \text{by Definition 7.15}\\
&- (g_{1/2}|V_p)(f|\gamma_{2p^2}V_4) && \text{after dividing by } (F_0|V_p)\\
=& (f|\gamma_{p^2}) - (g_\infty|V_{p,p,4})f - (g_{1/2}|V_{p,p,4})(f|\gamma_{2p^2}) && \text{by applying } V_{1,1,4}\\
=& (f|\gamma_{p^2}\gamma_4) - (g_\infty|\xi_1)(f|\gamma_4) + 2(g_{1/2}|\xi_1)(f|\gamma_{2p^2}\gamma_4) && \text{by applying } \gamma_4\\
=& (f|\gamma_{p^2}\gamma_4) - (g_\infty|\xi_2V_{1,-x_\infty,4p})(f|\gamma_4)\\
&+ 2(g_{1/2}|\xi_2V_{1,-x_\infty,4p})(f|\gamma_{2p^2}\gamma_4) && \text{because of (7.30)}
\end{aligned}$$

$=(f|\gamma_{p^2}\gamma_4) - (g_\infty|V_{1,-x_\infty,4p})(f|\gamma_4)$
$\quad + 2(g_{1/2}|V_{1,-x_\infty,4p})(f|\gamma_{2p^2}\gamma_4)$ because $g_\infty, g_{1/2} \in A_0(16)$
$=(f|\gamma_0 B_{-x_\infty}) - (g_\infty|V_{1,-x_\infty,4p})(f|\gamma_4)$
$\quad + 2(g_{1/2}|V_{1,-x_\infty,4p})(f|\gamma_2 B_{x_{1/2}-x_\infty})$ by Proposition 7.18
$=(f|\gamma_0 V_{4p}) - g_\infty(f|\gamma_4 V_{4p,x_\infty})$
$\quad + 2g_{1/2}(f|\gamma_2 V_{4p,x_{1/2}})$ by applying V_{4p,x_∞}
$=F_0(f|\gamma_0 V_{4p}) - F_\infty(f|\gamma_4 V_{4p,x_\infty})$ after multiplying by F_0
$\quad + 2(F_{1/2}|V_p)(f|\gamma_2 V_{4p,x_{1/2}})$ and using Def. 7.15. □

Corollary 7.21. *Let $p \geq 5$ be a prime and $x_{1/2}, x_\infty$ as in Lemma 7.20. Then if $f \in R_1(p)$*

$$\tilde{F}_0(q)(f|\gamma_0 V_{4p}) = \tilde{F}_\infty(q^4)(f|\gamma_4 V_{4p,x_\infty}) + 2q\tilde{F}_{1/2}(q^4)(f|\gamma_2 V_{4p,x_{1/2}}) \tag{7.31}$$

and if $f \in R_2(p)$

$$\tilde{F}_0(q^p)(f|\gamma_0 V_{4p}) = \tilde{F}_\infty(q^{4p})(f|\gamma_4 V_{4p,x_\infty}) + 2q^p\tilde{F}_{1/2}(q^{4p})(f|\gamma_2 V_{4p,x_{1/2}}). \tag{7.32}$$

Proof. The result follows immediately from Lemma 7.20, Lemma 7.19 and (2.26). □

Definition 7.22. *For $x, m \in \mathbb{Z}$ we define $\xi_{p,x} := \begin{pmatrix} m & x \\ 4m & 4x+1 \end{pmatrix}$.*

Lemma 7.23. *Let $p \geq 5$ be a prime, $i \in \{1,2\}$ and $f \in R_i(p)$. Let $x_\infty, x_{1/2}$ be any integers such that $4x_\infty + 1 \equiv 0 \pmod{p}$, $x_\infty + 1 \equiv 0 \pmod{4}$ and $2x_{1/2} + 1 \equiv 0 \pmod{p}$. Then $g := f|\xi_{p,x_\infty} \in R_{j(i)}(p)$ where $(j(1), j(2)) := (2,1)$. In addition we have*

$$f|\gamma_0 V_{4p} = g|\gamma_{p^2} V_{4,-1}; \tag{7.33}$$
$$f|\gamma_4 V_{4p,x_\infty} = g|V_4; \tag{7.34}$$
$$f|\gamma_2 V_{4p,x_{1/2}} = g|\gamma_{2p^2} V_4; \tag{7.35}$$
$$g|\xi_{p,x_\infty} = f \Leftrightarrow f|\xi_{p,x_\infty}^2 = f. \tag{7.36}$$

Proof. First we note that for any $\gamma \in \Gamma_0(4p)$ there exists $\gamma' \in \Gamma_0(4p)$ such that $\xi_{p,x_\infty}\gamma = \gamma'\xi_{p,x_\infty}$. Hence $f|\xi_{p,x_\infty}\gamma = f|\gamma'\xi_{p,x_\infty} = f|\xi_{p,x_\infty}$. Showing that $g \in A_0(4p)$. After substituting (7.33)-(7.35) in (7.29) one can verify that $g \in R_2(p)$ with Definition 7.7.

Proof of (7.33). Because of $\xi_{p,x_\infty}^{-1}\gamma_0 V_{4p} V_{4,-1}^{-1}\gamma_{p^2}^{-1} \in \Gamma_0(4p)$ we have

$$f|\gamma_0 V_{4p} = g|\xi_{p,x_\infty}^{-1}\gamma_0 V_{4p} = g|(\xi_{p,x_\infty}^{-1}\gamma_0 V_{4p} V_{4,-1}^{-1}\gamma_{p^2}^{-1})\gamma_{p^2} V_{4,-1} = g|\gamma_{p^2} V_{4,-1}.$$

Proof of (7.34). Because of $\xi_{p,x_\infty}^{-1}\gamma_4 V_{4p,x_\infty} V_4^{-1} = \mathrm{id} \in \Gamma_0(4p)$ we have

$$f|\gamma_4 V_{4p,x_\infty} = g|(\xi_{p,x_\infty}^{-1}\gamma_4 V_{4p,x_\infty} V_4^{-1})V_4 = g|V_4.$$

Proof of (7.35). Because of $\xi_{p,x_\infty}^{-1}\gamma_2 V_{4p,x_{1/2}} V_4^{-1}\gamma_{2p^2}^{-1} \in \Gamma_0(4p)$ we have

$$f|\gamma_2 V_{4p,x_{1/2}} = g|(\xi_{p,x_\infty}^{-1}\gamma_2 V_{4p,x_{1/2}} V_4^{-1}\gamma_{2p^2}^{-1})\gamma_{2p^2} V_4 = g|\gamma_{2p^2} V_4.$$

Proof of (7.36). Because of $\xi_{p,x_\infty}^2 = V_{p,0,p}\Gamma_0(4p)$ and because of $(f|V_{p,0,p}) = f$ we have (7.36).

The proof when $f \in R_2(p)$ is analogous. □

Lemma 7.24. *Let $p \geq 5$ be a prime. Let $x_\infty, x_{1/2}$ be any integers such that $4x_\infty + 1 \equiv 0 \pmod{p}$, $x_\infty + 1 \equiv 0 \pmod{4}$ and $2x_{1/2} + 1 \equiv 0 \pmod{p}$. If $f \in R_1(p)$ then*

$$\begin{aligned} \mathrm{ord}(f|\gamma_0 V_{4p}) &\leq 4 \cdot \mathrm{ord}(f|\gamma_4 V_{p,x_\infty}), & (7.37)\\ \mathrm{ord}(f|\gamma_0 V_{4p}) &\leq 4 \cdot \mathrm{ord}(f|\gamma_2 V_{p,x_{1/2}}) + 1, & (7.38)\end{aligned}$$

with equality in either (7.37) *or* (7.38). *If $f \in R_2(p)$ then*

$$\begin{aligned} \mathrm{ord}(f|\gamma_0 V_{4p}) &\leq 4 \cdot \mathrm{ord}(f|\gamma_4 V_{p,x_\infty}), & (7.39)\\ \mathrm{ord}(f|\gamma_0 V_{4p}) &\leq 4 \cdot \mathrm{ord}(f|\gamma_2 V_{p,x_{1/2}}) + p, & (7.40)\end{aligned}$$

with equality in either (7.39) *or* (7.40).

Proof. Let $g := f|\xi_{p,x_\infty}$. Then by (7.33)-(7.35) we see that proving (7.37)-(7.38) is equivalent to proving

$$\begin{aligned} \mathrm{ord}(g|\gamma_{p^2} V_{4,-1}) &\leq 4 \cdot \mathrm{ord}(g), & (7.41)\\ \mathrm{ord}(g|\gamma_{p^2} V_{4,-1}) &\leq 4 \cdot \mathrm{ord}(g|\gamma_{2p^2}) + 1, & (7.42)\end{aligned}$$

with one of them being an equality. The truth of (7.41)-(7.42) is implied by (7.13)-(7.14) because by Lemma 7.23 $g \in R_2(p)$. The proof of (7.39)-(7.40) is analogous. □

Lemma 7.25. *Let $p \geq 5$ a prime, $f \in A_0(p)$ and $x \in \mathbb{Z}$ such that $4x + 1 \equiv 0 \pmod{p}$. Then $f|\xi_{p,x} \in A_0(p)$.*

Proof. Let $\gamma = \begin{pmatrix} a & b \\ pc & d \end{pmatrix}$. Then one easily verifies that there exists a $\gamma' \in \Gamma_0(p)$ such that $\xi_{p,x}\gamma\xi_{p,x}^{-1} = \gamma'$ which implies that

$$(f|\xi_{p,x})|\gamma = (f|\xi_{p,x}\gamma) = (f|\gamma'\xi_{p,x}) = (f|\xi_{p,x}).$$

This verifies condition *(ii)* of Definition 2.6. Condition *(i)* of Definition 2.6 is immediate from Proposition 2.7 and condition *(iii)* of Definition 2.6 follows by *(i)* in Proposition 2.8. □

Definition 7.26. *Let $p \geq 5$ be a prime and $i \in \{1, 2\}$. Then $(R_i(p), \odot)$ denotes the $A_0(p)$-module with $g \odot f := (g|\xi_{p,x_\infty})f$ for $g \in A_0(p)$, $f \in R_i(p)$, and where x_∞ is any integer such that $4x_\infty + 1 \equiv 0 \pmod{p}$. By $(R_i(p), \cdot)$ we denote the $A_0(p)$-module with $g \cdot f := gf$ for $g \in A_0(p)$ and $f \in R_i(p)$.*

Note that $\odot$ is well defined because for $h \in \mathbb{Z}$ we have $\begin{pmatrix} p & x_\infty \\ 4p & 4x_\infty+1 \end{pmatrix}\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} p & ph+x_\infty \\ 4p & 4(x_\infty+ph)+1 \end{pmatrix}$, which because of $g|\xi_{p,x_\infty} \in A_0(p)$ implies $g|\xi_{p,x_\infty}|\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} = g|\xi_{p,x_\infty}$ by Lemma 7.25 showing that the choice of x_∞ does not result in any ambiguity. The goal of this section is fulfilled by the following lemma that gives the module isomorphism between $R_1(p)$ and $R_2(p)$.

Lemma 7.27. *Let $p \geq 5$ be a prime and x_∞ a integer such that $4x_\infty + 1 \equiv 0 \pmod{p}$ and $x_\infty + 1 \equiv 0 \pmod{4}$. Then the mapping $\phi : R_1(p) \to R_2(p)$ given by $\phi(f) := f|\xi_{p,x_\infty}$ is a module isomorphism from the module $(R_1(p), \cdot)$ to the module $(R_2(p), \odot)$.*

Proof. By Lemma 7.23 we see that $\phi(f) \in R_2(p)$ if $f \in R_1(p)$. So ϕ maps $R_1(p)$ to $R_2(p)$. That it is a homomorphism follows from Definition 7.26. Also because of (7.36) we see that $\phi^{-1}(f) = f|\xi_{p,x_\infty}$ showing that it is one to one and onto. □

Remark 7.28. *In particular, Lemma 7.27 shows that if* $b_1, \dots, b_n$ *is a* $A_0(p)$*-basis for* $R_1(p)$ *then* $\phi(b_1), \dots, \phi(b_n)$ *is* $A_0(p)$*-basis for* $R_2(p)$*. This gives us an algorithm to construct a basis for* $R_2(p)$ *once a basis for* $R_1(p)$ *is known which we will exploit in the last section of this chapter.*

7.4 $A_0(5) = \mathbb{C}[t, t^{-1}]$

In this section we prove that $A_0(5) = \mathbb{C}[t, t^{-1}]$. Recall form (6.7) that $t = \frac{\eta_5^6}{\eta^6}$. We need the following lemma

Lemma 7.29. *Let* δ *be a positive divisor of* 4*. Then*

$$(\eta_\delta|\gamma_0 V_{20})(\tau) = e^{-\pi i/4}(20\tau/\delta)^{1/2}\eta_{20/\delta}(\tau)$$

for $\tau \in \mathbb{H}$.

Proof. By Lemma 2.27 we have

$$(\eta_\delta|\gamma_0 V_{20})(\tau) = \eta\left(-\frac{1}{\frac{20}{\delta}\tau}\right) = \upsilon_\eta(0,-1,1,0)\eta_{20/\delta}(\tau)$$

and $\upsilon_\eta(0,-1,1,0) = e^{-\pi i/4}$ by (2.30). □

Corollary 7.30. *We have* $t|\gamma_0 V_5 = 5^{-3}t^{-1}$.

Proof. By Lemma 7.29,

$$(t|\gamma_0 V_{20})(\tau) = \frac{(\eta_5^6|\gamma_0 V_{20})(\tau)}{(\eta^6|\gamma_0 V_{20})(\tau)} = \frac{e^{-6\pi i/4}(20\tau/5)^3\eta_{20/5}^6(\tau)}{e^{-6\pi i/4}(5\tau)^3\eta_{20}^6(\tau)} = \frac{1}{5^3}\frac{\eta_4^6(\tau)}{\eta_{20}^6(\tau)} = (5^{-3}t^{-1}|V_4)(\tau) \tag{7.43}$$

for $\tau \in \mathbb{H}$. From (7.43) we find $(t|\gamma_0 V_5) = (t|\gamma_0 V_{20} V_4^{-1}) = (5^{-3}t^{-1}|V_4 V_4^{-1}) = 5^{-3}t^{-1}$. □

Lemma 7.31. *Let* $f \in A_0(5)$*. Then*

$$\mathrm{Ord}_{\Gamma_0(5)}(f, \gamma_0) = \mathrm{ord}(f|\gamma_0 V_5), \qquad \mathrm{Ord}_{\Gamma_0(5)}(f, \mathrm{id}) = \mathrm{ord}(f).$$

Proof. We use Lemma 2.37 to compute $\omega_{\Gamma_0(5),\gamma}$ for $\gamma \in \{\mathrm{id}, \gamma_0\}$ and then apply Lemma 2.15. □

Lemma 7.32. *Let* $f \neq 0 \in R_1(5)$*. Then* $\mathrm{Ord}_{\Gamma_0(5)}(f, \gamma_0) < 0$ *or* $\mathrm{Ord}_{\Gamma_0(5)}(f, \mathrm{id}) < 1$.

Proof. Assume by contradiction that $\mathrm{Ord}_{\Gamma_0(5)}(f, \gamma_0) \geq 0$ and $\mathrm{Ord}_{\Gamma_0(5)}(f, \mathrm{id}) \geq 1$. Then

$$\sum_{s \in R} \mathrm{Ord}_{\Gamma_0(20)}(f, s) \geq 1,$$

contradicting Lemma 2.18 because

$$R := \{\gamma_0, \mathrm{id}\}$$

is a complete set of representatives for the double cosets $\Gamma_0(5)\backslash\mathrm{SL}_2(\mathbb{Z})/\mathrm{SL}_2(\mathbb{Z})_\infty$ by Lemma 2.45. □

Lemma 7.33. *For $t = \frac{\eta_5^6}{\eta^6}$ we have*

$$\mathrm{Ord}_{\Gamma_0(5)}(t, \gamma_0) = -1 \quad \textit{and} \quad \mathrm{Ord}_{\Gamma_0(5)}(t, \mathrm{id}) = 1.$$

Proof. By Lemma 7.31 we have $\mathrm{Ord}_{\Gamma_0(5)}(t, \mathrm{id}) = \mathrm{ord}(t)$. By (2.26) we have $\mathrm{ord}(t) = 1$. By Lemma 7.31 we have $\mathrm{Ord}_{\Gamma_0(5)}(t, \gamma_0) = \mathrm{ord}(t|\gamma_0 V_5)$ and by Corollary 7.30 we have $(t|\gamma_0 V_5) = 5^{-3}t^{-1}$ and consequently $\mathrm{ord}(t|\gamma_0 V_5) = \mathrm{ord}(t^{-1}) = -1$ because of $\mathrm{ord}(t) = 1$. □

Lemma 7.34. *We have $\mathbb{C}[t, t^{-1}] = A_0(5)$.*

Proof. Let $f \in A_0(5)$. We must show that there is a Laurent polynomial $p(t) \in \mathbb{C}[t, t^{-1}]$ such that $f = p(t)$. We split the problem in two cases depending on the condition $\mathrm{Ord}_{\Gamma_0(5)}(f, \mathrm{id}) \geq 1$ or not.

Case 1. $\mathrm{Ord}_{\Gamma_0(5)}(f, \mathrm{id}) \geq 1$. By Lemma 7.33 we have $\mathrm{Ord}_{\Gamma_0(5)}(t, \mathrm{id}) = 1$, which implies that $\mathrm{Ord}_{\Gamma_0(5)}(f + p(t), \mathrm{id}) \geq 1$ for any $p(t) \in \mathbb{C}[t]$ with $p(0) = 0$. Next consider the set

$$O(f) := \{f + p(t) | p(t) \in \mathbb{C}[t], p(0) = 0\}. \tag{7.44}$$

We must prove that $0 \in O(f)$, because this implies that there exists $p(t) \in \mathbb{C}[t]$ (with $p(0) = 0$) such that $f = p(t)$ and this is what we want. Assume by contradiction $0 \notin O(f)$. Then there is an integer $n < 0$ such that

$$\max_{g \in O(f)} \mathrm{Ord}_{\Gamma_0(5)}(g, \gamma_0) = n, \tag{7.45}$$

because by Lemma 7.49, if $g \neq 0 \in A_0(5)$ and $\mathrm{Ord}_{\Gamma_0(5)}(g, \mathrm{id}) \geq 1$, then $\mathrm{Ord}_{\Gamma_0(5)}(g, \gamma_0) < 0$. Let $g \in O(f)$ be such that $\mathrm{Ord}_{\Gamma_0(5)}(g, \gamma_0) = n$. Then by Lemma 7.33 $\mathrm{Ord}_{\Gamma_0(5)}(t^{-n}, \gamma_0) = n$ and there exists a $c \in \mathbb{C}$ such that either

$$g + ct^{-n} = 0 \quad \text{or} \quad \mathrm{Ord}_{\Gamma_0(5)}(g + ct^{-n}, \gamma_0) > n.$$

Note that $g + ct^{-n} \in O(f)$ by (7.44) and because of $n < 0$. If $g + ct^{-n} = 0$, then $0 \in O(f)$. On the other hand if $\mathrm{Ord}_{\Gamma_0(5)}(g + ct^{-n}, \gamma_0) > n$ we have a contradiction to (7.45).

Case 2. $\mathrm{Ord}_{\Gamma_0(5)}(f, \mathrm{id}) < 1$. Assume that $\mathrm{Ord}_{\Gamma_0(5)}(f, \mathrm{id}) = n < 1$. Then by Lemma 7.33 we have $\mathrm{Ord}_{\Gamma_0(5)}(t^r f, \mathrm{id}) = r + n$. Consequently, for $r = 1 - n$ we have $\mathrm{Ord}_{\Gamma_0(5)}(t^r f, \gamma_{5^2}) \geq 1$ and we proved in *Case 1* that there is $p(t) \in \mathbb{C}[t]$ such that $t^r f = p(t)$ implying $f = t^{-r} p(t)$. □

7.5 Conditions that Yield Generators for the $A_0(5)$-Module $R_1(5)$

The structure of this section follows closely structure of the previous section and can thus be viewed as a generalization. The goal of this section is to prove the following lemma.

Lemma 7.35. *Let $P \in R_1(5)$, $t = \frac{\eta_5^6}{\eta^6}$ and assume that*

$$\mathrm{Ord}_{\Gamma_0(20)}(P, \gamma_0) = -7 \quad \textit{and} \quad \mathrm{Ord}_{\Gamma_0(20)}(P, \gamma_{5^2}) = 4.$$

Then $R_1(5)$ is generated by $\{1, P\}$ as a $\mathbb{C}[t, t^{-1}]$-module.

We need a couple of lemmas before we give the proof of (7.35).

Lemma 7.36. *Let $f \in A_0(20)$. Then*

$$\begin{array}{rclrcl} \mathrm{Ord}_{\Gamma_0(20)}(f,\gamma_0) &=& \mathrm{ord}(f|\gamma_0 V_{20}), & \mathrm{Ord}_{\Gamma_0(20)}(P,\gamma_{5^2}) &=& \mathrm{ord}(f|\gamma_{5^2}V_{4,-1}), \\ \mathrm{Ord}_{\Gamma_0(20)}(f,\gamma_2) &=& \mathrm{ord}(f|\gamma_2 V_{5,2}), & \mathrm{Ord}_{\Gamma_0(20)}(f,\gamma_{2\cdot 5^2}) &=& \mathrm{ord}(f|\gamma_{2\cdot 5^2}), \\ \mathrm{Ord}_{\Gamma_0(20)}(f,\gamma_4) &=& \mathrm{ord}(f|\gamma_4 V_{5,11}), & \mathrm{Ord}_{\Gamma_0(20)}(f,\mathrm{id}) &=& \mathrm{ord}(f). \end{array}$$

Proof. We use Lemma 2.37 to compute $\omega_{\Gamma_0(20),\gamma}$ for $\gamma \in \{\mathrm{id},\gamma_0,\gamma_2,\gamma_4,\gamma_{5^2},\gamma_{2\cdot 5^2}\}$ and then apply Lemma 2.15. □

Lemma 7.37. *Let $f \neq 0 \in R_1(5)$. Then $\mathrm{Ord}_{\Gamma_0(20)}(f,\gamma_0) < -3$ or $\mathrm{Ord}_{\Gamma_0(20)}(f,\gamma_{5^2}) < 4$.*

Proof. Assume by contradiction that $\mathrm{Ord}_{\Gamma_0(20)}(f,\gamma_0) \geq -3$ and $\mathrm{Ord}_{\Gamma_0(20)}(f,\gamma_{5^2}) \geq 4$. Then by Lemma 7.36 and 7.24 we have

$$-3 \leq \mathrm{Ord}_{\Gamma_0(20)}(f|\gamma_0) = \mathrm{ord}(f|\gamma_0 V_{20}) \leq 4\cdot \mathrm{ord}(f|\gamma_4 V_{5,11}) = 4\cdot \mathrm{Ord}_{\Gamma_0(20)}(f|\gamma_4)$$

and

$$-3 \leq \mathrm{Ord}_{\Gamma_0(20)}(f|\gamma_0) = \mathrm{ord}(f|\gamma_0 V_{20}) \leq 4\cdot \mathrm{ord}(f|\gamma_2 V_{5,2}) + 1 = 4\cdot \mathrm{Ord}_{\Gamma_0(20)}(f|\gamma_2) + 1$$

implying

$$0 = \lceil -3/4 \rceil \leq \mathrm{Ord}_{\Gamma_0(20)}(f|\gamma_4) \quad \text{and} \quad -1 \leq \mathrm{Ord}_{\Gamma_0(20)}(f|\gamma_2). \tag{7.46}$$

By Lemma 7.36 and 7.9 we have

$$4 \leq \mathrm{Ord}_{\Gamma_0(20)}(f,\gamma_{5^2}) = \mathrm{ord}(f|\gamma_{5^2}V_{4,-1}) \leq 4\cdot \mathrm{ord}(f) = 4\cdot \mathrm{Ord}_{\Gamma_0(20)}(f|\,\mathrm{id})$$

and

$$4 \leq \mathrm{Ord}_{\Gamma_0(20)}(f,\gamma_{5^2}) = \mathrm{ord}(f|\gamma_{5^2}V_{4,-1}) \leq 4\cdot \mathrm{ord}(f|\gamma_{2\cdot 5^2}) + 5 = 4\cdot \mathrm{Ord}_{\Gamma_0(20)}(f,\gamma_{2\cdot 5^2})$$

implying

$$0 = \lceil -1/4 \rceil \leq \mathrm{Ord}_{\Gamma_0(20)}(f,\gamma_{2\cdot 5^2}) \quad \text{and} \quad 1 \leq \mathrm{Ord}_{\Gamma_0(20)}(f|\,\mathrm{id}). \tag{7.47}$$

By (7.46) and (7.47) we find

$$\sum_{s\in R} \mathrm{Ord}_{\Gamma_0(20)}(f,s) \geq 1$$

contradicting Lemma 2.18 because

$$R := \{\gamma_0,\gamma_2,\gamma_4,\gamma_{5^2},\gamma_{2\cdot 5^2},\mathrm{id}\}$$

is a complete set of representatives for the double cosets $\Gamma_0(20)\backslash \mathrm{SL}_2(\mathbb{Z})/\mathrm{SL}_2(\mathbb{Z})_\infty$ by Lemma 2.45. □

Lemma 7.38. *For $t = \frac{\eta_5^6}{\eta^6}$ we have*

$$\mathrm{Ord}_{\Gamma_0(20)}(t,\gamma_0) = -4 \quad \textit{and} \quad \mathrm{Ord}_{\Gamma_0(20)}(t,\gamma_{5^2}) = 4.$$

Proof. By Lemma 7.36 we have $\mathrm{Ord}_{\Gamma_0(20)}(t,\gamma_{5^2}) = \mathrm{ord}(t|\gamma_{5^2}V_{4,-1})$. By Lemma 2.34 we have $t \in A_0(5)$ implying $\mathrm{ord}(t|\gamma_{5^2}V_{4,-1}) = \mathrm{ord}(t|V_{4,-1})$. By (2.26) we have $\mathrm{ord}(t) = 1$ and consequently $\mathrm{ord}(t|V_{4,-1}) = 4$. By Lemma 7.36 we have $\mathrm{Ord}_{\Gamma_0(20)}(t,\gamma_0) = \mathrm{ord}(t|\gamma_0 V_{20})$ and by Corollary 7.30 we have $(t|\gamma_0 V_5) = 5^{-3}t^{-1}$ which implies $(t|\gamma_0 V_{20}) = 5^{-3}(t^{-1}|V_4)$. Then $\mathrm{ord}(t|\gamma_0 V_{20}) = \mathrm{ord}(t^{-1}|V_4) = -4$ because of $\mathrm{ord}(t) = 1$. □

Proof of Lemma 7.35: Let $f \in R_1(5)$. We must show that there are Laurent polynomials $q_1(t)$, $q_2(t) \in \mathbb{C}[t, t^{-1}]$ such that $f = q_1(t) + Pq_2(t)$. We split the problem in two cases depending on if $\mathrm{Ord}_{\Gamma_0(20)}(f, \gamma_{5^2}) \geq 4$ or not.

Case 1. $\mathrm{Ord}_{\Gamma_0(20)}(f, \gamma_{5^2}) \geq 4$. By Lemma 7.38 we have $\mathrm{Ord}_{\Gamma_0(20)}(t, \gamma_{5^2}) = 4$ and by assumption $\mathrm{Ord}_{\Gamma_0(20)}(P, \gamma_{5^2}) = 4$, implying that for any $q_1(t)$, $q_2(t) \in \mathbb{C}[t]$ with $q_1(0) = 0$, we have $\mathrm{Ord}_{\Gamma_0(20)}(f + q_1(t) + Pq_2(t), \gamma_{5^2}) \geq 4$.

Next consider the set

$$O(f) := \{f + q_1(t) + Pq_2(t) | q_1(t), q_2(t) \in \mathbb{C}[t], q_1(0) = 0\}. \tag{7.48}$$

We must prove that $0 \in O(f)$, because then we have proven that there exist $q_1(t)$, $q_2(t) \in \mathbb{C}[t]$ (with $q_1(0) = 0$) such that $f = q_1(t) + Pq_2(t)$ and this is what we want. Let us assume that $0 \notin O(f)$. Then there is an integer $n < -3$ such that

$$\max_{g \in O(f)} \mathrm{Ord}_{\Gamma_0(20)}(g, \gamma_0) = n, \tag{7.49}$$

because by Lemma 7.37, if $g \neq 0 \in R_1(5)$ and $\mathrm{Ord}_{\Gamma_0(20)}(g, \gamma_{5^2}) \geq 4$, then $\mathrm{Ord}_{\Gamma_0(20)}(g, \gamma_0) < -3$.

Let $g \in O(f)$ be such that $\mathrm{Ord}_{\Gamma_0(20)}(g, \gamma_0) = n$. By Lemma 7.24 either $n \equiv 0 \pmod 4$ or $n \equiv 1 \pmod 4$.

If $n \equiv 0 \pmod 4$, then by Lemma 7.38, $\mathrm{Ord}_{\Gamma_0(20)}(t^{-\frac{n}{4}}, \gamma_0) = n$ and there exists a $c \in \mathbb{C}$ such that

$$g + ct^{-\frac{n}{4}} = 0 \quad \text{or} \quad \mathrm{Ord}_{\Gamma_0(20)}(g + ct^{-\frac{n}{4}}, \gamma_0) > n.$$

Note that $g + ct^{-\frac{n}{4}} \in O(f)$ by (7.48) and because of $-\frac{n}{4} \in \mathbb{N}^*$. If $g + ct^{-\frac{n}{4}} = 0$, then $0 \in O(f)$. On the other hand if $\mathrm{Ord}_{\Gamma_0(20)}(g + ct^{-\frac{n}{4}}, \gamma_0) > n$, we have a contradiction to (7.49).

If $n \equiv 1 \pmod 4$, then $n \leq -7$ and by Lemma 7.38 together with $\mathrm{Ord}_{\Gamma_0(20)}(P, \gamma_0) = -7$ by assumption, we see that $\mathrm{Ord}_{\Gamma_0(20)}(Pt^{-\frac{n+7}{4}}, \gamma_0) = n$ and there exists a $c \in \mathbb{Z}$ such that

$$g + ct^{-\frac{n+7}{4}} = 0 \quad \text{or} \quad \mathrm{Ord}_{\Gamma_0(20)}(g + ct^{-\frac{n+7}{4}}) > n.$$

Note that $g + ct^{-\frac{n+7}{4}} \in O(f)$ by (7.48) and because of $-\frac{n+7}{4} \in \mathbb{N}$. If $g + ct^{-\frac{n+7}{4}} = 0$, then $0 \in O(f)$. On the other hand if $\mathrm{Ord}_{\Gamma_0(20)}(g + ct^{-\frac{n+7}{4}}, \gamma_0) > n$, we again have a contradiction to (7.49).

Case 2. $\mathrm{Ord}_{\Gamma_0(20)}(f, \gamma_{5^2}) < 4$. Assume that $\mathrm{Ord}_{\Gamma_0(20)}(f, \gamma_{5^2}) = n < 4$. Then by Lemma 7.38 we have $\mathrm{Ord}_{\Gamma_0(20)}(t^r f, \gamma_{5^2}) = 4r + n$. Consequently, for $r = \lceil \frac{4-n}{4} \rceil$ we have $\mathrm{Ord}_{\Gamma_0(20)}(t^r f, \gamma_{5^2}) \geq 4$ and we proved in *Case 1* that there are $q_1(t)$, $q_2(t) \in \mathbb{C}[t]$ such that $t^r f = q_1(t) + Pq_2(t)$, implying $f = t^{-r} q_1(t) + Pt^{-r} q_2(t)$. □

7.6 The Proof of $p_0 \in R_1(5)$ and $p_1 \in R_2(5)$

The next lemma was inspired by [36, p. 5, Lemma 2]:

Lemma 7.39. *Let $f : \mathbb{H} \to \mathbb{C}$ be holomorphic, v, $m \in \mathbb{N}^*$, $r \in \mathbb{Z}$ and $\gamma = \begin{pmatrix} a & b \\ mc & d \end{pmatrix} \in \Gamma_0(m)$. Define $M := mc \gcd(cv, (m-1)(mcr + d))$ and*

$$\Gamma := \begin{cases} \Gamma_0(M), & \text{if } c \neq 0 \\ \mathrm{SL}_2(\mathbb{Z})_\infty, & \text{otherwise.} \end{cases}$$

Assume that $\gcd(v,m)=1$ *and* $f|\xi = f$ *for all* $\xi \in \Gamma$. *Then* $f|U_m\gamma V_{v,r} = f|\gamma V_{v,r}U_m$.

Proof. First we note that

$$\gamma V_{v,r}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}(\gamma V_{v,r})^{-1} = \begin{pmatrix} 1-mcav & a^2v \\ -m^2c^2v & 1+mcav \end{pmatrix} \in \Gamma. \tag{7.50}$$

Let v' be an integer such that $vv' \equiv 1 \pmod{m}$. Define $r_\lambda := v'(r+bd+\lambda d^2)$ for $\lambda \in \mathbb{Z}$. One verifies easily that $\{r_0,\dots,r_{m-1}\}$ is a complete set of representatives of the residue classes modulo m because $\lambda \mapsto r_\lambda$ is a bijection modulo m (because the inverse is given by $\lambda \equiv a^2vr_\lambda - a^2r - ab \pmod{m}$ because of $ad \equiv 1 \pmod{m}$). This implies by Lemma 3.21 and, because of $(f|\gamma V_{v,r})(\tau+1) = (f|\gamma V_{v,r})(\tau)$ by (7.50), that

$$f|\gamma V_{v,r}U_m = \frac{1}{m}\sum_{\lambda=0}^{m-1} f|\gamma V_{v,r}T_{r_\lambda} \tag{7.51}$$

and $T_\lambda := \begin{pmatrix} 1 & \lambda \\ 0 & m \end{pmatrix}$ for $\lambda \in \mathbb{Z}$. Next observe that for all $x,y \in \mathbb{Z}$ such that

$$ar+b+yd-avx \equiv 0 \pmod{m}, \tag{7.52}$$

we have

$$T_y\gamma V_{v,r}T_x^{-1}V_{v,r}^{-1}\gamma^{-1} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma, \tag{7.53}$$

where

$$\begin{aligned}
A &:= acvx + amcr + ad + ymc^2vx + yp^2c^2r + ymcd - car - bc - ymc^2r - cyd,\\
B &:= -a^2r - ab - ycavx - aymcr - mcyb + \frac{a}{m}(ar+b+yd-avx),\\
C &:= mc(m-1)(mcr+d) + m^2c^2vx,\\
D &:= -mcavx - m^2car - m^2cb + amcr + ad.
\end{aligned}$$

We see that the condition (7.52) is satisfied with $y=\lambda$ and $x=r_\lambda$ for $\lambda \in \{0,\dots,m-1\}$. By (7.53), (7.51) and because of $f|\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) = f$ by assumption we have

$$\begin{aligned}
f|U_m\gamma V_{v,r} &= \frac{1}{m}\sum_{\lambda=0}^{m-1} f|T_\lambda\gamma V_{v,r} = \frac{1}{m}\sum_{\lambda=0}^{m-1} f|\begin{pmatrix} A & B \\ C & D \end{pmatrix}\gamma V_{v,r}T_{r_\lambda}\\
&= \frac{1}{m}\sum_{\lambda=0}^{m-1} f|\gamma V_{v,r}T_{r_\lambda} = f|\gamma V_{v,r}U_m.
\end{aligned}$$

□

The following proposition is just a rewriting of Lemma 3.23.

Proposition 7.40. *Let* $p \geq 5$ *be a prime and* $f,g : \mathbb{H} \to \mathbb{C}$. *Then* $((f|V_p)g|U_p) = f(g|U_p)$.

Lemma 7.41. *Let* $p \geq 5$ *be a prime. Then* $f \in R_1(p^2)$ *implies* $f|U_p \in R_2(p)$.

Proof. By Lemma 3.24 we have $f|U_p \in A_0(4p)$ because $f \in A_0(4p^2)$ by Definition 7.6. It only remains to verify that $(f|U_p)$ satisfies (7.8). By Lemma 7.39 and Proposition 7.40 we see that:

$$\begin{aligned}0 =&(F_{r_0}|V_p)(f|\gamma_{p^2}V_{4,-1})|U_p - (F_{r_\infty}|V_{4p})(f|V_4)|U_p - 2(F_{r_{1/2}}|V_{4p})(f|\gamma_{2p^2}V_4)|U_p\\ =&F_{r_0}(f|\gamma_{p^2}V_{4,-1}U_p) - (F_{r_\infty}|V_4)(f|V_4U_p) - 2(F_{r_{1/2}}|V_4)(f|\gamma_{2p^2}V_4U_p)\\ =&F_{r_0}((f|U_p)|\gamma_{p^2}V_{4,-1}) - (F_{r_\infty}|V_4)((f|U_p)|V_4) - 2(F_{r_{1/2}}|V_4)((f|U_p)|\gamma_{2p^2}V_4) = 0. \quad \square\end{aligned}$$

Proposition 7.42. *Let $p \geq 5$ be a prime and $k, r \in \mathbb{Z}$. Assume that $p^2r + 1 \equiv 0 \pmod 4$. Then $f_{\infty,p}|\gamma_{p^2}V_{4,r} = f_{0,p}$, where $f_{0,p}$ and $f_{\infty,p}$ are as in Definition 7.5.*

Proof. Let δ be a divisor of 4. Define

$$\xi_1 := \begin{pmatrix} \delta & r \\ p^2 & \frac{p^2r+1}{\delta} \end{pmatrix} \quad \text{and} \quad \xi_2 := \begin{pmatrix} \delta & p^2r \\ 1 & \frac{p^2r+1}{\delta} \end{pmatrix}.$$

Then we have

$$V_\delta\gamma_{p^2}V_{4,r} = \xi_1V_{4,0,\delta} \quad \text{and} \quad V_{\delta p^2}\gamma_{p^2}V_{4,r} = \xi_2V_{4p^2,0,\delta}. \tag{7.54}$$

By Lemma 2.27 we have

$$(\eta|\xi_1V_{4,0,\delta})(\tau) = \left(\frac{4p^2\tau + p^2r + 1}{\delta}\right)^{1/2} \upsilon_\eta(\delta, r, p^2, (p^2r+1)/\delta)\eta\left(\frac{4\tau}{\delta}\right)$$

and

$$(\eta|\xi_2V_{4p^2,0,\delta})(\tau) = \left(\frac{4p^2\tau + p^2r + 1}{\delta}\right)^{1/2} \upsilon_\eta(\delta, p^2r, 1, (p^2r+1)/\delta)\eta\left(\frac{4p^2\tau}{\delta}\right)$$

for $\tau \in \mathbb{H}$, which implies that

$$\frac{\eta|\xi_1V_{4,0,\delta}}{\eta|\xi_2V_{4p^2,0,\delta}} = \frac{\eta_{4/\delta}}{\eta_{4p^2/\delta}} \tag{7.55}$$

because of $\frac{\upsilon_\eta(\delta,r,p^2,(p^2r+1)/\delta)}{\upsilon_\eta(\delta,rp^2,1,(p^2r+1)/\delta)} = 1$ by Lemma 2.29.

From (7.54) and (7.55) follows

$$\frac{\eta_\delta}{\eta_{\delta p^2}}|\gamma_{p^2}V_{4,r} = \frac{\eta|V_\delta}{\eta|V_{\delta p^2}}|\gamma_{p^2}V_{4,r} = \frac{\eta|V_\delta\gamma_{p^2}V_{4,r}}{\eta|V_{\delta p^2}\gamma_{p^2}V_{4,r}} = \frac{\eta|\xi_1V_{4,0,\delta}}{\eta|\xi_2V_{4p^2,0,\delta}} = \frac{\eta_{4/\delta}}{\eta_{4p^2/\delta}}. \tag{7.56}$$

Recalling that $f_{\infty,p} = \frac{\eta_2^5\eta_{4p^2}^2}{\eta_{2p^2}^5\eta_4^2}$ one gets from (7.56) that $f_{\infty,p}|\gamma_{p^2}V_{4,r} = \frac{\eta_2^5\eta_{p^2}^2}{\eta_{2p^2}^5\eta^2} = f_{0,p}$. $\square$

Proposition 7.43. *Let $p \geq 5$ be a prime. Then $f_{\infty,p}|\gamma_{2p^2} = f_{1/2,p}$.*

Proof. Let $\delta \neq 4$ be a divisor of 4. Then by Lemma 2.34 $\frac{\eta_\delta}{\eta_{\delta p^2}} \in A_0(2p^2)$ we have

$$\frac{\eta_\delta}{\eta_{\delta p^2}}|\gamma_{2p^2} = \frac{\eta_\delta}{\eta_{\delta p^2}}. \tag{7.57}$$

Let $\xi_1 := \begin{pmatrix} 2 & -p^2 \\ 1 & (1-p^2)/2 \end{pmatrix}$ and $\xi_2 := \begin{pmatrix} 2 & -1 \\ p^2 & (1-p^2)/2 \end{pmatrix}$. Then

$$V_{4p^2}\gamma_{2p^2} = \xi_1V_{p^2}V_{2,1,2} \quad \text{and} \quad V_4\gamma_{2p^2} = \xi_2V_{2,1,2}.$$

This relations imply:

$$\frac{\eta_{4p^2}}{\eta_4}|\gamma_{2p^2} = \frac{\eta|V_{4p^2}}{\eta|V_4}|\gamma_{2p^2} = \frac{\eta|V_{4p^2}\gamma_{2p^2}}{\eta|V_4\gamma_{2p^2}} = \frac{\eta|\xi_1 V_{p^2}V_{2,1,2}}{\eta|\xi_2 V_{2,1,2}}. \tag{7.58}$$

Next we use a derivation analog to the one used to prove (7.55) to show that

$$\frac{\eta|\xi_1 V_{p^2}V_{2,1,2}}{\eta|\xi_2 V_{2,1,2}} = \frac{\eta|V_{p^2}V_{2,1,2}}{\eta|V_{2,1,2}}. \tag{7.59}$$

By (7.59) and (7.58) we obtain

$$\frac{\eta_{4p^2}}{\eta_4}|\gamma_{2p^2} = \frac{\eta|\xi_1 V_{p^2}V_{2,1,2}}{\eta|\xi_2 V_{2,1,2}} = \frac{\eta|V_{p^2}V_{2,1,2}}{\eta|V_{2,1,2}}. \tag{7.60}$$

Next we exploit the well known relation (which is proven elementary after writing the eta products as infinite q-products):

$$\frac{e^{\pi i/24}\eta_2^3}{\eta_4\eta} = \eta|V_{2,1,2}. \tag{7.61}$$

Because of (7.60) and (7.61) we obtain:

$$\frac{\eta_{4p^2}^2}{\eta_4^2}|\gamma_{2p^2} = \frac{\eta^2|V_{p^2}V_{2,1,2}}{\eta^2|V_{2,1,2}} = \frac{(\eta|V_{2,1,2})^2|V_{p^2}}{(\eta|V_{2,1,2})^2} = \frac{\left(\frac{\eta_2^3}{\eta_4\eta}\right)^2|V_{p^2}}{\left(\frac{\eta_2^3}{\eta_4\eta}\right)^2} = \frac{\eta^2\eta_4^2\eta_{2p^2}^6}{\eta_{p^2}^2\eta_{4p^2}^2\eta_2^6}. \tag{7.62}$$

By definition $f_{\infty,p} = \frac{\eta_2^5\eta_{4p^2}^2}{\eta_{2p^2}^5\eta_4^2}$, which implies together with (7.57) and (7.62) that

$$f_{\infty,p}|\gamma_{2p^2} = \frac{\eta^2\eta_4^2\eta_{2p^2}}{\eta_{p^2}^2\eta_{4p^2}^2\eta_2} = f_{1/2,p}.$$

□

Lemma 7.44. *Let $p \geq 5$ and $f \in R_2(p^2)$, then $\frac{F_\infty f}{F_\infty|V_{p^2}}|U_p = f_{\infty,p}f|U_p \in R_1(p)$.*

Proof. We see that $\frac{F_\infty f}{F_\infty|V_{p^2}}|U_p \in A_0(4p)$ by Lemma 3.24. It only remains to show that (7.7) is satisfied. By Definition 7.7 and the Propositions 7.42 and 7.43 we obtain:

$$\begin{aligned}
0 =&F_0(f|\gamma_{p^2}V_{4,-1}) - (F_\infty|V_4)(f|V_4) - 2(F_{1/2}|V_4)(f|\gamma_{2p^2}V_4)\\
=&(F_0|V_{p^2})\frac{F_0}{F_{r_0}|V_{p^2}}(f|\gamma_{p^2}V_{4,-1}) - (F_\infty|V_{4p^2})\frac{F_\infty|V_4}{F_\infty|V_{4p^2}}(f|V_4)\\
&-2(F_{1/2}|V_{4p^2})\frac{F_{1/2}|V_4}{F_{1/2}|V_{4p^2}}(f|\gamma_{2p^2}V_4)\\
=&(F_0|V_{p^2})f_{0,p}(f|\gamma_{p^2}V_{4,-1}) - (F_\infty|V_{4p^2})(f_{\infty,p}|V_4)(f|V_4)\\
&-2(F_{1/2}|V_{4p^2})(f_{1/2,p}|V_4)(f|\gamma_{2p^2}V_4)\\
=&(F_0|V_{p^2})(f_{\infty,p}f|\gamma_{p^2}V_{4,-1}) - (F_\infty|V_{4p^2})(f_{\infty,p}f|V_4)\\
&-2(F_{1/2}|V_{4p^2})(f_{\infty,p}f|\gamma_{2p^2}V_4)\\
=&0.
\end{aligned}$$

Next we apply the U_p operator to the last line above and use Lemma 7.39 and Proposition 7.40 as in the proof of Lemma 7.41 to get the desired result. □

We are now ready to prove the main goal of this section.

Lemma 7.45. *$p_0 \in R_1(5)$ and $p_1 \in R_2(5)$.*

Proof. In the previous chapter, section 6.6 "The Fundamental Relations", the first relation of Group I is:

$$(Au^{-4}|U_5) = -5t + 25p_0 \tag{7.63}$$

where $A = \frac{\eta_2^5 \eta_{100}^2}{\eta_{50}^5 \eta_4^2}$ by (6.3), $u = \frac{\eta}{\eta^{25}}$ by (6.4) and $t = \frac{\eta_5^6}{\eta^6}$ by Definition 6.6. By Corollary 7.14

$$u^{-4} \in R_2(5^2) \quad \text{and} \quad t \in R_1(5) \tag{7.64}$$

because $u^{-4} \in A_0(5^2)$, $t \in A_0(5)$ by Lemma 2.34. By Lemma 7.44 we have

$$Au^{-4}|U_5 \in R_1(5) \tag{7.65}$$

because of (7.64) and because of $f_{\infty,5} = A$ by (7.5). Noting that $R_2(5)$ is a vector space over $\mathbb{C}$ we obtain by (7.63), (7.64) and (7.65)

$$\frac{1}{25}(Au^{-4}|U_5) + \frac{1}{5}t = p_0 \in R_1(5). \tag{7.66}$$

In the previous chapter, section 6.6, the second relation of Group IV is:

$$p_0 t^{-2}|U_5 = -5^5 t^2 - 14 \cdot 5^2 t + 7 - 5p_1. \tag{7.67}$$

By Lemma 7.10 we have

$$p_0 t^{-2} \in R_1(5) \tag{7.68}$$

because of (7.66) and because of $t^{-2} \in A_0(5)$ by Lemma 2.34. By Lemma 7.41 and because of (7.68) we have

$$p_0 t^{-2}|U_5 \in R_2(5) \tag{7.69}$$

Next observe by Corollary 7.14

$$-5^5 t^2 - 14 \cdot 5^2 t + 7 \in R_2(5) \tag{7.70}$$

because by Lemma 2.34 $1, t, t^2 \in A_0(5)$ and because $A_0(5)$ is a vector space over $\mathbb{C}$. Next we use that $R_2(5)$ is a vector space over $\mathbb{C}$ which implies by linearity together with (7.67), (7.69) and (7.70)

$$\frac{1}{5}(p_0 t^{-2}|U_5) + 5^4 t^2 + 14 \cdot 5t - \frac{7}{5} = p_1 \in R_2(5).$$

□

7.7 The $A_0(5)$-module $R_1(5)$ is Generated by $\{1, p_0\}$

Lemma 7.46. *Let δ be a positive divisor of 4. Then for all $\tau \in \mathbb{H}$ we have*

$$(\eta_\delta|\gamma_{5^2} V_{4,-1})(\tau) = e^{2\pi i(\delta - 24/\delta - 3)/24} \left(\frac{100\tau - 24}{\delta}\right)^{1/2} \eta_{4/\delta}(\tau)$$

and

$$(\eta_{5\delta}|\gamma_{5^2} V_{4,-1})(\tau) = \left(\frac{-24/\delta}{5}\right) e^{2\pi i \cdot 5(\delta - 24/\delta - 3)/24} \left(\frac{100\tau - 24}{\delta}\right)^{1/2} \eta_{20/\delta}(\tau).$$

Proof. Because of the matrix relations

$$\begin{pmatrix} 4\delta & -\delta \\ 4{\cdot}5^2 & -24 \end{pmatrix} = \begin{pmatrix} \delta & -1 \\ 5^2 & -24/\delta \end{pmatrix} V_{4,0,\delta} \quad \text{and} \quad \begin{pmatrix} 20\delta & -5\delta \\ 4{\cdot}5^2 & -24 \end{pmatrix} = \begin{pmatrix} \delta & -5 \\ 5 & -24/\delta \end{pmatrix} V_{20,0,\delta}$$

we have by Lemma 2.27:

$$\begin{aligned}(\eta_\delta|\gamma_{5^2}V_{4,-1})(\tau) &= \eta\left(\frac{4\delta\tau-\delta}{4{\cdot}5^2\tau-24}\right) = \eta\left(\frac{\delta\frac{4\tau}{\delta}-1}{5^2\frac{4\tau}{\delta}-24/\delta}\right) \\ &= \upsilon_\eta(\delta,-1,5^2,-24/\delta)\left(\frac{100\tau-24}{\delta}\right)^{1/2}\eta\left(\frac{4\tau}{\delta}\right)\end{aligned} \tag{7.71}$$

and

$$\begin{aligned}(\eta_{5\delta}|\gamma_{5^2}V_{4,-1})(\tau) &= \eta\left(\frac{20\delta\tau-5\delta}{4{\cdot}5^2\tau-24}\right) = \eta\left(\frac{\delta\frac{20\tau}{\delta}-5}{5\frac{20\tau}{\delta}-24/\delta}\right) \\ &= \upsilon_\eta(\delta,-5,5,-24/\delta)\left(\frac{100\tau-24}{\delta}\right)^{1/2}\eta\left(\frac{20\tau}{\delta}\right)\end{aligned} \tag{7.72}$$

where

$$\upsilon_\eta(\delta,-5,5,-24/\delta) = \left(\frac{-24/\delta}{5}\right)e^{2\pi i\cdot 5(\delta-24/\delta-3)/24}$$

and

$$\upsilon_\eta(\delta,-1,5^2,-24/\delta) = e^{2\pi i\cdot 25(\delta-24/\delta-3)/24}$$

because of (2.30). □

Corollary 7.47. *Let*

$$t := \frac{\eta_5^6}{\eta^6}, \quad \rho := \frac{\eta_2\eta_{10}^3}{\eta_4^3\eta_{20}}, \quad \sigma := \frac{\eta_2^2\eta_5^4}{\eta^4\eta_{10}^2}.$$

Then

$$\begin{aligned} t|\gamma_0V_{20} &= \tfrac{1}{5^3}t^{-1}|V_4, & t|\gamma_{5^2}V_{4,-1} &= t|V_4, \\ \rho|\gamma_0V_{20} &= \tfrac{2^2}{5}\tfrac{\eta_{10}\eta_2^3}{\eta_5^3\eta}, & \rho|\gamma_{5^2}V_{4,-1} &= -4\tfrac{\eta_2\eta_{10}^3}{\eta^3\eta_5}, \\ \sigma|\gamma_0V_{20} &= \tfrac{1}{5}\tfrac{\eta_{10}^2\eta_4^4}{\eta_{20}^4\eta_2^2}, & \sigma|\gamma_{5^2}V_{4,-1} &= \tfrac{\eta_2^2\eta_{20}^4}{\eta_4^4\eta_{10}^2}. \end{aligned}$$

Proof. By Corollary 7.30 we have $t|\gamma_0V_5 = 5^{-3}t^{-1}$ implying $t|\gamma_0V_{20} = \frac{1}{5^3}t^{-1}|V_4$ and $t|\gamma_{5^2}V_{4,-1} = t|V_4$ follows immediately from $t \in A_0(5)$ by Lemma 2.34. For the other cases we apply Lemma 7.29 and Lemma 7.46. By Lemma 7.29

$$\begin{aligned}(\rho|\gamma_0V_{20})(\tau) &= \frac{(\eta_2|\gamma_0V_{20})(\tau)(\eta_{10}^3|\gamma_0V_{20})(\tau)}{(\eta_4^3|\gamma_0V_{20})(\tau)(\eta_{20}|\gamma_0V_{20})(\tau)} \\ &= \frac{e^{-\pi i/4}(20\tau/2)^{1/2}\eta_{20/2}(\tau)e^{-3\pi i/4}(20\tau/10)^{3/2}\eta_{20/10}^3(\tau)}{e^{-3\pi i/4}(20\tau/4)^{3/2}\eta_{20/4}^3(\tau)e^{-\pi i/4}(20\tau/20)^{1/2}\eta_{20/20}(\tau)} \\ &= \frac{2^2}{5}\frac{\eta_{10}\eta_2^3}{\eta_5^3\eta}(\tau)\end{aligned}$$

and by Lemma 7.46

$$\begin{aligned}(\rho|\gamma_{5^2}V_{4,-1})(\tau) &= \frac{(\eta_2|\gamma_{5^2}V_{4,-1})(\tau)(\eta_{10}^3|\gamma_{5^2}V_{4,-1})(\tau)}{(\eta_4^3|\gamma_{5^2}V_{4,-1})(\tau)(\eta_{20}|\gamma_{5^2}V_{4,-1})(\tau)} \\ &= \frac{e^{-2\pi i\frac{13}{24}}\left(\frac{100\tau-24}{2}\right)^{1/2}\eta_{4/2}(\tau)(-1)^3e^{-2\pi i\frac{195}{24}}\left(\frac{100\tau-24}{2}\right)^{3/2}\eta_{20/2}^3(\tau)}{e^{-2\pi i\frac{15}{24}}\left(\frac{100\tau-24}{4}\right)^{3/2}\eta_{4/4}^3(\tau)e^{-2\pi i\frac{25}{24}}\left(\frac{100\tau-24}{4}\right)^{1/2}\eta_{20/4}(\tau)} \\ &= -4\frac{\eta_2\eta_{10}^3}{\eta^3\eta_5}(\tau)\end{aligned}$$

for $\tau \in \mathbb{H}$. The formulas corresponding to σ are proved analogously. □

Lemma 7.48. *Let p_0 be as in Definition 6.6. Then the elements in $\{1, p_0\}$ generate $R_1(5)$ as a $\mathbb{C}[t, t^{-1}]$-module.*

Proof. First we note by Lemma 7.45 that $p_0 \in R_1(5)$. We see by Definition 6.6 that p_0 is a polynomial in t, ρ and σ and we use Corollary 7.47 and (2.26) to compute

$$(p_0 - 25t^2|\gamma_{5^2}V_{4,-1}) = q^4 + 2q^5 - 6q^8 + 32q^9 - 4q^{10} - 105q^{12} + 286q^{13} - 64q^{14} + 8q^{15} + \cdots$$

and

$$(p_0 - 25t^2|\gamma_0 V_{20}) = 625^{-1}(-2q^{-7} + 4q^{-6} - 8q^{-5} + 25q^{-4} - 20q^{-3} + 32q^{-2} - 48q^{-1} + 6 + \cdots).$$

This implies that $\operatorname{ord}(p_0 - 25t^2|\gamma_{5^2}V_{4,-1}) = 4$ and $\operatorname{ord}(p_0 - 25t^2|\gamma_0 V_{20}) = -7$. Using this and Lemma 7.36 we find that $\operatorname{Ord}_{\Gamma_0(20)}(p_0 - 25t^2, \gamma_{5^2}) = 4$ and $\operatorname{Ord}_{\Gamma_0(20)}(p_0 - 25t^2, \gamma_0) = -7$. Then Lemma 7.35 shows that $\{1, p_0 - 25t^2\}$ generate $R_1(5)$ as a $\mathbb{C}[t, t^{-1}]$-module from which follow that also $\{1, p_0\}$ are generators. □

7.8 The $A_0(5)$-module $R_2(5)$ is Generated by $\{1, p_1\}$

The main result will be proved as a corollary of the next lemma. Recall t, p_0, p_1 from Definition 6.6 and ξ from Definition 7.22.

Lemma 7.49. *The following identities hold:*

$$p_0 = 25t^2 + 14t + 1 + 25(p_1|\xi_{5,11})t^2 \tag{7.73}$$

$$(t|\xi_{5,11}) = 5^{-3}t^{-1} \tag{7.74}$$

$$(p_0|\xi_{5,11}) = 5^{-4}t^{-2} + 14{\cdot}5^{-3}t^{-1} + 1 + 5^{-4}p_1t^{-2}. \tag{7.75}$$

Proof. Proof of (7.73)*:* By Lemma 7.23 $p_1|\xi_{5,11} \in R_1(5)$, implying

$$(p_1|\xi_{5,11})t^2 \in R_1(5) \tag{7.76}$$

by Lemma 7.10. By Corollary 7.14,

$$25t^2 + 14 + t + 1 \in R_1(5). \tag{7.77}$$

Finally by Lemma 7.45, $p_0 \in R_1(5)$, which implies together with (7.76) and (7.77) that $f := 25t^2 + 14t + 1 + 25(p_1|\xi_{5,11})t^2 - p_0 \in R_1(5)$ (because $R_1(5)$ is vector space over $\mathbb{C}$).

We must prove that $f = 0$. Assume by contradiction that $f \neq 0$. Then by Lemma 7.37 $\mathrm{Ord}_{\Gamma_0(20)}(f, \gamma_0) < -3$ or $\mathrm{Ord}_{\Gamma_0(20)}(f, \gamma_{5^2}) < 4$, which by Lemma 7.36 is equivalent to showing

$$\mathrm{ord}(f|\gamma_0 V_{20}) < -3 \quad \text{or} \quad \mathrm{ord}(f|\gamma_{5^2} V_{4,-1}) < 4. \tag{7.78}$$

We will compute explicitly the f below and show that (7.78) is not satisfied giving a contradiction to (7.78).

By (7.33) in Lemma 7.23 we have

$$(p_1|\gamma_0 V_{20}) = (p_1|\xi_{5,11})|\gamma_{5^2} V_{4,-1} \tag{7.79}$$

and $(p_1|\xi_{5,11})|\gamma_0 V_{20} = ((p_1|\xi_{5,11})|\xi_{5,11})|\gamma_{5^2} V_{4,1}$ which by (7.36) implies that

$$(p_1|\xi_{5,11})|\gamma_0 V_{20} = p_1|\gamma_{5^2} V_{4,-1}. \tag{7.80}$$

By (7.79)

$$f|\gamma_{5^2} V_{4,-1} = (25t^2 + 14t + 1|\gamma_{5^2} V_{4,-1}) + 25(p_1|\gamma_0 V_{20})(t^2|\gamma_{5^2} V_{4,-1}) - (p_0|\gamma_{5^2} V_{4,-1}) \tag{7.81}$$

and by (7.80)

$$f|\gamma_0 V_{20} = (25t^2 + 14t + 1|\gamma_0 V_{20}) + 25(p_1|\gamma_{5^2} V_{4,-1})(t^2|\gamma_0 V_{20}) - (p_0|\gamma_0 V_{20}). \tag{7.82}$$

Next by using Definition 6.6, Corollary 7.47 and (2.26) we find (by computer) that:

$$\begin{aligned}
5^3(t|\gamma_0 V_{20}) &= q^{-4} - 6 + 9q^4 + 10q^8 - 30q^{12} + \cdots, \\
5^6(t^2|\gamma_0 V_{20}) &= q^{-8} - 12q^{-4} + 54 - 88q^4 - 99q^8 + \cdots, \\
5^4(p_0|\gamma_0 V_{20}) &= q^{-8} - 2q^{-7} + 4q^{-6} - 8q^{-5} + 13q^{-4} - 20q^{-3} + 32q^{-2} + \cdots, \\
5^2(p_1|\gamma_0 V_{20}) &= -q^{-8} - q^{-4} + 2q^{-3} + 12 + 8q - 4q^2 + \cdots, \\
(t|\gamma_{5^2} V_{4,-1}) &= q^4 + 6q^8 + 27q^{12} + 98q^{16} + 315q^{20} + \cdots, \\
(t^2|\gamma_{5^2} V_{4,-1}) &= q^8 + 12q^{12} + 90q^{16} + 520q^{20} + 2535q^{24} + \cdots, \\
(p_0|\gamma_{5^2} V_{4,-1}) &= q^4 + 2q^5 + 19q^8 + 32q^9 - 4q^{10} + 195q^{12} + 286q^{13} + \cdots, \\
(p_1|\gamma_{5^2} V_{4,-1}) &= -2q + 4q^2 - 8q^3 - 45q^4 - 44q^5 + 80q^6 - 144q^7 - 739q^8 + \cdots,
\end{aligned}$$

which are plugged into (7.81) and (7.82) to verify $\mathrm{Ord}_{\Gamma_0(20)}(f, \gamma_{5^2}) \geq 4$ and $\mathrm{Ord}_{\Gamma_0(20)}(f, \gamma_0) \geq -3$ proving by Lemma 7.37 that $f = 0$.

Proof of (7.74)*:* One can easily verify that $\xi_{5,11} \in \Gamma_0(5)\gamma_0 V_5$ which together with $t \in A_0(5)$ by Lemma 2.34 implies $(t|\xi_{5,11}) = (t|\gamma_0 V_5)$ and by Corollary 7.30 $(t|\gamma_0 V_5) = 5^{-3}t^{-1}$.

Proof of (7.75)*:* This identity follows by applying $|\xi_{5,11}$ to (7.73) and using (7.74) together with (7.36) in Lemma 7.23. □

Corollary 7.50. *$R_2(5)$ is generated by $\{1, p_1\}$ as a $\mathbb{C}[t, t^{-1}]$-module.*

Proof. Let $f \in R_2(5)$. Then $f|\xi_{5,11} \in R_1(5)$ by Lemma 7.23. By Lemma 7.48 there exist Laurent polynomials $q_1(t), q_2(t) \in \mathbb{C}[t, t^{-1}]$ such that $f|\xi_{5,11} = q_1(t) + p_0 q_2(t)$ and by (7.36) in Lemma 7.23 we obtain

$$(f|\xi_{5,11})|\xi_{5,11} = f = q_1(t|\xi_{5,11}) + (p_0|\xi_{5,11})q_2(t|\xi_{5,11})$$

which by (7.74) and (7.75) rewrites into

$$f = q_1(5^{-3}t^{-1}) + (5^{-4}t^{-2} + 14{\cdot}5^{-3}t^{-1} + 1 + 5^{-4}p_1 t^{-2})q_2(5^{-3}t^{-1})$$

proving the desired representation for f. □

Bibliography

[1] G. E. Andrews. Generalized Frobenius Partitions. *Memoirs of the American Mathematical Society*, 49(301), 1984.

[2] G. E. Andrews and P. Paule. MacMahon's Partition Analysis XI: Broken Diamonds and Modular Forms. *Acta Arithmetica*, 126:281–294, 2007.

[3] A. O. L. Atkin. Proof of a Conjecture of Ramanujan. *Glasgow Mathematical Journal*, 8:14–32, 1967.

[4] A. O. L. Atkin. Ramanujan Congruences for $p_{-k}(n)$. *Canadian Journal of Mathematics*, 20:67–78, 1968.

[5] A. O. L. Atkin and J. Lehner. Hecke Operators on $\Gamma_0(M)$. *Mathematische Annalen*, 185:134–160, 1970.

[6] A. O. L. Atkin and J. N. O'Brien. Some Properties of $p(n)$ and $c(n)$ Modulo Powers of 13. *Transactions of the American Mathematical Society*, 126:442–459, 1967.

[7] B. C. Berndt. *Ramanujan's Notebooks Part III*. Springer, 1991.

[8] S. H. Chan. Some Congruences for Andrews-Paule's Broken 2-diamond Partitions. *Discrete Mathematics*, 308:5735–5741, 2008.

[9] D. Eichhorn and K. Ono. Congruences for Partition Functions. In: B. C. Berndt, H. G. Diamond and A. J. Hildebrand, editors, *Analytic Number Theory*, volume 1 of *Progress in Mathematics 138*, pages 309–321. Birkhäuser Boston, 1996.

[10] D. Eichhorn and J. A. Sellers. Computational Proofs of Congruences for 2-Colored Frobenius Partitions. *International Journal of Mathematics and Mathematical Sciences*, 29:333–340, 2002.

[11] D. A. Eichhorn. *Some Results on the Congruential and Gap-theoretic Study of Partition Functions*. PhD thesis, University of Illinois at Urbana-Champaign, 1999.

[12] D. Kim F. Garvan and D. Stanton. Cranks an T-cores. *Inventiones Mathematicae*, 101:1–17, 1990.

[13] P. V. Aiyar G. H. Hardy and B. M. Wilson. *Collected Papers of Srinivasa Ramanujan*. Chelsea Publishing, New York, 1962.

[14] F. Garvan. Modular functions, Maple and Andrews' 10th problem. In: S. D. Ahlgren, G. E. Andrews and K. Ono, editors, *Topics in Number Theory*, Mathematics and its Applications 467, pages 163–179. Kluwer Academic Publishers, 1999.

[15] B. Gordon. Ramanujan Congruences for $p_{-k} \pmod{11^r}$. *Glasgow Mathematical Journal*, 2:107–123, 1983.

[16] B. Gordon and K. Hughes. Congruences for $r_s(n)$. *Proceedings of London Mathematical Society*, 3:527–546, 1983.

[17] M. D. Hirschhorn and J. A. Sellers. On Recent Congruence Results of Andrews and Paule. *Bulletin of the Australian Mathematical Society*, 75:121–126, 2007.

[18] K. Hughes. Ramanujan Congruences for $p_{-k}(n)$ Modulo Powers of 17. *Canadian Journal of Mathematics*, 43:506–525, 1991.

[19] M. Kauers and B. Zimmermann. Computing the Algebraic Relations of C-finite Sequences and Multisequences. *Journal of Symbolic Computation*, 43(11):787–803, 2008.

[20] M. I. Knopp. *Modular Functions in Analytic Number Theory.* American Mathematical Society, 1993.

[21] O. Kolberg. Some Identities Involving the Partition Function. *Mathematica Scandinavica*, 5:77–92, 1957.

[22] O. Kolberg. An Elementary discussion of Certain Modular Forms. *UNIVERSITET I BERGEN ÅRBOK Naturvitenskapelig rekke*, 16, 1959.

[23] L. Kolitsch. A Congruence for Generalized Frobenius Partitions with 3 Colors Modulo Powers of 3. In: B. C. Berndt, H. G. Diamond, H. Halberstam and A. Hildebrand, editors, *Analytic Number Theory, Proceedings of a Conference in Honor of Paul Bateman*, Progress in Mathematics 85, pages 343–348. Birkhäuser Boston, 1990.

[24] J. Lehner. Divisibility Properties of the Fourier Coefficients of the Modular Invariant $j(\tau)$. *American Journal of Mathematics*, 71:136–148, 1949.

[25] R. Lewis. The Components of Modular Forms. *J. London Math. Soc.*, 52:245–254, 1995.

[26] J. Lovejoy. Ramanujan-Type Congruences for Three Colored Frobenius Partitions. *Journal of Number Theory*, 85:283–290, 2000.

[27] M. Newman. Construction and Application of a Class of Modular Functions. *Proceedings London Mathematical Society*, 3(7):334–350, 1957.

[28] M. Newman. Construction and Application of a Class of Modular Functions 2. *Proceedings London Mathematical Society*, 3(9):373–387, 1959.

[29] K. Ono. Congruences for Frobenius Partitions. *Journal of Number Theory*, 57(1):331–347, 1996.

[30] P. Paule and S. Radu. A Proof of Sellers' Conjecture. *Submitted for publication*, 2009.

[31] H. Pieper. *Variationen über ein zahlentheoretisches Thema von Carl Friedrich Gauß.* Birkhäuser, 1977.

[32] H. Rademacher. The Ramanujan Identities Under Modular Substitutions. *Transactions of the American Mathematical Society*, 51(3):609–636, 1942.

[33] S. Radu. An Algorithmic Approach to Ramanujan's Congruences. *Ramanujan Journal*, 20:215–251, 2009.

[34] S. Radu and J. Sellers. Parity Results for Broken k-diamond Partitions and $2k+1$-cores. *Submitted for publication*, 2009.

[35] R. A. Rankin. *Modular Forms and Functions.* Cambridge University Press, 1977.

[36] Ø. Rødseth. Congruence Properties of the Partition Functions $q(n)$ and $q_0(n)$. *UNIVERSITET I BERGEN ÅRBOK Naturvitenskapelig rekke*, 13, 1969.

[37] J. Sellers. Congruences Involving F-partition Functions. *International Journal of Mathematics and Mathematical Sciences*, 17:187–188, 1994.

[38] J. P. Serre. *A Course in Arithmetic.* Springer, 1996.

[39] W. A. Stein. Computing With Modular Forms. Available from `http://modular.fas.harvard.edu/257/notes/257.pdf`.

[40] J. Sturm. On the Congruence of Modular Forms. In: D. V. Chudnovsky, G. V. Chudnovsky, H. Cohn and M. B. Nathanson, editors, *Number Theory*, Lecture notes in Mathematics 1240, pages 275–280. Springer Berlin/Heidelberg, 1987.

[41] G. N. Watson. Ramanujan's Vermutung über Zerfällungsanzahlen. *Journal für die Reine und Angewandte Mathematik*, 179:97–128, 1938.

[42] E. T. Whittaker and G. N. Watson. *A Course of Modern Analysis.* Cambridge University Press, 1965.

Eidesstattliche Erklärung

Ich erkläre an Eides statt, daß ich die vorliegende Dissertation selbstständig und ohne fremde Hilfe verfaßt, andere als die angegebenen Quellen und Hilfsmittel nicht benutzt bzw. die wörtlich oder sinngemäß entnommenen Stellen als solche kenntlich gemacht habe.

Linz, April, 2010 Silviu-Cristian Radu

Curriculum Vitae

Personal data

Name	Silviu-Cristian Radu
Date and place of birth	June 13, 1977, Rosiorii de Vede, Romania
Nationality	Sweden

Contact

Affiliation	Research Institute for Symbolic Computation (RISC) Johannes Kepler University Linz Altenberger Straße 69 4040 Linz, Austria http://www.risc.uni-linz.ac.at
E-mail	sradu@risc.uni-linz.ac.at
Homepage	http://www.risc.uni-linz.ac.at/people/sradu/

Education

06/1995	Matura (high school graduation) at Kungsmadskolan, Växjö
10/2001–01/2006	Studying Technical Physics at Lund University of Sweden, Lund *Diploma thesis:* “A New Upper Bound on Rubik’s cube” *Thesis advisor:* Prof. Gert Almkvist
10/2006–04/2010	Doctorate studies at the Spezialforschungsbereich SFB013 “Numerical and Symbolic Scientific Computing”, Johannes Kepler University, Linz