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Symbolic summation assists particle physics

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FWF Der Wissenschaftsfonds

Feynman parameter integrals with one mass M and local operator insertions in $4+\varepsilon$-dimensional Minkowski space:

$$D_\varepsilon(n) = \int \frac{d^D p_1}{(2\pi)^D} \dots \int \frac{d^D p_k}{(2\pi)^D} \frac{N(p_1,\dots,p_k;p;M^2;\Delta,n)}{(-p_1^2+m_1^2)^{l_1}\dots(-p_k^2+m_k^2)^{l_k}} \prod_V \delta_V$$

$$D_\varepsilon(n) = \varepsilon^{-3} F_{-3}(n) + \varepsilon^{-2} F_{-2}(n) + \varepsilon^{-1}(n) F_{-1}(n) + \dots$$

Multi-summation and particle physics:

Feynman parameter integrals with one mass M and local operator insertions in $4+\varepsilon$-dimensional Minkowski space:

$$D_\varepsilon(n) = \int \frac{d^D p_1}{(2\pi)^D} \dots \int \frac{d^D p_k}{(2\pi)^D} \frac{N(p_1,\dots,p_k;p;M^2;\Delta,n)}{(-p_1^2+m_1^2)^{l_1}\dots(-p_k^2+m_k^2)^{l_k}} \prod_V \delta_V$$

$\downarrow$ (DESY, J. Blümlein)

Definite hypergeometric multi-sums:

$$D_\varepsilon(n) = \sum_{k_1=l_1}^{L_1(n)} \dots \sum_{k_v=l_v}^{L_v(n,k_1,\dots,k_{v-1})} \sum_{i=1}^{l} f_i(\varepsilon,n,k_1,\dots,k_v)$$

f_i: proper hypergeometric series in terms of Γ-functions
$L_v(n,k_1,\dots,k_{v-1})$: integer linear relation in the parameters (or ∞)

$\downarrow$ (symbolic summation)

$$D_\varepsilon(n) = \varepsilon^{-3}F_{-3}(n) + \varepsilon^{-2}F_{-2}(n) + \varepsilon^{-1}(n)F_{-1}(n) + \dots$$

Some of the available summation tools:

- Abramov, S.A.: On the summation of rational functions. Zh. vychisl. mat. Fiz. **11**, 1071–1074 (1971)
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- Chen, S., Kauers, M.: Order-Degree Curves for Hypergeometric Creative Telescoping. In: J. van der Hoeven, M. van Hoeij (eds.) Proceedings of ISSAC 2012, pp. 122–129 (2012)
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- M. Kauers and P. Paule. *The concrete tetrahedron*. Texts and Monographs in Symbolic Computation. SpringerWienNewYork, Vienna, 2011. Symbolic sums, recurrence equations, generating functions, asymptotic estimates.

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Some of the available summation tools:

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- Koornwinder, T.H.: On Zeilberger's algorithm and its q-analogue. J. Comp. Appl. Math. **48**, 91–111 (1993)
- Koutschan, C.: Creative telescoping for holonomic functions. In: C. Schneider, J. Blümlein (eds.) Computer Algebra in Quantum Field Theory: Integration, Summation and Special Functions, Texts and Monographs in Symbolic Computation, pp. 171–194. Springer (2013). ArXiv:1307.4554 [cs.SC]
- Paule, P.: Greatest factorial factorization and symbolic summation. J. Symbolic Comput. **20**(3), 235–268 (1995)
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- Petkovšek, M.: Hypergeometric solutions of linear recurrences with polynomial coefficients. J. Symbolic Comput. **14**(2-3), 243–264 (1992)
- Petkovšek, M., Wilf, H.S., Zeilberger, D.: $A = B$. A. K. Peters, Wellesley, MA (1996)
- Petkovšek, M., Zakrajšek, H.: Solving linear recurrence equations with polynomial coefficients. In: C. Schneider, J. Blümlein (eds.) Computer Algebra in Quantum Field Theory: Integration, Summation and Special Functions, Texts and Monographs in Symbolic Computation, pp. 259–284. Springer (2013)
- Pirastu, R., Strehl, V.: Rational summation and Gosper-Petkovšek representation. J. Symbolic Comput. **20**(5-6), 617–635 (1995)
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- Zeilberger, D., 1990. A holonomic systems approach to special functions identities. J. Comput. Appl. Math. 32, 321–368.
- Zeilberger, D.: The method of creative telescoping. J. Symbolic Comput. **11**, 195–204 (1991)

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Here I will restrict to the setting of difference rings/fields.

A warm up example

$$\text{GIVEN } F(n) = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \frac{e^{-e\gamma}}{\Gamma(\varepsilon+1)} \times$$

$$\times \Bigg(\frac{\Gamma(k+1)}{\Gamma(k+2+n)} \frac{\Gamma(\frac{\varepsilon}{2})\Gamma(1-\frac{\varepsilon}{2})\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+1+\frac{\varepsilon}{2})\Gamma(k+j+1+n)}{\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+2+n)\Gamma(k+j+2)}$$

$$\underbrace{+ \frac{\Gamma(k+1)}{\Gamma(k+2+n)} \frac{\Gamma(-\frac{\varepsilon}{2})\Gamma(1+\frac{\varepsilon}{2})\Gamma(j+1+\varepsilon)\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(k+j+1+\frac{\varepsilon}{2}+n)}{\Gamma(j+1)\Gamma(j+2+\frac{\varepsilon}{2}+n)\Gamma(k+j+2+\frac{\varepsilon}{2})}}_{f(n,k,j)} \Bigg).$$

Arose in the context of

I. Bierenbaum, J. Blümlein, and S. Klein, *Evaluating two-loop massive operator matrix elements with Mellin-Barnes integrals*. 2006

A warm up example

$$\text{GIVEN } F(n)=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{e^{-e\gamma}}{\Gamma(\varepsilon+1)}\times$$

$$\times\underbrace{\left(\frac{\Gamma(k+1)}{\Gamma(k+2+n)}\frac{\Gamma(\frac{\varepsilon}{2})\Gamma(1-\frac{\varepsilon}{2})\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+1+\frac{\varepsilon}{2})\Gamma(k+j+1+n)}{\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+2+n)\Gamma(k+j+2)}\right.
+\left.\frac{\Gamma(k+1)}{\Gamma(k+2+n)}\frac{\Gamma(-\frac{\varepsilon}{2})\Gamma(1+\frac{\varepsilon}{2})\Gamma(j+1+\varepsilon)\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(k+j+1+\frac{\varepsilon}{2}+n)}{\Gamma(j+1)\Gamma(j+2+\frac{\varepsilon}{2}+n)\Gamma(k+j+2+\frac{\varepsilon}{2})}\right)}_{f(n,k,j)}.$$

FIND the first coefficients of the ϵ-expansion

$$F(n)=F_0(n)+\varepsilon F_1(n)+\dots$$

Arose in the context of
I. Bierenbaum, J. Blümlein, and S. Klein, *Evaluating two-loop massive operator matrix elements with Mellin-Barnes integrals*. 2006

A warm up example

GIVEN $F(n)=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{e^{-e\gamma}}{\Gamma(\varepsilon+1)}\times$

$$\times\Bigg(\frac{\Gamma(k+1)}{\Gamma(k+2+n)}\frac{\Gamma(\frac{\varepsilon}{2})\Gamma(1-\frac{\varepsilon}{2})\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+1+\frac{\varepsilon}{2})\Gamma(k+j+1+n)}{\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+2+n)\Gamma(k+j+2)}$$

$$\underbrace{+\frac{\Gamma(k+1)}{\Gamma(k+2+n)}\frac{\Gamma(-\frac{\varepsilon}{2})\Gamma(1+\frac{\varepsilon}{2})\Gamma(j+1+\varepsilon)\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(k+j+1+\frac{\varepsilon}{2}+n)}{\Gamma(j+1)\Gamma(j+2+\frac{\varepsilon}{2}+n)\Gamma(k+j+2+\frac{\varepsilon}{2})}}_{f(n,k,j)}\Bigg).$$

Step 1: Compute the first coefficients of the ϵ-expansion

$$f(n,k,j)=\quad f_0(n,k,j)+\quad \varepsilon f_1(n,k,j)+\dots$$

Arose in the context of

I. Bierenbaum, J. Blümlein, and S. Klein, *Evaluating two-loop massive operator matrix elements with Mellin-Barnes integrals*. 2006

A warm up example

$$\text{GIVEN } F(n)=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{e^{-e\gamma}}{\Gamma(\varepsilon+1)}\times$$
$$\times\Bigg(\frac{\Gamma(k+1)}{\Gamma(k+2+n)}\frac{\Gamma(\frac{\varepsilon}{2})\Gamma(1-\frac{\varepsilon}{2})\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+1+\frac{\varepsilon}{2})\Gamma(k+j+1+n)}{\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+2+n)\Gamma(k+j+2)}$$
$$\underbrace{+\frac{\Gamma(k+1)}{\Gamma(k+2+n)}\frac{\Gamma(-\frac{\varepsilon}{2})\Gamma(1+\frac{\varepsilon}{2})\Gamma(j+1+\varepsilon)\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(k+j+1+\frac{\varepsilon}{2}+n)}{\Gamma(j+1)\Gamma(j+2+\frac{\varepsilon}{2}+n)\Gamma(k+j+2+\frac{\varepsilon}{2})}}_{f(n,k,j)}\Bigg).$$

Step 2: Simplify the sums in

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f(n,k,j)=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_0(n,k,j)+\varepsilon\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_1(n,k,j)+\dots$$

Arose in the context of

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A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}(\underbrace{\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(j)})$$

where

$$S_1(n)=\sum_{i=1}^{n}\frac{1}{i}\quad (=H_n)$$

Arose in the context of

I. Bierenbaum, J. Blümlein, and S. Klein, *Evaluating two-loop massive operator matrix elements with Mellin-Barnes integrals*. 2006

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\underbrace{\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}\Big)}_{f(j)}$$

FIND $g(j)$:

$$\boxed{f(j) = g(j+1) - g(j)}$$

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Big(\underbrace{\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(j)}\Big)$$

FIND $g(j)$:

$$\boxed{f(j) = g(j+1) - g(j)}$$

$\uparrow$ summation package Sigma

$$g(j) = \frac{(j+k+1)(j+n+1)j!k!(j+k+n)!\Big(S_1(j)-S_1(j+k)-S_1(j+n)+S_1(j+k+n)\Big)}{kn(j+k+1)!(j+n+1)!(k+n+1)!}$$

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Big(\underbrace{\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(j)}\Big)$$

FIND $g(j)$:

$$\boxed{f(j) = g(j+1) - g(j)}$$

Summing the telescoping equation over j from 0 to a gives

$$\sum_{j=0}^{a} f(j) = g(a+1) - g(0)$$

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Big(\underbrace{\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(j)}\Big)$$

FIND $g(j)$:

$$\boxed{f(j)=g(j+1)-g(j)}$$

Summing the telescoping equation over j from 0 to a gives

$$\sum_{j=0}^{a} f(j) = g(a+1)-g(0)$$

$$=\underbrace{\frac{(a+1)!(k-1)!(a+k+n+1)!(S_1(a)-S_1(a+k)-S_1(a+n)+S_1(a+k+n))}{n(a+k+1)!(a+n+1)!(k+n+1)!} + \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)n!} + \frac{(2a+k+n+2)a!k!(a+k+n)!}{(a+k+1)(a+n+1)(a+k+1)!(a+n+1)!(k+n+1)!}}_{a\to\infty}$$

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\underbrace{\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}\Big)}_{f(j)}$$

$$\sum_{j=0}^{\infty} f(j) = \qquad \frac{1}{n!}\frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}$$

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\underbrace{\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}\Big)}_{f(j)}$$

$$\sum_{k=1}^{\infty}\sum_{j=0}^{\infty}f(j)=\frac{1}{n!}\sum_{k=1}^{\infty}\frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}$$

Telescoping

GIVEN

$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}}_{=:\, f(k)}.$$

FIND $g(k)$:

$$\boxed{g(k+1)-g(k)} = \boxed{f(k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

no solution ☹

Zeilberger's creative telescoping paradigm

GIVEN
$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=: f(n,k)}.$$

FIND $g(n,k)$

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{f(n,k)}$$

for all $0 \le k \le n$ and all $n \ge 0$.

no solution ☹

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$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}}_{=:\, f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n), c_1(n)$:

$$\boxed{g(n,k+1)-g(n,k)} = \boxed{c_0(n) f(n,k) + c_1(n)\, f(n+1,k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

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$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n) f(n,k) + c_1(n)\, f(n+1,k)}$$

for all $0 \le k \le n$ and all $n \ge 0$.

Sigma *computes*: $c_0(n) = -n$, $c_1(n) = (n+2)$ and

$$g(n,k) = \frac{kS_1(k) + (-n-1)S_1(n) - kS_1(k+n) - 2}{(k+n+1)(n+1)^2}$$

Zeilberger's creative telescoping paradigm

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FIND $g(n,k)$ and $c_0(n), c_1(n)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n) f(n,k) + c_1(n)\, f(n+1,k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

Summing this equation over k from 1 to a gives:

$$\boxed{g(n,a+1) - g(n,1)} = \boxed{\sum_{k=1}^{a} \Big[c_0(n)\, f(n,k) + c_1(n)\, f(n+1,k)\Big]}$$

Zeilberger's creative telescoping paradigm

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$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}}_{=:\, f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n), c_1(n)$:

$$\boxed{g(n,k+1)-g(n,k)} = \boxed{c_0(n) f(n,k) + c_1(n)\, f(n+1,k)}$$

for all $0 \le k \le n$ and all $n \ge 0$.

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$$\boxed{g(n,a+1)-g(n,1)} = \boxed{\sum_{k=1}^{a} c_0(n)\, f(n,k) + \sum_{k=1}^{a} c_1(n)\, f(n+1,k)}$$

Zeilberger's creative telescoping paradigm

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$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=:\, f(n,k)}.$$

FIND $g(n, k)$ and $c_0(n), c_1(n)$:

$$\boxed{g(n, k+1) - g(n, k)} = \boxed{c_0(n) f(n, k) + c_1(n)\, f(n+1, k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

Summing this equation over k from 1 to a gives:

$$\boxed{g(n, a+1) - g(n, 1)} = \boxed{c_0(n) \sum_{k=1}^{a} f(n, k) + c_1(n) \sum_{k=1}^{a} f(n+1, k)}$$

Zeilberger's creative telescoping paradigm

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$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=:\, f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n), c_1(n)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n) f(n,k) + c_1(n)\, f(n+1,k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

Summing this equation over k from 1 to a gives:

$$\boxed{g(n,a+1) - g(n,1)} = \boxed{c_0(n)\,\mathsf{A}(n) + c_1(n)\,\mathsf{A}(n+1)}$$

Zeilberger's creative telescoping paradigm

GIVEN

$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}}_{=:\, f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n), c_1(n)$:

$$\boxed{g(n,k+1)-g(n,k)} = \boxed{c_0(n)f(n,k)+c_1(n)\,f(n+1,k)}$$

for all $0 \le k \le n$ and all $n \ge 0$.

Summing this equation over k from 1 to a gives:

$$\boxed{g(n,a+1)-g(n,1)} = \boxed{c_0(n)\,\mathsf{A}(n)+c_1(n)\,\mathsf{A}(n+1)}$$

$$\| \qquad\qquad\qquad\qquad \|$$

$$\frac{(a+1)(S_1(a)+S_1(n)-S_1(a+n))}{(n+1)^2(a+n+2)} + \frac{a(a+1)}{(n+1)^3(a+n+1)(a+n+2)} \qquad -n\mathsf{A}(n)+(2+n)\mathsf{A}(n+1)$$

$$(n+2)\mathbf{A}(n+1) - n\mathbf{A}(n) = \frac{(n+1)S_1(n)+1}{(n+1)^3}$$

recurrence finder

$$A(n) = \sum_{k=1}^{\infty} \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}$$

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recurrence solver

$$A(n) = \sum_{k=1}^{\infty} \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)} \in \left\{ c \times \frac{1}{n(n+1)} + \frac{S_1(n)^2+S_2(n)}{2n(n+1)} \middle| c \in \mathbb{R} \right\}$$

where

$$S_1(n) = \sum_{i=1}^{n} \frac{1}{i} \qquad S_2(n) = \sum_{i=1}^{n} \frac{1}{i^2}$$

$$(n+2)\mathbf{A}(n+1) - n\mathbf{A}(n) = \frac{(n+1)S_1(n)+1}{(n+1)^3}$$

Summation package Sigma

(based on difference field/ring algorithms/theory
see, e.g., Karr 1981, Bronstein 2000, Schneider 2001–)

$$A(n) = \sum_{k=1}^{\infty} \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)} = 0 \times \frac{1}{n(n+1)} + \frac{S_1(n)^2+S_2(n)}{2n(n+1)}$$

where

$$S_1(n) = \sum_{i=1}^{n} \frac{1}{i} \qquad S_2(n) = \sum_{i=1}^{n} \frac{1}{i^2}$$

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\underbrace{\sum_{j=0}^{\infty}\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}\Big)}_{f(j)}$$

$$\sum_{k=1}^{\infty}\sum_{j=0}^{\infty}f(j) = \frac{1}{n!}\sum_{k=1}^{\infty}\frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}$$
$$= \frac{1}{n!}\frac{S_1(n)^2+S_2(n)}{2n(n+1)}$$

where

$$S_1(n)=\sum_{i=1}^{n}\frac{1}{i} \qquad S_2(n)=\sum_{i=1}^{n}\frac{1}{i^2}$$

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$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} f(n,k,j) = \frac{S_1(n)^2+3S_2(n)}{2n(n+1)!}$$

where

$$S_1(n)=\sum_{i=1}^{n}\frac{1}{i} \qquad S_2(n)=\sum_{i=1}^{n}\frac{1}{i^2}$$

GIVEN

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{e^{-e\gamma}}{\Gamma(\varepsilon+1)}\Bigg(\frac{\Gamma(k+1)}{\Gamma(k+2+n)}\,\frac{\Gamma(\frac{\varepsilon}{2})\Gamma(1-\frac{\varepsilon}{2})\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+1+\frac{\varepsilon}{2})\Gamma(k+j+1+n)}{\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+2+n)\Gamma(k+j+2)}$$

$$+\frac{\Gamma(k+1)}{\Gamma(k+2+n)}\,\frac{\Gamma(-\frac{\varepsilon}{2})\Gamma(1+\frac{\varepsilon}{2})\Gamma(j+1+\varepsilon)\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(k+j+1+\frac{\varepsilon}{2}+n)}{\Gamma(j+1)\Gamma(j+2+\frac{\varepsilon}{2}+n)\Gamma(k+j+2+\frac{\varepsilon}{2})}\Bigg).$$

$$=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_0(n,k,j)+$$

Sigma computes

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_0(n,k,j)=\frac{S_1(n)^2+3S_1(n)}{2n(n+1)!}.$$

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$$=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_0(n,k,j)+\varepsilon\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_1(n,k,j)+$$

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$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_0(n,k,j)=\frac{S_1(n)^2+3S_1(n)}{2n(n+1)!}.$$

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_1(n,k,j)=\frac{-S_1(n)^3-3S_2(n)S_1(n)-8S_3(n)}{6n(n+1)!}.$$

GIVEN

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{e^{-e\gamma}}{\Gamma(\varepsilon+1)}\Bigg(\frac{\Gamma(k+1)}{\Gamma(k+2+n)}\,\frac{\Gamma(\frac{\varepsilon}{2})\Gamma(1-\frac{\varepsilon}{2})\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+1+\frac{\varepsilon}{2})\Gamma(k+j+1+n)}{\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+2+n)\Gamma(k+j+2)}$$

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$$=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_0(n,k,j)+\varepsilon\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_1(n,k,j)+\varepsilon^2\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_2(n,k,j)+$$

Sigma computes

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_2(n,k,j)=\frac{1}{96n(n+1)}\Big(S_1(n)^4+(12\zeta_2+54S_2(n))S_1(n)^2$$

$$+104S_3(n)S_1(n)-48S_{2,1}(n)S_1(n)+51S_2(n)^2+36\zeta_2S_2(n)$$

$$+126S_4(n)-48S_{3,1}(n)-96S_{1,1,2}(n)\Big)$$

GIVEN

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{e^{-e\gamma}}{\Gamma(\varepsilon+1)}\Bigg(\frac{\Gamma(k+1)}{\Gamma(k+2+n)}\,\frac{\Gamma(\frac{\varepsilon}{2})\Gamma(1-\frac{\varepsilon}{2})\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+1+\frac{\varepsilon}{2})\Gamma(k+j+1+n)}{\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+2+n)\Gamma(k+j+2)}$$

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Sigma computes

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_3(n,k,j)=\frac{1}{960n(n+1)}\Big(S_1(n)^5+(20\zeta_2+130S_2(n))S_1(n)^3+$$
$$(40\zeta_3+380S_3(n))S_1(n)^2+\left(135S_2(n)^2+60\zeta_2S_2(n)+510S_4(n)\right)S_1(n)$$
$$-240S_{3,1}(n)S_1(n)-240S_{1,1,2}(n)S_1(n)+160\zeta_2S_3(n)+S_2(n)(120\zeta_3$$
$$+380S_3(n))+624S_5(n)+\left(-120S_1(n)^2-120S_2(n)\right)S_{2,1}(n)$$
$$-240S_{4,1}(n)-240S_{1,1,3}(n)+240S_{2,2,1}(n)\Big)$$

Toolbox 1: Indefinite summation

Toolbox 2: Definite summation

Toolbox 3: Special function algorithms

Toolbox 1: Indefinite summation

Recall some basic notions

$(\mathbb{K}, +, \cdot, 0, 1)$ is a field iff

- $(\mathbb{K}, +, 0)$ is a commutative group
- $(\mathbb{K} \setminus \{0\}, \cdot, 1)$ is a commutative group
- for all $a, b, c \in \mathbb{K}:\ a \cdot (b + c) = a \cdot b + a \cdot c$ (distributivity law)

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Examples:

- the rational numbers $\mathbb{Q}$
- the rational function field $\mathbb{Q}(k)$:
  k is transcendental over $\mathbb{Q}$ and

$$\mathbb{Q}(k) = \left\{ \frac{f_0 + f_1\, k + \cdots + f_r\, k^r}{\underbrace{g_0 + g_1\, k + \cdots + g_s\, k^s}_{\neq 0}} \mid f_i, g_i \in \mathbb{Q} \right\}$$

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Example: $3\, p^{-2} + 2\, p^{-1} + 1 + 2\, p + 3\, p^2 \in \mathbb{Q}[p, p^{-1}]$

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Be aware of zero-divisors:

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- The algebraic ring

$$\mathbb{A}[x] = \{f_0 + f_1\,x \mid f_0, f_1 \in \mathbb{A}\}$$

subject to the relation $x^2 = 1$

- We can construct a tower of such ring extensions

$$\mathbb{Q}(k)[p_1, p_1^{-1}][p_2, p_2^{-1}] \ldots [p_e, p_e^{-1}][x][s_1][s_2] \ldots [s_u]$$

Recall some basic notions

Let $\mathbb{A}$ be a commutative ring with 1

(like a field, but not all elements w.r.t. the operation $\cdot$ are invertible).

- A map $\sigma : \mathbb{A} \to \mathbb{A}$ is a ring automorphism iff
  - for all $a, b \in \mathbb{A}$: $\sigma(a + b) = \sigma(a) + \sigma(b)$
  - for all $a, b \in \mathbb{A}$: $\sigma(a\, b) = \sigma(a)\, \sigma(b)$
  - $\sigma(0) = 0$ and $\sigma(1) = 1$
  - σ is bijective

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  - σ is bijective

Definition. A difference ring $(\mathbb{A}, \sigma)$ is a ring $\mathbb{A}$ equipped with a ring automorphism $\sigma : \mathbb{A} \to \mathbb{A}$.

Telescoping

GIVEN $f(k) = S_1(k)$.

FIND $g(k)$:

$$\boxed{f(k) = g(k+1) - g(k)}$$

for all $1 \leq k \leq n$ and $n \geq 0$.

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Sigma compute

$$g(k) = (S_1(k) - 1)k.$$

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Summing this equation over k from 1 to n gives

$$\boxed{\sum_{k=1}^{n} S_1(k)} = \boxed{g(n+1) - g(1)}$$
$$= (S_1(n+1) - 1)(n+1).$$

Telescoping in the given difference ring

FIND a closed form for

$$\sum_{k=1}^{n} S_1(k).$$

A difference ring for the summand

Consider a ring

$$\mathbb{A}$$

with the automorphism $\sigma : \mathbb{A} \to \mathbb{A}$ defined by

Telescoping in the given difference ring

FIND a closed form for

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A difference ring for the summand

Consider a ring

$$\mathbb{A} := \mathbb{Q}$$

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$$\sigma(c) = c \quad \forall c \in \mathbb{Q},$$

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A difference ring for the summand

Consider a ring

$$\mathbb{A} := \mathbb{Q}(k)[h]$$

with the automorphism $\sigma : \mathbb{A} \to \mathbb{A}$ defined by

$$\sigma(c) = c \quad \forall c \in \mathbb{Q},$$

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$$\sigma(h) = h + \frac{1}{k+1}, \qquad \mathcal{S}\, S_1(k) = S_1(k) + \frac{1}{k+1}.$$

Telescoping in the given difference ring

FIND $g \in \mathbb{A}$:

$$\boxed{\sigma(g) - g = h.}$$

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$$g(k) = (S_1(k) - 1)k.$$

Hence,

$$(S_1(n+1) - 1)(n+1) = \sum_{k=1}^{n} S_1(k).$$

Toolbox 1: Indefinite summation – the basic tactic

(inspired by Karr's algorithm, 1981)

CONSTRUCT a difference ring $(\mathbb{A}, \sigma)$ for $f(k)$:

- a ring (containing $\mathbb{Q}$)

$$\mathbb{A} := \mathbb{K}$$

- with an automorphism where $\sigma(c) = c$ for all $c \in \mathbb{K}$ and where

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CONSTRUCT a difference ring $(\mathbb{A}, \sigma)$ for $f(k)$:

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$$\boxed{\mathbb{A} := \mathbb{K}(k)[p_1, p_1^{-1}]}$$

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Sk!=(k+1)k! $\quad \leftrightarrow \quad \sigma(p_1) = (k+1)p_1$

CONSTRUCT a difference ring $(\mathbb{A},\sigma)$ for $f(k)$:

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hypergeometric products $\quad \leftrightarrow \quad \sigma(p_1) = a_1\, p_1 \qquad a_1 \in \mathbb{K}(k)^*$

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hypergeometric products $\leftrightarrow$

$$\begin{aligned}
\sigma(p_1) &= a_1\, p_1 && a_1 \in \mathbb{K}(k)^* \\
\sigma(p_2) &= a_2 p_2 && a_2 \in \mathbb{K}(k)[p_1, p_1^{-1}]^* \\
&\vdots \\
\sigma(p_e) &= a_e p_e && a_e \in \mathbb{K}(k)[p_1, p_1^{-1}] \dots [p_{e-1}, p_{e-1}^{-1}]^*
\end{aligned}$$

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		$\sigma(p_2)=a_2p_2$	$a_2\in\mathbb{K}(k)[p_1,p_1^{-1}]^*$
		$\vdots$	
		$\sigma(p_e)=a_ep_e$	$a_e\in\mathbb{K}(k)[p_1,p_1^{-1}]\dots[p_{e-1},p_{e-1}^{-1}]^*$
$\mathbf{(-1)^k}$	$\leftrightarrow$	$\sigma(\mathbf{x})=-\mathbf{x}$	$\mathbf{x^2=1}$

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$$\mathbb{A} := \mathbb{K}(k)[p_1, p_1^{-1}][p_2, p_2^{-1}] \dots [p_e, p_e^{-1}][x][s_1]$$

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$$\mathcal{S} S_1(k) = S_1(k) + \frac{1}{k+1} \quad \leftrightarrow \quad \sigma(s_1) = s_1 + \frac{1}{k+1}$$

CONSTRUCT a difference ring $(\mathbb{A},\sigma)$ for $f(k)$:

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(nested) sum	$\leftrightarrow$	$\sigma(s_1)=s_1+f_1$	$f_1\in\mathbb{K}(k)[p_1,p_1^{-1}]\dots[p_e,p_e^{-1}][x]$

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$$\sigma(s_u) = s_u + f_u \qquad f_u \in \mathbb{K}(k)[p_1, p_1^{-1}] \dots [p_e, p_e^{-1}][x][s_1] \dots [s_u]$$

CONSTRUCT a $R\Pi\Sigma^*$-ring $(\mathbb{A}, \sigma)$ for $f(k)$: (Karr81,CS14,CS16,CS17)

- a ring (containing $\mathbb{Q}$)

$$\mathbb{A} := \mathbb{K}(k)[p_1, p_1^{-1}][p_2, p_2^{-1}] \dots [p_e, p_e^{-1}][x][s_1][s_2] \dots [s_u]$$

- with an automorphism where $\sigma(c) = c$ for all $c \in \mathbb{K}$ and where

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such that $\text{const}_\sigma \mathbb{A} = \{c \in \mathbb{A} | \sigma(c) = c\} = \mathbb{K}$.

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(nested) sum … $, p_e^{-1}][x]$

… $, p_e^{-1}][x][s_1]$

> **GIVEN** $f\in\mathbb{A}$;
> **FIND**, in case of existence, a $g\in\mathbb{A}$ such that
> $$\sigma(g)-g=f.$$

$$\vdots$$

$$\sigma(s_u)=s_u+f_u \qquad f_u\in\mathbb{K}(k)[p_1,p_1^{-1}]\dots[p_e,p_e^{-1}][x][s_1]\dots[s_u]$$

such that $\mathrm{const}_\sigma\mathbb{A}=\{c\in\mathbb{A}|\sigma(c)=c\}=\mathbb{K}$.

Telescoping in the given difference ring

FIND a closed form for

$$\sum_{k=1}^{n} S_1(k).$$

A $R\Pi\Sigma^*$-ring for the summand

$\boxed{\mathrm{const}_\sigma \mathbb{A} = \mathbb{Q}}$

Consider a ring

$$\mathbb{A} := \mathbb{Q}(k)[h]$$

with the automorphism $\sigma : \mathbb{A} \to \mathbb{A}$ defined by

$$\sigma(c) = c \quad \forall c \in \mathbb{Q},$$

$$\sigma(k) = k + 1, \qquad\qquad \mathcal{S}\, k = k + 1,$$

$$\sigma(h) = h + \frac{1}{k+1}, \qquad\qquad \mathcal{S}\, S_1(k) = S_1(k) + \frac{1}{k+1}.$$

FIND $g \in \mathbb{Q}(k)[h]$:

$$\sigma(g) - g = h.$$

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Degree bound: COMPUTE $b \geq 0$:

$$\forall g \in \mathbb{Q}(k)[h] \quad \sigma(g) - g = h \quad \Rightarrow \quad \deg(g) \leq b.$$

FIND $g \in \mathbb{Q}(k)[h]$:

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Degree bound: COMPUTE $b \geq 0$: $\boxed{b = 2}$

$$\forall g \in \mathbb{Q}(k)[h] \quad \sigma(g) - g = h \quad \Rightarrow \quad \deg(g) \leq b.$$

FIND $g \in \mathbb{Q}(k)[h]$:

$$\sigma(g) - g = h.$$

Degree bound: COMPUTE $b \geq 0$: $b = 2$

$$\forall g \in \mathbb{Q}(k)[h] \quad \sigma(g) - g = h \quad \Rightarrow \quad \deg(g) \leq b.$$

Polynomial Solution: FIND

$$g = g_2\, h^2 + g_1\, h + g_0 \in \mathbb{Q}(k)[h].$$

ANSATZ $g = g_2\,h^2 + g_1\,h + g_0 \in \mathbb{Q}(k)[h]$

$$\sigma(g) - g = h$$

ANSATZ $g = g_2\, h^2 + g_1\, h + g_0 \in \mathbb{Q}(k)[h]$

$$\big[\sigma(g_2\, h^2 + g_1 h + g_0)\big] \\ -\big[g_2\, h^2 + g_1 h + g_0\big] = h$$

ANSATZ $g = g_2\,h^2 + g_1\,h + g_0 \in \mathbb{Q}(k)[h]$

$$\Big[\sigma(g_2\,h^2) + \sigma(g_1 h + g_0)\Big] \\ -\Big[g_2\,h^2 + g_1 h + g_0\Big] = h$$

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$$\Big[\sigma(g_2)\,\sigma(h^2) + \sigma(g_1 h + g_0)\Big] \\ -\Big[g_2\, h^2 + g_1 h + g_0\Big] = h$$

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ANSATZ $g = g_2\, h^2 + g_1\, h + g_0 \in \mathbb{Q}(k)[h]$

$$\Big[\sigma(g_2)\big(h + \tfrac{1}{k+1}\big)^2 + \sigma(g_1 h + g_0)\Big] \\ -\Big[g_2\, h^2 + g_1 h + g_0\Big] = h$$

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$$\Big[\sigma(g_2)\big(h + \tfrac{1}{k+1}\big)^2 + \sigma(g_1 h + g_0)\Big] - \Big[g_2\, h^2 + g_1 h + g_0\Big] = h$$

coeff. comp.

$$\boxed{\sigma(g_2) - g_2 = 0}$$

ANSATZ $g = g_2\, h^2 + g_1\, h + g_0 \in \mathbb{Q}(k)[h]$

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$g_2 = c \in \mathbb{Q}$

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$$\boxed{\sigma(g_2) - g_2 = 0}$$

$g_2 = c \in \mathbb{Q}$

$$\Big[c\big(h + \tfrac{1}{k+1}\big)^2 + \sigma(g_1 h + g_0)\Big] - \big[c\, h^2 + g_1 h + g_0\big] = h$$

ANSATZ $g = g_2\,h^2 + g_1\,h + g_0 \in \mathbb{Q}(k)[h]$

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$g_2 = c \in \mathbb{Q}$

$$\sigma(g_1\,h + g_0) - (g_1\,h + g_0) = h - c\Big[\tfrac{2h(k+1)+1}{(k+1)^2}\Big]$$

ANSATZ $g = g_2\, h^2 + g_1\, h + g_0 \in \mathbb{Q}(k)[h]$

$$\Big[\sigma(g_2)\big(h + \tfrac{1}{k+1}\big)^2 + \sigma(g_1 h + g_0)\Big] - \big[g_2\, h^2 + g_1 h + g_0\big] = h$$

coeff. comp.

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$$\boxed{\sigma(g_0) - g_0 = -1 - d\frac{1}{k+1}}$$

ANSATZ $g = g_2\,h^2 + g_1\,h + g_0 \in \mathbb{Q}(k)[h]$

$$\Big[\sigma(g_2)\big(h+\tfrac{1}{k+1}\big)^2 + \sigma(g_1 h + g_0)\Big] - \big[g_2\,h^2 + g_1 h + g_0\big] = h$$

coeff. comp. → g=hk-k

$$\sigma(g_2) - g_2 = 0$$

$g_2 = c \in \mathbb{Q}$

$$\sigma(g_1\,h + g_0) - (g_1\,h + g_0) = h - c\Big[\tfrac{2h(k+1)+1}{(k+1)^2}\Big]$$

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$$\sigma(g_1) - g_1 = 1 - c\,\tfrac{2}{k+1}$$

$c = 0$, $g_1 = k + d$, $d \in \mathbb{Q}$

$$\sigma(g_0) - g_0 = -1 - d\frac{1}{k+1}$$

← $g_0 = -k$, $d = 0$

Telescoping in the given difference ring

FIND $g \in \mathbb{A}$:

$$\boxed{\sigma(g) - g = h.}$$

We compute

$$g = (h-1)k \in \mathbb{A}.$$

This gives

$$\boxed{g(k+1) - g(k) = S_1(k)}$$

with

$$g(k) = (S_1(k) - 1)k.$$

Hence,

$$(S_1(n+1) - 1)(n+1) = \sum_{k=1}^{n} S_1(k).$$

Toolbox 1: Improved indefinite summation – symbolic simplification

For algorithmic details see:

- CS. Symbolic summation with single-nested sum extensions. In J. Gutierrez, editor, *Proc. ISSAC'04*, pages 282–289. ACM Press, 2004.
- CS. Product representations in $\Pi\Sigma$-fields. *Ann. Comb.*, 9(1):75–99, 2005.
- CS. Simplifying Sums in $\Pi\Sigma$-Extensions. *J. Algebra Appl.*, 6(3):415–441, 2007.
- CS. A refined difference field theory for symbolic summation. *J. Symbolic Comput.*, 43(9):611–644, 2008. [arXiv:0808.2543v1].
- S.A. Abramov, M. Petkovšek. Polynomial ring automorphisms, rational (w,σ)-canonical forms, and the assignment problem. *J. Symbolic Comput.*, 45(6): 684–708, 2010.
- CS, A Symbolic Summation Approach to Find Optimal Nested Sum Representations. In: A. Carey, D. Ellwood, S. Paycha, S. Rosenberg (eds.) Motives, Quantum Field Theory, and Pseudodifferential Operators, Clay Mathematics Proceedings, vol. 12, pp. 285–308. Amer. Math. Soc (2010). ArXiv:0808.2543
- CS, Parameterized Telescoping Proves Algebraic Independence of Sums. Ann. Comb. **14**(4), 533–552 (2010). [arXiv:0808.2596]
- CS. Structural Theorems for Symbolic Summation. *Appl. Algebra Engrg. Comm. Comput.*, 21(1):1–32, 2010.
- CS. Fast Algorithms for Refined Parameterized Telescoping in Difference Fields. To appear in *Computer Algebra and Polynomials*, Lecture Notes in Computer Science (LNCS), Springer, 2014. arXiv:1307.7887 [cs.SC].

For special cases see:

- S.A. Abramov. On the summation of rational functions. Zh. vychisl. mat. Fiz., 11: 1071-1074, 1971.
- P. Paule. Greatest factorial factorization and symbolic summation, J. Symbolic Comput., 20(3): 235-268, 1995.

The basic difference ring approach

GIVEN a $R\Pi\Sigma^*$-ring $(\mathbb{A}, \sigma)$ with $f \in \mathbb{A}$.

FIND $g \in \mathbb{A}$:

$$\sigma(g) - g = f.$$

A symbolic summation approach

1. FIND an appropriate $R\Pi\Sigma^*$-ring $(\mathbb{A}, \sigma)$ with $f \in \mathbb{A}$.

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2. FIND an appropriate extension $\mathbb{E}>\mathbb{A}$ with $g\in\mathbb{E}$:

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appropriate = degrees in denominators minimal

Example:

$$\sum_{k=1}^{a}\Big(\frac{-2+k}{10(1+k^2)}+\frac{(1-4k-2k^2)S_1(k)}{10(1+k^2)(2+2k+k^2)}+\frac{(1-4k-2k^2)S_3(k)}{5(1+k^2)(2+2k+k^2)}\Big)$$

$$=?$$

A symbolic summation approach

1. FIND an appropriate $R\Pi\Sigma^*$-ring $(\mathbb{A}, \sigma)$ with $f \in \mathbb{A}$.

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$$= \frac{a^2+4a+5}{10(a^2+2a+2)}S_1(a) - \frac{(a-1)(a+1)}{5(a^2+2a+2)}S_3(a) - \frac{2}{5}\sum_{k=1}^{a}\frac{1}{k^2}$$

A symbolic summation approach

1. FIND an appropriate $R\Pi\Sigma^*$-ring $(\mathbb{A}, \sigma)$ with $f \in \mathbb{A}$.

2. FIND an appropriate extension $\mathbb{E} > \mathbb{A}$ with $g \in \mathbb{E}$:

$$\sigma(g) - g = f.$$

appropriate = sum representations with optimal nesting depth

Example:

$$\sum_{k=1}^{n} \frac{\sum_{j=1}^{k} \frac{\sum_{i=1}^{j} \frac{1}{i}}{j}}{k} = ?$$

A symbolic summation approach

1. FIND an appropriate $R\Pi\Sigma^*$-ring $(\mathbb{A}, \sigma)$ with $f \in \mathbb{A}$.

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Example:

$$\sum_{k=1}^{n} \frac{\sum_{j=1}^{k} \frac{\sum_{i=1}^{j} \frac{1}{i}}{j}}{k} = \frac{1}{6}\left(\sum_{i=1}^{n}\frac{1}{i}\right)^3 + \frac{1}{2}\left(\sum_{i=1}^{n}\frac{1}{i^2}\right)\left(\sum_{i=1}^{n}\frac{1}{i}\right) + \frac{1}{3}\sum_{i=1}^{n}\frac{1}{i^3}$$

depth 3 depth 1

A symbolic summation approach

1. FIND an appropriate $R\Pi\Sigma^*$-ring $(\mathbb{A},\sigma)$ with $f\in\mathbb{A}$.

2. FIND an appropriate extension $\mathbb{E}>\mathbb{A}$ with $g\in\mathbb{E}$:

$$\sigma(g)-g=f.$$

appropriate = sum representations with minimal number of objects

Example:

$$\sum_{k=0}^{a}(-1)^k S_1(k)^2\binom{n}{k}=?$$

A symbolic summation approach

1. FIND an appropriate $R\Pi\Sigma^*$-ring $(\mathbb{A},\sigma)$ with $f\in\mathbb{A}$.

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appropriate = sum representations with minimal number of objects

Example:

$$\sum_{k=0}^{a}(-1)^k S_1(k)^2\binom{n}{k} = -\frac{1}{n}\sum_{j=1}^{a}\frac{(-1)^j}{j}\binom{n}{j} - (a-n)\big(n^2S_1(a)^2+2nS_1(a)+2\big)\frac{(-1)^a\binom{n}{a}}{n^3}-\frac{2}{n^2}$$

Simplification of nested product-sum expressions

$A(k)$: nested product-sum expression (sums/products not in the denominator)

$\downarrow$ `SigmaReduce[A,k]`

$B(k)$: nested product-sum expression (sums/products not in the denominator)

- such that

$A(\lambda) = B(\lambda)$ for all $\lambda \in \mathbb{N}$ with $\lambda \geq \delta$
(δ can be computed explicitly)

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- such that all the sums in $B(k)$ are **simplified** as above

- and such that

  the arising sums in $B(k)$ are **algebraically independent**
  (i.e., they do not satisfy any polynomial relation)

Simplification of nested product-sum expressions

$A(k)$: nested product-sum expression (sums/products not in the denominator)

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$B(k)$: nested product-sum expression (sums/products not in the denominator)

Application 1: the expression $B(k)$ is usually much smaller

Simplification of nested product-sum expressions

$A(k)$: nested product-sum expression (sums/products not in the denominator)

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$B(k)$: nested product-sum expression (sums/products not in the denominator)

Application 1: the expression $B(k)$ is usually much smaller

Application 2: We solve the zero-recognition problem.

$A(k)$ evaluates to 0 from a certain point on $\quad\Leftrightarrow\quad B(k)=0$

CS, J. Symb. Comput. 72, pp. 82-127. 2016. arXiv:1408.2776 [cs.SC]

CS, J. Symb. Comput. 80(3), pp. 616-664. 2017. arXiv:1603.04285 [cs.SC]

Toolbox 2: Definite summation

$$(n+2)\mathbf{A}(n+1) - n\mathbf{A}(n) = \frac{(n+1)S_1(n)+1}{(n+1)^3}$$

Summation package Sigma

(based on difference field/ring algorithms/theory
see, e.g., Karr 1981, Bronstein 2000, Schneider 2001–)

$$A(n) = \sum_{k=1}^{\infty} \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)} = 0 \times \frac{1}{n(n+1)} + \frac{S_1(n)^2+S_2(n)}{2n(n+1)}$$

where

$$S_1(n) = \sum_{i=1}^{n} \frac{1}{i} \qquad S_2(n) = \sum_{i=1}^{n} \frac{1}{i^2}$$

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$A(n) = \sum_{k=1}^{n} f(n,k);$$

$f(n,k)$: indefinite nested product-sum in k;
n: extra parameter

FIND a recurrence for $A(n)$

Zeilberger's creative telescoping paradigm

GIVEN

$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}}_{=:\, f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n), c_1(n)$:

$$\boxed{g(n,k+1)-g(n,k)} = \boxed{c_0(n)f(n,k)+c_1(n)\,f(n+1,k)}$$

for all $0 \le k \le n$ and all $n \ge 0$.

Sigma *computes*: $c_0(n) = -n$, $c_1(n) = (n+2)$ and

$$g(n,k) = \frac{kS_1(k)+(-n-1)S_1(n)-kS_1(k+n)-2}{(k+n+1)(n+1)^2}$$

Zeilberger's creative telescoping paradigm

GIVEN

$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=:\, f(n,k)}.$$

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for all $0 \le k \le n$ and all $n \ge 0$.

Summing this equation over k from 1 to a gives:

$$\boxed{g(n,a+1) - g(n,1)} = \boxed{c_0(n)\,\mathsf{A}(n) + c_1(n)\,\mathsf{A}(n+1)}$$

$$||\qquad\qquad\qquad ||$$

$$\frac{(a+1)(S_1(a)+S_1(n)-S_1(a+n))}{(n+1)^2(a+n+2)} + \frac{a(a+1)}{(n+1)^3(a+n+1)(a+n+2)} \qquad -n\mathsf{A}(n) + (2+n)\mathsf{A}(n+1)$$

Zeilberger's creative telescoping paradigm

GIVEN

$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=:\, f(n,k)}.$$

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$$\lim_{a\to\infty} \quad || \qquad\qquad\qquad\qquad ||$$

$$\frac{(n+1)S_1(n)+1}{(n+1)^3} \qquad\qquad -n\mathsf{A}(n) + (2+n)\mathsf{A}(n+1)$$

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$A(n) = \sum_{k=1}^{n} f(n,k);$$

$f(n,k)$: indefinite nested product-sum in k;
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FIND a recurrence for $A(n)$

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FIND a recurrence for $A(n)$

2. Recurrence solving

GIVEN a recurrence

$a_0(n), \dots, a_d(n), h(n)$:
indefinite nested product-sum expressions in n.

$$a_0(n)A(n) + \cdots + a_d(n)A(n+d) = h(n);$$

FIND all solutions expressible by indefinite nested products/sums in n.
(d'Alembertian solutions)
(Abramov/Bronstein/Petkovšek/CS, in preparation)

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Note: the sum solutions are highly nested
(possibly with denominators of high degrees)

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FIND all solutions expressible by indefinite nested products/sums in n.
(d'Alembertian solutions)
(Abramov/Bronstein/Petkovšek/CS, in preparation)

3. Simplify the solutions (using difference ring/field theory) s.t.
   - the sums are algebraically independent;
   - the sums are flattened;
   - the sums can be given in terms of special functions.

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

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4. Find a "closed form"

A(n)=combined solutions in terms of indefinite nested sums in n.

In[1]:= $<<$ **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= $$\mathbf{mySum} = \sum_{k=1}^{A} \frac{S[1,k] + S[1,n] - S[1,k+n]}{kn(k+n+1)};$$

In[1]:= $<<$ **Sigma.m**

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Compute a recurrence

In[3]:= $\mathbf{rec = GenerateRecurrence[mySum, n][[1]]}$

Out[3]= $-n\mathsf{SUM}[n] + (1+n)(2+n)\mathsf{SUM}[n+1] == \frac{(a+1)(S[1,a]+S[1,n]-S[1,a+n])}{(n+1)^2(a+n+2)n!} + \frac{a(a+1)}{(n+1)^3(a+n+1)(a+n+2)n!}$

In[4]:= $\mathbf{rec = LimitRec[rec, SUM[n], \{n\}, A]}$

Out[4]= $$-\mathtt{n}\mathsf{SUM}[\mathtt{n}] + (1+\mathtt{n})(2+\mathtt{n})\mathsf{SUM}[\mathtt{n}+1] == \frac{(\mathtt{n}+1)\mathtt{S}[1,\mathtt{n}]+1}{(\mathtt{n}+1)^3}$$

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In[4]:= $\mathbf{rec = LimitRec[rec, SUM[n], \{n\}, A]}$

Out[4]= $-n\mathrm{SUM}[n] + (1+n)(2+n)\mathrm{SUM}[n+1] == \frac{(n+1)S[1,n]+1}{(n+1)^3}$

Solve a recurrence

In[5]:= $\mathbf{recSol = SolveRecurrence[rec, SUM[n], IndefiniteSummation \to False]}$

In[1]:= $<<$ **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

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Solve a recurrence

In[5]:= $\mathbf{recSol = SolveRecurrence[rec, SUM[n], IndefiniteSummation \to False]}$

Out[5]= $\{\{0, \frac{1}{n(n+1)}\}, \{1, \frac{\sum_{i=1}^{n} \frac{S[1,i]}{i}}{n(n+1)}\}\}$

In[1]:= $<<$ **Sigma.m**

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In[2]:= $\mathbf{mySum} = \sum_{k=1}^{A} \frac{S[1,k]+S[1,n]-S[1,k+n]}{kn(k+n+1)};$

Compute a recurrence

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In[3]:= $\mathbf{rec = GenerateRecurrence[mySum, n][[1]]}$

Out[3]= $n\mathsf{SUM}[n]+(1+n)(2+n)\mathsf{SUM}[n+1] == \frac{(a+1)(S[1,a]+S[1,n]-S[1,a+n])}{(n+1)^2(a+n+2)n!} + \frac{a(a+1)}{(n+1)^3(a+n+1)(a+n+2)n!}$

In[4]:= $\mathbf{rec = LimitRec[rec, SUM[n], \{n\}, A]}$

Out[4]= $-\mathtt{n}\mathsf{SUM}[\mathtt{n}]+(1+\mathtt{n})(2+\mathtt{n})\mathsf{SUM}[\mathtt{n}+1] == \frac{(\mathtt{n}+1)\mathtt{S}[1,\mathtt{n}]+1}{(\mathtt{n}+1)^3}$

Solve a recurrence

In[5]:= $\mathbf{recSol = SolveRecurrence[rec, SUM[n], IndefiniteSummation \to True]}$

Out[5]= $\{\{0, \frac{1}{\mathtt{n}(\mathtt{n}+1)}\}, \{1, \frac{\mathtt{S}[1,\mathtt{n}]^2 + \sum_{\mathtt{i}=1}^{\mathtt{n}} \frac{1}{\mathtt{i}^2}}{2\mathtt{n}(\mathtt{n}+1)}\}\}$

In[1]:= $<<$ **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= $$\mathbf{mySum} = \sum_{\mathbf{k=1}}^{\mathbf{A}} \frac{\mathbf{S[1,k] + S[1,n] - S[1,k+n]}}{\mathbf{kn(k+n+1)}};$$

Compute a recurrence

In[3]:= $\mathbf{rec = GenerateRecurrence[mySum, n][[1]]}$

Out[3]= $$nSUM[n] + (1+n)(2+n)SUM[n+1] == \frac{(a+1)(S[1,a]+S[1,n]-S[1,a+n])}{(n+1)^2(a+n+2)n!} + \frac{a(a+1)}{(n+1)^3(a+n+1)(a+n+2)n!}$$

In[4]:= $\mathbf{rec = LimitRec[rec, SUM[n], \{n\}, A]}$

Out[4]= $$-nSUM[n] + (1+n)(2+n)SUM[n+1] == \frac{(n+1)S[1,n]+1}{(n+1)^3}$$

Solve a recurrence

In[5]:= $\mathbf{recSol = SolveRecurrence[rec, SUM[n], IndefiniteSummation \to True]}$

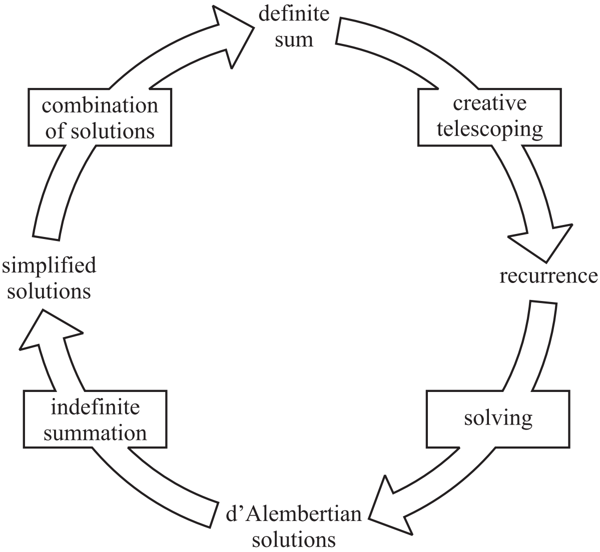
Out[5]= $$\{\{0, \frac{1}{n(n+1)}\}, \{1, \frac{S[1,n]^2 + \sum_{i=1}^{n} \frac{1}{i^2}}{2n(n+1)}\}\}$$

Combine the solutions

In[6]:= $\mathbf{FindLinearCombination[recSol, \{1, \{1/2\}, n, 2]}$

Out[6]= $$\frac{S[1,n]^2 + \sum_{i=1}^{n} \frac{1}{i^2}}{2n(n+1)}$$

Sigma's summation spiral



Example

Toolbox 3: Special function algorithms

Computer algebra and special functions:

Harmonic sums (Borwein, Hoffman, Broadhurst, Vermaseren, Remmiddi, Blümlein,. . .)

$$\boxed{\sum_{i=1}^{n}\frac{1}{i^2}\sum_{j=1}^{i}\frac{1}{j}}$$

Computer algebra and special functions:

Harmonic sums (Borwein, Hoffman, Broadhurst, Vermaseren, Remmiddi, Blümlein,. . .)

$$\boxed{\sum_{i=1}^{n} \frac{1}{i^2} \sum_{j=1}^{i} \frac{1}{j}}$$

Integral representation:

$$= \int_0^1 \frac{x^n - 1}{1-x} \left(\int_0^x \frac{\int_0^y \frac{1}{1-z} dz}{y} dy - \zeta_2 \right) dx, \qquad \zeta_z := \sum_{i=1}^{\infty} 1/i^z$$

Computer algebra and special functions:

Harmonic sums (Borwein, Hoffman, Broadhurst, Vermaseren, Remmiddi, Blümlein,. . .)

$$\boxed{\sum_{i=1}^{n}\frac{1}{i^2}\sum_{j=1}^{i}\frac{1}{j}}$$

Integral representation:

$$=\int_0^1 \frac{x^n-1}{1-x}\left(\int_0^x \frac{\int_0^y \frac{1}{1-z}dz}{y}dy-\zeta_2\right)dx, \qquad \zeta_z := \sum_{i=1}^{\infty} 1/i^z$$

Asymptotic expansion:

$$=\left(\frac{1}{30n^5}-\frac{1}{6n^3}+\frac{1}{2n^2}-\frac{1}{n}\right)\ln(n)$$
$$-\tfrac{1}{100n^5}-\tfrac{1}{6n^4}+\tfrac{13}{36n^3}-\tfrac{1}{4n^2}-\tfrac{1}{n}+2\zeta_3+O(\tfrac{\ln(n)}{n^6}).$$

limit computations **numerical evaluation**

- Generalized algorithms for generalized harmonic sums

$$\sum_{k=1}^{n} \frac{2^k \sum_{i=1}^{k} \frac{2^{-i} \sum_{j=1}^{i} \frac{S_1(j)}{j}}{i}}{k} = -\frac{21\zeta_2^2}{20}\frac{1}{n} + \frac{1}{8n^2} + \frac{295}{216n^3} - \frac{1115}{96n^4} + O(n^{-5})$$
$$+ \big(\frac{1}{2n} - \frac{3}{4n^2} + \frac{19}{12n^3} - \frac{5}{n^4} + O(n^{-5})\big)\zeta_2$$
$$+ 2^n\big(\frac{3}{2n} + \frac{3}{2n^2} + \frac{9}{2n^3} + \frac{39}{2n^4} + O(n^{-5})\big)\zeta_3$$
$$+ \big(\frac{1}{n} + \frac{3}{4n^2} - \frac{157}{36n^3} + \frac{19}{n^4} + O(n^{-5})\big)(\log(n) + \gamma)$$
$$+ \big(\frac{1}{2n} - \frac{3}{4n^2} + \frac{19}{12n^3} - \frac{5}{n^4} + O(n^{-5})\big)(\log(n) + \gamma)^2)$$

[Ablinger, Blümlein, CS, J. Math. Phys. 54, 2013, arXiv:1302.0378 [math-ph]]

- Generalized algorithms for cyclotomic harmonic sums

$$\sum_{k=1}^{n}\frac{\sum_{j=1}^{k}\frac{\sum_{i=1}^{j}\frac{1}{1+2i}}{j^2}}{(1+2k)^2} = \Big(-3+\frac{35\zeta_3}{16}\Big)\zeta_2-\frac{31\zeta_5}{8}$$
$$+\frac{1}{n}-\frac{33}{32n^2}+\frac{17}{16n^3}-\frac{4795}{4608n^4}+O(n^{-5})$$
$$+\log(2)\Big(6\zeta_2-\frac{1}{n}+\frac{9}{8n^2}-\frac{7}{6n^3}+\frac{209}{192n^4}+O(n^{-5})\Big)$$
$$+\Big(-\frac{7}{4}-\frac{7}{16n}+\frac{7}{16n^2}-\frac{77}{192n^3}+\frac{21}{64n^4}+O(n^{-5})\Big)\zeta_3$$
$$+\Big(\frac{1}{16n^2}-\frac{1}{8n^3}+\frac{65}{384n^4}+O(n^{-5})\Big)(\log(n)+\gamma)$$

[Ablinger, Blümlein, CS, J. Math. Phys. 52, 2011, arXiv:1302.0378 [math-ph]]

- Generalized algorithms for nested binomial sums

$$\sum_{j=1}^{n} \frac{4^j S_1(j-1)}{\binom{2j}{j} j^2} = 7\zeta_3 + \sqrt{\pi}\sqrt{n}\Bigg\{\Bigg[-\frac{2}{n} + \frac{5}{12n^2} - \frac{21}{320n^3} - \frac{223}{10752n^4} + \frac{671}{49152n^5} + \frac{11635}{1441792n^6} - \frac{1196757}{136314880n^7} - \frac{376193}{50331648n^8} + \frac{201980317}{18253611008n^9} + O(n^{-10})\Bigg]\ln(\bar{n}) - \frac{4}{n} + \frac{5}{18n^2} - \frac{263}{2400n^3} + \frac{579}{12544n^4} + \frac{10123}{1105920n^5} - \frac{1705445}{71368704n^6} - \frac{27135463}{11164188672n^7} + \frac{197432563}{7927234560n^8} + \frac{405757489}{775778467840n^9} + O(n^{-10})\Bigg\}$$

Ablinger, Blümlein, CS, ACAT 2013, arXiv:1310.5645 [math-ph]

Ablinger, Blümlein, Raab, CS, J. Math. Phys. 55, 2014. arXiv:1407.1822 [hep-th]

The full machinery:

Toolbox 1 + Toolbox 2 + Toolbox 3

The full machinery:

In[1]:= $<<$ **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= $<<$ **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[3]:= $<<$ **EvaluateMultiSums.m**

EvaluateMultiSums by Carsten Schneider © RISC-Linz

In[4]:= **EvaluateMultiSum[**

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Big(\frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!}\Big),\{n\},\{1\}]$$

The full machinery:

In[1]:= $<<$ **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= $<<$ **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[3]:= $<<$ **EvaluateMultiSums.m**

EvaluateMultiSums by Carsten Schneider © RISC-Linz

In[4]:= **EvaluateMultiSum**[

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Big(\frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!}\Big), \{n\}, \{1\}]$$

Out[4]= $\dfrac{S_1(n)^2+3S_2(n)}{2n(n+1)!}$

Back to Feynman integrals

joint work (RISC–DESY) with

J. Ablinger, A. Behring, I. Bierenbaum, J. Blümlein, A. Hasselhuhn,
A. de Freitas, A. von Manteuffel, C.G. Raab, M. Round, S. Klein, F. Wißbrock

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\underbrace{\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}\Big)}$$

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} f(n,k,j) = \frac{S_1(n)^2+3S_2(n)}{2n(n+1)!}$$

where

$$S_1(n)=\sum_{i=1}^{n}\frac{1}{i} \qquad S_2(n)=\sum_{i=1}^{n}\frac{1}{i^2}$$

Arose in the context of

I. Bierenbaum, J. Blümlein, and S. Klein, *Evaluating two-loop massive operator matrix elements with Mellin-Barnes integrals*. 2006

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

Simple sum

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\boxed{\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}}$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\boxed{\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}}$$

$$||$$

$$\boxed{\binom{j+1}{r}\Big(\frac{(-1)^r(-j+n-2)!r!}{(n+1)(-j+n+r-1)(-j+n+r)!}+\frac{(-1)^{n+r}(j+1)!(-j+n-2)!(-j+n-1)_r r!}{(n-1)n(n+1)(-j+n+r)!(-j-1)_r(2-n)_j}\Big)}$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\boxed{\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\binom{j+1}{r}\Big(\frac{(-1)^r(-j+n-2)!r!}{(n+1)(-j+n+r-1)(-j+n+r)!}+\frac{(-1)^{n+r}(j+1)!(-j+n-2)!(-j+n-1)_r r!}{(n-1)n(n+1)(-j+n+r)!(-j-1)_r(2-n)_j}\Big)}$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\boxed{\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\binom{j+1}{r}\Big(\frac{(-1)^r(-j+n-2)!r!}{(n+1)(-j+n+r-1)(-j+n+r)!}+\frac{(-1)^{n+r}(j+1)!(-j+n-2)!(-j+n-1)_r r!}{(n-1)n(n+1)(-j+n+r)!(-j-1)_r(2-n)_j}\Big)}$$

$$||$$

$$\boxed{\Big(\frac{n^2-n+1}{(n-1)^2n^2(n+1)(2-n)_j}+\frac{\sum_{i=1}^{j}\frac{(2-n)_i}{(-i+n-1)^2(i+1)!}}{(n+1)(2-n)_j}+\frac{(-1)^{j+n}(-j-2)(-j+n-2)!}{(j-n+1)(n+1)^2n!}\Big)(j+1)!-\frac{1}{(n+1)^2(-j+n-1)}}$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\sum_{j=0}^{n-2}\left(\Big(\frac{n^2-n+1}{(n-1)^2n^2(n+1)(2-n)_j}+\frac{\sum_{i=1}^{j}\frac{(2-n)_i}{(-i+n-1)^2(i+1)!}}{(n+1)(2-n)_j}+\right.$$

$$\left.\frac{(-1)^{j+n}(-j-2)(-j+n-2)!}{(j-n+1)(n+1)^2n!}\Big)(j+1)!-\frac{1}{(n+1)^2(-j+n-1)}\right)$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

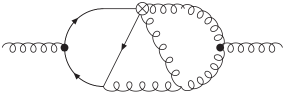
$$||$$

$$\sum_{j=0}^{n-2}\Bigg(\Big(\frac{n^2-n+1}{(n-1)^2n^2(n+1)(2-n)_j}+\frac{\sum_{i=1}^{j}\frac{(2-n)_i}{(-i+n-1)^2(i+1)!}}{(n+1)(2-n)_j}+$$
$$\frac{(-1)^{j+n}(-j-2)(-j+n-2)!}{(j-n+1)(n+1)^2n!}\Big)(j+1)!-\frac{1}{(n+1)^2(-j+n-1)}\Bigg)$$

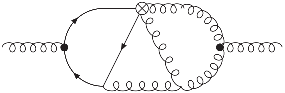
$$||$$

$$\frac{-n^2-n-1}{n^2(n+1)^3}+\frac{(-1)^n\left(n^2+n+1\right)}{n^2(n+1)^3}-\frac{2S_{-2}(n)}{n+1}+\frac{S_1(n)}{(n+1)^2}+\frac{S_2(n)}{-n-1}$$

Note: $S_a(n)=\sum_{i=1}^{N}\frac{\mathsf{sign}(a)^i}{i^{|a|}}$, $a\in\mathbb{Z}\setminus\{0\}$.



$$= F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + F_{-1}(n)\varepsilon^{-1} + \boxed{F_0(n)} + \dots$$



$$= F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + F_{-1}(n)\varepsilon^{-1} + \boxed{F_0(n)} + \dots$$

Simplify

$$||$$

$$\sum_{j=0}^{n-3}\sum_{k=0}^{j}\sum_{l=0}^{k}\sum_{q=0}^{-j+n-3}\sum_{s=1}^{-l+n-q-3}\sum_{r=0}^{-l+n-q-s-3}(-1)^{-j+k-l+n-q-3}\times$$

$$\times\frac{\binom{j+1}{k+1}\binom{k}{l}\binom{n-1}{j+2}\binom{-j+n-3}{q}\binom{-l+n-q-3}{s}\binom{-l+n-q-s-3}{r}r!(-l+n-q-r-s-3)!(s-1)!}{(-l+n-q-2)!(-j+n-1)(n-q-r-s-2)(q+s+1)}$$

$$\Big[4S_1(-j+n-1)-4S_1(-j+n-2)-2S_1(k)$$

$$-(S_1(-l+n-q-2)+S_1(-l+n-q-r-s-3)-2S_1(r+s))$$

$$+2S_1(s-1)-2S_1(r+s)\Big] + \textbf{3 further 6–fold sums}$$

$$
\boxed{F_0(n)} =
$$

$$
\begin{aligned}
&\frac{7}{12}S_1(n)^4+\frac{(17n+5)S_1(n)^3}{3n(n+1)}+\Big(\frac{35n^2-2n-5}{2n^2(n+1)^2}+\frac{13S_2(n)}{2}+\frac{5(-1)^n}{2n^2}\Big)S_1(n)^2\\
&+\Big(-\frac{4(13n+5)}{n^2(n+1)^2}+\Big(\frac{4(-1)^n(2n+1)}{n(n+1)}-\frac{13}{n}\Big)S_2(n)+\Big(\frac{29}{3}-(-1)^n\Big)S_3(n)\\
&+\big(2+2(-1)^n\big)S_{2,1}(n)-28S_{-2,1}(n)+\frac{20(-1)^n}{n^2(n+1)}\Big)S_1(n)+\Big(\frac{3}{4}+(-1)^n\Big)S_2(n)^2\\
&-2(-1)^nS_{-2}(n)^2+S_{-3}(n)\Big(\frac{2(3n-5)}{n(n+1)}+\big(26+4(-1)^n\big)S_1(n)+\frac{4(-1)^n}{n+1}\Big)\\
&+\Big(\frac{(-1)^n(5-3n)}{2n^2(n+1)}-\frac{5}{2n^2}\Big)S_2(n)+S_{-2}(n)\Big(10S_1(n)^2+\Big(\frac{8(-1)^n(2n+1)}{n(n+1)}\\
&+\frac{4(3n-1)}{n(n+1)}\Big)S_1(n)+\frac{8(-1)^n(3n+1)}{n(n+1)^2}+\big(-22+6(-1)^n\big)S_2(n)-\frac{16}{n(n+1)}\Big)\\
&+\Big(\frac{(-1)^n(9n+5)}{n(n+1)}-\frac{29}{3n}\Big)S_3(n)+\Big(\frac{19}{2}-2(-1)^n\Big)S_4(n)+\big(-6+5(-1)^n\big)S_{-4}(n)\\
&+\Big(-\frac{2(-1)^n(9n+5)}{n(n+1)}-\frac{2}{n}\Big)S_{2,1}(n)+\big(20+2(-1)^n\big)S_{2,-2}(n)+\big(-17+13(-1)^n\big)S_{3,1}(n)\\
&-\frac{8(-1)^n(2n+1)+4(9n+1)}{n(n+1)}S_{-2,1}(n)-\big(24+4(-1)^n\big)S_{-3,1}(n)+\big(3-5(-1)^n\big)S_{2,1,1}(n)\\
&+32S_{-2,1,1}(n)+\left(\frac{3}{2}S_1(n)^2-\frac{3S_1(n)}{n}+\frac{3}{2}(-1)^nS_{-2}(n)\right)\zeta(2)
\end{aligned}
$$

$$\boxed{F_0(n)} =$$

$$\frac{7}{12}S_1(n)^4 + \frac{(17n+5)S_1(n)^3}{\ldots} + \Big(\frac{35n^2-2n-5}{2n^2(n+1)^2} + \frac{13S_2(n)}{2} + \frac{5(-1)^n}{2n^2}\Big)S_1(n)^2$$

$$S_1(n) = \sum_{i=1}^{n} \frac{1}{i}$$

$$+ \Big(\ldots \frac{\ldots 1)^n(2n+1)}{n(n+1)} - \frac{13}{n}\Big)S_2(n) + \Big(\frac{29}{3} - (-1)^n\Big)S_3(n)$$

$$+ \Big(2 + \ldots - 28S_{-2,1}(n) + \frac{20(-1)^n}{n^2(n+1)}\Big)S_1(n) + \Big(\frac{3}{4} + (-1)^n\Big)S_2(n)^2$$

$$- 2(-1)^n S_{-2}(n)^2 + S_{-3}(n)\Big(\frac{2(3n-5)}{n(n+1)} + \big(26 + 4(-1)^n\big)S_1(n) + \frac{4(-1)^n}{n+1}\Big)$$

$$+ \Big(\frac{(-1)^n(5-3n)}{2n^2(n+1)} - \frac{5}{2n^2}\Big)S_2(n) + S_{-2}(n)\Big(10S_1(n)^2 + \Big(\frac{8(-1)^n(2n+1)}{n(n+1)}$$

$$+ \frac{4(3n-1)}{n(n+1)}\Big)S_1(n) + \frac{8(-1)^n(3n+1)}{n(n+1)^2} + \big(-22 + 6(-1)^n\big)S_2(n) - \frac{16}{n(n+1)}\Big)$$

$$+ \Big(\frac{(-1)^n(9n+5)}{n(n+1)} - \frac{29}{3n}\Big)S_3(n) + \Big(\frac{19}{2} - 2(-1)^n\Big)S_4(n) + \big(-6 + 5(-1)^n\big)S_{-4}(n)$$

$$+ \Big(-\frac{2(-1)^n(9n+5)}{n(n+1)} - \frac{2}{n}\Big)S_{2,1}(n) + \big(20 + 2(-1)^n\big)S_{2,-2}(n) + \big(-17 + 13(-1)^n\big)S_{3,1}(n)$$

$$- \frac{8(-1)^n(2n+1) + 4(9n+1)}{n(n+1)}S_{-2,1}(n) - \big(24 + 4(-1)^n\big)S_{-3,1}(n) + \big(3 - 5(-1)^n\big)S_{2,1,1}(n)$$

$$+ 32S_{-2,1,1}(n) + \left(\frac{3}{2}S_1(n)^2 - \frac{3S_1(n)}{n} + \frac{3}{2}(-1)^n S_{-2}(n)\right)\zeta(2)$$

$$\boxed{F_0(n)} =$$

$$\frac{7}{12}S_1(n)^4 + \frac{(17n+5)S_1(n)^3}{\cdots} + \Big(\frac{35n^2-2n-5}{2n^2(n+1)^2} + \frac{13S_2(n)}{2} + \frac{5(-1)^n}{2n^2}\Big)S_1(n)^2$$

$$+ \Big(\cdots \frac{\cdots 1)^n(2n+1)}{n(n+1)} - \frac{13}{n}\Big)S_2(n) + \Big(\frac{29}{3} - (-1)^n\Big)S_3(n)$$

$$+ \big(2 + \cdots - 28S_{-2,1}(n) + \frac{20(-1)^n}{n^2(n+1)}\big) \cdots \big)S_2(n)^2$$

$$- 2(-1)^n S_{-2}(n)^2 + S_{-3}(n)\Big(\frac{2(3n-5)}{n(n+1)} + \big(26 + 4(\cdots\Big)$$

$$+ \Big(\frac{(-1)^n(5-3n)}{2n^2(n+1)} - \frac{5}{2n^2}\Big)S_2(n) + S_{-2}(n)\Big(10S_1(n)^2 + \Big(\frac{8(-1)^n(2n+1)}{n(n+1)}$$

$$+ \frac{4(3n-1)}{n(n+1)}\Big)S_1(n) + \frac{8(-1)^n(3n+1)}{n(n+1)^2} + \big(-22 + 6(-1)^n\big)S_2(n) - \frac{16}{n(n+1)}\Big)$$

$$+ \Big(\frac{(-1)^n(9n+5)}{n(n+1)} - \frac{29}{3n}\Big)S_3(n) + \Big(\frac{19}{2} - 2(-1)^n\Big)S_4(n) + \big(-6 + 5(-1)^n\big)S_{-4}(n)$$

$$+ \Big(-\frac{2(-1)^n(9n+5)}{n(n+1)} - \frac{2}{n}\Big)S_{2,1}(n) + \big(20 + 2(-1)^n\big)S_{2,-2}(n) + \big(-17 + 13(-1)^n\big)S_{3,1}(n)$$

$$- \frac{8(-1)^n(2n+1) + 4(9n+1)}{n(n+1)}S_{-2,1}(n) - \big(24 + 4(-1)^n\big)S_{-3,1}(n) + \big(3 - 5(-1)^n\big)S_{2,1,1}(n)$$

$$+ 32S_{-2,1,1}(n) + \left(\frac{3}{2}S_1(n)^2 - \frac{3S_1(n)}{n} + \frac{3}{2}(-1)^n S_{-2}(n)\right)\zeta(2)$$

$$S_1(n) = \sum_{i=1}^{n}\frac{1}{i}$$

$$S_2(n) = \sum_{i=1}^{n}\frac{1}{i^2}$$

$F_0(n)$ =

$$\frac{7}{12}S_1(n)^4 + \frac{(17n+5)S_1(n)^3}{\dots} + \Big(\frac{35n^2-2n-5}{2n^2(n+1)^2} + \frac{13S_2(n)}{2} + \frac{5(-1)^n}{2n^2}\Big)S_1(n)^2$$

$$+ \Big(\dots \frac{\dots 1)^n(2n+1)}{n(n+1)} - \frac{13}{n}\Big)S_2(n) + \Big(\frac{29}{3} - (-1)^n\Big)S_3(n)$$

$$+ \Big(2 + \dots - 28S_{-2,1}(n) + \frac{20(-1)^n}{n^2(n+1)}\Big) \dots \Big)S_2(n)^2$$

$$- 2(-1)^n S_{-2}(n)^2 + S_{-3}(n)\Big(\frac{2(3n-5)}{n(n+1)} + \big(26 + 4(\dots\Big)$$

$$+ \Big(\frac{(-1)^n(5-3n)}{2n^2 \dots} - \frac{5}{\dots}\Big)S_2(n) \dots \big(10S_1(n)^2 + \Big(\frac{8(-1)^n(2n+1)}{n(n+1)}$$

$$+ \frac{4(3n \dots}{n(n+\dots} \dots -1)^n\big)S_2(n) - \frac{16}{n(n+1)}\Big)$$

$$+ \Big(\frac{(-1)\dots}{n(\dots} \dots n) + \big(-6 + 5(-1)^n\big)S_{-4}(n)$$

$$+ \Big(- \dots S_{2,-2}(n) + \big(-17 + 13(-1)^n\big)S_{3,1}(n)$$

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$$S_{-2,1,1}(n) = \sum_{i=1}^{n} \frac{(-1)^i \sum_{j=1}^{i} \frac{\sum_{k=1}^{j} \frac{1}{k}}{j}}{i^2}$$

Further tools

Find a recurrence for the integral/sum

$$D_\varepsilon(n) = \int_0^1 \cdots \int_0^1 \Phi(\varepsilon, n, x_1, x_2, \ldots, x_7) dx_1 dx_2 \ldots dx_7$$
$$\stackrel{?}{=} F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + F_{-1}(n)\varepsilon^{-1} + \ldots$$

multivariate
Almquist/Zeilberger
(Jakob Ablinger)

$$a_0(\varepsilon, n) D_\varepsilon(n) + \ldots + a_d(\varepsilon, n) D_\varepsilon(n+d) = h(\varepsilon, n)$$

Find a recurrence for the integral/sum

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$$\sum_{i_1} \cdots \sum_{i_7} f(\varepsilon, n, i_1, i_2, \ldots, i_7)$$

MultiSum Package
(Flavia Stan)

$$a_0(\varepsilon, n) D_\varepsilon(n) + \ldots + a_d(\varepsilon, n) D_\varepsilon(n+d) = h(\varepsilon, n)$$

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Holonomic/difference field Approach
(Mark Round)

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ε-recurrence solver

multivariate
Almquist/Zeilberger
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$$\sum_{i_1} \cdots \sum_{i_7} f(\varepsilon, n, i_1, i_2, \ldots, i_7)$$

MultiSum Package
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Holonomic/difference field Approach
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$$a_0(\varepsilon, n) D_\varepsilon(n) + \ldots + a_d(\varepsilon, n) D_\varepsilon(n + d) = h(\varepsilon, n)$$

Ansatz (for power series)

$$
\begin{aligned}
&a_0(\varepsilon,n)\Big[D_\varepsilon(n)\Big]\\
&+a_1(\varepsilon,n)\Big[D_\varepsilon(n+1)\Big]\\
&+\\
&\ \vdots\\
&+a_d(\varepsilon,n)\Big[D_\varepsilon(n+d)\Big]\\
&\qquad\qquad = h_0(n)+h_1(n)\varepsilon+h_2(n)\varepsilon^2+\dots
\end{aligned}
$$

given (in terms of indefinite nested sums and products)

Ansatz (for power series)

$$
\begin{aligned}
&a_0(\varepsilon,n)\Big[F_0(n)+F_1(n)\varepsilon+F_2(n)\varepsilon^2+\dots\Big]\\
&+a_1(\varepsilon,n)\Big[D_\varepsilon(n+1)\Big]\\
&+\\
&\ \vdots\\
&+a_d(\varepsilon,n)\Big[D_\varepsilon(n+d)\Big]\\
&\qquad\qquad\qquad\qquad= h_0(n)+h_1(n)\varepsilon+h_2(n)\varepsilon^2+\dots
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+&a_1(\varepsilon,n)\Big[F_0(n+1)+F_1(n+1)\varepsilon+F_2(n+1)\varepsilon^2+\dots\Big]\\
+&\\
&\vdots\\
+&a_d(\varepsilon,n)\Big[D_\varepsilon(n+d)\Big]\\
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+&\\
&\vdots\\
+&a_d(\varepsilon,n)\Big[F_0(n+d)+F_1(n+d)\varepsilon+F_2(n+d)\varepsilon^2+\dots\Big]\\
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+& \\
& \vdots\\
+& a_d(\varepsilon,n)\Big[F_0(n+d)+F_1(n+d)\varepsilon+F_2(n+d)\varepsilon^2+\dots\Big]\\
& \qquad\qquad = h_0(n)+h_1(n)\varepsilon+h_2(n)\varepsilon^2+\dots
\end{aligned}
$$

$\Downarrow$ lowest terms must agree

$$
\boxed{a_0(0,n)F_0(n)+a_1(0,n)F_0(n+1)+\dots+a_d(0,n)F_0(n+d)=h_0(n)}
$$

Ansatz (for power series)

$$\begin{aligned}
&a_0(\varepsilon,n)\Big[F_0(n)+F_1(n)\varepsilon+F_2(n)\varepsilon^2+\dots\Big]\\
+&a_1(\varepsilon,n)\Big[F_0(n+1)+F_1(n+1)\varepsilon+F_2(n+1)\varepsilon^2+\dots\Big]\\
+&\\
&\vdots\\
+&a_d(\varepsilon,n)\Big[F_0(n+d)+F_1(n+d)\varepsilon+F_2(n+d)\varepsilon^2+\dots\Big]\\
&\qquad\qquad\qquad\qquad = h_0(n)+h_1(n)\varepsilon+h_2(n)\varepsilon^2+\dots
\end{aligned}$$

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$$\boxed{a_0(0,n)F_0(n)+a_1(0,n)F_0(n+1)+\dots+a_d(0,n)F_0(n+d)=h_0(n)}$$

REC solver: Given the initial values $F_0(1), F_0(2), \dots, F_0(d)$, **decide** if $F_0(n)$ can be written in terms of indefinite nested sums and products.

Ansatz (for power series)

$$
\begin{aligned}
& a_0(\varepsilon,n)\Big[F_0(n) + F_1(n)\varepsilon + F_2(n)\varepsilon^2 + \dots\Big] \\
+ & a_1(\varepsilon,n)\Big[F_0(n+1) + F_1(n+1)\varepsilon + F_2(n+1)\varepsilon^2 + \dots\Big] \\
+ & \\
\vdots & \\
+ & a_d(\varepsilon,n)\Big[F_0(n+d) + F_1(n+d)\varepsilon + F_2(n+d)\varepsilon^2 + \dots\Big] \\
& \qquad = h_0(n) + h_1(n)\varepsilon + h_2(n)\varepsilon^2 + \dots
\end{aligned}
$$

$\Downarrow$ lowest terms must agree

$$
\boxed{a_0(0,n)F_0(n) + a_1(0,n)F_0(n+1) + \cdots + a_d(0,n)F_0(n+d) = h_0(n)}
$$

Ansatz (for power series)

$$
\begin{aligned}
& a_0(\varepsilon,n)\Big[F_0(n)+F_1(n)\varepsilon+F_2(n)\varepsilon^2+\dots\Big]\\
+& a_1(\varepsilon,n)\Big[F_0(n+1)+F_1(n+1)\varepsilon+F_2(n+1)\varepsilon^2+\dots\Big]\\
+& \\
& \vdots\\
+& a_d(\varepsilon,n)\Big[F_0(n+d)+F_1(n+d)\varepsilon+F_2(n+d)\varepsilon^2+\dots\Big]\\
&\qquad = h_0(n)+h_1(n)\varepsilon+h_2(n)\varepsilon^2+\dots
\end{aligned}
$$

$\Downarrow$ lowest terms must agree

$$\boxed{a_0(0,n)F_0(n)+a_1(0,n)F_0(n+1)+\dots+a_d(0,n)F_0(n+d)=h_0(n)}$$

$$
\begin{aligned}
&\quad a_0(\varepsilon,n)\Big[F_1(n)\varepsilon + F_2(n)\varepsilon^2 + \dots\Big] \\
&+a_1(\varepsilon,n)\Big[F_1(n+1)\varepsilon + F_2(n+1)\varepsilon^2 + \dots\Big] \\
&+ \\
&\ \vdots \\
&+a_d(\varepsilon,n)\Big[F_1(n+d)\varepsilon + F_2(n+d)\varepsilon^2 + \dots\Big] \\
&\qquad\qquad\qquad\qquad = h_0'(n) + h_1'(n)\varepsilon + h_2'(n)\varepsilon^2 + \dots
\end{aligned}
$$

$$
\begin{aligned}
& a_0(\varepsilon,n)\Big[F_1(n)\varepsilon + F_2(n)\varepsilon^2 + \dots\Big] \\
+ & a_1(\varepsilon,n)\Big[F_1(n+1)\varepsilon + F_2(n+1)\varepsilon^2 + \dots\Big] \\
+ & \\
\vdots & \\
+ & a_d(\varepsilon,n)\Big[F_1(n+d)\varepsilon + F_2(n+d)\varepsilon^2 + \dots\Big] \\
& \qquad = \underbrace{h_0'(n)}_{\mathbf{=0}} + h_1'(n)\varepsilon + h_2'(n)\varepsilon^2 + \dots
\end{aligned}
$$

Devide by ε

$$
\begin{aligned}
& a_0(\varepsilon,n)\Big[F_1(n)+F_2(n)\varepsilon+\dots\Big] \\
+ & a_1(\varepsilon,n)\Big[F_1(n+1)+F_2(n+1)\varepsilon+\dots\Big] \\
+ & \\
\vdots & \\
+ & a_d(\varepsilon,n)\Big[F_1(n+d)+F_2(n+d)\varepsilon+\dots\Big] = h'_1(n)+h'_2(n)\varepsilon+\dots
\end{aligned}
$$

Now repeat for $F_1(n), F_2(n), \dots$ Example

$$
\begin{aligned}
&a_0(\varepsilon,n)\Big[F_1(n)+F_2(n)\varepsilon+\dots\Big]\\
+&a_1(\varepsilon,n)\Big[F_1(n+1)+F_2(n+1)\varepsilon+\dots\Big]\\
+&\\
\vdots&\\
+&a_d(\varepsilon,n)\Big[F_1(n+d)+F_2(n+d)\varepsilon+\dots\Big]=h'_1(n)+h'_2(n)\varepsilon+\dots
\end{aligned}
$$

Now repeat for $F_1(n), F_2(n), \dots$ Example

Extension 1: Works also for Laurent series.

J. Blümlein, S. Klein, CS, F. Stan. J. Symbolic Comput. 47, 2012; arXiv:1011.2656v2
J. Ablinger, J. Blümlein, M. Round, CS. LL2012, 2012. arXiv:1210.1685 [cs.SC]

$$
\begin{aligned}
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&+\\
&\ \vdots\\
&+a_d(\varepsilon,n)\Big[F_1(n+d)+F_2(n+d)\varepsilon+\dots\Big]=h_1'(n)+h_2'(n)\varepsilon+\dots
\end{aligned}
$$

Now repeat for $F_1(n), F_2(n), \dots$ Example

Extension 2: Can be generalized to solve coupled systems of difference and differential equations

A. De Freitas, J. Blümlein, CS. PoS(LL2014)017 , pp. 1-13. 2014. arXiv:1407.2537 [cs.SC]

J. Ablinger, J. Blümlein, A. de Freitas, CS. PoS(RADCOR2015)060, pp. 1-13. 2015. arXiv:1601.01856 [cs.SC]

J. Ablinger, A. Behring, J. Blümlein, A. de Freitas, C. Schneider. PoS(LL2016)005, pp. 1-15. 2016. arXiv:1608.05376 [cs.SC]

So far derived results (in Nucl. Phys. B and Phys. Review D)

1. I. Bierenbaum, J. Blümlein, S. Klein, and C. Schneider. Two–Loop Massive Operator Matrix Elements for Unpolarized Heavy Flavor Production to $O(\epsilon)$. *Nucl.Phys. B* 803(1-2):1–41, 2008.
2. J. Ablinger, J. Blümlein, S. Klein, C. Schneider, F. Wissbrock. The $O(\alpha_s^3)$ Massive Operator Matrix Elements of $O(n_f)$ for the Structure Function $F_2(x,Q^2)$ and Transversity. Nucl. Phys. B, 844: 26-54, 2011.
3. J. Ablinger, J. Blümlein, A. Hasselhuhn, S. Klein, C. Schneider, F. Wissbrock Massive 3-loop Ladder Diagrams for Quarkonic Local Operator Matrix Elements. Nuclear Physics B. 864: 52-84, 2012.
4. J. Blümlein, A. Hasselhuhn, S. Klein, C. Schneider. The $O(\alpha_s^3 n_f T_F^2 C_{A,F})$ Contributions to the Gluonic Massive Operator Matrix Elements. Nuclear Physics B: 866: 196-211, 2013.
5. J. Ablinger, J. Blümlein, A. De Freitas A. Hasselhuhn, A. von Manteuffel, M. Round, C. Schneider, F. Wissbrock. The Transition Matrix Element $A_{gq}(N)$ of the Variable Flavor Number Scheme at $O(\alpha_s^3)$. Nuclear Physics B 882, pp. 263-288. 2014.
6. J. Ablinger, J. Blümlein, C. Raab, C. Schneider, F. Wissbrock. Calculating Massive 3-loop Graphs for Operator Matrix Elements by the Method of Hyperlogarithms. Nuclear Physics B 885, pp. 409-447. 2014.
7. J. Ablinger, J. Blümlein, A. De Freitas, A. Hasselhuhn, A. von Manteuffel, M. Round, C. Schneider. The $O(\alpha_s^3 T_F^2)$ Contributions to the Gluonic Operator Matrix Element. Nuclear Physics B 885, pp. 280-317. 2014.
8. J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. Hasselhuhn, A. von Manteuffel, M. Round, C. Schneider, F. Wissbrock. The 3-Loop Non-Singlet Heavy Flavor Contributions and Anomalous Dimensions for the Structure Function $F_2(x,Q^2)$ and Transversity. Nuclear Physics B 886, pp. 733-823. 2014.
9. J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. von Manteuffel, C. Schneider The 3-Loop Pure Singlet Heavy Flavor Contributions to the Structure Function $F_2(x,Q^2)$ and the Anomalous Dimension. Nuclear Physics B 890, pp. 48-151. 2015. arXiv:1409.1135 [hep-ph].
10. A. Behring, J. Blümlein, A. De Freitas, A. von Manteuffel, C. Schneider. The 3-Loop Non-Singlet Heavy Flavor Contributions to the Structure Function $g_1(x,Q^2)$ at Large Momentum Transfer. Nucl. Phys. B 897, pp. 612-644. 2015. arXiv:1504.08217 [hep-ph].
11. A. Behring, J. Blümlein, A. De Freitas, A. Hasselhuhn, A. von Manteuffel, C. Schneider. The $O(\alpha_s^3)$ Heavy Flavor Contributions to the Charged Current Structure Function $xF_3(x,Q^2)$ at Large Momentum Transfer. Physical Review D 92(114005), pp. 1-19. 2015. arXiv:1508.01449 [hep-ph].
12. J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. von Manteuffel, C. Schneider. Calculating three loop ladder and V-topologies for massive operator matrix elements by computer algebra. *Comput. Phys. Comm.*, 202, pp. 33-112. 2016. arXiv:1509.08324 [hep-ph].
13. A. Behring, J. Blmlein, G. Falcioni, A. De Freitas, A. von Manteuffel, C. Schneider. The Asymptotic 3-Loop Heavy Flavor Corrections to the Charged Current Structure Functions $F_L^{W^+-W^-}(x,Q^2)$ and $F_2^{W^+-W^-}(x,Q^2)$. Physical Review D, in press, 2016. arXiv:1609.06255 [hep-ph].

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6. J. Ablinger, ... Operator Matrix Elements by
7. J. Ablinger, ... he $O(\alpha_s^3 T_F^2)$ Contribution
8. J. Ablinger, ... chneider, F. Wissbrock. ... ructure Function $F_2(x, Q^2)$

Determine the coupling constant of the strong force (5 % error → 1% error)

One (of many) motivations:
the central value hints if and how the fundamental forces unite to one elementary force

9. J. Ablinger, A ... Singlet Heavy Flavor Contributions to the Structure Function $F_2(x, Q^2)$ and the Anomalous Dimension. Nuclear Physics B 890, pp. 48-151. 2015. arXiv:1409.1135 [hep-ph].
10. A. Behring, J. Blümlein, A. De Freitas, A. von Manteuffel, C. Schneider. The 3-Loop Non-Singlet Heavy Flavor Contributions to the Structure Function $g_1(x, Q^2)$ at Large Momentum Transfer. Nucl. Phys. B 897, pp. 612-644. 2015. arXiv:1504.08217 [hep-ph].
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- efficient implementations for large expressions
- reliable software based on proof certificates