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December, 2016

Axiomatic Description of Gröbner Reduction

Dissertation

zur Erlangung des akademischen Grades

Doktor der Technischen Wissenschaften

im Doktoratsstudium der

Technischen Wissenschaften

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JOHANNES KEPLER UNIVERSITY LINZ

RESEARCH INSTITUTE FOR SYMBOLIC COMPUTATION

– DOCTORAL THESIS –

Axiomatic Description of Gröbner Reduction

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Gewidmet meiner Familie

I don't know where I'm going from
here, but I promise it won't be boring.

David Bowie

Acknowledgements

In this doctoral thesis, we present results that have been obtained in Hagenberg in the last years. The first, and really deep intended, thanks go to my advisor Prof. Dr. Franz Winkler, who accepted me as Ph.D. student, and gave me an intriguing topic to work on. Having his good advices in mind, he guided me how to extend this branch of mathematics. It has been my privilege to learn a lot from Franz, who always managed to motivate me to this research. Further, he organized many occasions to present my research at conferences, as well as to my second advisor Prof. Dr. Meng Zhou. I'm very thankful for his motivating words as well, and his willingness to serve as referee for this thesis.

Further I want to express my gratitude to Dr. Günter Landsmann, who helped me at so many technical disussions, publishing several papers, and staying along the correct way in finding all this remarkable results. He always allowed me to present, discuss and improve the results, and contributed majorly to the final state of this thesis. Besides that, he always had an open ear, in non-mathematical issues, and i am really grateful to had the joy of working with him and benefiting from his experiences.

Many more people contributed, countless discussions allowed me to view this work from different perspectives, and reflecting the topics in this thesis from several viewpoints. I want to thank Prof. Alexander Levin for his willingness in writing a joint paper. The Algebra community of Linz, who was the major audience to whom this results have been presented, have always warmly integrated me in their middle. A huge thanks to all the people who helped me at my mathematical way. With the words of Joe Cocker, everything is possible *With a little Help from my Friends*.

Also, I want to thank the administrative team, who helped me in organizing travels, working material, and the warm-hearted working environment they've provided to me.

I've been an associated Ph.D. student at the Doctoral Program 'Computational Mathematics' W1214. This work was supported by the Austrian Science Fund (FWF) grant W1214-N15, project DK11, who allowed my to travel to the ISSAC conference in Bath, as well as the ACA conference in Kassel. I am very grateful for this support. I also want to thank Prof. Peter Paule, who was the advisor of my master thesis, and paved the way for my mathematical career in so many ways.

For the last words, I want to point to my family, *meiner lieben Mutter, meinem lieben Vater und meiner lieben Schwester*, to whom I want to dedicate this thesis. Without their support, I never would have managed to make my dream come true.

Summary

In this doctoral thesis we treat modules over rings of difference- and differential operators. To that end, we use concepts of Gröbner bases, and generalize the well-known notions to non-commutative ground domains.

While in recent literature concrete instances have been examined, we explicitly point out the essential ingredients, for the computation of the multivariate Hilbert function. The interplay of the considered rings already indicates that a common methodology is applicable. Having the right definitions available, allowed us to unify the different approaches present, and to formulate them in the most general fashion.

In particular, for the ring of difference-differential operators, we take *relative reduction*, for the ring of Ore-polynomials we have *reduction with respect to several term orderings* and for the Weyl-algebra we can consider (x, ∂)*-reduction* as the appropriate specialization of our concepts. But not only the non-commutative case is of interest, also for the usual commutative polynomial ring our concepts are applicable.

While the motives in the considered papers follow the same ideas, the details differ. This is mainly due to the different nature of the underlying rings. After introducing the notion of Gröbner Reduction and preparing the algebraic setup, we examine and extend the papers having this topic, in particular, we point out the relation to Gröbner reduction.

Finally, we present a Buchberger-type algorithm for computing Gröbner bases in multi-filtered rings, and introduce the new concept of set-relative reduction. This reduction provides a treatment for a wide class of rings.

Zusammenfassung

In dieser Dissertation betrachten wir Moduln über Ringe von Differenzen- und Differential Operatoren. Dafür verwenden wir die Konzepte von Gröbner Basen, and verallgemeinern die wohlbekannten Notationen auf nicht kommutativer Grunddomäne.

Während in der aktuellen Literatur konkrete Instanzen untersucht wurden, betrachten wir explizit die wesentlichen Bestandteile, die notwendig sind für die Berechnung der multivariaten Hilbertfunktion. Das Zusammenspiel der betrachteten Ringe zeigt auf, dass ein gemeinsamer Algorithmus anwendbar ist. Als wir die korrekten Definitionen zur Verfügung hatten, waren wir in der Lage die verschiedenen Ansätze zu vereinen, und in der allgemeinsten Formulierung zu präsentieren.

Insbesondere, für Differenz-Differential Operatoren betrachten wir *Relative Reduktion*, für den Ring der Ore-Polynome *Reduktion mit mehreren Termordnungen* und für Weyl-Algebren kann (x, ∂)-*Reduktion* als die angemessene Spezialisierung unserer Konzepte betrachtet werden kann. Aber nicht nur der nicht-kommutative Fall ist interessant, auch für den üblichen Ring von kommutativen Polynomen können unsere Überlegungen spezialisiert and angewendet werden.

Obwohl die Motive in den betrachteten Arbeiten die gleichen sind, sind die Details doch unterschiedlich, was sich hauptsächlich auf die verschiedenen Eigenschaften in den konkreten Ringen zurückführen lässt. Nach Einführung der Definition von Gröbner Reduktion, und dem Vorbereiten des algebraischen Setups, betrachten und erweitern wir die Arbeiten die dieses Thema haben. Insbesondere zeigen wir die Beziehung zu Gröbner Reduktion auf.

Zum Abschluss präsentieren wir eine Formulierung des Buchberger-Algorithmus zur Berechnung von Gröbnerbasen für multi-filtrierte Ringe, and präsentieren das neue Konzept von Set-relativer Reduktion. Diese Reduktion stellt ein Verfahren für eine große Klasse von Ringen dar.

Contents

1. Basic Setting

The first part of this doctoral thesis is to clarify the general setting, that will guide the reader through this thesis.

The first section starts this thesis with the basic setting in differential algebra. The theory of differential algebra was, amongst many contributors, essentially established by Ritt and Kolchin [Rit50, Kol73]. They contributed the notion of characteristic sets as a tool to describe (radical) differential ideals. With characteristic sets they were in the position to prove the existence of (univariate) differential dimension polynomials, to describe the size of a differential field extension.

Choosing the right notation is depending on the point of view, and not rigorously consistent throughout literature. We have decided to use the one presented in this chapter. We want to fix some standard assumptions, as well as more specialized conventions that are used throughout. We neglect concrete application for the moment, as we will return to it in subsequent chapters (with the exception, that we present generating function identities, to get a feeling how the size of a difference-differential field could be measured).

A major topic in this thesis is the theory of Gröbner bases in modules over non-commutative rings. To that end, we sketch considerations of Gröbner bases over modules in section 1.2. We recall the theory of relative Gröbner bases over difference-differential modules. In the upcoming chapters, we want to set up a general theory on Gröbner bases over filtered rings and extend the work of [ZW06, Lev07, ZW08b, Dön12].

The relations between the rings are sketched in section 1.4. At the heart of our considerations, we consider modules of difference-differential operators, Ore operators and degenerate versions of this rings, like differential operators (with polynomial coefficients), difference-operators and the usual commutative polynomial ring.

By taking differential and difference algebra as special case, results from the difference-differential algebra carry over and prove (implicit) theorems in differential and difference algebra.

1.1. Introduction

Throughout this thesis we shall denote by $\mathbb{N}$ the natural numbers including zero, $\mathbb{Z}$ the ring of integers, $\mathbb{Q}$ resp. $\mathbb{R}$ the field of rational resp. real numbers. The letter R shall always denote a (not necessarily commutative) ring with one, containing a field

$\mathbb{K}$ as a subring, i.e. $\mathbb{K} \subseteq R$. We assume that the field $\mathbb{K}$ contains a subfield that has characteristic zero.

Definition 1 (Derivation)**.**
Let R be a commutative ring. An R-map $\delta : R \to R$ is called a *derivation* on R if and only if it satisfies for all a, b in R:

$$\begin{aligned} \delta(a+b) &= \delta(a) + \delta(b), && \text{(Linearity)} \\ \delta(ab) &= a\delta(b) + b\delta(a). && \text{(Leibniz rule)} \end{aligned}$$

The set of derivations on R is denoted by $\mathrm{Der}(R)$.

Note the difference to the notion of *skew-derivation* as introduced on page 35.

Lemma 1 (Elementary Properties of Derivations)**.**
If δ is a derivation on R, then for $a, b \in R$ it holds:

- $\delta(0) = \delta(1) = 0$*;*
- $\delta(a^k) = ka^{k-1}\delta(a)$ *where* $k > 0$*;*
- $\delta(a^{-1}) = -\delta(a)/a^2$ *for* $a \neq 0$*;*
- *For* $k > 0$ *we have*

$$\delta^k(ab) = \sum_{i=0}^{k} \binom{k}{i} \delta^{k-i}(a)\delta^i(b) \qquad \text{(General Leibniz rule)}$$

- $\delta(a/b) = (b\delta(a) - a\delta(b))/b^2$ *for* $b \neq 0$*.*

Definition 2 (Differential Ring)**.**
Starting from the commutative ring R, we adjoin a fixed set of pairwise commutative derivations on R,

$$\Delta := \{\delta_1, \ldots, \delta_m\}, \qquad \delta_k \in \mathrm{Der}(R).$$

We call (R, Δ) a *(partial) differential ring*. If $m = 1$ it is called an *ordinary differential ring*. The commutative semigroup Θ_m of formal products (which we call monomials) in a differential ring is defined by

$$\Theta_m := \{\theta := \delta_1^{k_1} \ldots \delta_m^{k_m} : \ (k_1, \ldots, k_m) \in \mathbb{N}^m\}, \qquad (1.1)$$

elements in Θ_m are called *derivation monomials or differential monomials.*

We define the *order* of a derivation monomial θ by

$$\mathrm{ord}_{\Theta_m}(\theta) := \mathrm{ord}_{\Theta_m}(\delta_1^{k_1} \ldots \delta_m^{k_m}) = k_1 + \ldots + k_m, \qquad \theta \in \Theta_m.$$

The subsets $\Theta_m(s)$ and $\Theta_m^{=}(s)$ of Θ_m are the set of bounded derivation monomials, i.e. the set of differential monomials of order less or equal to s, resp. of order exactly s

$$\Theta_m(s) := \{\theta \in \Theta_m : \ \mathrm{ord}_{\Theta_m}(\theta) \leq s\}, \qquad \Theta_m^{=}(s) := \{\theta \in \Theta_m : \mathrm{ord}_{\Theta_m}(\theta) = s\}.$$

Obviously we have the inclusions on the set Θ_m

$$\Theta_m(s) \subseteq \Theta_m(t), \qquad \text{for} \ \ s \leq t. \tag{1.2}$$

A differential ring gives rise to the associated *ring of differential operators*. Let K be a subring of R, a *differential operator* is a K-linear combination of derivation monomials

$$f = \sum_{\theta \in \Theta_m} a_\theta \theta, \qquad a_\theta \in K,$$

where at most finitely many a_θ are not zero. The *order of a differential operator* is given by

$$\mathrm{ord}_{\Theta_m}(f) = \mathrm{ord}_{\Theta_m}\left(\sum_{\theta \in \Theta_m} a_\theta \theta\right) := \max\{\mathrm{ord}_{\Theta_m}(\theta) : \ a_\theta \neq 0\}.$$

Remark. Aistleitner [Ais10] pointed out that derivation monomials, as well as differential operators might be derivations themselves, but don't need to be.

The product of a differential operator with an element in a ring is given by

$$\left(\sum_{k=0}^{n} a_k \delta^k\right) \cdot x = a_0 \cdot x + \sum_{k=1}^{n} a_k \cdot \delta^{k-1} \cdot (x \cdot \delta + \delta(x)), \qquad n > 0, \ x \in R,$$

and iterated application of the Leibniz rule. The rule for $\theta \in \Theta_m$ for a partial differential ring is similar.

For a differential ring, we will next describe the number of differential monomials of certain order. Obviously, the sets $\Theta_m(s)$ and $\Theta_m^{=}(s)$ are related by

$$\Theta_m(s) = \bigcup_{k=0}^{s} \Theta_m^{=}(k), \qquad s \in \mathbb{N}.$$

This correspondence can be translated into a correspondence between the sequences $|\Theta(s)|$ and $|\Theta^{=}(s)|$. In fact, we can apply the analogous construction to difference-differential rings and difference-rings, that will appear later in this section.

Let $\mathbb{M}$ be a set of monomials, and let the map $v : \mathbb{M} \to \mathbb{N}$ be a functional describing the monomials (such as $\deg(\cdot), \mathrm{ord}_{\Theta_m}(\cdot), \ldots$). For $s \in \mathbb{N}$, let

$$\mathbb{M}(s) := \{\mathfrak{m} \in \mathbb{M} : v(\mathfrak{m}) \leq s\}, \qquad \mathbb{M}^{=}(s) := \{\mathfrak{m} \in \mathbb{M} : v(\mathfrak{m}) = s\},$$

and suppose that $|\mathbb{M}(0)| = |\mathbb{M}^{=}(0)| = 1$. Then,

$$|\mathbb{M}(s)| = \sum_{k=0}^{s} |\mathbb{M}^{=}(k)|, \qquad s \geq 0. \tag{1.3}$$

A Few Words on Generating Functions

In the following part, we will consider certain sequences, that describe the number of monomials ('power products') in a ring. An adequate way of mathematically describing a sequence is, to view it as a function from the non-negative integers to the field of real numbers. This is closely related to the study of generating functions. But what exactly is meant by a generating function? Quoting the reference book [Wil06],

A generating function is a clothesline on which
we hang up a sequence of numbers for display.

Let be given a sequence $(a_k)_{k\geq 0} \in \mathbb{R}^{\mathbb{N}}$. We can assign to the sequence $(a_k)_{k\geq 0}$ its *ordinary generating function* which is given by

$$\begin{aligned} \mathcal{G} &: \mathbb{R}^{\mathbb{N}} \to \mathbb{R}[\![x]\!] \\ (a_k)_{k\geq 0} &\mapsto \mathcal{G}\left((a_k)_{k\geq 0}\right) := \sum_{k=0}^{\infty} a_k x^k, \end{aligned}$$

the infinite *formal power series*, where the element a_k is the coefficient of x^k for all k. So far, this is only a fancy rewriting of the input sequence. But the power of generating functions lies in the fact, that by giving an appropriate definition of addition and multiplication, we can turn this structure into a ring, called the *ring of formal power series.*

So, let $\mathbb{R}[\![x]\!]$ be formally defined by

$$\mathbb{R}[\![x]\!] := \left\{(a_0, a_1, \ldots) \in \mathbb{R}^{\mathbb{N}}\right\},$$

and define the operation $+$ for two sequences $(a_k)_{k\geq 0}$ and $(b_k)_{k\geq 0}$ by

$$(a_0, a_1, a_2, \ldots) + (b_0, b_1, b_2, \ldots) := (a_0 + b_0, a_1 + b_1, a_2 + b_2, \ldots)$$

and the multiplication $\times$ by

$$(a_0, a_1, a_2, \ldots)\times(b_0, b_1, b_2, \ldots) := (c_0, c_1, c_2, \ldots) = (a_0b_0, a_0b_1+a_1b_0, a_2b_0+a_1b_1+a_0b_2, \ldots),$$

i.e. the coefficient c_k in the product is given by the *Cauchy product*

$$c_k := a_0b_k + a_1b_{k-1} + \ldots + a_kb_0 = \sum_{i=0}^{k} a_ib_{k-i}, \qquad k \geq 0.$$

With this operations, the tuple $(\mathbb{R}[\![x]\!], +, \times)$ forms a commutative ring with one. But more can be said.

Lemma 2. *The ring of formal power series form an integral domain.*

Based on that result, we could continue and construct the quotient field of $\mathbb{R}[\![x]\!]$, to obtain the field of (formal) Laurent series $\mathbb{R}((x))$. As a short outlook to Theorem 5, we want to remark right now, that the sequence $(a_k)_{k\geq 0}$ is a polynomial in k if and only if the generating function of $(a_k)_{k\geq 0}$ is a rational function. However, we won't need formal Laurent series in the upcoming chapters, and therefore skip it. Also, we will skip the notion of limit in $\mathbb{R}[\![x]\!]$, that is defined different compared to analytic functions. We refer the interested reader to [Wil06, KP11, Sta13]. Instead, we continue by looking at the possible analytic interpretation of generating functions.

In analysis, we have for all $|x| < 1$ in $\mathbb{R}[\![x]\!]$ the equality

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}. \tag{1.4}$$

More precisely, consider two maps:

$$\begin{aligned} l : (-1,1) &\to \mathbb{R} & \qquad r : \mathbb{R}\backslash\{1\} &\to \mathbb{R} \\ x &\mapsto \sum_{k=0}^{\infty} x^k = \lim_{N\to\infty} \sum_{k=0}^{N} x^k & x &\mapsto \frac{1}{1-x}. \end{aligned}$$

Then (1.4) means the equality $l(x) = r(x)$ holds for all $x \in (-1,1)$.

From that viewpoint, if we take the sequence $(a_k)_{k\geq 0}$ with $a_k = 1$ for all $k \geq 0$, we assign

$$(1,1,1,\ldots) \mapsto \sum_{k=0}^{\infty} x^k = \frac{1}{1-x} = (1-x)^{-1}, \tag{1.5}$$

the element $(1-x)^{-1}$ is identified as multiplicative inverse of the generating function $1 - x = (1,-1,0,\ldots)$. therefore, we could formally write

$$\begin{aligned} (1,1,1,\ldots) &= (1,-1,0,\ldots)^{-1} &\Rightarrow (1,1,1,\ldots) &\times (1,-1,0,\ldots) &= (1,0,0,\ldots) \\ \left(\frac{1}{1-x}\right) &= (1-x)^{-1} &\Rightarrow \left(\frac{1}{1-x}\right) &\times (1-x) &= 1. \end{aligned}$$

Hence, we can either manipulate the sequence a_k with the definitions of addition and multiplication, or the generating function $\mathcal{G}(a_k)$. At generating function level, we might are more flexible, by defining further operations, such as

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\sum_{k=0}^{\infty} a_k x^k\right) := \sum_{k=0}^{\infty}(k+1)a_{k+1}x^k,$$

which reminds to the theory of holomorphic functions, but without its analytic interpretation. Here, we concentrate on the formal treatment of certain sequences, hence, we could have a look at the product

$$\left(\frac{1}{1-x}\right)\times\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{1-x}\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\sum_{k=0}^{\infty} x^k\right) = \sum_{k=0}^{\infty}(k+1)x^k, \quad |x| < 1,$$

which could be confirmed by direct calculation, as we have

$$(a_k)_{k\geq 0}\times(b_k)_{k\geq 0}=(1,1,1,\ldots)\times(1,1,1,\ldots)=(1,2,3,\ldots),\qquad c_k=\sum_{i=0}^{k}a_ib_{k-i}=k+1.$$

Our first application is to prove a Lemma on Cauchy product at the ring of formal power series.

Lemma 3 (Partial sum Cauchy product)**.** *For any real sequence $(a_k)_{k\geq 0}$, in the ring of formal power series $\mathbb{R}[\![x]\!]$, we have:*

$$\frac{1}{1-x}\sum_{k=0}^{\infty}a_kx^k=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}a_k\right)x^n \tag{1.6}$$

Proof. In the preceding discussion, we've already encountered

$$\sum_{k=0}^{n}x^k=\frac{1}{1-x},\qquad |x|<1.$$

Therefore, the left hand side of (1.6) is interpreted as

$$\frac{1}{1-x}\left(\sum_{k=0}^{\infty}a_kx^k\right)=\left(\sum_{k=0}^{\infty}x^k\right)\left(\sum_{k=0}^{\infty}a_kx^k\right)=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}a_k\right)x^n$$

where the last step is the Cauchy product. □

The proof strategy will consist of applying (1.3) and Lemma 3 to $|\mathbb{M}^{=}(s)|$ and $|\mathbb{M}(s)|$, where $\mathbb{M}$ is specialized to the set of monomials in the concrete ring. This will allow us, to prove the upcoming Lemma 4 and Lemma 5, as well as Theorem 1 and Theorem 2, and Theorem 3 and Theorem 4 "pairwise", i.e. just proving the generating function identity for the sequence $|\mathbb{M}^{=}(s)|$, and passing to $|\mathbb{M}(s)|$ by application of the last Lemma. At the proof of Lemma 5 we will demonstrate this in more detail.

Lemma 4 (Number of differential monomials in $\Theta_m(s)$)**.**
Let (R,Δ) be a differential ring with m derivations. The number of differential monomials contained in $\Theta_m(s)$ is

$$|\Theta_m(s)|=\binom{m+s}{s}, \tag{1.7}$$

the generating function of $|\Theta_m(s)|$ is given by

$$\sum_{s=0}^{\infty}|\Theta_m(s)|x^s=\frac{1}{(1-x)^{m+1}}\qquad |x|<1. \tag{1.8}$$

Proof. Identity (1.7) is proven by a combinatorial argument, for the generating function identity we use the binomial coefficient identity

$$\binom{m+s}{s} = \binom{m+s-1}{s-1} + \binom{m+s-1}{s}, \qquad m, s \geq 0.$$

Denoting the generating function

$$\begin{aligned} F_m(x) &:= \sum_{s=0}^{\infty} |\Theta_m(s)| x^s = \sum_{s=0}^{\infty} \binom{m+s}{s} x^s = \sum_{s=0}^{\infty} \binom{m+s-1}{s-1} x^s + \sum_{s=0}^{\infty} \binom{m+s-1}{s} x^s \\ &= x \cdot \left(\sum_{s=1}^{\infty} |\Theta_m(s-1)| x^{s-1} \right) + \sum_{s=0}^{\infty} |\Theta_{m-1}(s)| x^s = x \cdot F_m(x) + F_{m-1}(x), \end{aligned}$$

it is possible to derive the functional equation

$$F_m(x) = \frac{1}{1-x} F_{m-1}(x), \qquad m > 0.$$

The initial value $F_0(x)$ reduces to a simple geometric series

$$F_0(x) = \sum_{s=0}^{\infty} |\Theta_0(s)| x^s = \frac{1}{1-x}, \qquad |x| < 1,$$

proving the generating function identity (1.8). □

Lemma 5 (Number of differential monomials in $\Theta_m^{=}(s)$)**.**
In the setting of Lemma 4, we have

$$|\Theta_m^{=}(s)| = \binom{m+s-1}{s}, \tag{1.9}$$

the generating function of $|\Theta_m^{=}(s)|$ is given by

$$\sum_{s=0}^{\infty} |\Theta_m^{=}(s)| x^s = \frac{1}{(1-x)^m} \qquad |x| < 1.$$

Proof. Formula (1.9) is shown by a combinatorial argument. By (1.3) and Lemma 3 we can view

$$\frac{1}{(1-x)^{m+1}} = \sum_{s=0}^{\infty} |\Theta_m(s)| x^s = \sum_{s=0}^{\infty} \left(\sum_{k=0}^{s} |\Theta_m^{=}(k)| \right) x^s = \frac{1}{1-x} \sum_{s=0}^{\infty} |\Theta_m^{=}(s)| x^s,$$

and therefore

$$\sum_{s=0}^{\infty} |\Theta_m^{=}(s)| x^s = \frac{1}{(1-x)^m}$$

□

Based on a Differential Ring, we can next consider the notion of Difference-Differential Ring.

Definition 3 (Difference-Differential Ring)**.**
Consider a differential ring R with set of derivations $\Delta := \{\delta_1, \dots, \delta_m\}$. If we adjoin a set of unitary R-automorphisms,

$$\Sigma := \{\sigma_1, \dots, \sigma_n\}, \qquad \sigma_j \in \mathrm{Aut}(R),$$

on R, which are pairwise commutative (i.e. $\alpha \circ \beta = \beta \circ \alpha$ for all $\alpha, \beta \in \Delta \cup \Sigma$), we obtain the *difference-differential ring* (R, Δ, Σ). The commutative *group* Γ_n of Σ consists of formal expressions of the form

$$\Gamma_n := \{\gamma := \sigma_1^{l_1} \dots \sigma_n^{l_n} : \ (l_1, \dots, l_n) \in \mathbb{Z}^n\}. \tag{1.10}$$

The commutation rule of an automorphism with elements in the ring R is now given by

$$\sigma^l \cdot x = \sigma^l(x)\sigma^l, \qquad l \in \mathbb{Z}^n, x \in R.$$

A difference-differential ring gives rise to the associated *ring of difference-differential operators* . In the difference-differential ring (R, Δ, Σ) we fix the field $\mathbb{K} \subseteq R$ as coefficient domain for difference-differential operators.

General Assumption 1. Throughout this thesis, we consider the difference-differential ring (R, Δ, Σ). We denote by D the ring of difference-differential operators over the field $\mathbb{K} \subseteq R$, with set of derivations Δ and set of unitary automorphisms Σ, given by

$$\Delta := \{\delta_1, \dots, \delta_m\}, \qquad \Sigma := \{\sigma_1, \dots, \sigma_n\},$$

all elements in $\Delta \cup \Sigma$ are pairwise commutative. Monomials in a difference-differential ring are of the form $\theta\gamma$, where $\theta \in \Theta_m$, $\gamma \in \Gamma_n$ (the sets Θ_m and Γ_n as defined by (1.1) and (1.10)), and therefore are formal expressions of the form

$$\Lambda_{m,n} := \{\delta^k \sigma^l = \delta_1^{k_1} \dots \delta_m^{k_m} \sigma_1^{l_1} \dots \sigma_n^{l_n}, \qquad k = (k_1, \dots, k_m) \in \mathbb{N}^m, l = (l_1, \dots, l_n) \in \mathbb{Z}^n\}, \tag{1.11}$$

the ring elements are $\mathbb{K}$-linear combination of monomials of this form[1].

We distinguish the identity elements $\theta_{\mathrm{id}} \in \Theta_m$ and $\sigma_{\mathrm{id}} \in \Gamma_n$, as well as the difference-differential operators $\lambda_{\mathrm{id}} \in \Lambda_{m,n}$, whose exponents are all zero, defined by

$$\theta_{\mathrm{id}} := \delta_1^0 \dots \delta_m^0, \qquad \sigma_{\mathrm{id}} := \sigma_1^0 \dots \sigma_n^0, \qquad \lambda_{\mathrm{id}} = \theta_{\mathrm{id}}\sigma_{\mathrm{id}} = \delta_1^0 \dots \delta_m^0 \cdot \sigma_1^0 \dots \sigma_n^0,$$

[1]Previously, at the ring of differential operators, we've considered K-linear combinations (where K is a commutative subring of R). In principle, we could require the coefficient domain to be a subring as well. However, in applications it will turn out being a useful assumption, to consider only $\mathbb{K}$-linear combinations, due to the existence of multiplicative inverse elements in the coefficient domain. Obviously, a commutative field $\mathbb{K}$ is a ring, hence we could also define the ring D as the free $\mathbb{K}$-module with set of generators $\Lambda_{m,n}$.

that act as multiplicative neutral element to ring elements, by satisfying the commutation rule

$$\forall x \in D : \alpha_{\mathrm{id}} \cdot x = x \cdot \alpha_{\mathrm{id}} = x, \qquad \alpha \in \{\theta, \sigma, \lambda\}.$$

Obviously, we obtain

$$\Theta_m(0) = \Theta_m^{=}(0) = \{\theta_{\mathrm{id}}\}, \qquad \Gamma_n(0) = \Gamma_n^{=}(0) = \{\gamma_{\mathrm{id}}\}, \qquad \Lambda_{m,n}(0) = \Lambda_{m,n}^{=}(0) = \{\lambda_{\mathrm{id}}\},$$

in particular, the assumption $|\mathbb{M}(0)| = |\mathbb{M}^{=}(0)| = 1$ (where $\mathbb{M}$ is a set of monomials), from (1.3) is fulfilled at this examples.

Lemma 6 (Non-Commutative Multiplication in a Δ-Σ-Ring)**.**
The product of a difference-differential operator with an element $a \in D$ is given by

$$\lambda \cdot a = \delta^{\mathbf{k}} \sigma^{\boldsymbol{\ell}} \cdot a = \delta^{\mathbf{k}} \cdot \overline{a} \cdot \sigma^{\boldsymbol{\ell}} = \sum_{S \subseteq [|\mathbf{k}|]} \left(\prod_{i \in S} \delta^{[i]}(\overline{a}) \right) \cdot \left(\prod_{j \in [|\mathbf{k}|] \setminus S} \delta^{[j]} \sigma^{\boldsymbol{\ell}} \right), \qquad \mathbf{k} \in \mathbb{N}^m, \boldsymbol{\ell} \in \mathbb{Z}^n,$$

where we abbreviate

$$\delta^{\mathbf{k}} = \delta_1^{k_1} \dots \delta_m^{k_m}, \quad \sigma^{\boldsymbol{\ell}} = \sigma_1^{l_1} \dots \sigma_n^{l_n}, \qquad [|\mathbf{k}|] = \{1, \dots, k_1 + \dots + k_m\}.$$

The element $\overline{a}$ is computed by $\overline{a} = \sigma^{\boldsymbol{\ell}}(a)$.

Remark on Notation. The differential monomial $\delta^{\mathbf{k}} = \delta_1^{k_1} \dots \delta_m^{k_m}$ in its expanded form is the product

$$\delta^{\mathbf{k}} = \underbrace{\delta_1 \dots \delta_1}_{k_1-\text{times}} \cdot \underbrace{\delta_2 \dots \delta_2}_{k_2-\text{times}} \cdot \ldots \cdot \underbrace{\delta_m \dots \delta_m}_{k_m-\text{times}},$$

of $|\mathbf{k}| = k_1 + \dots + k_m$ (non-distinct) derivations. So we associate the indices $1 \le i \le k_1$ to δ_1, the indices $k_1 + 1 \le i \le k_1 + k_2$ to δ_2 and so on. In particular, we have

$$\delta^{[i]} := \begin{cases} \delta_1, & 1 \le i \le k_1; \\ \delta_2, & k_1 + 1 \le i \le k_1 + k_2; \\ \vdots & \\ \delta_m, & k_1 + \dots + k_{m-1} + 1 \le i \le k_1 + \dots + k_m. \end{cases}$$

By writing

$$\prod_{i \in S} \delta^{[i]}(\bar{a}),$$

we mean that we *apply*[2] the derivation δ_j associated to $\delta^{[i]}$ with $i \in S$ to $\bar{a}$. Consider for instance three derivations $\delta_1, \delta_2, \delta_3$, and the monomial $\delta_1^3 \delta_2^2 \delta_3$. The set $[|\mathbf{k}|]$ is given by $\{1, 2, 3, 4, 5, 6\}$. If we choose for example the set $S := \{1, 3, 5, 6\}$, the above symbol can be re-written to

$$\prod_{i \in \{1,3,5,6\}} \delta^{[i]}(\bar{a}) = \delta_1(\delta_1(\delta_2(\delta_3(\bar{a})))).$$

[2]Recall that the derivations are pairwise commutative, making the product well defined.

The subset $[|\mathbf{k}|]\backslash S \subseteq [|\mathbf{k}|]$ holds "the complement of S", so in this case, we would get $\{1,2,3,4,5,6\}\backslash\{1,3,5,6\} = \{2,4\}$. This gives as one summand $\delta_1(\delta_1(\delta_2(\delta_3(\bar{a}))))\delta_1\delta_2$.

The number of subsets of an m-element set is given by 2^m, so the sum contains at most 2^m terms, and exactly 2^m terms if and only if $\delta_i(\bar{a}) \neq 0$ for all i.

To make it even more obvious, we state an example before we give the proof.

Example 1. *Let $R = \mathbb{Q}(x,y,z)$ be the field of rational functions in $\{x,y,z\}$, endowed with the sets Δ and Σ given by*

$$\Delta := \left\{\delta_k := \frac{\mathrm{d}}{\mathrm{d}k} : k \in \{x,y,z\}\right\}, \qquad \Sigma := \{\sigma_k := k \mapsto k+1 : k \in \{x,y,z\}\}.$$

By that choice, the tuple (R,Δ,Σ) forms a difference-differential ring. Consider a rational function $a := r(x,y,z)/s(x,y,z)$, suppose we want to compute the product

$$\delta_x\delta_y\delta_z\sigma_x\sigma_y^2\sigma_z^{-1} \cdot a = \delta_x\delta_y\delta_z \cdot \frac{r(x+1,y+2,z-1)}{s(x+1,y+2,z-1)} \cdot \sigma_x\sigma_y^2\sigma_z^{-1}.$$

We denote by $\bar{a} := r(x+1,y+2,z-1)/s(x+1,y+2,z-1)$, and look for the subsets of $\{x,y,z\}$[3]. Using the above formula, we find that $\delta_x\delta_y\delta_z \cdot \bar{a} \cdot \sigma_x\sigma_y^2\sigma_z$ is calculated by

$$\begin{aligned}
&[\delta_x(\delta_y(\delta_z(\bar{a})))+ && \text{(three-element subset } \{x,y,z\})\\
&\ \delta_x(\delta_y(\bar{a}))\delta_z + \delta_x(\delta_z(\bar{a}))\delta_y + \delta_y(\delta_z(\bar{a}))\delta_x+ && \text{(two-element subsets } \{x,y\},\{x,z\},\{y,z\})\\
&\ \delta_x(\bar{a})\delta_y\delta_z + \delta_y(\bar{a})\delta_x\delta_z + \delta_z(\bar{a})\delta_x\delta_y+ && \text{(one-element subsets } \{x\},\{y\},\{z\})\\
&\ \bar{a}\cdot\delta_x\delta_y\delta_z]\cdot\sigma_x\sigma_y^2\sigma_z^{-1}. && \text{(zero-element subset } \emptyset)
\end{aligned}$$

As observed above, the sum has exactly $8 = 2^3$ terms, coming from the fact that

$$|\mathcal{P}(\Delta)| = 2^{|\Delta|},$$

the symbol $\mathcal{P}(\cdot)$ denoting the power set of its input argument. Further, it should be noted, that the automorphisms $\sigma^{\boldsymbol{\ell}}$ appear in every summand with a non-zero coefficient.

Proof. The first equality

$$\delta^{\mathbf{k}}\sigma^{\boldsymbol{\ell}} \cdot a = \delta^{\mathbf{k}} \cdot \bar{a} \cdot \sigma^{\boldsymbol{\ell}}, \qquad \mathbf{k} \in \mathbb{N}^m, \boldsymbol{\ell} \in \mathbb{Z}^n$$

is obvious. For the second part, we use induction on $|\mathbf{k}|$. If $|\mathbf{k}| = 1$, then $\delta = \delta_i$ for some index i and

$$\delta_i \cdot \bar{a} \cdot \sigma^{\boldsymbol{\ell}} = \bar{a} \cdot \delta_i\sigma^{\boldsymbol{\ell}} + \delta_i(\bar{a})\sigma^{\boldsymbol{\ell}} = \delta^{\emptyset}(\bar{a})\delta_i\sigma^{\boldsymbol{\ell}} + \delta_i(\bar{a})\delta^{\emptyset}\sigma^{\boldsymbol{\ell}} = \sum_{S\subseteq\{i\}} \delta^S(\bar{a})\delta^{\{i\}\backslash S}\sigma^{\boldsymbol{\ell}}$$

[3] We have got three derivations, each appearing with an exponent of 1. In that particular situation, we have $\delta^{[1]} = \delta_x, \delta^{[2]} = \delta_y$ and $\delta^{[3]} = \delta_z$. For the sense of readability, we refer to this set by writing $\{x,y,z\}$ instead of $\{1,2,3\}$.

Suppose now, the statement holds for $|\mathbf{k}|$, and consider the case $|\mathbf{k}|+1$:

$$\begin{aligned}
\prod_{i=1}^{|\mathbf{k}|+1} \delta^{[i]} \cdot \overline{a} \cdot \sigma^{\boldsymbol{\ell}} = \delta^{[|\mathbf{k}|+1]} \cdot \prod_{i=1}^{|\mathbf{k}|} \delta^{[i]} \cdot \overline{a} \cdot \sigma^{\boldsymbol{\ell}} &= \delta^{[|\mathbf{k}|+1]} \cdot \sum_{S \subseteq [|\mathbf{k}|]} \left(\prod_{i \in S} \delta^{[i]}(\overline{a}) \right) \cdot \left(\prod_{j \in [|\mathbf{k}|] \setminus S} \delta^{[j]} \sigma^{\boldsymbol{\ell}} \right) \\
&= \sum_{S \subseteq [|\mathbf{k}|]} \delta^{[|\mathbf{k}|+1]} \cdot \left(\prod_{i \in S} \delta^{[i]}(\overline{a}) \right) \cdot \left(\prod_{j \in [|\mathbf{k}|] \setminus S} \delta^{[j]} \sigma^{\boldsymbol{\ell}} \right) \\
&= \sum_{S \subseteq [|\mathbf{k}|]} \delta^{S}(\overline{a}) \delta^{[|\mathbf{k}|+1]} \delta^{[|\mathbf{k}|] \setminus S} \sigma^{\boldsymbol{\ell}} + \delta^{[|\mathbf{k}|+1]} \left(\delta^{S}(\overline{a}) \right) \delta^{[|\mathbf{k}|] \setminus S} \sigma^{\boldsymbol{\ell}} \\
&= \sum_{S \subseteq [|\mathbf{k}|]} \left(\delta^{S}(\overline{a}) \delta^{[|\mathbf{k}|+1] \setminus S} \sigma^{\boldsymbol{\ell}} + \delta^{S \cup [|\mathbf{k}|+1]}(\overline{a}) \delta^{[|\mathbf{k}|+1] \setminus (S \cup \{|\mathbf{k}|+1\})} \sigma^{\boldsymbol{\ell}} \right) \\
&= \sum_{S \subseteq [|\mathbf{k}|+1]} \delta^{S}(\overline{a}) \delta^{[|\mathbf{k}|+1] \setminus S} \sigma^{\boldsymbol{\ell}}
\end{aligned}$$

□

Remark. We've constructed a difference-differential ring starting from a differential ring and adjoining a set of unitary automorphisms Σ. Specializing the set of derivations Δ to be the empty set, we obtain a so called *difference ring*.

In particular, we will use the notion $(R, \Sigma) := (R, \emptyset, \Sigma)$ for a *difference-ring*, i.e. a difference-differential ring where the set of derivations is the empty set. The *ring of difference-operators* $\mathcal{D}$ consists of *difference-monomials*, that are elements of the set Γ_n, we've encountered at (1.10), and *difference-operators* that are formal expressions of the form

$$\sum_{\gamma \in \Gamma_n} c_\gamma \, \gamma, \qquad c_\gamma \in K, \text{ the set } K \text{ a subring of } R, \; \Gamma_n \text{ as in } (1.10)$$

From that point of view, it is reasonable to consider a theory of difference-differential rings, as difference- and differential-rings embed as special cases into this framework.

$$\begin{tikzcd}
 & \text{Difference-Differential Ring } (R, \Delta, \Sigma) \arrow[dl, dash, "\Sigma = \emptyset"'] \arrow[dr, dash, "\Delta = \emptyset"] & \\
\text{Differential Ring } (R, \Delta) & & \text{Difference-Ring } (R, \Sigma)
\end{tikzcd}$$

But even more is possible. We will later on consider the ring of Ore polynomials, as well as the Weyl-algebra and the commutative polynomial ring $\mathbb{K}[x_1, ..., x_n]$. Therefore, we will extend this picture in section 1.4.

Lets stay for one more moment at the difference-ring. The *order of a difference-monomial* is defined by

$$\operatorname{ord}_{\Gamma_n}(\gamma) := \operatorname{ord}_{\Gamma_n}(\sigma_1^{l_1} \dots \sigma_n^{l_n}) = |l_1| + \dots + |l_n|, \qquad \gamma \in \Gamma_n.$$

Difference-monomials of order s and order less or equal to s are collected in

$$\Gamma_n(s) := \{\gamma \in \Gamma_n : \operatorname{ord}_{\Gamma_n}(\gamma) \leq s\}, \qquad \Gamma_n^{=}(s) := \{\gamma \in \Gamma_n : \operatorname{ord}_{\Gamma_n}(\gamma) = s\}.$$

Next, we will give the generating functions of $|\Gamma_n(s)|$ and $|\Gamma_n^{=}(s)|$.

To that end, we will use the following fact: In [KLAV98] it is shown, that the number of solutions to the equation $|x_1| + \dots + |x_n| = k$ in integers is

$$\sum_{j=0}^{n} 2^j \binom{n}{j} \binom{k-1}{j-1}, \qquad n \geq 1, k \geq 1. \tag{1.12}$$

We mention explicit the exceptional case, where $k = 0$, because (1.12) is only valid for $k \geq 1$. In this case there is only one possibility where we set $x_1 = \dots = x_n = 0$.

Theorem 1 (Generating Function of $|\Gamma_n(s)|$)**.**
Consider a difference ring (R, Σ) *with* n *automorphisms. Then, in the ring* $\mathbb{R}[\![x]\!]$ *we have the identity:*

$$\sum_{s=0}^{\infty} |\Gamma_n(s)| x^s = \frac{(1+x)^n}{(1-x)^{n+1}}, \qquad |x| < 1.$$

Theorem 2 (Generating Function of $|\Gamma_n^{=}(s)|$)**.**
In the setting as in Theorem 1, we have

$$\sum_{s=0}^{\infty} |\Gamma_n^{=}(s)| x^s = \left(\frac{1+x}{1-x}\right)^n, \qquad |x| < 1.$$

Proof. Doing Taylor expansion around $x_0 = 0$ on the right hand side, yields the coefficient of x^0 to be 1. There is a single difference-monomial of order 0, namely σ_{id}. For $s \geq 1$, we use the Mathematica package developed by Koutschan [Kou09, Kou10]. The package is loaded by typing

```
In[1]:= << HolonomicFunctions.m
        HolonomicFunctions package by Christoph Koutschan, RISC-Linz, Version 1.6 (12.04.2012)
        –>Type ?HolonomicFunctions for help
```

We plug in sum (1.12) and want to get rid of the sum quantifier. This is performed by creative telescoping.

```
In[1]:= summand[s_] := 2^j * Binomial[n, j] * Binomial[s - 1, j - 1]

In[2]:= First[CreativeTelescoping[summand[s], S[j] - 1, {S[s]}]]
```

Out[2]= $\{(-2-s)\,\mathrm{S_s}^2+2n\,\mathrm{S_s}+s\}$

In[3]:= **ApplyOreOperator[%, a[s]]**

Out[3]= $s\mathrm{a}[s]+2n\mathrm{a}[1+s]+(-2-s)\mathrm{a}[2+s]$

The initial values of the sequence $a_s=|\Gamma_n^{=}(s)|$ are given by $a_0=1, a_1=2n$.

For the right hand side, we use the convolution

$$\left(\frac{1+x}{1-x}\right)^n=\left(\sum_{s=0}^{\infty}\binom{n}{s}x^s\right)\cdot\left(\sum_{s=0}^{\infty}\binom{n+s-1}{s}x^s\right)=\sum_{s=0}^{\infty}\left(\sum_{k=0}^{s}\binom{n+k-1}{k}\binom{n}{s-k}\right)x^s.$$

Equipped with this sum representation, we can again apply creative telescoping

In[1]:= **summand[s_] := Binomial[n + k − 1, k] ∗ Binomial[n, s − k]**

In[2]:= **First[CreativeTelescoping[summand[s], S[k] − 1, {S[s]}]]**

Out[2]= $\{(-2-s)\mathrm{S_s^2}+2n\mathrm{S_s}+s\}$

In[3]:= **ApplyOreOperator[%, b[s]]**

Out[3]= $s\mathrm{b}[s]+2n\mathrm{b}[1+s]+(-2-s)\mathrm{b}[2+s]$

It remains to check the initial values, given by $b_0=1$ and $b_1=2n$. Hence, both sequences satisfy the same recurrence equation with identical initial values, and therefore agree. □

The notion $\mathrm{ord}_{\Theta_m}(\cdot)$ from the differential ring, as well as the notion $\mathrm{ord}_{\Gamma_n}(\cdot)$ from the difference ring, can now be lifted to difference-differential ring. To that end, we will now introduce the notion of *order of a difference-differential monomial*.

Definition 4. The *order of a difference-differential monomial* λ is defined by

$$\mathrm{ord}_{\Lambda_{m,n}}(\lambda):=\mathrm{ord}_{\Lambda_{m,n}}(\delta_1^{k_1}\ldots\delta_m^{k_m}\sigma_1^{l_1}\ldots\sigma_n^{l_n})=k_1+\ldots+k_m+|l_1|+\ldots+|l_n|,\qquad\lambda\in\Lambda_{m,n}.$$

Obviously, if $\lambda=\theta\gamma$ where $\theta\in\Theta_m$ and $\gamma\in\Gamma_n$, we have

$$\mathrm{ord}_{\Lambda_{m,n}}(\lambda)=\mathrm{ord}_{\Lambda_{m,n}}(\theta\gamma)=\mathrm{ord}_{\Theta_m}(\theta)+\mathrm{ord}_{\Gamma_n}(\gamma),\qquad\lambda\in\Lambda_{m,n}.$$

The set $\Lambda_{m,n}(s)$ is the subset of monomials in $\Lambda_{m,n}$ whose order is bounded by s, i.e.

$$\Lambda_{m,n}(s):=\{\lambda\in\Lambda_{m,n}:\ \lambda=\delta_1^{k_1}\ldots\delta_m^{k_m}\sigma_1^{l_1}\ldots\sigma_n^{l_n}\ \text{ and }\ \mathrm{ord}_{\Lambda_{m,n}}(\lambda)\le s\}.$$

The identity element λ_{id} is the unique monomial in a difference-differential ring of order zero. Similar to (1.2), we have for integers $s\le t$:

$$\Lambda_{m,n}(s)\subseteq\Lambda_{m,n}(t).$$

It is obvious that the number of solutions in the integers to the diophantine inequality given by

$$x_1+\ldots+x_m+|y_1|+\ldots+|y_n|\le s,\qquad(x_1,\ldots,x_m)\in\mathbb{N}^m,(y_1,\ldots,y_n)\in\mathbb{Z}^n$$

equals $|\Lambda_{m,n}(s)|$.

Further, we will be interested in the set of operators of order exactly s defined by

$$\Lambda^{=}_{m,n}(s) := \{\lambda \in \Lambda_{m,n} : \ \lambda = \delta_1^{k_1} \ldots \delta_m^{k_m} \sigma_1^{l_1} \ldots \sigma_n^{l_n} \ \text{ and } \ \mathrm{ord}_{\Lambda_{m,n}}(\lambda) = s\}.$$

As before, a different perspective on the set $\Lambda^{=}_{m,n}(s)$ is the diophantine equation

$$x_1 + \ldots + x_m + |y_1| + \ldots + |y_n| = s, \qquad (x_1, \ldots, x_m) \in \mathbb{N}^m, (y_1, \ldots, y_n) \in \mathbb{Z}^n$$

where the number of solutions is given by $|\Lambda^{=}_{m,n}(s)|$.

We will now give the generating functions of $|\Lambda_{m,n}(s)|$ and $|\Lambda^{=}_{m,n}(s)|$. It will turn out, that this generating functions are rational functions of the form $p(x)/(1-x)^d$ for some integer d, and a classic result in the theory of generating functions (Theorem 5) will then imply a recurrence representation for the coefficient sequences (Corollary 1 and Corollary 2). The validity of our Theorems will be shown at Example 2. Finally Corollary 3 holds the essence of our considerations. At the end of this section, we summarize the points that characterize the rings (R, Δ), (R, Σ) and (R, Δ, Σ).

Theorem 3 (Generating Function of $|\Lambda_{m,n}(s)|$)**.** *Let (R, Δ, Σ) be a difference-differential ring, with set of derivations $\Delta := \{\delta_1, \ldots, \delta_m\}$ and set of automorphisms $\Sigma := \{\sigma_1, \ldots, \sigma_n\}$. Then, in the ring of formal power series $\mathbb{R}[\![x]\!]$:*

$$\sum_{s=0}^{\infty} |\Lambda_{m,n}(s)| x^s = \frac{(1+x)^n}{(1-x)^{m+n+1}}, \qquad |x| < 1. \tag{1.13}$$

Theorem 4 (Generating Function of $|\Lambda^{=}_{m,n}(k)|$)**.** *In the setting as in Theorem 3,*

$$\sum_{s=0}^{\infty} |\Lambda^{=}_{m,n}(s)| x^s = \frac{(1+x)^n}{(1-x)^{m+n}}, \qquad |x| < 1. \tag{1.14}$$

Proof. Starting from (1.13), we obtain

$$\begin{aligned} \frac{(1+x)^n}{(1-x)^{m+n+1}} = \sum_{s=0}^{\infty} |\Lambda_{m,n}(s)| x^s &= \sum_{s=0}^{\infty} \left(\sum_{k=0}^{s} |\Lambda^{=}_{m,n}(k)| \right) x^s \\ &= \frac{1}{1-x} \sum_{s=0}^{\infty} |\Lambda^{=}_{m,n}(s)| x^s, \end{aligned}$$

where we used Lemma 3. Next we show (1.14). To that end, we calculate the cardinality of $|\Lambda^{=}_{m,n}(s)|$. The order is split up between the derivations and the automorphisms. It is shown in [KLAV98] that the number of solutions to the equation $x_1 + \ldots + x_m = k$ in non-negative integers is given by

$$\binom{m+k-1}{k}, \qquad m \geq 1, k \geq 0.$$

The no. of solutions to $|x_1| + \ldots + |x_n| = k$ has already been met at (1.12). Both identities can be verified by a combinatorial argument. We are looking for all splits of orders (for the derivations resp. automorphisms) that sum up to s. Therefore, we have to prove that

$$\sum_{s=0}^{\infty}\left(\sum_{i=0}^{s}\binom{m+i-1}{i}\left(\sum_{j=0}^{n}\binom{n}{j}\binom{s-i-1}{j-1}2^j\right)\right)x^s = \frac{(1+x)^n}{(1-x)^{m+n}}, \qquad |x|<1.$$

Bringing the sum quantifiers to the left, we get as a summand a product of binomial coefficients. We can get rid of the inner sums (on i and j) by first computing the annihilator of the summand, and afterwards applying creative telescoping. Hence, we are now considering

$$a_s = \sum_{i=0}^{s}\sum_{j=0}^{n}\binom{m+i-1}{i}\binom{n}{j}\binom{s-i-1}{j-1}2^j,$$

The annihilator can be computed by Koutschan's package:

In[4]:= **summand[s_] := Binomial[m+i−1, i]∗Binomial[n, j]∗Binomial[s−i−1, j−1]∗2^j**

In[5]:= **First[CreativeTelescoping[summand[s], S[j] − 1, {S[i], S[s]}]]**

Out[5]= $\{(1+2i+i^2-s-is)\mathrm{S_i} + (i-i^2+m-im+is+ms)\mathrm{S_s} + (-2in-2mn), (-2+i-s)\mathrm{S_s^2} + 2n\mathrm{S_s} + (-i+s)\}$

In[6]:= **First[CreativeTelescoping[%, S[i] − 1, {S[s]}]]**

Out[6]= $\{(-2-s)\mathrm{S_s^2} + (m+2n)\mathrm{S_s} + (m+s)\}$

In[7]:= **ApplyOreOperator[%, a[s]]**

Out[7]= $(m+s)\mathrm{a}[s] + (m+2n)\mathrm{a}[1+s] + (-2-s)\mathrm{a}[2+s]$

We have got a second order recurrence equation with polynomial coefficients. The initial values are given by: $a_0 = 1$ and $a_1 = m+2n$. On the other hand, we compute the convolution of

$$\sum_{s=0}^{\infty}\binom{n}{s}x^s = (1+x)^n, \qquad \sum_{s=0}^{\infty}\binom{m+n+s-1}{s}x^s = \frac{1}{(1-x)^{m+n}},$$

which would yield

$$\frac{(1+x)^n}{(1-x)^{m+n}} = \sum_{s=0}^{\infty}\left(\sum_{k=0}^{s}\binom{m+n+k-1}{k}\binom{n}{s-k}\right)x^s =: \sum_{s=0}^{\infty} b_s x^s. \tag{1.15}$$

We claim, that the generating function (1.15) satisfies the same recurrence equation. For the proof we use again Koutschan's package:

In[1]:= **summand[s_] := Binomial[$m + n + k - 1, k$] ∗ Binomial[$n, s - k$]**

In[2]:= **First[CreativeTelescoping[summand[s], S[k] − 1, {S[s]}]]**

Out[2]= $\{(-2-s)\mathrm{S}_{\mathrm{s}}^2 + (m+2n)\mathrm{S}_{\mathrm{s}} + (m+s)\}$

In[3]:= **ApplyOreOperator[%, b[s]]**

Out[3]= $(m+s)\mathrm{b}[s] + (m+2n)\mathrm{b}[1+s] + (-2-s)\mathrm{b}[2+s]$

To conclude the proof, we check initial values, which gives $b_0 = 1$ and $b_1 = m + 2n$. Therefore, we have for all $s \geq 0$ that $a_s = b_s$. This proves our claim. $\square$

So, the generating functions of both $|\Lambda_{m,n}(s)|$ and $|\Lambda^{=}_{m,n}(s)|$ are rational functions. We quote a famous result [Sta13, Cor. 4.3.1, p. 543], about the relation between a rational generating function and its coefficient sequence, and will apply it to our setting.

Theorem 5 (Rational Generating Functions)**.** *Consider the sequence* $(a_k)_{k\geq 0} \in \mathbb{R}^{\mathbb{N}}$*, and let* d *be a non-negative integer. The following is equivalent:*

- *For a polynomial* $p(x)$ *of degree less or equal to* d
$$\sum_{k=0}^{\infty} a_k x^k = \frac{p(x)}{(1-x)^{d+1}}$$
- *For all* $k \geq 0$*,*
$$\sum_{i=0}^{d+1} (-1)^{(d+1-i)} \binom{d+1}{i} a_{k+i} = 0.$$
- a_k *is a polynomial function of* k *of degree at most* d*. Moreover,* a_k *has degree exactly* d *if and only if* $p(1) \neq 0$*, in that case the leading coefficient of* a_k *is* $p(1)/d!$

Corollary 1 (Recurrence representation for $|\Lambda_{m,n}(k)|$)**.** *The sequence* $(|\Lambda_{m,n}(k)|)_{k\geq 0}$ *satisfies*

$$\sum_{i=0}^{m+n+1} (-1)^{(m+n+1-i)} \binom{m+n+1}{i} |\Lambda_{m,n}(k+i)| = 0, \qquad k \geq 0.$$

Hence, for fixed values m*,*n *the sequence is uniquely determined given the first* $m+n+1$ *values.* $|\Lambda_{m,n}(k)|$ *is a polynomial in* k *of degree exactly* $m+n$ *with leading coefficient* $2^n/(m+n)!$

Example 2 (Taylor Expansion of Rational Generating Function)**.**
Consider the sets

$$\Delta := \{\delta_1, \delta_2\}, \qquad \Sigma := \{\sigma_1\}$$

over a field $\mathbb{K}$*. We want to consider all operators of order* ≤ 2*. They are given by:*

$$\Lambda_{2,1}(2) = \{\lambda_{id}, \sigma_1, \delta_1, \delta_2, \sigma_1^{-1}, \sigma_1^{-2}, \sigma_1^2, \delta_1^2, \delta_2^2, \delta_1\sigma_1, \delta_1\sigma_1^{-1}, \delta_2\sigma_1, \delta_2\sigma_1^{-1}, \delta_1\delta_2\}.$$

Our formula gives:

$$\frac{(1+x)^1}{(1-x)^{2+1+1}} = \sum_{k=0}^{\infty} |\Lambda_{2,1}(k)| x^k = 1 + 5x + 14x^2 + 30x^3 + 55x^4 + 91x^5 + \ldots$$

hence, there are 14 *operators of degree less equal* 2*. The coefficient sequence* $(|\Lambda_{2,1}(k)|)_{k\geq 0}$ *satisfies:*

$$|\Lambda_{2,1}(k)| - 4|\Lambda_{2,1}(k+1)| + 6|\Lambda_{2,1}(k+2)| - 4|\Lambda_{2,1}(k+3)| + |\Lambda_{2,1}(k+4)| = 0,$$

with initial values $|\Lambda_{2,1}(0)| = 1$*,* $|\Lambda_{2,1}(1)| = 5$*,* $|\Lambda_{2,1}(2)| = 14$ *and* $|\Lambda_{2,1}(3)| = 30$*.*
$|\Lambda_{2,1}(k)|$ *can be expressed as polynomial like:*

$$|\Lambda_{2,1}(k)| = \frac{2^1}{(2+1)!}k^3 + a_2k^2 + a_1k + a_0 = \frac{k^3}{3} + a_2k^2 + a_1k + a_0,$$

and with initial values plugged in

$$|\Lambda_{2,1}(k)| = \frac{1}{6}(2k^3 + 9k^2 + 13k + 6). \tag{1.16}$$

Corollary 2 ([Recurrence representation for $|\Lambda^{=}_{m,n}(k)|$)**.** *The sequence* $\left(|\Lambda^{=}_{m,n}(k)|\right)_{k\geq 0}$ *satisfies*

$$\sum_{i=0}^{m+n} (-1)^{(m+n-i)} \binom{m+n}{i} |\Lambda^{=}_{m,n}(k+i)| = 0, \qquad k \geq 0.$$

Hence, for fixed values m*,*n *the sequence is uniquely determined given the first* $m+n$ *values.* $|\Lambda^{=}_{m,n}(k)|$ *is a polynomial in* k *of degree exactly* $m+n-1$ *with leading coefficient* $2^n/(m+n-1)!$*.*

Example 2 (continued)**.** *Continuing from before, we find the generating function*

$$\sum_{s=0}^{\infty} |\Lambda^{=}_{2,1}(s)|x^s = \frac{(1+x)^1}{(1-x)^{2+1}} = 1 + 4x + 9x^2 + 16x^3 + 25x^4 + 36x^5 + 49x^6 + \ldots,$$

the sequence $\left(|\Lambda^{=}_{2,1}(k)|\right)_{k\geq 0}$ *satisfies:*

$$-|\Lambda^{=}_{2,1}(k)| + 3|\Lambda^{=}_{2,1}(k+1)| - 3|\Lambda^{=}_{2,1}(k+2)| + |\Lambda^{=}_{2,1}(k+3)| = 0,$$

with initial values $|\Lambda^{=}_{2,1}(0)| = 1$*,* $|\Lambda^{=}_{2,1}(1)| = 4$ *and* $|\Lambda^{=}_{2,1}(2)| = 9$*.*
Viewn as polynomial, we find that

$$|\Lambda^{=}_{2,1}(k)| = \frac{2^1}{(2+1-1)!}k^2 + a_1k + a_0,$$

with initial values plugged in, we get

$$|\Lambda^{=}_{2,1}(k)| = k^2 + 2k + 1 = (k+1)^2.$$

Observe the relation (plugging in (1.16)*)*

$$\begin{aligned} |\Lambda_{2,1}(k)| - |\Lambda_{2,1}(k-1)| &= \frac{1}{6}\left(2k^3 + 9k^2 + 13k + 6\right) - \ldots \\ &\frac{1}{6}\left((2(k-1)^3 + 9(k-1)^2 + 13(k-1) + 6)\right) = k^2 + 2k + 1 = |\Lambda^{=}_{2,1}(k)|. \end{aligned}$$

The following Corollary is a summary about the above considerations.

Corollary 3 (Polynomial growth of $|\Lambda^{=}_{m,n}(t)|$ and $|\Lambda_{m,n}(t)|$)**.** *There exist polynomials $p(t)$, $q(t) \in \mathbb{Q}[t]$ such that*

$$\deg(p(t)) = m + n - 1, \qquad \deg(q(t)) = m + n, \qquad m + n \geq 1,$$

with leading coefficients

$$\mathrm{LC}(p(t)) = \frac{2^n}{(m+n-1)!}, \qquad \mathrm{LC}(q(t)) = \frac{2^n}{(m+n)!},$$

such that for $t \in \mathbb{N}$

$$|\Lambda^{=}_{m,n}(t)| = p(t), \qquad |\Lambda_{m,n}(t)| = q(t),$$

Moreover, p and q are related by

$$p(0) = q(0) = 1, \qquad p(t) = q(t) - q(t-1), \qquad t > 0.$$

In particular, the last line implies that for $t > 0$:

$$q(t) = \sum_{s=0}^{t} p(s).$$

Further, the polynomials $p(t)$ and $q(t)$ satisfy recurrence equations with constant coefficients such that

$$c_0 \cdot P(k) + \cdots + c_T \cdot P(k+T) = 0, \qquad k \geq 0,$$

where the coefficients are given by

$$c_i = \begin{cases} (-1)^{(m+n-i)} \binom{m+n}{i}, & P(t) = p(t), \quad 0 \leq i \leq T = m+n; \\ (-1)^{(m+n+1-i)} \binom{m+n+1}{i}, & P(t) = q(t), \quad 0 \leq i \leq T = m+n+1. \end{cases}$$

Summary

All the rings R, we've considered so far, contain a set of monomials $\mathbb{M}$. We've defined the order of a monomial in each ring, that has lead us to the consideration of $\mathbb{M}^{=}(s)$ and $\mathbb{M}(s)$. We've given the cardinality of $|\mathbb{M}^{=}(s)|$ and $|\mathbb{M}(s)|$ by considering the generating function.

Ring	$\mathbb{M}$	Eq.	Generating Function $\lvert\mathbb{M}^{=}(s)\rvert$	Generating Function $\lvert\mathbb{M}(s)\rvert$
(R, Δ)	Θ_m	(1.1)	$1/(1-x)^m$	$1/(1-x)^{m+1}$
(R, Σ)	Γ_n	(1.10)	$(1+x)^n/(1-x)^n$	$(1+x)^n/(1-x)^{n+1}$
(R, Δ, Σ)	$\Lambda_{m,n}$	(1.11)	$(1+x)^n/(1-x)^{m+n}$	$(1+x)^n/(1-x)^{m+n+1}$

Note the "factorization" of the generating function of $|\Lambda^{=}_{m,n}(k)|$ as follows:

Difference-Differential Ring (R, Δ, Σ)

$\sum\limits_{k=0}^{\infty} |\Lambda^{=}_{m,n}(k)| x^k = \frac{(1+x)^n}{(1-x)^{m+n}}$

$n = 0$ $\qquad$ $m = 0$

Differential Ring (R, Δ)

$\sum\limits_{k=0}^{\infty} |\Theta^{=}_{m}(k)| x^k = \frac{1}{(1-x)^{m}}$

Difference-Ring (R, Σ)

$\sum\limits_{k=0}^{\infty} |\Gamma^{=}_{n}(k)| x^k = \frac{(1+x)^n}{(1-x)^{n}}$

This comes as no surprise, due to the fact that

$$\begin{aligned}\sum_{k=0}^{\infty} |\Lambda^{=}_{m,n}(k)| x^k &= \sum_{k=0}^{\infty} \left(\sum_{i=0}^{k} |\Theta^{=}_{m}(i)| \cdot |\Gamma^{=}_{n}(k-i)| \right) x^k \\ &= \left(\sum_{k=0}^{\infty} |\Theta^{=}_{m}(k)| x^k \right) \cdot \left(\sum_{k=0}^{\infty} |\Gamma^{=}_{n}(k)| x^k \right),\end{aligned} \tag{1.17}$$

in particular, a difference-differential operator of order k consisting of a differential operator of order i (where $0 \le i \le k$) is multiplied by a difference-operator of order $k-i$ for all k.

Each of the quantities $|\mathbb{M}^{=}(s)|$ and $|\mathbb{M}(s)|$ can be measured by a polynomial $p \in \mathbb{Q}[s]$ characterized by the following data.

Ring	$\mathbb{M}$	$\deg(\lvert\mathbb{M}^{=}(s)\rvert)$	$\mathrm{LC}(\lvert\mathbb{M}^{=}(s)\rvert)$	$\deg(\lvert\mathbb{M}(s)\rvert)$	$\mathrm{LC}(\lvert\mathbb{M}(s)\rvert)$
(R, Δ)	Θ_m	$m-1$	$1/(m-1)!$	m	$1/m!$
(R, Σ)	Γ_n	-	-	n	$2^n/n!$
(R, Δ, Σ)	$\Lambda_{m,n}$	$m+n-1$	$2^n/(m+n-1)!$	$m+n$	$2^n/(m+n)!$

It is not possible to make statements about $\deg(|\Gamma^{=}_{n}(s)|)$ and $\mathrm{LC}(|\Gamma^{=}_{n}(s)|)$, based on Theorem 5, because the assumption of Theorem is not fulfilled.

However, Theorem 5 allows us to conclude that each of $|\mathbb{M}^{=}(s)|$ and $|\mathbb{M}(s)|$ satisfies a recurrence equation with constant coefficients of the form

$$c_0 \cdot P(s) + \ldots + c_T \cdot P(s+T) = 0, \qquad s \ge 0,$$

where T and c_i are given as in the following table.

Once again, note that $|\Gamma^{=}_{n}(s)|$ is not covered by Theorem 5.

Ring	$\mathbb{M}$	$P(s)$	T	c_i
(R,Δ)	Θ_m	$\lvert\Theta_m^{=}(s)\rvert$	$m-1$	$(-1)^{(m-1-i)}\binom{m-1}{i}$
(R,Δ)	Θ_m	$\lvert\Theta_m(s)\rvert$	m	$(-1)^{(m-i)}\binom{m}{i}$
(R,Σ)	Γ_n	$\lvert\Gamma_n(s)\rvert$	n	$(-1)^{(n-i)}\binom{n}{i}$
(R,Δ,Σ)	$\Lambda_{m,n}$	$\lvert\Lambda_{m,n}^{=}(s)\rvert$	$m+n$	$(-1)^{(m+n-i)}\binom{m+n}{i}$
(R,Δ,Σ)	$\Lambda_{m,n}$	$\lvert\Lambda_{m,n}(s)\rvert$	$m+n+1$	$(-1)^{(m+n+1-i)}\binom{m+n+1}{i}$

In [KLAV98, Section II.], it is shown at the set of monomials Θ_m and Γ_n, how this polynomial can be expressed using binomial coefficients. The reasoning is done by combinatorial arguments. The equation (1.17) gives the link to difference-differential monomials.

Ring	$\mathbb{M}$	$\lvert\mathbb{M}^{=}(s)\rvert$
(R,Δ)	Θ_m	$\binom{m+s-1}{m-1}, \quad s \geq 0$
(R,Σ)	Γ_n	$\sum_{i=0}^{n} 2^i \binom{n}{i}\binom{s-1}{i-1}, \quad s \geq 1$; $\quad 1, \quad s = 0.$
(R,Δ,Σ)	$\Lambda_{m,n}$	$\binom{m+s-1}{m-1} + \sum_{\ell=0}^{s-1}\binom{m+\ell-1}{m-1}\left(\sum_{i=0}^{n} 2^i\binom{n}{i}\binom{s-\ell-1}{i-1}\right), \quad s \geq 1$; $\quad 1 \quad s = 0$

The relation between $\lvert\mathbb{M}(s)\rvert$ and $\lvert\mathbb{M}^{=}(s)\rvert$ is given by (1.3).

Ring	$\mathbb{M}$	$\lvert\mathbb{M}(s)\rvert$
(R, Δ)	Θ_m	$\binom{m+s}{s}, \qquad s \geq 0$
(R, Σ)	Γ_n	$\sum_{i=0}^{n} 2^i \binom{n}{i}\binom{s}{i}, \qquad s \geq 1$ $1, \qquad s = 0$
(R, Δ, Σ)	$\Lambda_{m,n}$	$\sum_{k=0}^{s} \left(\binom{m+k-1}{m-1} + \sum_{\ell=0}^{k-1} \binom{m+\ell-1}{m-1} \left(\sum_{i=0}^{n} 2^i \binom{n}{i} \binom{k-\ell-1}{i-1} \right) \right), \qquad s \geq 1$ $1 \qquad s = 0$

Example 2 (continued)**.** *Let us one last time revisit Example 2. We've got a difference-differential ring, with two derivations and one automorphism, giving us the parameters* $m = 2, n = 1$*. Then,* $|\Lambda^{=}_{m,n}(s)|$ *simplifies for* $s = 2$ *to*

$$\begin{aligned} |\Lambda^{=}_{2,1}(2)| &= |\Theta^{=}_2(0)| \cdot |\Gamma^{=}_1(2)| + |\Theta^{=}_2(1)| \cdot |\Gamma^{=}_1(1)| + |\Theta^{=}_2(2)| \cdot |\Gamma^{=}_1(0)| \\ &= 1 \cdot 2 + 2 \cdot 2 + 3 \cdot 1 = 9 \text{ (\textit{confirming our result} } = (2+1)^2). \end{aligned}$$

Alternatively, we could use the formula from above table to obtain

$$\begin{aligned} |\Lambda^{=}_{m,n}(s)| &= \binom{m+s-1}{m-1} + \sum_{\ell=0}^{s-1} \binom{m+\ell-1}{m-1} \left(\sum_{i=0}^{n} 2^i \binom{n}{i} \binom{s-\ell-1}{i-1} \right) \\ |\Lambda^{=}_{2,1}(2)| &= \binom{3}{1} + \sum_{\ell=0}^{1} \binom{\ell+1}{1} \left(\sum_{i=0}^{1} 2^i \binom{1}{i} \binom{1-\ell}{i-1} \right) \\ &= 3 + \sum_{\ell=0}^{1} (\ell+1) \left(2^0 \binom{1}{0} \binom{1-\ell}{-1} + 2^1 \binom{1}{1} \binom{1-\ell}{0} \right) = 3 + 2 \sum_{\ell=0}^{1} (\ell+1) = 9. \end{aligned}$$

To obtain $|\Lambda_{2,1}(2)|$ *we plug in the formula*

$$\begin{aligned} |\Lambda_{m,n}(s)| &= \sum_{k=0}^{s} \left(\binom{m+k-1}{m-1} + \sum_{\ell=0}^{k-1} \binom{m+\ell-1}{m-1} \left(\sum_{i=0}^{n} 2^i \binom{n}{i} \binom{k-\ell-1}{i-1} \right) \right) \\ |\Lambda_{2,1}(2)| &= \sum_{k=0}^{2} \left(\binom{k+1}{1} + \sum_{\ell=0}^{k-1} \binom{\ell+1}{1} \left(\sum_{i=0}^{1} 2^i \binom{1}{i} \binom{k-\ell-1}{i-1} \right) \right) \\ &= \sum_{k=0}^{2} \left((k+1) + \sum_{\ell=0}^{k-1} (\ell+1) \left(2^0 \binom{1}{0} \binom{k-\ell-1}{-1} + 2^1 \binom{1}{1} \binom{k-\ell-1}{0} \right) \right) \\ &= \sum_{k=0}^{2} \left((k+1) + 2 \left(\sum_{\ell=1}^{k} \ell \right) \right) = \sum_{k=0}^{2} (k+1)^2 = 1 + 4 + 9 = 14. \end{aligned}$$

Remark. Let now $s > 0$. If we specialize $\Sigma = \emptyset \Leftrightarrow |\Sigma| = n = 0$ in the formulas for $|\Lambda^{=}_{m,n}(s)|$ and $|\Lambda_{m,n}(s)|$ we obtain

$$\binom{m+s-1}{m-1} + \sum_{\ell=0}^{s-1} \binom{m+\ell-1}{m-1} \left(\sum_{i=0}^{n} 2^i \binom{n}{i} \binom{s-\ell-1}{i-1} \right) \Bigg|_{n=0} = \binom{m+s-1}{m-1},$$

and similar

$$\sum_{k=0}^{s} \left(\binom{m+k-1}{m-1} + \sum_{\ell=0}^{k-1} \binom{m+\ell-1}{m-1} \left(\sum_{i=0}^{n} 2^i \binom{n}{i} \binom{k-\ell-1}{i-1} \right) \right) \Bigg|_{n=0} = \binom{m+s}{s},$$

giving as the formulas from the differential ring.

Analog, we can specialize $\Delta = \emptyset \Leftrightarrow |\Delta| = m = 0$, to obtain

$$\binom{m+s-1}{m-1} + \sum_{\ell=0}^{s-1} \binom{m+\ell-1}{m-1} \left(\sum_{i=0}^{n} 2^i \binom{n}{i} \binom{s-\ell-1}{i-1} \right) \Bigg|_{m=0} = \sum_{i=0}^{n} 2^i \binom{n}{i} \binom{s-1}{i-1},$$

and similar

$$\sum_{k=0}^{s} \left(\binom{m+k-1}{m-1} + \sum_{\ell=0}^{k-1} \binom{m+\ell-1}{m-1} \left(\sum_{i=0}^{n} 2^i \binom{n}{i} \binom{k-\ell-1}{i-1} \right) \right) \Bigg|_{m=0} = \sum_{i=0}^{n} 2^i \binom{n}{i} \binom{s}{i},$$

demonstrating the consistency of our considerations.

1.2. Gröbner Bases for Modules

We start this section with a brief historical overview about the progress how the idea of Gröbner bases developed over the past five decades and put particular emphasis on the most important steps on that journey. There exist an enormous amount of literature on Gröbner bases, the interested reader is referred to the Gröbner bases Bibliography [BZ12] as the ultimate reference to any work related to the theory of Gröbner Bases. The main presentation of this chapter is then the theory of relative Gröbner bases for difference-differential operators, that is later on abstracted by the Concept of Gröbner Reduction.

Buchberger [Buc65, Buc70, Buc85, BW98] has introduced the concept of Gröbner bases (also called standard bases) in his Ph.D. thesis for ideals in a commutative multivariate polynomial ring over a field $\mathbb{K}$. This was the starting point of manifold considerations of related ideas over various domains. The polynomial ring $\mathbb{K}[x_1, ..., x_n]$ forms a commutative noetherian ring[4].

[4] A *noetherian ring* is a ring in which every ideal is finitely generated. We will later on use the term *noetherian* in various contexts, e.g. in Hilbert's Basis Theorem (compare Theorem 7) as well as for finite reduction relations (compare Definition 17). The meaning, if not explicit clear from the context, will be emphasized at occurrence.

Generalizations in several directions were considered over the years, and it is almost impossible to give a comprehensive description containing all the important milestones in development. Therefore, we make here a cut, and come to the point of interest for this thesis, namely a generalization to a non-commutative ground domain, as it is the case in modules of differential operators.

Treating differential operators with a Gröbner basis method has been considered in [Gal85, OS94, SST00]. Applications of Gröbner bases in a differential ring are given at [Tak89].

Pauer [IP98, Pau07] formulated Gröbner Bases over not necessarily commutative rings, including rings of differential operators and polynomial rings over commutative noetherian rings.

Gröbner bases in a multivariate Ore-Algebra was studied by [Kou09]. He considered the subclass of so called holonomic modules and provided algorithmic applications, known as Zeilberger's holonomic system approach. A prominent example of the application of this theory is the recently proven q-TSPP conjecture [KKZ11] appearing as q-analogue of a theorem appearing at the enumeration of Totally Symmetric Plane Paritions in the combinatorial discipline of partition analysis.

In this thesis we are mainly interested in modules of difference-differential operators over a field. The question, how to treat difference-differential operators is of general interest, not only in pure mathematics but as well in mathematical physics [DL12b].

Several authors investigated how to set up a theory of Gröbner bases in this general setting, most notable A. Levin, F. Winkler and M. Zhou. Major application of Gröbner bases in this setting is the computation of univariate and multivariate dimension polynomials for finitely generated modules of difference- and differential operators.

In this section, we will recall the theory of relative Gröbner bases. The theory of relative Gröbner bases is introduced and developed in [ZW08a]. We will later on present a generalized notion of relative Gröbner bases and its applications, so we will recall the original formulation here. The contents presented in this section appeared in a series of papers [ZW06, ZW08b, Dön12].

The common backbone of every developed Gröbner basis theory is the existence of monomials in the considered ring R. As coefficient domain, we presume given a commutative ring $K \subseteq R$. We fix the set of monomials in R to be $\mathbb{M}$, and require that every element of R has an unique representation of the form

$$f \in R: \quad f = \sum_{\mathfrak{m} \in \mathbb{M}} a_{\mathfrak{m}} \cdot \mathfrak{m}, \qquad a_{\mathfrak{m}} \in K, \, \mathbb{M} \subseteq R, \text{ at most finitely many } a_{\mathfrak{m}} \text{ not zero.}$$

We can add elements in R in the usual way, and multiply from the left by elements in

$\mathbb{M}$ by taking into account the possible non-commutative product of $\mathbb{M}$ and elements in the coefficient domain K.

Notation. Throughout this thesis, we study the set of monomials that appear in a ring/module element. Therefore, if a formal expression

$$f = \sum_{\mathfrak{m}\in\mathbb{M}} a_{\mathfrak{m}} \cdot \mathfrak{m}, \qquad a_{\mathfrak{m}} \in K \text{ a subring of the ring } R, \tag{1.18}$$

where $\mathbb{M}$ is a set of monomials, appears in our considerations, we will denote the set

$$\mathrm{T}\left(\sum_{\mathfrak{m}\in\mathbb{M}} a_{\mathfrak{m}} \cdot \mathfrak{m}\right) := \{\mathfrak{m} \in \mathbb{M} : a_{\mathfrak{m}} \neq 0\} \subseteq \mathbb{M},$$

as the *term set of* f, that is commonly also known as the *support of* f . This set $\mathrm{T}(f)$ is always a finite set. If the element f is given by (1.18), we will denote the coefficient $a_{\mathfrak{m}}$ of $\mathfrak{m} \in \mathrm{T}(f)$ by $f_{\mathfrak{m}}$. This applies also to products, such as $(ag)_{\mathfrak{m}}$, by which we mean the coefficient of $\mathfrak{m}$ in ag. Sometimes, we will denote the set of monomials of a ring element by Λ, this will be emphasized at occurence.

Example 3. *Consider the ring $R := \mathbb{Q}[x_1, x_2, x_3]$. Let $f \in R$, and let $n_i := \deg_{x_i}(f)$. Then, f can be written as*

$$f = \sum_{i=0}^{n_1}\sum_{j=0}^{n_2}\sum_{k=0}^{n_3} a_{i,j,k} \cdot x_1^i x_2^j x_3^k, \qquad a_{i,j,k} \in K = \mathbb{Q},\ \mathbb{M} := \{x_1^i x_2^j x_3^k : (i,j,k) \in \mathbb{N}^3\}.$$

For example, if $f = 3x_1^2x_3 - 2x_2x_3^4 - x_1x_2x_3$ we have

$$\mathrm{T}(f) = \{x_1^2x_3, x_2x_3^4, x_1x_2x_3\}.$$

We then have that

$$f_{x_1^2x_3} = 3, \qquad f_{x_2x_3^4} = -2, \qquad (2x_1 \cdot f)_{x_1^2x_2x_3} = -2.$$

Example 4. *We consider differential operators with polynomial coefficients. To that end, let $K := \mathbb{Q}[x]$ and d_x be the differential operator w.r.t. x, that is, the element d_x satisfies*

$$d_x \cdot x = x \cdot d_x + 1.$$

If we now consider an element in this ring,

$$f = \sum_{i=0}^{k} a_i(x) d_x^i = \sum_{i=0}^{k}\left(\sum_{j=0}^{m_i} a_{j,i} \cdot x^j\right) d_x^i, \qquad a_i(x) \in K = \mathbb{Q}[x],\ a_k(x) \neq 0,$$

we obtain

$$\mathrm{T}(f) = \left\{d_x^i :\ a_i(x) \neq 0\right\}$$

One could also think of choosing $K = \mathbb{Q}$ and obtaining

$$\mathrm{T}(f) = \left\{x^j d_x^i :\ a_{j,i} \neq 0\right\}.$$

From that, we see that the choice of the ring K has crucial influence on the set $\mathrm{T}(f)$.

Based on the ring R, we can construct the free R-module F, that is generated by $E := \{e_1, ..., e_q\}$. From that, we have

$$F = Re_1 \oplus \cdots \oplus Re_q,$$

hence, every $f \in F$ can be represented as

$$f = \sum_{i=1}^{q} \sum_{\mathfrak{m} \in \mathbb{M}} (a_{\mathfrak{m},i} \cdot \mathfrak{m}) \cdot e_i, \qquad a_{\mathfrak{m},i} \in K,$$

and monomials in F are of the form $\mathbb{M}E := \{(\mathfrak{m}, e) : \mathfrak{m} \in \mathbb{M}, e \in E\}$. We can think of the set E to consist of $\{e_1, \ldots, e_q\}$ where e_i might be identified by the i-th unit vector.

In Lemma 6 we've encountered the non-commutative product of a difference-differential operator with an element from the ring D. Rephrasing Lemma 6 by using the set of terms we obtain

$$\mathrm{T}(\lambda \cdot a) = \mathrm{T}(\delta^k \sigma^l \cdot a) \subseteq \{\delta^{k'} \sigma^l : \ k' \leq_\pi k\}, \qquad a \in D, \ k \in \mathbb{N}^m, l \in \mathbb{Z}^n, \tag{1.19}$$

where $\leq_\pi$ denotes the (partial) product order, i.e.

$$k = (k_1, \ldots, k_m) \leq_\pi l = (l_1, \ldots, l_m) :\Leftrightarrow k_i \leq l_i, \quad 1 \leq i \leq m. \tag{1.20}$$

We assume that the monomials in our considered rings are ordered with respect to $\preccurlyeq \subseteq \mathbb{M} \times \mathbb{M}$. Once, an order has been defined, it makes sense to designate the *maximal term* or the *leading term* in a set of monomials. We will use the symbol $\mathrm{LT}_\prec$ for the leading term, respectively $\mathrm{LC}_\prec$ to denote its coefficient, or just LT and LC if $\prec$ is clear from the context.

For commutative polynomials this order is usually established by considering the exponent vector in $\mathbb{N}^n$ and then put order on the n-fold non-negative integers.

For difference-differential operators we face monomials containing integer exponents. Obviously, we can relate $\Lambda_{m,n} \cong \mathbb{N}^m \times \mathbb{Z}^n$ in the natural way. Zhou and Winkler [ZW06] suggest to use a generalization of term order concept on $\mathbb{N}^m \times \mathbb{Z}^n$. The handling of negative exponents is done by decomposing $\mathbb{Z}^n$ into so called orthants, that cover the whole plane and overlap only trivially.

Definition 5 (Orthant Decomposition, [ZW06, ZW08a]).
A decomposition of $\mathbb{Z}^n$ into k parts is called an *orthant decomposition* and $\mathbb{Z}^n_j$ is called the j-th orthant of this decomposition, if

$$\mathbb{Z}^n = \bigcup_{j=1}^{k} \mathbb{Z}^n_j,$$

and for all j we have that:

1. $(0,\ldots,0) \in \mathbb{Z}_j^n$, and $c := (c_1,\ldots,c_n)$ in $\mathbb{Z}_j^n$ implies that $-c := (-c_1,\ldots,-c_n)$ is not in $\mathbb{Z}_j^n$,
2. $\mathbb{Z}_j^n$ is a finitely generated subgroup of $\mathbb{Z}^n$, which is isomorphic to $\mathbb{N}^n$ as a semigroup
3. the group generated by $\mathbb{Z}_j^n$ is $\mathbb{Z}^n$

So, after all, what is an orthant decomposition? One possible view of orthant decompositions of the plane is to consider it as a finite family of monoid homomorphisms $\phi_u : \mathbb{N}^n \to \mathbb{Z}^n$ each of whose images generate the group $\mathbb{Z}^n$ and being such that

$$\bigcup_{u\in\mathbb{N}^n} \mathrm{im}(\phi_u) = \mathbb{Z}^n.$$

Consequently the set of monomials $\Lambda_{m,n}E$ of F is covered by finitely many isomorphic copies of $\mathbb{N}^m \times \mathbb{N}^n \times E$ in which term orders are well founded and reduction is supposed to behave well. Remark that only the automorphisms, i.e. the exponent vector that lies in $\mathbb{Z}^n$ determines the orthant of a monomial $\delta^r\sigma^s \cdot e_i$.

Given such an orthant decomposition of $\mathbb{Z}^n$ and $m \in \mathbb{N}$, the family $\{\mathbb{N}^m \times \mathbb{Z}_j^n : 1 \le j \le k\}$ is said to be an orthant decomposition of $\mathbb{N}^m \times \mathbb{Z}^n$. A standard example of an orthant decomposition of $\mathbb{Z}^n$ is a family $\{\mathbb{Z}_1^n,\ldots,\mathbb{Z}_{2^n}^n\}$ of all cartesian products of n sets each of which is either $\mathbb{N}$ or $-\mathbb{N}$.

If we now can reduce elements in every orthant to zero, a characterization similar to S-polynomials as in the commutative Gröbner basis theory can be proven. An essential ingredient is the concept of generalized term order, as considered in [ZW06, ZW08a].

Definition 6 (Generalized Term order)**.**
Given an orthant decomposition $\{\mathbb{Z}_j^n : 1 \le j \le k\}$ of $\mathbb{Z}^n$, let $E := \{e_1,\ldots,e_q\}$ be a set of generators of a free module F. A total order $\prec$ on $\mathbb{N}^m \times \mathbb{Z}^n \times E$ is called a *generalized term order* on $\mathbb{N}^m \times \mathbb{Z}^n \times E$ with respect to the decomposition, if and only if the following conditions hold

1. $(0,\ldots,0,e_i)$ is minimal in $\mathbb{N}^m \times \mathbb{Z}^n \times \{e_i\}$, $e_i \in E$
2. If $(a,e_i) \prec (b,e_j)$, then for any c such that c and b are in the same orthant,

$$(a+c,e_i) \prec (b+c,e_j), \qquad \text{where } a,b,c \in \mathbb{N}^m \times \mathbb{Z}^n,\ e_i,e_j \in E.$$

With the notation introduced at the beginning of this section, let F be a free D-module with a set of free generators $E = \{e_1,\ldots,e_q\}$. As we have seen, $\Lambda_{m,n}E = \{\lambda e_i : \lambda \in \Lambda_{m,n}, 1 \le i \le q\}$ is a set of monomials of F which is in natural one-to-one correspondence with the set $\mathbb{N}^m \times \mathbb{Z}^n \times E$ ($\delta_1^{k_1}\ldots\delta_m^{k_m}\sigma_1^{l_1}\ldots\sigma_n^{l_n}e \leftrightarrow (k_1,\ldots,k_m,l_1,\ldots,l_n,e)$). A total order $\prec$ of the set of monomials $\Lambda_{m,n}E$ is a generalized term order on $\Lambda_{m,n}E$ if the corresponding order of the set $\mathbb{N}^m \times \mathbb{Z}^n \times E$ is a generalized term order in the above sense.

If $\mu = \delta_1^{k_1} \dots \delta_m^{k_m} \sigma_1^{l_1} \dots \sigma_n^{l_n}$ and $\nu = \delta_1^{k'_1} \dots \delta_m^{k'_m} \sigma_1^{l'_1} \dots \sigma_n^{l'_n}$, we say that μ *divides* ν and write $\mu|\nu$ if and only if the $(m+n)$-tuples $(k_1, \dots, k_m, l_1, \dots, l_n)$ and $(k'_1, \dots, k'_m, l'_1, \dots, l'_n)$ lie in the same orthant of $\mathbb{N}^m \times \mathbb{Z}^n$ and $k_i \leq k'_i$ for $1 \leq i \leq m$ and $|l_j| \leq |l'_j|$ for $1 \leq j \leq n$.

If $t_1 = \mu e_i$ and $t_2 = \nu e_j$ are elements of $\Lambda_{m,n}E$, we say that t_1 divides t_2 and write $t_1|t_2$ if and only if $\mu|\nu$ and $i = j$.

In what follows, if the exponent vectors of two elements $\mu, \nu \in \Lambda_{m,n}$ lie in the same orthant of $\mathbb{N}^m \times \mathbb{Z}^n$, we write $\mu \sim \nu$. If $t_1 = \mu e_i, t_2 = \nu e_j$ we write $t_1 \sim t_2$ if $\mu \sim \nu$. Also, we write $\mu \sim t_2$ if $\mu \sim \nu$.

Since $\Lambda_{m,n}E$ is a free basis of F as a K-module, every element $f \in F$ has a unique representation of the form

$$f = a_1\lambda_1 e_{j_1} + \dots + a_d\lambda_d e_{j_d}, \qquad a_i \in K, 1 \leq i \leq d,$$

where $\lambda_1 e_{j_1}, \dots, \lambda_d e_{j_d}$ are distinct elements of $\Lambda_{m,n}E$. Given a generalized term order $\prec$ on $\Lambda_{m,n}E$, it is easy to see that if $\mathrm{LT}_\prec(f) = \mu e_i$ (for $\mu \in \Lambda_{m,n}$, $1 \leq i \leq q$), and $\mu \in \Lambda_{m,n}$, then $\mathrm{LT}_\prec(\nu f) = \nu \mathrm{LT}_\prec(f)$ if and only if $\nu \sim \mu$.

The above mentioned reduction is a generalization of division to (non-commutative) ring elements by taking into account multiple quotients. As already mentioned, the paper [ZW08a] is talking about the ring of difference-differential operators D by taking into account a bivariate filtration as introduced in Chapter 2.

Theorem 6 (Relative Reduction [ZW08a, p. 731, Thm. 3.1])**.**
Let "$\prec_1$" and "$\prec_2$" be two generalized term orders on $\Lambda_{m,n}E$. Let $g_1, \dots, g_p \in F\backslash\{0\}$ and $f \in F$. Then

$$f = h_1 g_1 + \dots + h_p g_p + r, \tag{1.21}$$

for some elements $h_1, \dots, h_p \in D$ and $r \in F$ such that

1. *$h_i = 0$ or $\mathrm{LT}_{\prec_1}(h_i g_i) \preccurlyeq_1 \mathrm{LT}_{\prec_1}(f)$, $i = 1, \dots, p$;*
2. *$r = 0$ or $\mathrm{LT}_{\prec_1}(r) \prec_1 \mathrm{LT}_{\prec_1}(f)$ such that*

$$\mathrm{LT}_{\prec_1}(r) \notin \{\mathrm{LT}_{\prec_1}(\lambda g_i) : \ \mathrm{LT}_{\prec_2}(\lambda g_i) \preccurlyeq_2 \mathrm{LT}_{\prec_2}(r), \ \lambda \in \Lambda_{m,n}, \ i = 1, \dots, p\}. \tag{1.22}$$

Definition 7. Let "$\prec_1$" and "$\prec_2$" be two generalized term orders on $\Lambda_{m,n}E$. We say that f *$\prec_1$-reduces relative to $\prec_2$ to r*, and call this procedure *relative reduction*, if f and r are represented as in (1.21) and the conditions of Theorem 6 apply.

Hence, we perform a reduction of f modulo $\{g_1, \dots, g_p\}$ but require additionally for the reducing element $g_k \in \{g_1, \dots, g_p\}$ that

$$\mathrm{LT}_{\prec_2}(\lambda g_k) \preccurlyeq_2 \mathrm{LT}_{\prec_2}(r), \qquad \lambda \in \Lambda_{m,n}. \tag{1.23}$$

Condition (1.34) appears strange at first sight, but is explained later on, where relative reduction is applied to filtered rings. It is a stronger assumption on the reduction relation compared to polynomial reduction, where the second condition is dropped and only the first assumption is kept.

The characteristic property of a Gröbner basis now lies in its behaviour with respect to reduction as introduced at Theorem 6. Of course, there are a lot of equivalent ways to define Gröbner bases, known in literature as Buchberger's S-polynomial criterion[5], in terms of divisibility of leading term ideals[6] or even by considering the syzygy-module [7]. A summary is given at [BWK93, Proposition 5.38].

At the commutative multivariate polynomial ring, Hilbert's basis theorem holds.

Theorem 7 (Hilbert's Basis Theorem)**.** *If the ring R is noetherian (or, what is equivalent, every ideal of R is finitely generated) then the polynomial ring $R[x]$ is also noetherian.*

Hilbert's Basis Theorem provides the induction base to prove that any ideal in $R[X]$ has a finite system of generators if R has. Gröbner bases now provide unique remainders, called normal-forms, with respect to generalized division.

Definition 8 (Relative Gröbner Basis)**.**
Let N be a D-module of difference-differential operators, $\prec_1$ and $\prec_2$ be two generalized term orders on $\Lambda_{m,n}E$. Then, the set $G := \{g_1, \dots, g_t\} \subseteq N \backslash \{0\}$ is a $\prec_1$- Gröbner basis relative to $\prec_2$, if and only if G generates the D-module N, and f in N implies that it can be $\prec_1$-reduced relative to $\prec_2$ to zero modulo G.

To decide whether a given set of polynomials is a Gröbner basis one can apply Buchberger's Theorem [BWK93, Theorem 5.64]. The generalization to the ring D was presented in [ZW08a, Theorem 3.2].

Theorem 8 (Buchberger Theorem at D)**.**
Let F be a free D-module, $\prec$ a generalized term order on $\Lambda_{m,n}E$, the notions LT *and* LC *understood with respect to $\prec$. Consider $G \subseteq F \backslash \{0\}$, and N the submodule of F spanned*

[5]Compare [AL94, Theorem 1.7.4]: Define the S-polynomial $S(g_i, g_j)$ by

$$S(g_i, g_j) := \frac{\mathrm{lcm}(\mathrm{LT}(g_i), \mathrm{LT}(g_j))}{\mathrm{LT}(g_i)} g_i - \frac{\mathrm{lcm}(\mathrm{LT}(g_i), \mathrm{LT}(g_j)}{\mathrm{LT}(g_j)} g_j.$$

Let $G := \{g_1, \dots, g_t\}$ be a set of non-zero polynomials in $\mathbb{K}[X]$. Then G is a Gröbner basis for the ideal $\langle g_1, ..., g_t \rangle$ if and only if for all $i \neq j$ the S-polynomial $S(g_i, g_j)$ can be reduced modulo G to zero.

[6]Compare [CLO97, Proposition 3]: The leading term ideal $\langle \mathrm{LT}(I) \rangle = \langle \mathrm{LT}(f) : f \in I \rangle$ can be described by a finite set of polynomials $g_1, \dots, g_t \in I$ as $\langle \mathrm{LT}(I) \rangle = \langle \mathrm{LT}(g_1), \dots, \mathrm{LT}(g_t) \rangle$.

[7]Compare [CLO98, Proposition 1.9]: Let $(f_1, ..., f_t) \in \mathbb{K}[X]^t$. The set of all $(a_1, \dots, a_t) \in \mathbb{K}[X]^t$ such that $a_1 f_1 + \dots + a_t f_t = 0$ is an $\mathbb{K}[X]$-submodule of $\mathbb{K}[X]^t$, called the first syzygy-module of $(f_1, \dots, f_t)$. With the help of Buchberger's algorithm we can do computations for the first syzygy module: The reduction to zero of the S-polynomial of a pair of polynomials in a Gröbner basis is a syzygy. These syzygies generate the first syzygy-module.

by G. The orthant decomposition on $\mathbb{Z}^n$ (i.e. the exponent vector of the automorphisms) gives motivation to consider

$$\Lambda^{(j)} := \{\delta^k \sigma^l : \ k \in \mathbb{N}^m, l \in \mathbb{Z}^n_j\} \tag{1.24}$$

The sets $\Lambda^{(j)}$ are now considered as the orthants of Λ. The set G is a Gröbner basis for N if and only if for all orthants $\Lambda^{(j)}$, for all $f, g \in G$ and for all

$$v \in \ _{R[\Lambda^{(j)}]}\langle \tau \in \Lambda^{(j)} E \mid \exists\, \lambda \in \Lambda : \ \tau = \mathrm{LT}(\lambda f)\rangle \quad \cap \quad _{R[\Lambda^{(j)}]}\langle \xi \in \Lambda^{(j)} E \mid \exists\, \eta \in \Lambda : \ \xi = \mathrm{LT}(\eta g)\rangle$$

the S-polynomial

$$S(j, f, g, v) := \frac{v}{\mathrm{LC}(f)} \frac{f}{\mathrm{LT}(f)} - \frac{v}{\mathrm{LC}(g)} \frac{g}{\mathrm{LT}(g)}$$

can be reduced modulo G to zero.

1.3. Associated Rings of Operators

Throughout in literature, systems of partial differential or difference equations are described by operators. In this thesis we restrict our attention to linear operators.

We are interested in certain rings R containing a commutative subring $K \subseteq R$ (not necessarily central), such that R is a free module over K. In particular there exists a K-basis $\mathbb{M} \subseteq R$ (which we call monomials). If this is the case, we write $R = K^{(\mathbb{M})}$, and define the symbol by

$$R = K^{(\mathbb{M})} :\Longleftrightarrow \text{ every } r \in R \text{ has a representation as } r = \sum_{\mathfrak{m} \in \mathbb{M}} r_{\mathfrak{m}} \cdot \mathfrak{m}, \tag{1.25}$$

where $r_{\mathfrak{m}} \in K,\ r_{\mathfrak{m}} = 0$ almost everywhere. Because R is free over K, this representation is unique.

We want to identify certain systems of difference-/ differential-equations with elements in an *operator-ring*. We will carry out the construction of the operator-ring on the example of the ring of differential operators, the other examples we've encountered are analogous.

Let R be a ring and $\Delta := \{\delta_1, \ldots, \delta_m\} \subseteq \mathrm{End}_{\mathbb{Z}}(R)$ be a set of pairwise commutative derivations on R. We use the embedding $R \to \mathrm{End}_{\mathbb{Z}}(R)$, $r \mapsto \bar{r}$, where for all $x \in R$ we have $\bar{r}(x) := r \cdot x$. This is, we denote by $\bar{r} \in \mathrm{End}_{\mathbb{Z}}(R)$ the endomorphism associated to the element $r \in R$.

Next, we extend $\bar{R}$ (i.e. the image of R) by $\{\delta_1, \ldots, \delta_m\}$, that is, we consider the smallest subring of $\mathrm{End}_{\mathbb{Z}}(R)$ containing $\bar{R} \cup \{\delta_1, \ldots, \delta_m\}$. This is the *ring of differential operators* induced by Δ on R. Elements in this ring are of the form

$$\sum_{j=1}^{n} \beta_{i_1} \circ \ldots \circ \beta_{i_{d(j)}}, \qquad \beta_{i_k} \in \bar{R} \cup \{\delta_1, \ldots, \delta_m\}. \tag{1.26}$$

We now have

$$(\delta_i \circ \bar{a})(x) = \delta_i(a \cdot x) = a \cdot \delta_i(x) + \delta_i(a) \cdot x = (\bar{a} \circ \delta_i + \overline{\delta_i(a)})(x),$$

i.e. we have

$$\delta_i \circ \bar{a} = \bar{a} \circ \delta_i + \overline{\delta_i(a)}, \qquad \delta_i \in \Delta,$$

that corresponds to the Leibniz rule. This justifies to view operator (1.26) as

$$\sum_{\alpha \in \mathbb{N}^m} \overline{a_\alpha} \circ \delta_1^{\alpha_1} \circ \ldots \circ \delta_m^{\alpha_m}, \qquad a_\alpha \in R, \alpha = (\alpha_1, \ldots, \alpha_m), \tag{1.27}$$

only finitely many a_α not zero. By Lemma 6 a similar reasoning can be applied to rings of difference and difference-differential operators.

It turns out, representation (1.27) is not unique, due to possible relations among the endomorphisms. To overcome this, we define the *free ring of operators* $\mathsf{Op}(R)$ as follows.

Consider a differential ring (R, Δ) where $\Delta = \{\delta_1, \ldots, \delta_m\} \subseteq \mathrm{End}_{\mathbb{Z}}(R)$ is a set of pairwise commutative derivations. A differential monomial is identified by its exponent vector

$$\Theta_m \to \mathbb{N}^m, \qquad \delta_1^{k_1} \ldots \delta_m^{k_m} \mapsto (k_1, \ldots, k_m), \quad \text{hence } \Theta_m \cong \mathbb{N}^m.$$

From that point of view, the additive group of $\mathsf{Op}(R)$ is given by the free R-module $R^{(\mathbb{N}^m)}$, i.e. we have 'point-wise addition' of monomials with same exponent vector[8]. Hence, elements of $\mathsf{Op}(R)$ are mappings $f : \mathbb{N}^m \to R$ that are zero almost everywhere. With the mappings

$$\mathrm{E}_k : \mathbb{N}^m \to R, \qquad \mathrm{E}_k(l) = \begin{cases} 1 \ldots & k = l, \\ 0 \ldots & \text{else.} \end{cases}, \qquad k \in \mathbb{N}^m$$

we can write

$$f = \sum_{k \in \mathbb{N}^m} f(k)\mathrm{E}_k, \qquad f(k) \in R,\ f(k) = 0 \text{ almost everywhere.}$$

We now have to define multiplication in a way that we can consider the mapping

$$\delta^k = \delta_1^{k_1} \circ \ldots \circ \delta_m^{k_m}, \qquad k = (k_1, \ldots, k_m) \in \mathbb{N}^m,$$

that we've met at (1.27). To that end, we consider

$$\partial_i := \mathrm{E}_{e_i}, \quad (e_i = (0, \ldots, 0, 1, 0, \ldots, 0) - 1 \text{ at position } i).$$

[8]Note that actually not $\mathbb{N}^m$ is contained in R, but $\Theta_m \subseteq R$, i.e. the monomials in a differential ring are elements of the set Θ_m. The notion is justified by the above mentioned isomorphism between the sets Θ_m and $\mathbb{N}^m$.

As the derivations are pairwise commutative, the products (in the sense of composition), build 'power products', i.e. we take the commutative monoid $\mathbb{N}^m$ as basis. So we require $\mathbb{N}^m \to R^{(\mathbb{N}^m)}$ to be a homomorphism of monoids from $(\mathbb{N}^m, +)$ to $(\mathsf{Op}(R), \cdot)$:

$$\mathrm{E}_{k_1,\ldots,k_m} = \mathrm{E}_{k_1 e_1 + \ldots + k_m e_m} = \mathrm{E}_{e_1}^{k_1} \ldots \mathrm{E}_{e_m}^{k_m} = \partial_1^{k_1} \ldots \partial_m^{k_m}.$$

Elements in $\mathsf{Op}(R)$ can now be written as

$$f = \sum_{k \in \mathbb{N}^m} f(k) \partial_1^{k_1} \ldots \partial_m^{k_m} = \sum_{k \in \mathbb{N}^m} f(k) \partial^k, \qquad f(k) \in R,\ f(k) = 0 \text{ almost everywhere.}$$

For a final definition of the multiplication, we set for $r \in R$

$$\partial_i \cdot r = r \cdot \partial_i + \delta_i(r).$$

By requiring the multiplication to be distributive, we get the following Theorem.

Theorem 9. $(\mathsf{Op}(R), +, \cdot)$ *forms a ring, in which every element has an unique representation.*

We call this ring *the free ring of operators* $\mathsf{Op}(R)$.

Let S be a ring. We consider an S-module F (F is called a *function space*)[9]. The set of operators that applied to $f \in F$ give 0 is called the *annihilator of* f

$$\operatorname{ann}_S(f) := \{r \in S : r \cdot f = 0\},$$

which forms a left S-module. The annihilator of f gives an implicit description of the element f.

Example 5. *Consider the differential ring* $R = \mathbb{R}[x]$ *with* $\Delta := \{d_x := \mathrm{d}/\mathrm{d}x\}$. *Elements in* $S := \mathsf{Op}(R)$ *are of the form*

$$\sum_{i=0}^{k} a_i(x) d_x^i, \qquad a_i(x) \in \mathbb{R}[x],\ a_k(x) \neq 0.$$

A suitable function space F would be the set of smooth[10] functions over the reals $C^\infty(\mathbb{R})$. The domain F is a left S-module. The ring S acts on F by

$$(a_0(x) + a_1(x) d_x + \ldots + a_k(x) d_x^k) \cdot f(x) := a_0(x) f(x) + a_1(x) f'(x) + \ldots + a_k(x) f^{(k)}(x),$$

where $a_i(x) \in \mathbb{R}[x]$, $f \in C^\infty(\mathbb{R})$. *Because*

$$\frac{\mathrm{d}}{\mathrm{d}x} e^{2x^2-x} = (4x-1) e^{2x^2-x} \Rightarrow \left(\frac{\mathrm{d}}{\mathrm{d}x} - (4x-1) \right) \cdot e^{2x^2-x} = 0,$$

it follows that

$$d_x - (4x-1) \in \operatorname{ann}_S\left(e^{2x^2-x}\right).$$

[9]Later on, we will specialize the ring S to be $S = \mathsf{Op}(R)$. However, the developed theory is valid for general rings S, and is therefore formulated as general as possible.

[10]A function is called *smooth* if and only if it is infinitely often differentiable. The set of real functions $\mathbb{R} \to \mathbb{R}$ that are smooth is denoted by $C^\infty(\mathbb{R})$.

At [Zei90, Kou09] it is shown, how one can do algorithmic computation of generators of the annihilator module, and decide questions such as module membership, to prove that a given identity is a consequence of defining relations, such as Gröbner bases witness that a given polynomial is contained in a polynomial ideal.

Let us now consider the (non-commutative) ring $S := \mathsf{Op}(R)$, and let M be a left S-module. The ring element $a \in S$ appears as additive (i.e. $\mathbb{Z}$-linear) operator $\bar{a} : M \to M$, $m \mapsto \bar{a}(m) := a \cdot m$. So for $X \subseteq S$ we may ask for the solution set

$$\mathbb{V}_M(X) := \{m \in M : \forall a \in X : a \cdot m = 0\},$$

sometimes called *the solution variety*, that is obviously given by

$$\mathbb{V}_M(X) = \bigcap_{a \in X} \ker(\bar{a}) \subseteq M.$$

If S is commutative, this abelian group is an S-module. Deciding membership in $\mathbb{V}_M(X)$ where X is finite, is obvious. But in general, it is not clear to decide whether for given $U \subseteq M$ we have that $\mathbb{V}_M(X) = U$, in particular (for $U = \emptyset$) if there exists any solution at all.

From that, it seems reasonable to define *the module of* X as S/SX. SX is the left ideal generated by X, i.e. a submodule of S considered as a left S-module; thus S/SX is a left S-module too.

The projection $\pi : S \to S/SX$, defined by $\pi(1) =: u$, is the way to describe $\mathbb{V}_M(X)$ adequate. If $a \in X$ we have that $\pi(a) = 0$ and so

$$a \cdot u = a \cdot \pi(1) = \pi(a \cdot 1) = \pi(a) = 0,$$

meaning that u is a solution of X in S/SX. Further, it is an *universal solution*:

Lemma 7. *If M is an arbitrary left S-module and m is a solution of X in M, then there exists exactly one homomorphism $\phi : S/SX \to M$ such that $\phi(u) = m$.*

Proof. Consider $\psi : S \to M$, defined by $\psi(s) := s \cdot m$. This is a S-homomorphism and, since $m \in \mathbb{V}_M(X)$ we get $X \subseteq \ker(\psi)$. Therefore, $SX \subseteq \ker(\psi)$ and so ψ factors as

$$\begin{tikzcd} S \arrow[r, "\pi"] \arrow[dr, "\psi"'] & S/SX \arrow[d, "\phi"] \\ & M \end{tikzcd}$$

Now, this ϕ is what we want:

$$\phi(u) = \phi(\pi(1)) = \psi(1) = 1 \cdot m = m.$$

Assume there is another map $\phi' : S/SX \to M$ with $\phi'(u) = m$. Then, for arbitrary $t \in S$ we can conclude, that

$$\phi'(\pi(t)) = (\phi' \circ \pi)(t \cdot 1) = t \cdot (\phi' \circ \pi)(1) = t \cdot \phi'(u) = t \cdot m = t \cdot \phi(u) = t \cdot \phi(\pi(1)) = \phi(\pi(t)).$$

Because t was chosen arbitrary in S, the image $\pi(t)$ ranges over the entire S/SX we have $\phi = \phi'$. □

But we can even say more.

Lemma 8. *If M is a S-module and $m \in M$, then*

$$m \in \mathbb{V}_M(X) \Leftrightarrow \exists f \in {}_S\operatorname{Hom}(S/SX, M) : \ f(u) = m.$$

Proof. '⇒' is already proved. For the converse take $x \in X$. Then

$$0 = (f \circ \pi)(x) = (f \circ \pi)(x \cdot 1) = x \cdot (f \circ \pi)(1) = x \cdot f(u) = x \cdot m.$$

□

In principle we could now replace R by a differential-, difference- or a difference-differential ring as considered in section 1.1. However, we want to introduce yet another ring, which is a prominent example of representing operators in a polynomial algebra, namely the ring of *Ore-polynomials.*

Difference-Differential Rings and the Ore-Algebra

We've considered the ring D of difference-differential operators over a field $\mathbb{K}$. In this section we want to describe another approach of treating difference- and differential operators. At the introduction, we've already encountered the commutation rules

$$\delta x = x\delta + \delta(x), \qquad \sigma x = \sigma(x)\sigma. \tag{1.28}$$

In this section we want to describe the theory of Ore-polynomials, that generalize both commutation rules at the same time. These non-commutative polynomials were first studied by Øystein Ore [Ore33], who considered the univariate polynomial ring $\mathbb{K}[\partial; \sigma, \delta]$, where elements are of the form

$$f = \sum_{k=0}^{t} a_k \cdot \partial^k = a_t\partial^t + a_{t-1}\partial^{t-1} + \ldots + a_0, \qquad a_k \in \mathbb{K},\ a_t \neq 0,\ \deg(f) = t.$$

with the usual addition (as in the algebra of commutative polynomials), and a product that has to satisfy

$$\deg(a \cdot b) \leq \deg(a) + \deg(b), \qquad a, b \in \mathbb{K}[\partial; \sigma, \delta]. \tag{1.29}$$

In [Mid11, Chapter 3], the ring of Ore-Polynomials is constructed from a ring R' that is a coefficient domain, and a ring extension ∂ whose powers generate the ring of Ore-polynomials. In principle, very general constructions are available, that we won't need in this thesis. We restrict our attention to polynomials with coefficients in a field $\mathbb{K}$.

If one considers the product of $\partial = 1 \cdot \partial^1 + 0$ with $x \in \mathbb{K}$ we get by (1.29):

$$\deg(\partial \cdot x) \leq \deg(\partial) + \deg(x) = 1 + 0.$$

Therefore, the product is a polynomial of degree less equal 1, hence it can be written as

$$\partial \cdot x = a_1 \cdot \partial + a_0, \qquad a_1, a_0 \in \mathbb{K},$$

and a_1 and a_0 depend on x. From that viewpoint it is reasonable to view $a_1 = \sigma(x)$ and $a_0 = \delta(x)$[11]. Let us now collect properties of σ and δ. To ensure that $\partial \cdot 1 = 1 \cdot \partial = \partial$, we have to impose the condition

$$\partial \cdot 1 = \sigma(1) \cdot \partial + \delta(1) \stackrel{!}{=} 1 \cdot \partial + 0 \Rightarrow \sigma(1) = 1, \delta(1) = 0,$$

hence the map σ is an unitary homomorphism. A very similar argument shows, that δ and σ are *additive* homomorphisms by requiring distributivity

$$\partial \cdot (x + y) \stackrel{!}{=} \partial \cdot x + \partial \cdot y, \qquad x, y \in \mathbb{K}.$$

Likewise, we can derive relations for the product by considering

$$\partial \cdot (x \cdot y) \stackrel{!}{=} (\partial \cdot x) \cdot y, \qquad x, y \in \mathbb{K}.$$

This arguments, and very similar properties describe σ and δ. We summarize in the following Lemma. We use the symbol $\mathbb{O}^{(1)} := \mathbb{K}[\partial; \sigma, \delta]$, or just $\mathbb{O}$, when it is clear that we consider one element ∂, for this construction.

Lemma 9 (Properties of skew-derivations)**.**
For the univariate Ore-Algebra $\mathbb{O}$ over the field $\mathbb{K}$, a,b in $\mathbb{K}$:

- $\delta(0) = \delta(1) = 0$;
- $\delta(ab) = \sigma(a)\delta(b) + \delta(a)b$;
- $\delta(a/b) = -(\sigma(a)\delta(b))/(\sigma(b)b) + \delta(a)/b$;
- $\sigma(a^{-1}) = \sigma(a)^{-1}$;
- $\delta(a^{-1}) = -\delta(a)/(\sigma(a)a)$.

[11] Note that δ is *not* a derivation in the classic sense. The following discussion shows that we derive at so called *σ-skew derivations*. By the choice $\sigma = \mathrm{id}$ we derive at the classic case, therefore the symbol δ is not misleading.

Collecting all points that are required from the initially presented properties, we can uniquely define an Ore-operator ∂ by fixing an unitary endomorphism σ and a σ-skew-derivation δ, meaning that

$$\delta(x \cdot y) = \sigma(x) \cdot \delta(y) + \delta(x) \cdot y, \qquad x, y \in \mathbb{K}.$$

Recently, [KJJ14] have produced an implementation in the Sage [S$^+$13] computer algebra system of algorithms for the univariate case. For its multivariate generalization[12]

$$\mathbb{O}^{(n)} := \mathbb{K}[\partial_1; \sigma_1, \delta_1][\partial_2; \sigma_2, \delta_2] \dots [\partial_n; \sigma_n, \delta_n],$$

one has to be particularly careful. The commutation rule of the Ore-operator ∂_1 with elements from $\mathbb{K}$ is governed by the two maps δ_1 and σ_1. But already for the operator ∂_2 we have as a base ring not $\mathbb{K}$ but $\mathbb{K}[\partial_1; \sigma_1, \delta_1]$, which makes it necessary to consider the product of ∂_1 with ∂_2. To overcome this, we assume that the product of two different Ore-operators is commutative, i.e.

$$\partial_i \cdot \partial_j := \partial_j \cdot \partial_i, \qquad 1 \le i, j \le s.$$

Using this convention, we've got all the possible constellations of products in $\mathbb{O}$, which can be used to construct a difference-differential ring (R, Δ, Σ). First of all, instead of considering a set of m derivations Δ, we consider a set of *skew*-derivations $\{\partial_1, \dots, \partial_m\}$, that are governed by fixing unitary endomorphisms σ and fixing σ-skew derivations δ. Powers of ∂ are defined recursively for $k > 0$ by

$$\partial^k x = \partial^{k-1}(\partial x) = \partial^{k-1}(\sigma(x)\partial + \delta(x)), \qquad x \in \mathbb{K}.$$

Lemma 10 (Product of Ore-operators with ring elements)**.** *For the univariate Ore-Algebra $\mathbb{O}$ over the field $\mathbb{K}$, and x in $\mathbb{K}$:*

$$\partial^k x = S_{k,0}(x)\partial^k + S_{k,1}(x)\partial^{k-1} + \dots + S_{k,k}(x)\partial^0, \qquad k > 0, \tag{1.30}$$

where

$$S_{k,0}(x) = \sigma^k(x), \qquad S_{k,k}(x) = \delta^k(x),$$

and

$$S_{k,i}(x) = \delta^i(\sigma^{k-i}(x)) + \dots + \delta^{k-i}(\sigma^i(x)), \qquad 0 < i < k.$$

A proof is given in [Ore33]. The choice $\sigma := \sigma_{\text{id}}$, defined by

$$\sigma_{\text{id}} : \mathbb{K} \to \mathbb{K}, \qquad x \mapsto \sigma_{\text{id}}(x) := x,$$

allows to model the commutation property

$$\sigma_{\text{id}} \cdot x = \sigma_{\text{id}}(x) \cdot \sigma_{\text{id}} = x \cdot \sigma_{\text{id}}, \qquad x \in \mathbb{K}.$$

[12]Again we use the convention, that we just write $\mathbb{O}$ for $\mathbb{O}^{(n)}$ if n is clear from the context

If we choose the derivation δ as the trivial derivation δ_0, mapping any element to 0

$$\delta_0 : \mathbb{K} \to \mathbb{K}, \qquad x \mapsto \delta_0(x) := 0,$$

we can then consider the ring $\mathbb{K}[\partial; \delta_0, \sigma_{\text{id}}]$. With this choice we get the very important special case

$$\partial \cdot x = \sigma_{\text{id}}(x) \cdot \partial + \delta_0(x) = x \cdot \partial, \qquad x \in \mathbb{K},$$

which is the usual commutative polynomial ring.

The next question we pose, is how to handle negative exponents of the automorphisms. First we have to distinguish the symbols $\sigma^{-1}(x)$, rather than $\sigma(x)^{-1}$. The symbol $\sigma^{-1}(x)$ is the unique element x' in R that satisfies

$$\sigma(x') = \sigma(\sigma^{-1}(x)) = x = \sigma^{-1}(\sigma(x)),$$

and therefore represents the inverse with respect to composition. On the other hand, σ is an unitary automorphism, meaning $\sigma(1) = 1$, allowing us to gather

$$1 = \sigma(1) = \sigma(x \cdot x^{-1}) = \sigma(x) \cdot \sigma(x^{-1}), \qquad x \in \mathbb{K},$$

and therefore we conclude

$$\sigma(x^{-1}) = \sigma(x)^{-1}, \qquad x \in \mathbb{K},$$

i.e. the image of the multiplicative inverse of x is the multiplicative inverse of the *image* of x. When talking about a negative exponent, say $-k$, we usually mean applying the compositorial inverse k times, i.e. we want to express $\sigma^{-k}(x)$ that is defined as

$$\sigma^{-k}(x) := \sigma^{-(k-1)}(\sigma^{-1}(x)), \qquad k > 0.$$

For our Ore-Algebra construction, suppose we want to represent n automorphisms. An automorphism $\sigma_i, (0 < i \leq n)$ can be viewed as Ore-operator ∂_i, that is governed by the the commutation rule

$$\partial_i x = \sigma_i(x)\partial_i, \qquad x \in \mathbb{K},$$

and iterated application gives

$$\partial_i^k x = \partial_i^{k-1}\left(\partial_i x\right) = \partial_i^{k-1}\left(\sigma_i(x)\partial_i\right) = \ldots = \sigma_i^k(x)\partial_i^k, \qquad k > 0,$$

i.e., for positive powers, the Ore operator is represented by the tuple $(\partial_i, \sigma_i, \delta_0)$. To represent inverse application, we consider the Ore-operator ∂_{m+i} that determines the inverse tuple $(\partial_{n+i}, \sigma_i^{-1}, \delta_0)$, the action on ring elements is given by

$$\partial_{n+i} x = \sigma_i^{-1}(x)\partial_{n+i}, \qquad x \in \mathbb{K},$$

and its powers

$$\partial_{n+i}^k x = \partial_{n+i}^{k-1}\left(\partial_{n+i} x\right) = \partial_{n+i}^{k-1}\left(\sigma_i^{-1}(x)\partial_{n+i}\right) = \ldots = \sigma_i^{-k}(x)\partial_{n+i}^k, \qquad k > 0.$$

If we consider the Ore-algebra

$$\mathbb{O} = \mathbb{K}[\partial_1; \delta_0, \sigma_1] \dots [\partial_n; \delta_0, \sigma_n][\partial_{n+1}; \delta_0, \sigma_1^{-1}] \dots [\partial_{2n}; \delta_0, \sigma_n^{-1}],$$

then the Ore-operators are related by its application to elements x in $\mathbb{K}$. Namely,

$$\partial_i \cdot \partial_{n+i} \cdot x = \partial_i \cdot \sigma_i^{-1}(x)\partial_{n+i} = \sigma_i(\sigma_i^{-1}(x)) \cdot \partial_i \cdot \partial_{n+i} = x \cdot \partial_i \cdot \partial_{n+i}. \tag{1.31}$$

Often in literature it is implicitly stated that difference-differential rings can be described by positive exponents exclusively. In this part, we are going to show explicitly that the ring of operator monomials in a difference-differential ring is a multivariate non-commutative polynomial ring in $|\Delta| + 2 \cdot |\Sigma|$ variables. To that end, we consider a field $\mathbb{K}$ endowed with a set of derivations $\Delta := \{\delta_1, \dots, \delta_m\}$ and a set of automorphisms $\Sigma := \{\sigma_1, \dots, \sigma_n\}$. We assume that the elements of $\Delta \cup \Sigma$ are pairwise commutative, i.e. for all $u, v \in \Delta \cup \Sigma$ we have $u \cdot v = v \cdot u$, which is not necessarily the case for the product with field elements $x \in \mathbb{K}$. We define the set $\mathbb{T}$ as

$$\mathbb{T} := \{\tau_1, \dots, \tau_n\} \subseteq \mathrm{Aut}(\mathbb{K}), \qquad 1 \leq k \leq n : \tau_k := \sigma_k^{-1},$$

i.e. the operator τ_k is the compositoral inverse of σ_k. We will consider the non-commutative polynomial rings

$$R := \mathbb{K}\left[\Delta \ \cup \ \Sigma^{\pm 1} \ \cup \ \mathbb{T}^{\pm 1}\right], \qquad S := \mathbb{K}\left[\Delta \ \cup \ \Sigma^{\pm 1}\right].$$

We will use the following symbols to abbreviate power products in a way that

$$\delta^r := \delta_1^{r_1} \dots \delta_m^{r_m}, \qquad r \in \mathbb{N}^m.$$

and similar for σ and τ. Elements in R are $\mathbb{K}$-linear combinations of elements

$$\{\delta^r \sigma^s \tau^t : \ (r, s, t) \in \mathbb{N}^m \times \mathbb{Z}^n \times \mathbb{Z}^n\},$$

whereas monomials in S are of the form

$$\{\delta^r \sigma^s : \ (r, s) \in \mathbb{N}^m \times \mathbb{Z}^n\},$$

i.e. elements of S are of the form

$$S := \left\{ a := \sum_{(r,s)} a_{r,s} \delta^r \sigma^s : \ (r, s) \in \mathbb{N}^m \times \mathbb{Z}^n \right\},$$

that corresponds to difference-differential operators. Further let the set R^+ be defined by

$$R^+ := \left\{ a := \sum_{(r,s,t)} a_{r,s,t} \ \delta^r \sigma^s \tau^t \in R : \ (r, s, t) \in \mathbb{N}^m \times \mathbb{N}^n \times \mathbb{N}^n \right\}.$$

Obviously R^+ is a subring of R too. We define a $\mathbb{K}$-linear map

$$\varphi : \ R^+ \to S, \qquad \delta^r \sigma^s \tau^t \mapsto \varphi(\delta^r \sigma^s \tau^t) := \delta^r \sigma^{s-t}.$$

Lemma 11 (Characterization of φ). *$\varphi : R^+ \to S$ is a surjection of rings.*

Proof. Consider an element $w := \delta^r\sigma^p \in S$, where $r \in \mathbb{N}^m, p \in \mathbb{Z}^n$. Choose non-negative integer vectors $s, t \in \mathbb{N}^n$ such that $s - t = p$. Then, $\varphi(\delta^r\sigma^s\tau^t) = w$. Thus, $R^+ \to S$ is a $\mathbb{K}$-linear epimorphism. We must now show that φ is a homomorphism of rings. First we show that φ is unitary, i.e. $\varphi(1) = 1$. This is obvious by

$$\varphi(1) = \varphi(\delta^0\sigma^0\tau^0) = \delta^0\sigma^{0-0} = 1.$$

As a next step, we consider $\varphi(\delta^s \cdot a)$ for $s \in \mathbb{N}^m$ and $a \in R^+$. Assume that

$$a = \sum_{(r,s,t)} a_{r,s,t} \cdot \delta^r\sigma^s\tau^t \in R^+, \qquad (r,s,t) \in \mathbb{N}^m \times \mathbb{N}^n \times \mathbb{N}^n,\ a_{r,s,t} \in \mathbb{K}.$$

If we apply a derivation δ_i to a, then

$$\delta_i \cdot a = \sum_{(r,s,t)} \left(a_{r,s,t}\delta_i + \delta_i(a_{r,s,t})\right)\delta^r\sigma^s\tau^t = \sum_{(r,s,t)} a_{r,s,t}\delta^{r+e_i}\sigma^s\tau^t + \sum_{(r,s,t)} \delta_i(a_{r,s,t})\delta^r\sigma^s\tau^t,$$

where e_i denotes the i-th unit vector $e_i := (0,\ldots,0,1,0,\ldots,0) \in \mathbb{N}^m$, the 1 appearing at i-th position. The map φ applied to $\delta_i \cdot a$ gives

$$\begin{aligned}
\varphi(\delta_i \cdot a) &= \sum_{(r,s,t)} a_{r,s,t}\delta^{r+e_i}\sigma^{s-t} + \sum_{(r,s,t)} \delta_i(a_{r,s,t})\delta^r\sigma^{s-t} \\
&= \sum_{(r,s,t)} \left(a_{r,s,t}\delta_i + \delta_i(a_{r,s,t})\right)\delta^r\sigma^{s-t} \\
&= \delta_i \cdot \sum_{(r,s,t)} a_{r,s,t}\delta^r\sigma^{s-t} = \delta_i \cdot \varphi(a),
\end{aligned}$$

proving the identity

$$\varphi(\delta^s \cdot a) = \delta^s \cdot \varphi(a).$$

To overcome the non-commutativity with field elements $x \in \mathbb{K}$ in the product, we show

$$\varphi(\sigma^s\tau^t \cdot x \cdot \delta^u\sigma^v\tau^w) = \varphi(\sigma^s\tau^t) \cdot \varphi(x \cdot \delta^u\sigma^v\tau^w).$$

This follows from

$$\sigma^s\tau^t \cdot x \cdot \delta^u\sigma^v\tau^w = \sigma^{s-t}(x)\sigma^s\tau^t\delta^u\sigma^v\tau^w = \sigma^{s-t}(x)\delta^u\sigma^{s+v}\tau^{t+w},$$

therefore

$$\begin{aligned}
\varphi(\sigma^s\tau^t \cdot x \cdot \delta^u\sigma^v\tau^w) &= \sigma^{s-t}(x)\delta^u\sigma^{s+v-t-w} &&= \sigma^{s-t}(x)\sigma^{s-t}\delta^u\sigma^{v-w} \\
&= \sigma^{s-t} \cdot x \cdot \delta^u\sigma^{v-w} &&= \varphi(\sigma^s\tau^t) \cdot \varphi(x \cdot \delta^u\sigma^v\tau^w)
\end{aligned}$$

Combining the last two points, it is an easy matter to prove for $x, y \in \mathbb{K}$, that

$$\begin{aligned}
\varphi(x \cdot \delta^r\sigma^s\tau^t \cdot y \cdot \delta^u\sigma^v\tau^w) &= x \cdot \delta^r \cdot \varphi(\sigma^s\tau^t \cdot y \cdot \delta^u\sigma^v\tau^w) = x \cdot \delta^r \cdot \varphi(\sigma^s\tau^t) \cdot \varphi(y \cdot \delta^u\sigma^v\tau^w) \\
&= \varphi(x \cdot \delta^r\sigma^s\tau^t) \cdot \varphi(y \cdot \delta^u\sigma^v\tau^w)
\end{aligned}$$

Considering now the properties gives the multiplicative property. We set

$$\mu_{r,s,t} := a_{r,s,t}\delta^r\sigma^s\tau^t, \quad \nu_{u,v,w} := b_{u,v,w}\delta^u\sigma^v\tau^w$$

to get for $a, b \in R^+$

$$\begin{aligned}
\varphi(ab) = \ & \varphi\left(\sum_{(r,s,t)} \mu_{r,s,t} \cdot \sum_{(u,v,w)} \nu_{u,v,w}\right) = \varphi\left(\sum_{\substack{(r,s,t)\\(u,v,w)}} \mu_{r,s,t} \cdot \nu_{u,v,w}\right) \\
= \ & \sum_{\substack{(r,s,t)\\(u,v,w)}} \varphi(\mu_{r,s,t}) \cdot \varphi(\nu_{u,v,w}) = \sum_{(r,s,t)} \varphi(\mu_{r,s,t}) \cdot \sum_{(u,v,w)} \varphi(\nu_{u,v,w}) \\
= \ & \varphi\left(\sum_{(r,s,t)} \mu_{r,s,t}\right) \cdot \varphi\left(\sum_{(u,v,w)} \nu_{u,v,w}\right) = \varphi(a) \cdot \varphi(b)
\end{aligned}$$

□

Lemma 12 (Characterization of $\ker(\varphi)$)**.**

$$\ker(\varphi) = \left\{ a \in R^+ : \ \forall r \in \mathbb{N}^m \ \forall d \in \mathbb{Z}^n \ \sum_{r-s=d} a_{d,r,s} = 0 \right\}.$$

Proof. Consider an element

$$a = \sum_{(r,s,t)} a_{r,s,t}\delta^r\sigma^s\tau^t \in R^+, \qquad (r,s,t) \in \mathbb{N}^m \times \mathbb{N}^n \times \mathbb{N}^n.$$

This element $a \in \ker(\varphi)$ if and only if

$$\begin{aligned}
a \in \ker(\varphi) \Leftrightarrow \ & \sum_{(r,s,t)} a_{r,s,t}\delta^r\sigma^{s-t} = 0 \\
\Leftrightarrow \ & \sum_{r\in\mathbb{N}^m} \sum_{d\in\mathbb{Z}^n} \sum_{s-t=d} a_{r,s,t}\delta^r\sigma^d = 0 \\
\Leftrightarrow \ & \forall r \in \mathbb{N}^m \ \forall d \in \mathbb{Z}^n \ \sum_{r-s=d} a_{d,r,s} = 0.
\end{aligned}$$

□

Theorem 10 (Representation of $\ker(\varphi)$)**.** $\ker(\varphi) = \langle \sigma_1\tau_1 - 1, \ldots, \sigma_n\tau_n - 1 \rangle \triangleleft R^+$.

Proof. φ is a morphism of rings, hence $\ker(\varphi)$ is an ideal in R^+. From

$$\varphi(\sigma_j\tau_j - 1) = 0, \qquad 1 \le j \le n,$$

we get that $\ker(\varphi) \supseteq \langle \sigma_1\tau_1 - 1, \ldots, \sigma_n\tau_n - 1\rangle$.

For the converse let $a \in \ker(\varphi)$. Thus

$$\forall r \in \mathbb{N}^m \ \forall d \in \mathbb{Z}^n \ \sum_{s-t=d} a_{r,s,t} = 0. \tag{1.32}$$

Let now a be given by

$$a = \sum_{(r,s,t)} a_{r,s,t} \cdot \delta^r \sigma^s \tau^t = \sum_{r\in\mathbb{N}^m} \sum_{d\in\mathbb{Z}^n} \underbrace{\sum_{s-t=d} a_{r,s,t}\delta^r\sigma^s\tau^t}_{S_{r,d}(a)} = \sum_{r\in\mathbb{N}^m} \sum_{d\in\mathbb{Z}^n} S_{r,d}(a),$$

where the sum ranges over $(r,s,t) \in \mathbb{N}^m \times \mathbb{N}^n \times \mathbb{N}^n$. We consider such a summand $S_{r,d}(a)$ where (1.32) holds. Thus

$$\begin{aligned} S_{r,d}(a) = & \sum_{t\in\mathbb{N}^n} a_{r,t+d,t}\sigma^{t+d}\tau^t\delta^r = \sum_{t\in\mathbb{N}^n} a_{r,t+d,t}\sigma^t\tau^t\delta^r\sigma^d - \sum_{t\in\mathbb{N}^n} a_{r,t+d,t}\delta^r\sigma^d \\ = & \sum_{t\in\mathbb{N}^n} a_{r,t+d,t}(\sigma^t\tau^t - 1)\delta^r\sigma^d, \end{aligned}$$

where in the last expression the expression

$$\sigma^t\tau^t - 1 = \sigma_1^{t_1}\ldots\sigma_n^{t_n} \cdot \tau_1^{t_1}\ldots\tau_n^{t_n} - 1 = (\sigma_1\tau_1)^{t_1}\ldots(\sigma_n\tau_n)^{t_n} - 1.$$

Now remark that the elements $\sigma_i\tau_i$ are central elements in R^+:

$$\sigma_i\tau_i \cdot x = \sigma_i \cdot \tau_i(x)\tau_i = \sigma_i(\tau_i(x)) \cdot \sigma_i\tau_i = x \cdot \sigma_i\tau_i, \qquad x \in \mathbb{K},$$

which is just (1.31). Therefore we can find representations

$$(\sigma_i\tau_i)^{t_i} = (\sigma_i\tau_i - 1)\gamma_i + 1, \qquad 1 \le i \le n$$

where $\gamma_i = 1 + \sigma_i\tau_i + (\sigma_i\tau_i)^2 + \ldots + (\sigma_i\tau_i)^{t_i-1}$ where $1 \le i \le n$. Multiplying together we obtain

$$(\sigma_1\tau_1)^{t_1}\ldots(\sigma_n\tau_n)^{t_n} = \gamma + 1,$$

where γ is a sum of terms each of which contains some expressions $\sigma_i\tau_i - 1$ as a factor. It follows that $\sigma^t\tau^t - 1 = \gamma \in \langle \sigma_1\tau_1 - 1, \ldots, \sigma_n\tau_n - 1\rangle$. Consequently

$$S_{r,d}(a) \in \langle \sigma_1\tau_1 - 1, \ldots, \sigma_n\tau_n - 1\rangle \Rightarrow a = \sum_{r\in\mathbb{N}^n} \sum_{d\in\mathbb{Z}^m} S_{r,d}(a) \in \langle \sigma_1\tau_1 - 1, \ldots, \sigma_n\tau_n - 1\rangle.$$

□

Corollary 4 (Correspondence to positive exponents). *Summarizing above considerations, we get the isomorphism*

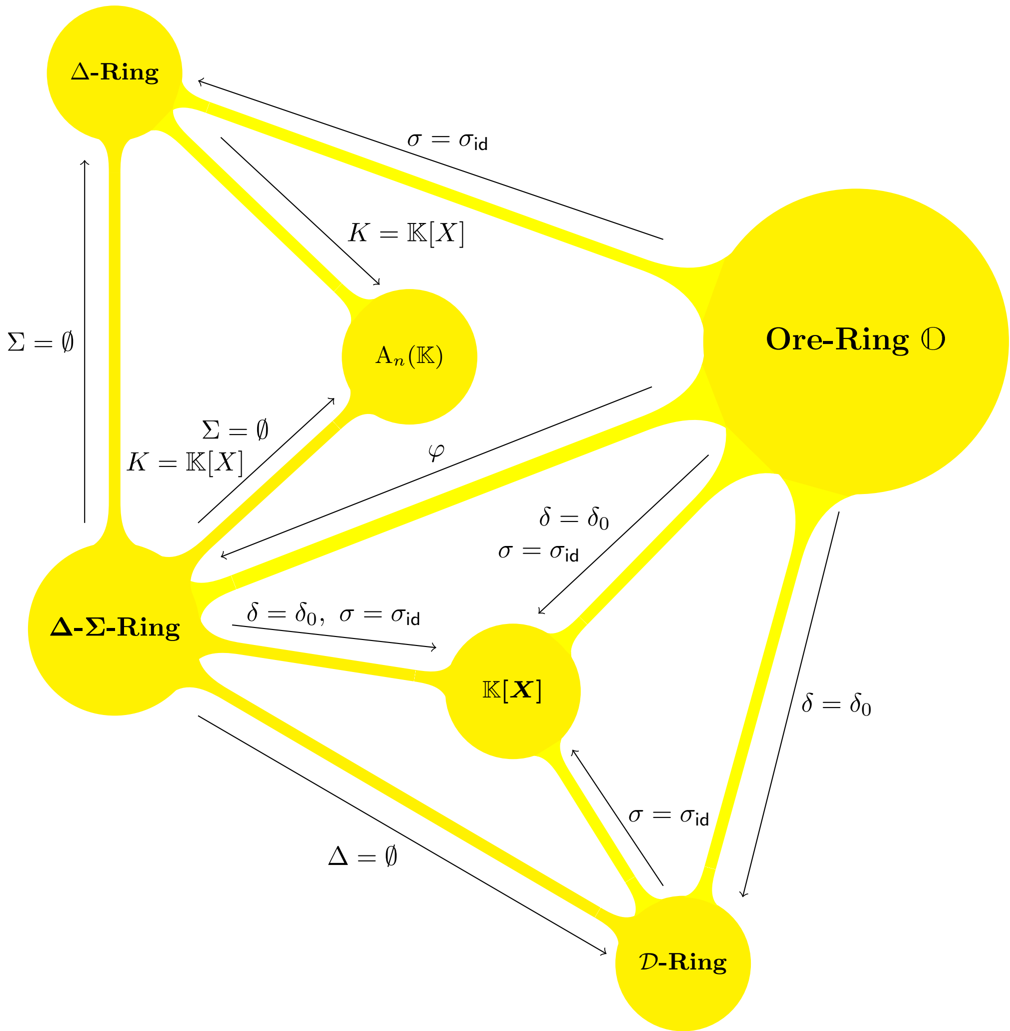
$$
\begin{aligned}
R^+ = & \left\{ a := \sum_{(r,s,t)} a_{r,s,t}\, \delta^r \sigma^s \tau^t \in R : \ (r,s,t) \in \mathbb{N}^m \times \mathbb{N}^n \times \mathbb{N}^n \right\} \Big/ \underbrace{\langle \sigma_i \tau_i - 1 : 1 \leq i \leq n \rangle}_{\ker(\varphi)} \\
\cong & \left\{ a := \sum_{(r,s)} a_{r,s} \delta^r \sigma^s : \ (r,s) \in \mathbb{N}^m \times \mathbb{Z}^n \right\} = S,
\end{aligned}
$$

i.e. the ring of difference-differential operators can be viewn as a quotient of a polynomial ring, where the dependence of the automorphisms and their compositorial inverses is factored out.

1.4. The Interplay of the considered Rings

In this section, we want to give an overview, how the rings, that are considered in this thesis relate to each other. In literature each of the rings is studied mostly independent of each other. Appropriate choices of “the variable-part of the definition” makes it possible to pass from one ring to another.

The advantage of this abstraction is, that theorems proven in a larger class apply to the smaller class. For instance, a theorem on multivariate dimension polynomials in difference-differential rings applies to all rings that could be obtained from a difference-differential ring.



The figure shows the relation between

- the ring of Ore-polynomials $\mathbb{O}$,
- the ring of difference-differential operators D,
- the ring of differential operators with set of derivations Δ,
- the ring of difference operators $\mathcal{D}$,
- the ring of commutative polynomials $\mathbb{K}[X]$.

At each of this rings, a theory of Gröbner bases for the computation of univariate and multivariate dimension polynomials has been developed. The connection of this rings already leads to the suspicion, that a common theory applies to all this rings.

In the picture arrows indicate how one can pass from one ring to the other. For example, the arrow from the ring of Ore-polynomials $\mathbb{O}$ to a differential ring indicates, that by choosing σ as the identity operator σ_{id}, an Ore variable ∂ acts as derivation.

Section 1.3. shows how to construct the ring of difference-differential operators from the ring of Ore-polynomials by the fundamental isomorphism φ and derive the characterization Corollary 4. From that, it is evident that we can model differential rings and difference-rings by choosing the set of automorphisms resp. the set of derivations as the empty set, and therefore connect this concepts.

As we have seen in section 1.3, an Ore-Variable ∂ is fully characterized by fixing an unitary endomorphism σ and a σ-skew derivation δ (see Lemma 9), and using the commutation rule

$$\partial \cdot x = \sigma(x)\partial + \delta(x), \qquad x \in \mathbb{K}.$$

To pass from the Ore-Ring to a differential ring, the choice $\sigma = \sigma_{\mathrm{id}}$ and derivation δ allows the Ore-variable ∂ to act by derivation and therefore gives rise to a differential ring.

Similar, by choosing the derivation δ as zero-derivation $\delta = \delta_0$ and σ arbitrary we get a difference-operator

$$\partial \cdot x = \sigma(x)\partial, \qquad x \in \mathbb{K},$$

and therefore a difference ring $\mathcal{D}$.

If at the same time $\delta = \delta_0$ and $\sigma = \sigma_{\mathrm{id}}$ we've got the commutation rule

$$\partial \cdot x = x\partial,$$

and therefore the model of a commutative polynomial.

Starting now from the difference-differential ring (R, Δ, Σ), difference-rings and differential-rings are easily obtained by choosing the set of derivations Δ respectively the set of automorphisms Σ as the empty set.

Commutative polynomials are obtained by setting the set of derivations $\Delta := \{\delta_0\}$ and the set of automorphisms $\Sigma := \{\sigma_{\mathrm{id}}\}$. Using multiple copies of δ_0 resp. σ_{id} allows to model a multivariate polynomial ring $\mathbb{K}[X]$.

Finally, a difference-operator acts by $\sigma \cdot x = \sigma(x)\sigma$, hence $\sigma = \sigma_{\mathrm{id}}$ gives rise to commutative polynomials.

The Weyl-Algebra $\mathrm{A}_n(\mathbb{K})$, the ring of differential operators with polynomial coefficients, can be constructed in at least two ways. One way would be to view it as a sub-algebra of the algebra of linear operators of $\mathbb{K}[X]$, namely $\mathrm{End}_{\mathbb{K}}(\mathbb{K}[X])$. In particular, $\mathrm{A}_n(\mathbb{K})$ is generated by the operators $\hat{x_1}, \ldots, \hat{x_n}$ and $d_1, \ldots, d_n$, where $\hat{x_i},\ d_i : \mathbb{K}[X] \to \mathbb{K}[X]$ act naturally by

$$\hat{x}_i(f) := x_i \cdot f, \qquad d_i(f) := \frac{\mathrm{d}}{\mathrm{d}x_i} f, \qquad f \in \mathbb{K}[X].$$

Another view would be that $\mathrm{A}_n(\mathbb{K})$ is the free algebra

$$\mathbb{K}\langle x_1, \ldots, x_n, d_1, \ldots, d_n \rangle$$

whose generators satisfy the commutation rule

$$x_i x_j = x_j x_i, \qquad d_i d_j = d_j d_i, \qquad d_i x_j = x_j d_i + \delta_{i,j} \tag{1.33}$$

where $\delta_{i,j}$ is the Kronecker symbol, defined by $\delta_{i,j} = 1$ if $i = j$ and zero otherwise.

To clarify the last missing connectors from the (difference-)differential-ring, we observe that $\mathrm{A}_n(\mathbb{K})$ a differential ring (and therefore as a particular difference-differential ring), where the underlying ring is $\mathbb{K}[X]$, and monomials are of the form d^l with $l \in \mathbb{N}^n$.

The Relation to Algebras of Solvable Type

In literature, the notion of algebras of solvable type occured in [KRW90]. If $\mathbb{K}$ is a (commutative) field of characteristic zero, they consider $R = \mathbb{K}\langle X_1, \ldots, X_n \rangle$, the free associative algebra in the variables X_i over $\mathbb{K}$. Words w of length p in R can be represented as $X_{i_1} \cdots X_{i_p}$. A word w is said to be a *standard monomial* if and only if

$$w = X_{i_1} \cdots X_{i_p} \wedge 1 \leq i_1 \leq i_2 \leq \ldots \leq i_p \leq n.$$

In R, we can add terms, but we restrict the multiplication X_i and X_j to be the product $X_i X_j := X_i \cdot X_j$ only if $1 \leq i \leq j \leq n$. For the product of X_j with X_i where i, j are as in the last sentence, we encounter the operation $\star$.

Suppose we have a linear order $<$ on the set of words in R. On the generators $X_1, \ldots, X_n$ of R, we introduce a new, non-commutative, product $\star$ that satisfies for all choices of i, j such that $1 \leq i \leq j \leq n$ the condition

$$\exists c_{ij} \in \mathbb{K}\backslash\{0\} \; \exists p_{ij} \in R : X_j \star X_i = c_{ij} X_i X_j + p_{ij} \wedge \mathrm{LT}(p_{ij}) < X_i X_j. \tag{1.34}$$

In particular, it is possible to express every $f \in R$ as a $\mathbb{K}$-linear combination of standard monomials (and hence, remove any occurence of the $\star$-product, leaving only the usual product).

Definition 9. The algebra $R = \mathbb{K}\langle X_1, \ldots, X_n \rangle$ is called *an algebra of solvable type* if and only if

- R is an associative ring with 1
- for all $a, b \in \mathbb{K}$ and indices $1 \leq h \leq i \leq j \leq k \leq n$, and t is a monomial in $X_i, \ldots, X_j$

1. $a \star bt = bt \star a = abt$
2. $X_h \star bt = bX_h t$
3. $bt \star X_k = btX_k$

- for $1 \leq i \leq j \leq n$ there exist $0 \neq c_{ij} \in \mathbb{K}$ and $p_{ij} \in R$ such that

$$X_j \star X_i = c_{ij} X_i X_j + p_{ij}, \qquad \mathrm{LT}(p_{ij}) < X_i X_j.$$

In particular, we can consider $f, g \in R$ and calculate w.r.t. the new product $\star$ as follows:

$$f \star g = c \cdot fg + h, \qquad c \in \mathbb{K},\ \mathrm{LT}(h) < \mathrm{LT}(fg).$$

The axioms we've considered so far, allows us to find that for $f, g \in R$ we have:

- $\mathrm{LT}(f \star g) = c \cdot \mathrm{LT}(f) \cdot \mathrm{LT}(g)$ for some $c \in \mathbb{K}$;
- For $h \in R$, $\mathrm{LT}(f) < \mathrm{LT}(g)$ implies
  1. $\mathrm{LT}(f \star h) < \mathrm{LT}(g \star h)$
  2. $\mathrm{LT}(h \star f) < \mathrm{LT}(h \star g)$.

So, there is quite some structure available. Let us consider the situation where we have a solvable algebra with two generators, i.e. $n = 2$. This situation has been studied in [LKM11]. It turns out, that condition (1.34) translates as follows

$$X_2 \star X_1 = c_{12} \cdot X_1 X_2 + \alpha \cdot X_1 + \beta \cdot X_2 + \gamma, \qquad c_{12}, \alpha, \beta, \gamma \in \mathbb{K}.$$

In this setting we have the following ([LKM11, Theorem 1.]):

Theorem 11. *Consider the free algebra*

$$A(c, \alpha, \beta, \gamma) := \mathbb{K}\langle x, y \mid y \star x = c \cdot xy + \alpha x + \beta y + \gamma \rangle,$$

Let q be transcendental over $\mathbb{K}$. Then, $A(c, \alpha, \beta, \gamma)$ is isomorphic to one of the five algebras:

- *the commutative algebra* $\mathbb{K}[x, y]$*;*
- *the first Weyl algebra* $\mathrm{A}_1 = \mathbb{K}\langle x, d \mid dx = xd + 1 \rangle$*;*
- *the shift algebra* $S_1 = \mathbb{K}\langle s, x \mid sx = xs + s \rangle$*;*
- *the q-commutative algebra* $\mathbb{K}_q[x, y] = \mathbb{K}(q)\langle x, y \mid yx = q \cdot xy \rangle$*;*
- *the first q-Weyl algebra* $\mathrm{A}_1^{(q)} = \mathbb{K}(q)\langle x, \partial \mid \partial x = q \cdot x\partial + 1 \rangle$*.*

From that, we see that algebras of solvable type in two generators are also contained in our picture, and therefore related to our construction. We will in the upcoming chapters introduce an algorithmic treatment of rings of that type.

2. The Concept of Gröbner Reduction

At the introduction we have encountered a variety of non-commutative polynomial rings that can be used to model physical problems in a suitable operator algebra.

By careful inspection of the procedures in current literature, we've got the increasing evidence that the interplay of filtrations and Gröbner bases must have a key role. Therefore, in this section, we are going to equip the considered rings with a certain type of filtration (so called "monomial filtrations"). The filtration on the ring extends to a filtration on the considered module. This "filter-space" (which we are defining more precisely in the next section), viewn as a $\mathbb{K}$-vector space, has as most important invariant its dimension.

For several rings a theory of computation of dimension has been considered. At [FL15a], the author, in joint work with Günter Landsmann, has developed the concept of Gröbner Reduction. This work was presented at ISSAC 2015 in Bath.

Gröbner reduction will provide an uniform approach of computing the vector space dimension of filter-spaces of left modules. The introduced concept is an axiomatic approach to related techniques, characterizing the properties of a binary reduction relation.

With this layer of abstraction, we are in the position to prove several known characterizations from a different point of view then in current literature, without explicit knowing the reduction relation forehead.

2.1. Filtered Modules over Filtered Rings

As indicated in the introductory chapter, we let throughout this thesis denote R an arbitrary ring with one, containing a commutative ring K as a subring. Sometimes, K will coincide with a field $\mathbb{K}$ of characteristic zero, this will be emphasized at occurence. Hence, we will use the symbol K if we want to express that K is a commutative ring, and $\mathbb{K}$ to express that K equals a field $\mathbb{K}$ of characteristic zero. The symbol $\leq_\pi$ is understood as in (1.20).

Definition 10 (Filtered Ring).
Let R be a ring. By a p-fold filtration on R we mean a family of additive subgroups

$$R_r \subseteq R, \qquad r \in \mathbb{N}^p,$$

such that

1. $R_r \cdot R_s \subseteq R_{r+s}, \qquad r, s \in \mathbb{N}^p$;
2. $R_r \subseteq R_s, \qquad r \leq_\pi s \in \mathbb{N}^p$;
3. $R = \bigcup_{r \in \mathbb{N}^p} R_r$;
4. $1 \in R_0$

Definition 11 (Monomial Filtration).
A filtration of R is called *monomial* if and only if

1. $R_0 = K$;
2. $f \in R_r \Rightarrow \mathrm{T}(f) \subseteq R_r, \qquad r \in \mathbb{N}^p$;

General p-fold filtrations on rings and modules, with an application of computing the Gelfand-Kirillov-Dimension, were studied in [Tor99].

Lemma 13 (R is left- and right-R_0-module).
If R is a filtered ring, R_0 is a subring and each R_r is a left and a right R_0-module.

We will give now some examples of filtrations on rings appearing in literature. Later on, we will use the filtrations appearing in this examples in applications involving this filtered rings.

Example 6 (Filtration on $\mathbb{K}[X]$).
If $R := \mathbb{K}[x_1, \ldots, x_n]$, one possible (n-fold) filtration is given by

$$F_r^{(1)} := \{f \in R : \ \forall i : \ \deg_{x_i}(f) \leq r_i\}, \qquad r = (r_1, \ldots, r_n) \in \mathbb{N}^n.$$

This filtration on R is an example of a monomial filtration.

Another example of a monomial filtration would be, for a term order $\preccurlyeq$, to consider the subset of R

$$F_r^{(2)} := \{f \in \mathbb{K}[x_1, \ldots, x_n] : \ \mathrm{LT}(f) \preccurlyeq x^r\}, \qquad r \in \mathbb{N}^n.$$

For the commutative semigroup in the multivariate polynomial ring, we will use the common symbol $T^n(X)$ (e.g. [AL94]), defined by

$$T^n(X) := \{x_1^{k_1} \ldots x_n^{k_n} : \ (k_1, \ldots, k_n) \in \mathbb{N}^n\}.$$

(i.e. the commutative semigroup consisting of power products), to indicate monomials. The first filtration contains at most finitely many monomials, the total number is given by the multinomial coefficient

$$\#|F_r^{(1)} \cap T^n(X)| = \binom{r_1 + \ldots + r_n}{r_1, \ldots, r_n} = \frac{(r_1 + \ldots + r_n)!}{r_1! \cdot r_2! \cdots r_n!}, \qquad r = (r_1, \ldots, r_n) \in \mathbb{N}^n,$$

The second filtration might consists of infinitely many monomials m that satisfy $m \preccurlyeq x^r$.

Example 7 (Filtration on D)**.**
For the ring D, in [ZW08a] the following situation is considered: Let

$$\lambda = \delta_1^{k_1} \ldots \delta_m^{k_m} \sigma_1^{l_1} \ldots \sigma_n^{l_n} \in \Lambda_{m,n} \subseteq D, \qquad (k_1, \ldots, k_m) \in \mathbb{N}^m, (l_1, \ldots, l_n) \in \mathbb{Z}^n,$$

and set

$$|\lambda|_1 := k_1 + \ldots + k_m, \qquad |\lambda|_2 := |l_1| + \ldots + |l_n|.$$

For a general operator $f \in D$ we define

$$|f|_\nu = |\sum_{\lambda \in \Lambda_{m,n}} f_\lambda \lambda|_\nu := \max\{|\lambda|_\nu : \ f_\lambda \neq 0\}, \qquad \nu = 1, 2.$$

Then, a bivariate filtration on D is given by

$$D_{r,s} := \{f \in D : \ |f|_1 \leq r \ \wedge \ |f|_2 \leq s\}, \qquad r, s \in \mathbb{N}.$$

To prove that

$$f \in D_p \cdot D_q \Rightarrow f \in D_{p+q}, \qquad p = (p_1, p_2) \ q = (q_1, q_2) \in \mathbb{N}^2,$$

take $f \in D_p \cdot D_q$, i.e. there exists $a \in D_p$ and $b \in D_q$ such that $f = ab$. Let

$$a = \sum_{\eta \in \Lambda_{m,n}} a_\eta \eta \in D_p, \qquad b = \sum_{\mu \in \Lambda_{m,n}} b_\mu \mu \in D_q, \qquad a_\eta, \ b_\mu \in \mathbb{K},$$

such that $\mathrm{T}(a) \subseteq D_p$ and $\mathrm{T}(b) \subseteq D_q$, or what is equivalent,

$$\forall \eta \in \mathrm{T}(a) : |\eta|_1 \leq p_1 \ \wedge \ |\eta|_2 \leq p_2, \qquad \forall \mu \in \mathrm{T}(b) : |\mu|_1 \leq q_1 \ \wedge \ |\mu|_2 \leq q_2.$$

By Lemma 6, we have that

$$\mathrm{T}(ab) \subseteq \{\lambda' \mu : \mu \in \mathrm{T}(b)\},$$

where

$$\lambda' \in \{\delta^{k'} \sigma^l : \exists \ \delta^k \sigma^l \in \mathrm{T}(a) \ \textit{such that} \ k' \leq_\pi k\}.$$

Hence, for all monomials $\eta \in \mathrm{T}(a)$ with $|\eta|_1 = |a|_1$, we get for all $\mu \in \mathrm{T}(b)$ with $|\mu|_1 = |b|_1$ that

$$|\lambda' \mu|_1 = |\lambda'|_1 + |\mu|_1 \leq |\eta|_1 + |\mu|_1 = |a|_1 + |b|_1 = p_1 + q_1.$$

For $|\cdot|_2$ we use the property that the automorphisms have the same exponent vector regardless of non-commutativity, i.e.

$$\forall x \in \mathbb{K} : \sigma^l \cdot x = \sigma^l(x)\sigma^l \Rightarrow |\sigma^l \cdot x|_2 = |\sigma^l(x) \cdot \sigma^l|_2 = |\sigma^l|_2,$$

hence by triangle inequality

$$|\lambda' \cdot \mu|_2 = |\eta\mu|_2 \leq |\eta|_2 + |\mu|_2 = |a|_2 + |b|_2 = p_2 + q_2.$$

The remaining properties

$$D_{r,s} \subseteq D_{r',s'} \quad (r,s) \leq_\pi (r',s') \qquad D = \bigcup_{r,s} D_{r,s} \qquad 1 \in D_{0,0}$$

are plain.

Example 8 (Filtration on $\mathbb{O}$).
In [Lev07], the multivariate ring of Ore-polynomials in $\mathcal{O}^{(n)} := \{\partial_1, \ldots, \partial_n\}$ over the field $\mathbb{K}$, denoted by $\mathbb{O}$, with commutation rule

$$\partial_i \cdot x = \sigma_i(x)\partial_i + \delta_i(x), \qquad x \in \mathbb{K},$$

for σ_i an injective $\mathbb{K}$-homomorphism and δ_i a σ_i-skew derivation is considered. Ring elements are $\mathbb{K}$-linear combinations of power products of the form

$$T^n\left(\mathcal{O}^{(n)}\right) := \{\partial^k = \partial_1^{k_1} \ldots \partial_n^{k_n},\ k = (k_1, \ldots, k_n) \in \mathbb{N}^n\}.$$

If now partition the set $\mathcal{O}^{(n)}$ into p disjoint subsets $\mathcal{O}_1, \ldots, \mathcal{O}_p$ such that

$$\begin{aligned} &\mathcal{O}_1 := \{\partial_1, \ldots, \partial_{n_1}\} \\ &\mathcal{O}_k := \{\partial_{n_1+\ldots+n_{k-1}+1}, \ldots, \partial_{n_1+\ldots+n_k}\}, \qquad 1 < k \leq p, \end{aligned}$$

and $n_1 + \ldots + n_p = n$, then we denote

$$|\partial^k|_{\mathcal{O}_i} := |\partial_1^{k_1} \ldots \partial_n^{k_n}|_{\mathcal{O}_i} = \sum_{\partial_t \in \mathcal{O}_i} k_t.$$

A p-fold filtration on $\mathbb{O}$ is given by

$$\mathbb{O}_r := \{f \in \mathbb{O} :\ |f|_{\mathcal{O}_i} \leq r_i\}, \qquad r = (r_1, \ldots, r_p) \in \mathbb{N}^p,$$

where

$$|f|_{\mathcal{O}_i} := \max\{|\eta|_{\mathcal{O}_i} :\ \eta \in \mathrm{T}(f)\}, \qquad 1 \leq i \leq p.$$

In [Lev07], the monomials in $\mathbb{O}_r$ are denoted by the symbol $\Theta(r_1, \ldots, r_p)$. However, at the first section, we reserved the symbol Θ for differential monomials. Therefore, we suggest to use the symbol

$$\mathcal{O}(r_1, \ldots, r_p) := \{o \in T^n\left(\mathcal{O}^{(n)}\right) :\ \forall i : |o|_{\mathcal{O}_i} \leq r_i\}, \qquad (r_1, \ldots, r_p) \in \mathbb{N}^p.$$

instead. A different view on that would be, that we can build up this p-fold filtration as an intersection of p univariate filtrations. This is achieved by setting

$$\mathcal{O}^{(j)}_{k_i} := \{f \in \mathbb{O} : |f|_{\mathcal{O}_j} \leq k_i\}, \qquad k_i \in \mathbb{N}, 1 \leq j \leq p,$$

and considering the filtration

$$\mathcal{O}(k_1, \dots, k_p) := \mathcal{O}^{(1)}_{k_1} \cap \dots \cap \mathcal{O}^{(p)}_{k_p}, \qquad k = (k_1, \dots, k_p) \in \mathbb{N}^p.$$

Writing the p-fold filtration in that way, reflects in the structure of a partial product order. However, not every p-fold filtration has the structure of the product order.

Many more filtrations are know throughout literature, such as the Bernstein filtration at the Weyl algebra, the ring of differential operators with polynomial coefficients.

Given now a ring R, the obvious question is how to construct a filtration. We are going to answer the question in two steps. As a first step, we restrict ourselves to the case $p = 1$. Later on, we will consider the general case. It is obvious that we consider certain subsets $R_k \subseteq R$, that satisfy

- $\forall k, l \in \mathbb{N} : k \leq l \Rightarrow R_k \subseteq R_l$;
- $\bigcup\limits_{k=0}^{\infty} R_k = R$.

Let S denote the set of subsets R_k of R that fulfill this two rules.

One possible choice to design a filtration on R is to consider the family of subsets

$$R^{(u)}_k := \{r \in R : u(r) \leq k\} \subseteq R,$$

where u is about to be characterized. This choice of $R^{(u)}_k$ obviously satisfies the two characteristic properties for the set S, i.e. $R^{(u)}_k \in S$. The following Lemma gives a full characterization of univariate filtrations.

Lemma 14 (Characterization of one-dimensional filtrations)**.** *With the above notation, the family*

$$R^{(u)}_k := \{f \in R : \ u(f) \leq k\}, \quad k \in \mathbb{N},$$

is an univariate filtration of R if and only if the map u satisfies the following conditions:

1. *If $x \in R$, then $u(x) = 0$ if and only if $x \in K$;*
2. *$\forall x, y \in R : u(x + y) \leq \max\{u(x), u(y)\}$;*
3. *$\forall x, y \in R : u(xy) \leq u(x) + u(y)$;*

Furthermore, for any univariate filtration R_r, there exists a mapping $u : R \to \mathbb{N}$, satisfying conditions 1.-3. such that $R_r = R^{(u)}_r$.

Proof. Clearly, if $u : R \to \mathbb{N}$ is a mapping satisfying the above conditions and $R_k^{(u)} = \{x \in R : u(x) \le k\}$ (where $k \in \mathbb{N}$), then the family $\{R_k^{(u)} : k \in \mathbb{N}\}$ satisfies conditions 1.-4. of Definition 10 (with $p = 1$). Note that, if $x \in R_k^{(u)}$ and $c \in K$, then $u(cx) \le u(c) + u(x) = u(x)$. This observation and property 2. imply that every $R_k^{(u)}$ is a K-module. Conversely, suppose that u is a mapping from R to $\mathbb{N}$ such that the family $R_k^{(u)} = \{x \in R : u(x) \le k\}$, $k \in \mathbb{N}$, satisfies conditions 1.-4. of Definition 10. Since $R_0^{(u)} = K$ and $u(x) \ge 0$ for any $x \in R$, we obtain that $x \in K$ is equivalent to $u(x) = 0$. The other properties of the map u follow from the fact that every $R_k^{(u)}$ is a K-module and from the first two conditions of Definition 10.

In order to prove the last part of the statement, consider a univariate filtration $\{R_r : r \in \mathbb{N}\}$ of R and define the mapping $u : R \to \mathbb{N}$ by setting $u(x) = \min\{k : x \in R_k\}$. It is easy to check that u satisfies conditions 1.-3. Indeed, since $R_0 = K$, we have that $u(a) = 0$ for any $a \in K$ and, conversely, the equality $u(x) = 0$ (where $x \in R$) implies that $x \in R_0 = K$. Furthermore, the fact that every R_k is a K-module and the first two properties of a filtration imply that the mapping satisfies conditions 2 and 3.

It remains to show that $R_r^{(u)} = R_r$ for all $r \in \mathbb{N}$. As we have already seen, $R_0^{(u)} = K = R_0$. Let $x \in R_r$. Then $u(x) \le r$, hence, $x \in R_r^{(u)}$. Conversely, let $y \in R_r^{(u)}$. Then $u(y) \le r$, so $y \in R_r$. This completes the proof of the lemma.

□

Remark. The first part of Lemma 14 can be generalized to p-fold filtrations ($p > 1$) as follows. Let us consider a mapping $u : R \to \mathbb{N}^p$ and let $u_i := \pi_i \circ u : R \to \mathbb{N}$ ($1 \le i \le p$), where π_i is the projection of $\mathbb{N}^p$ onto its i-th component: $(a_1, \dots, a_p) \mapsto a_i$. For any $r = (r_1, \dots, r_p) \in \mathbb{N}^p$, let $R_r^{(u)} = \{x \in R : u_i(x) \le r_i \text{ for } 1 \le i \le p\}$. Then, one can mimic the corresponding part of the proof of Lemma 14 to obtain that $\{R_r^{(u)} : r \in \mathbb{N}^p\}$ is a p-fold filtration of R if and only if the mapping u satisfies the following conditions:

1. If $x \in R$, then $u(x) = 0$ if and only if $x \in K$;
2. $u(x + y) \le_\pi (\max\{u_1(x), u_1(y)\}, \dots, \max\{u_p(x), u_p(y)\})$ for all $x, y \in R$;
3. $u(xy) \le_\pi (u_1(x) + u_1(y), \dots, u_p(x) + u_p(y))$ for all $x, y \in R$.

At the same time, if $p > 1$, then not every p-fold filtration is of the form $\{R_r^{(u)} \;\; r \in \mathbb{N}^p\}$ with a mapping $u : R \to \mathbb{N}^p$ satisfying the above conditions. It follows from the fact that the same element of R can belong to different components R_r and R_s with incomparable (with respect to $\le_\pi$) p-tuples $r, s \in \mathbb{N}^p$.

Example 9. *Let $\mathbb{K}[x, y]$ be a polynomial ring in two variables over a field $\mathbb{K}$, equipped with a natural bifiltration*

$$R_{r,s} := \{f \in \mathbb{K}[x, y] : \deg_x(f) \le r \wedge \deg_y(f) \le s\}, \qquad (r, s) \in \mathbb{N}^2,$$

and let the factor ring $R = \mathbb{K}[x,y]/\langle x^3 - y^2\rangle$ be equipped with the canonical image $\overline{R}_{r,s}$ of the filtration $\{R_r :\ r \in \mathbb{N}^2\}$. Denoting the coset of a polynomial $f \in \mathbb{K}[x,y]$ by $\overline{f}$, we obtain that, say, $\overline{xy^2} \in \overline{R}_{1,2} \cap \overline{R}_{4,0}$ (e.g. $\overline{x^2} \in \overline{R}_{4,0} \backslash \overline{R}_{1,2}$).

Definition 12 (Filter valuation)**.** Let $\mathbb{M}$ be a set of monomials contained in R, $\nu : \mathbb{M} \to \mathbb{N}$. We extend ν to R by setting:

$$\nu : R \to \mathbb{N}, \quad f \mapsto \nu(f) := \{\max\{\nu(\mathfrak{m})\} : \mathfrak{m} \in \mathrm{T}(f)\}.$$

We call ν a *filter-valuation on* R if and only if

$$R_k := \{f \in R : \nu(f) \leq k\}$$

defines a filtration on R.

Lemma 15 (Characterization of filter valuations)**.** *Let R be a ring containing a set of monomials $\mathbb{M} \subseteq R$, and let $\mathbb{K}$ be a field of characteristic zero. Further, let ν be a filter valuation on R. Set*

$$R_k^{(\nu)} := \{f \in R : \nu(f) \leq k\}, \qquad R_0 = \mathbb{K}.$$

Then, for $\alpha, \beta, \gamma, \lambda, \eta \in \mathbb{M}$, $c \in \mathbb{K}\backslash\{0\}$, $r, s \in \mathbb{N}$, $a, b \in R$, for the statements

(1) $\nu(\lambda \cdot c \cdot \eta) \leq \nu(\lambda) + \nu(\eta)$;

(2) $\nu(\lambda\eta) \leq \nu(\lambda) + \nu(\eta)$;

(3) $\nu(\lambda \cdot c) = \nu(\lambda)$;

(4) $(\mathbb{M} \cap R_r^{(\nu)}) \cdot R_s^{(\nu)} \subseteq R_{r+s}^{(\nu)}$;

(5) $\exists \alpha : \nu(\alpha\beta) \leq r \wedge \nu(\gamma) \leq s \Rightarrow \nu(\beta\gamma) \leq r + s$;

(6) $\exists \alpha : \nu(\alpha\beta) \leq r \Rightarrow \nu(\beta) \leq r$;

(7) $\nu(\beta) \leq \nu(\alpha\beta)$;

(8) $\nu(ab) \leq \nu(a) + \nu(b)$;

the following implications hold:

- (1) $\Leftrightarrow$ (8);
- (1) $\Rightarrow$ (2) $\wedge$ (3) $\wedge$ (4);
- (5) $\Rightarrow$ (2) $\wedge$ (6);
- (6) $\Leftrightarrow$ (7);
- ((2) $\wedge$ (7) $\wedge$ $\mathbb{M} \cdot \mathbb{M} \subseteq \mathbb{M}$) $\Rightarrow$ (5);

Proof. First, we assume (1) and show (8). To that end,

$$a \cdot b = \left(\sum_{\mu \in \mathbb{M}} a_\mu \mu \right) \cdot \left(\sum_{\eta \in \mathbb{M}} b_\eta \eta \right) = \sum_{\mu, \lambda \in \mathbb{M}} a_\mu \mu \cdot b_\eta \eta, \qquad a_\mu, b_\eta \in \mathbb{K},$$

hence

$$\nu(ab) \leq \max_{\mu, \eta \in \mathbb{M}} \{\nu(a_\mu \mu \cdot b_\eta \eta)\} = \max_{\mu, \eta \in \mathbb{M}} \{\nu(\mu \cdot b_\eta \eta)\}.$$

Now, by (1), we conclude

$$\nu(ab) \leq \nu(\mu \cdot b_\eta \eta) \leq \nu(\mu) + \nu(\eta) \leq \nu(a) + \nu(b).$$

For the converse, assume (8) and specialize $a = \lambda$ and $b = c \cdot \eta$. Then:

$$\nu(ab) = \nu(\lambda \cdot c \cdot \eta) \leq \nu(\lambda) + \nu(c \cdot \eta) = \nu(\lambda) + \nu(\eta).$$

(1) $\Rightarrow$ (2) is seen by specializing $c = 1$. Similar, setting η to 1, shows that (1) $\Rightarrow$ (3). For proving (1) $\Rightarrow$ (4) take

$$a \in (\mathbb{M} \cap R_r^{(\nu)}) \cdot R_s^{(\nu)} \Rightarrow a = \lambda \cdot b = \lambda \cdot \sum_{\mu \in \mathbb{M}} b_\mu \mu = \sum_{\mu \in \mathbb{M}} \lambda \cdot b_\mu \mu,$$

$$\nu(a) \leq \max\{\nu(\lambda \cdot b_\mu \mu) : b_\mu \neq 0\} \leq \nu(\lambda) + \nu(\mu) \leq r + s \Rightarrow a \in R_{r+s}^{(\nu)}.$$

(5) $\Rightarrow$ (2) is shown by taking $\alpha = 1$. For (5) $\Rightarrow$ (6) take $\gamma = 1$ and observe that $\nu(1) = 0$. Next, we will show (6) $\Rightarrow$ (7). To that end, set $r := \nu(\alpha\beta)$ to get

$$\nu(\beta) \leq r \Rightarrow \nu(\beta) \leq \nu(\alpha\beta).$$

For the converse, we take the chain of inequalities

$$\nu(\beta) \leq \nu(\alpha\beta) \leq r \Rightarrow \nu(\beta) \leq r.$$

Finally, we assume $(2), (7)$ and that the monomials form a multiplicative monoid, we want to show (5). We have

$$\nu(\beta\gamma) \leq \nu(\alpha\beta\gamma) \leq \nu(\alpha\beta) + \nu(\gamma) = r + s.$$

□

Definition 13 (Set of p-fold filtrations)**.** We denote *the set of all p-fold filtrations on the ring R* by the symbol $\mathcal{F}_p(R)$, meaning that there is a $\mathbb{N}^p$-indiced family of subsets, satisfying the conditions of Definition 10. For filtrations F and G we set

$$(F \cap G)_{(r,s)} := F_r \cap G_s, \qquad F \in \mathcal{F}_p(R), G \in \mathcal{F}_q(R), r \in \mathbb{N}^p, s \in \mathbb{N}^q.$$

Lemma 16 (Intersection of Filtrations)**.** *Given a ring R, let $F \in \mathcal{F}_p(R)$ be a p-fold filtration, $G \in \mathcal{F}_q(R)$ a q-fold filtration on R, such that F and G are distinct. Then $F \cap G \in \mathcal{F}_{p+q}(R)$.*

Proof. To prove property 1. take

$$a \in (F \cap G)_{(r,s)} \ \wedge \ b \in (F \cap G)_{(t,u)} \Leftrightarrow a \in F_r \cap G_s \ \wedge \ b \in F_t \cap G_u.$$

From that, we get that the product $ab \in F_r F_t \subseteq F_{r+t}$ and $ab \in G_s G_u \subseteq G_{s+u}$, hence

$$ab \in F_{r+t} \cap G_{s+u} = (F \cap G)_{(r+t,s+u)}.$$

For property 2. assume that $(r,s) \leq_\pi (r',s')$. Then, $F_r \subseteq F_{r'}$ and $G_s \subseteq G_{s'}$. Therefore,

$$(F \cap G)_{(r,s)} = F_r \cap G_s \subseteq F_{r'} \cap G_{s'} = (F \cap G)_{(r',s')}.$$

To show property 3., we take the two-fold union

$$\bigcup_{r,s} (F \cap G)_{(r,s)} = \bigcup_r \bigcup_s (F_r \cap G_s) = \bigcup_r F_r \cap \bigcup_s G_s = R \cap R = R.$$

Finally, $1 \in (F \cap G)_{(0,0)}$. $\square$

Lemma 17. *Given a p-fold filtration $F \in \mathcal{F}_p(R)$ of R, and a partition $(p_1, \ldots, p_n)$ of p s.t. $p = p_1 + \ldots + p_n$. We define n projections*

$$\pi_i : \mathcal{F}_p(R) \to \mathcal{F}_{p_i}(R), \qquad F \mapsto \pi_i(F)_t := \bigcup_{i \in \mathcal{I}} F_{r_1,\ldots,r_{i-1},t,r_{i+1},\ldots,r_n} \qquad 1 \leq i \leq n,$$

where the union ranges over all tuples

$$\mathcal{I} := \{i := (r_1, \ldots, r_{i-1}, r_{i+1}, \ldots, r_n) \in \mathbb{N}^{p_1} \times \ldots \times \mathbb{N}^{p_{i-1}} \times \mathbb{N}^{p_{i+1}} \times \ldots \times \mathbb{N}^{p_n}\}.$$

Then, $\pi_i(F)$ is a p_i-fold filtration of R.

Proof. Throughout this proof, we abbreviate

$$(t, \hat{r}) := (r_1, \ldots, r_{i-1}, t, r_{i+1}, \ldots, r_n).$$

Further, we set

$$\begin{aligned}\sup\{(t,\hat{r}),(t,\hat{s})\} &:= (\max\{r_1,s_1\}, \ldots, \max\{r_{i-1},s_{i-1}\}, t, \max\{r_{i+1},s_{i+1}\}, \ldots, \max\{r_n,s_n\}) \\ &= (t, \sup\{\hat{r},\hat{s}\}).\end{aligned}$$

To show that $\pi(F)_t$ are vector spaces, take $a, b \in \pi_i(F)_t$. Then, there exists $\hat{r}, \hat{s}$ such that

$$a \in F_{(t,\hat{r})} \quad \wedge \quad b \in F_{(t,\hat{s})} \Rightarrow a, b \in F_{\sup\{(t,\hat{r}),(t,\hat{s})\}} = F_{(t,\sup\{\hat{r},\hat{s}\})} \subseteq \pi_i(F)_t.$$

If $a \in \pi_i(F)_t, b \in \pi_i(F)_{t'}$ then there exist $\hat{r}$, $\hat{s}$ such that $a \in F_{(t,\hat{r})}$ and $b \in F_{(t,\hat{s})}$. Thus, the product

$$ab \in F_{(t+t',\hat{r}+\hat{s})} \subseteq \pi_i(F)_{t+t'}.$$

The remaining properties are obvious. $\square$

A special case is that each p-fold filtration of R has p projections onto $\mathcal{F}_1(R)$

$$\pi_i : \mathcal{F}_p(R) \to \mathcal{F}_1(R), \qquad 1 \leq i \leq p.$$

Now, we consider the converse.

Lemma 18 (Projection of Filtrations)**.** *For each integer tuple $(p_1, \ldots, p_n)$, there is a map*

$$\begin{aligned} \varphi : \mathcal{F}_{p_1}(R) \times \ldots \times \mathcal{F}_{p_n}(R) &\to \mathcal{F}_{p_1+\cdots+p_n}(R) \\ \left(F^{(1)}, \ldots, F^{(n)}\right) &\mapsto \varphi\left(\left(F^{(1)}, \ldots, F^{(n)}\right)\right) := F^{(1)} \cap \ldots \cap F^{(n)}. \end{aligned}$$

The map φ is injective.

Proof. We generalize the notion $(F \cap G)_{(r,s)}$ from before,

$$\left(F^{(1)}, \ldots, F^{(n)}\right)_{(r_1,\ldots,r_n)} := F^{(1)}_{r_1} \cap \ldots \cap F^{(n)}_{r_n}$$

where each $F^{(i)}$ is a p_i-fold filtration and $r_i \in \mathbb{N}^{p_i}$. Assume that

$$\varphi\left(\left(F^{(1)}, \ldots, F^{(n)}\right)\right) = \varphi\left(\left(G^{(1)}, \ldots, G^{(n)}\right)\right),$$

i.e. we have for $(r_1, \ldots, r_n) \in \mathbb{N}^n$ that

$$\left(F^{(1)}, \ldots, F^{(n)}\right)_{(r_1,\ldots,r_n)} = \left(G^{(1)}, \ldots, G^{(n)}\right)_{(r_1,\ldots,r_n)}.$$

We obtain for each $1 \leq i \leq n$:

$$\begin{aligned} F^{(i)}_{r_i} = F^{(i)}_{r_i} \cap R = F^{(i)}_{r_i} \cap \bigcup_{r_j \neq r_i} \left(\bigcap_{j=0}^{n} F^{(j)}_{r_j}\right) &= \bigcup_{j=0}^{n} \left(\bigcap_{j=0}^{n} F^{(j)}_{r_j}\right) = \bigcup_{j=0}^{n} \left(\bigcap_{j=0}^{n} G^{(j)}_{r_j}\right) \\ &= G^{(i)}_{r_i} \cap \bigcup_{r_j \neq r_i} \left(\bigcap_{j=0}^{n} G^{(j)}_{r_j}\right) = G^{(i)}_{r_i} \cap R = G^{(i)}_{r_i}. \end{aligned}$$

□

Corollary 5. *Given a partition $p = p_1 + \ldots + p_n$,*

1. *For $1 \leq i \leq n$ the association $F \mapsto \pi_i(F)$ provides a map*

$$\pi_i : \mathcal{F}_p(R) \to \mathcal{F}_{p_i}(R)$$

2. *The association φ is a map*

$$\varphi : \mathcal{F}_{p_1}(R) \times \ldots \times \mathcal{F}_{p_n}(R) \to \mathcal{F}_p(R).$$

3. Let $\pi := (\pi_1, \ldots, \pi_n)$*, i.e.* $\pi : \mathcal{F}_p(R) \to \mathcal{F}_{p_1} \times \ldots \times \mathcal{F}_{p_n}(R)$*. Then*

$$\pi \circ \varphi = id.$$

Consequently the map φ *is a section and* π *is the corresponding retraction.*

Having now fixed a filtration on a ring R, we consider now a finitely generated left R-module M, that is generated by $\{h_1, \ldots, h_q\}$, such that M inherits a filtration on R in the natural way, by setting

$$M_r := R_r h_1 + \ldots + R_r h_q, \qquad r \in \mathbb{N}^p.$$

Definition 14 (Filtered Module)**.** Let M be a left R-module. A p-fold filtration of the module M with respect to the p-fold filtered ring R is a family

$$M_r \subseteq M, \qquad r \in \mathbb{N}^p$$

of additive subgroups of M with the properties

1. $R_r \cdot M_s \subseteq M_{r+s}, \qquad r, s \in \mathbb{N}^p$;
2. $M_r \subseteq M_s, \qquad r \leq_\pi s \in \mathbb{N}^p$;
3. $M = \bigcup_{r \in \mathbb{N}^p} M_r$.

M together with such a filtration is called a *filtered module* over the filtered ring R.

Notation. If X is an arbitrary subset of a filtered module

$$M = \bigcup_{r \in \mathbb{N}^p} M_r$$

wet set

$$X_r := X \cap M_r, \qquad r \in \mathbb{N}^p.$$

Definition 15 (Artinian Ring)**.** We call a ring *left-artinian* if and only if it satisfies the descending chain condition on left-ideals (i.e. every strictly descending chain of left-ideals must terminate).

Definition 16 (Noetherian Module)**.** A left R-module M is called *noetherian* if and only for submodules of M the ascending chain condition holds (i.e. every strictly ascending chain of submodules must terminate).

In the following Lemma we want to give an intuition about the nature of noetherian modules. An overview is given in [RM87].

Lemma 19 (Characterization of noetherian modules)**.**
The following statements are equivalent:

- M *is a noetherian* R*-module;*

- *Every submodule of M is finitely generated;*
- *Every collection of submodules of M has a maximal element;*

Moreover, for given a filtered ring R and a filtration M_r of the R-module M

- *If R is a noetherian ring and M is a finitely generated R-module, then M is noetherian;*
- *For each $r \in \mathbb{N}^p$, M_r is a left R_0-module.*

Obviously, if M is a left module over a left-artinian ring R then M is noetherian is equivalent to saying M is a finitely generated left R-module, and satisfies the ascending chain condition.

Example 10. *We consider, the skew polynomial ring $S = R[x; \sigma, \delta]$ where x acts on R by*

$$x \cdot r = \sigma(x) \cdot x + \delta(r), \qquad \delta(rs) = \sigma(r)\delta(s) + s\delta(r), \qquad r, s \in R.$$

In [RM87, Theorem 2.9 (iv)], it is shown [1], that if σ is an automorphism and the ring R is (left-) noetherian, then S is (left-) noetherian. But remark, that σ bijective is an additional assumption, as condition (1.29) *only allows to deduce that σ is injective. However, in applications, the assumption that σ is an automorphism is often fulfilled, and induction allows to prove a multivariate analog of Hilbert's basis theorem (Theorem 7).*

A p-fold filtration of R extends naturally to a p-fold filtration on free modules: Let R be a filtered ring, and

$$F = Re_1 \oplus \ldots \oplus Re_q$$

the free R-module on the set $E := \{e_1, \ldots, e_q\}$. Then,

$$F_r = R_r e_1 \oplus \ldots \oplus R_r e_q, \qquad r \in \mathbb{N}^p, \tag{2.1}$$

defines a filtration of the free module F. If a filtration is monomial with respect to the basis $\mathbb{M}$, then so is the extended filtration of F with respect to the basis $\mathbb{M}E$, meaning that always

$$f \in F_r \Rightarrow \mathrm{T}(f) \subseteq F_r.$$

By using above notation, we immediately get that if $R = K^{(\mathbb{M})}$ then $F = R^{(E)} = K^{(\mathbb{M}E)}$.

Example 11 (Filtration on free D-module F)**.**
We extend the order functions of the difference-differential ring D to the free module $F = D^q$: For $\lambda e \in \Lambda_{m,n}E$ and $\nu = 1, 2$ let

$$|\lambda e|_\nu := |\lambda|_\nu$$

[1] While in [RM87] the skew-polynomial ring $S = R[x; \sigma, \delta]$ is considered as right R-module, i.e. elements in S are written as $\sum_{k=0}^{n} x^k b_k$ with b_k in R, we reformulate their approach to left-modules.

and for a module element let

$$|f|_\nu := \max\{|t|_\nu : t \in \mathrm{T}(f)\}, \qquad f = \sum_{t \in \Lambda_{m,n} E} f_t t \in F.$$

This gives the extended filtration on F - for $r, s \in \mathbb{N}$

$$F_{r,s} = D_{r,s}e_1 \oplus \cdots \oplus D_{r,s}e_q = \{f \in F : |f|_1 \leq r \wedge |f|_2 \leq s\}.$$

From $|e_j|_\nu = 0$ it is clear that $E \subseteq F_{0,0}$ whence $(F_{r,s})$ is a bivariate filtration. Since the ring filtration is monomial, the extended filtration is so too. From that, we see that the following is equivalent

- $f \in F_{r,s}$
- $\forall t \in \mathrm{T}(f):\ |t|_1 \leq r \wedge |t|_2 \leq s$
- $\mathrm{T}(f) \subseteq F_{r,s}$

Let R be a filtered ring, and M, N filtered R-modules. An R-homomorphism $\varphi \colon M \to N$ is called a *filter respecting homomorphism* (or simply a *morphism*) if it respects the filter structure, that is, if

$$\varphi(M_r) \subseteq N_r \qquad r \in \mathbb{N}^p.$$

A morphism induces R_0-linear maps $M_r \to N_r$ for all $r \in \mathbb{N}^p$.

Lemma 20 (Homomorphic images of Filtration)**.**
Let R be a filtered ring and $\varphi \colon M \to N$ a homomorphism of R-modules.

1. *If M is filtered over R then $\mathrm{im}(\varphi)$ is filtered by setting $\mathrm{im}(\varphi)_r = \varphi(M_r)$. φ is then a morphism $M \to \mathrm{im}(\varphi)$.*
2. *If N is filtered over R then M is filtered by setting $M_r = \varphi^{-1}(N_r)$. φ is then a morphism $M \to N$.*

Thus, each finitely generated R-module $M = Rh_1 + \cdots + Rh_q$ inherits a filtration by first extending the family R_r to the free module $F \cong R^q$ and then pushing down with a map

$$\pi \colon F \to M, \qquad e_i \mapsto \pi(e_i) := h_i, \qquad 1 \leq i \leq q.$$

By specializing Lemma 20 to inclusion $N \hookrightarrow M$ any submodule $N \subseteq M$ naturally inherits a filtration from M via

$$N_r = N \cap M_r.$$

2.2. Reduction Relations

The key ingredient of Gröbner basis techniques is the notion of reduction. A reduction relation ρ on X is a binary relation $\rho \subseteq X \times X$. We write $f \longrightarrow h$ to indicate that $(f, h) \in \rho$, and $f \longrightarrow^{\star} h$ when there is a chain of finite length

$$f = f_0 \longrightarrow f_1 \longrightarrow \cdots \longrightarrow f_k = h, \qquad k \in \mathbb{N}.$$

Note, that also $k = 0$ is allowed in this setting, that indicates that f reduces to itself. An equivalent characterization is given by

$$f \longrightarrow^{\star} h :\Leftrightarrow (f, h) \in \bigcup_{k \in \mathbb{N}} \rho^k.$$

We give now a couple of examples of reduction relations.

Example 12 (Polynomial reduction)**.**
For the multivariate polynomial ring over the field $\mathbb{K}$, the polynomial f reduces modulo g to h in one step, if and only if a monomial $m \in T^n(X)$ exists such that

$$\mathrm{LT}(f) = \mathrm{LT}(m \cdot g) \quad \wedge \quad h = f - m \cdot g, \qquad f, g, h \in R,\ m \in T^n(X).$$

Again, $T^n(X)$ is the commutative semigroup of power products in $x_1, \ldots, x_n$.

Winkler and Zhou [ZW08a] give a reduction relation for the ring D.

Example 13 (Relative reduction)**.**
Let $\prec$ and $\prec'$ be generalized term orders on $\Lambda_{m,n}E$, F a finitely generated free D-module. For $f, g, h \in F$,

$$\begin{aligned} f \xrightarrow{\mathrm{rel}}_g h \quad \Leftrightarrow \quad \exists \lambda \in \Lambda_{m,n} :\ & \mathrm{LT}_{\prec}(\lambda g) = \mathrm{LT}_{\prec}(f) \ \wedge\ \mathrm{LT}_{\prec'}(\lambda g) \preccurlyeq' \mathrm{LT}_{\prec'}(f) \quad \wedge \\ & h = f - \mathrm{LC}_{\prec}(f)/\mathrm{LC}_{\prec}(\lambda g) \cdot \lambda g. \end{aligned}$$

Therefore, writing $\longrightarrow_g$ for ordinary leading term reduction w.r.t. $\prec$ by g, we obtain

$$f \xrightarrow{\mathrm{rel}}_g h :\Leftrightarrow\ f \longrightarrow_g h \ \wedge\ \mathrm{LT}_{\prec'}(\lambda g) \preccurlyeq' \mathrm{LT}_{\prec'}(f).$$

With I we denote the set of ρ-irreducible elements, that is

$$I := \{x \in X : \ \nexists y \in X \text{ such that } (x, y) \in \rho\}. \tag{2.2}$$

A subset $Y \subseteq X$ is called ρ-stable if $y \in Y$ and $y \longrightarrow z$ implies that $z \in Y$.

For a Gröbner reduction, we will later on consider stable reduction relations on filtered free modules.

If $\rho \subseteq M \times M$ is a relation on a R-module M, then for $k \in \mathbb{N}$ we set

$$Z_k := \{f : \ (f, 0) \in \rho^k\}, \qquad Z_{\leq k} := \bigcup_{l \leq k} Z_l, \qquad Z := \bigcup_{k=0}^{\infty} Z_k, \tag{2.3}$$

i.e. the set Z_k holds the elements $f \in M$ that reduce to zero in k steps, the set $Z_{\leq k}$ hold elements that reduce in *at most* k steps to zero, and Z all elements that reduce to 0. It is plain, that

$$Z = \bigcup_{k=0}^{\infty} Z_{\leq k} = \{f \in M : \ f \longrightarrow^{\star} 0\}.$$

Obviously $0 \in Z$, because 0 reduces in zero steps to 0, hence $0 \in Z_0 \subseteq Z$.

In the following we set up a list of axioms, which make a relation appropriate for reducing module elements to normal forms.

Definition 17 (Strong Reduction)**.**
Let M be an R-module, $N \subseteq M$ a submodule and ρ a binary relation on M. ρ is called a *reduction for* N provided that

1. ρ is noetherian, i.e. every sequence

$$f_1 \longrightarrow f_2 \longrightarrow \cdots$$

   terminates;

2. the set of irreducibles I (compare (2.2)) is a monomial $\mathbb{K}$-linear subspace of M, that is, I is a $\mathbb{K}$-vector space and

$$\forall f \in M : \ f \in I \Rightarrow \mathrm{T}(f) \subseteq I;$$

3. $f \longrightarrow h \Rightarrow f \equiv h \pmod{N}$;

ρ is called a *strong* reduction for N if it satisfies in addition

4. $I \cap N = 0$, that is, every non-zero element in N is reducible.

Lemma 21. *A relation that satisfies axioms 1-4, i.e. a strong reduction, is noetherian and confluent*[2]*.*

Proof. The case where $f \in I$ is trivial, since f can't be reduced further. On the other hand, if $f \in N$ it reduces by axiom 4. in a finite number of steps (axiom 1.) to an irreducible element i (which we call normal form). Obvious, $i \in I$ and by axiom 3. it is in N. Therefore we conclude $i = 0$. Suppose, there exist two different normal forms i_1 and i_2, then their difference $i_1 - i_2$ is contained in $I \cap N$ and therefore zero. This proves that $i_1 = i_2$, or what is equivalent, the reduction relation $\longrightarrow$ is confluent. □

[2]A reduction relation $\rho \subseteq X \times X$ is called *confluent* if and only if for all $a, b, c \in X$ with b not equal to c, such that $a \to b$ and $a \to c$, implies the existence of $d \in X$ such that $b \to^{\star} d$ and $c \to^{\star} d$. A noetherian reduction relation that is confluent is said to have the *Church-Rosser property.* In literature, this property is sometimes also called *diamond property.* The theory of polynomial Gröbner basis is developed upon this presentation in [Win96].

Notation. If F is a free module, and $\longrightarrow$ is strong reduction, we will write $\mathrm{NF}(f)$ for the unique normal form of $f \in F$. Thus, we always have

$$f \longrightarrow^{\star} \mathrm{NF}(f).$$

In the following we will characterize a reduction relation by describing the behavior on arbitrary modules.

Lemma 22 (Additive Decomposition by strong reduction)**.**
Let M be an R-module, $N \subseteq M$ be a submodule of M, and the relation $\rho \subseteq M \times M$ be a (noetherian) reduction on M. Then, we have

1. $M = N + I$*;*
2. $I \cap N \subseteq 0 \Leftrightarrow Z = N$*.*

Consequently, if F is a free module and ρ is a strong reduction for $N \subseteq F$, then

$$F = N \oplus I \quad \text{ and } \quad Z = N.$$

Proof. The additive decomposition is plain. To prove that $I \cap N \subseteq 0 \Rightarrow Z = N$, observe that by axiom 1. and 3. the zero set Z is contained in N. Assume now, that $I \cap N \subseteq 0$, and let $n \in N$. Then, there exists an irreducible element $r \in N$ such that $n \longrightarrow^{\star} r$. Thus, $r \in I \cap N \subseteq 0$ and so $n \longrightarrow^{\star} 0$, i.e. $n \in Z$. For the converse, assume that $x \in I \cap N$, $x \longrightarrow^{\star} 0$ and x is irreducible. Therefore $x = 0$. Consequently $I \cap N \subseteq 0$. □

One might is now led to the question how such a reduction relation might look like. We want to stay as general as possible, but give the reader the intuition that this concept is actually available on an arbitrary module.

Lemma 23 (Existence of strong reduction)**.**
Let N be an arbitrary submodule of the free R-module F. Then, there exists a strong reduction for N.

Proof. Assume that $N \subset F$ whence $\mathbb{M}E \not\subseteq N$. Choose a set S being maximal in the non-empty inductively ordered set $\{T \subseteq \mathbb{M}E : \mathbb{K} \cdot T \cap N = 0\}$. Put $C = \mathbb{K} \cdot S$. Obviously

$$F = \mathbb{K}^{(\mathbb{M}E)} = N \oplus C,$$

so consider projection $p_C : N \oplus C \longrightarrow C$ and define a reduction relation

$$\rho = \{(f, h) \in F \times F : f \notin C \wedge h = p_C(f)\}.$$

It is clear that ρ terminates. The set I of ρ-irreducible elements in C which is a monomial $\mathbb{K}$-linear space. If $f \longrightarrow h$ then

$$f = n + c \in N \oplus C$$

and $h = c$ which shows that $f \equiv h \pmod{N}$. Finally, $I \cap N = C \cap N = 0$, and thus ρ is a strong reduction for N. □

For applications one might wants to consider restricted reduction relations, where we do not leave a subspace of the considered module. This is of particular interest for filtered modules. Therefore we conclude the definition started before by giving the defining property for Gröbner reduction.

Definition 17 (Continued)**.** Let R be a filtered ring, F the free R-module generated by $E := \{e_1, \dots, e_q\}$. Let $N \subseteq F$ be a submodule. A strong reduction $\rho \subseteq F \times F$ for N is called a *Gröbner reduction* for N if it satisfies the axiom

5. F_r is ρ-stable for all $r \in \mathbb{N}^p$.

Lemma 24 (Additive Decomposition by Gröbner reduction)**.**
Let R be a filtered ring, F the free R-module generated by $\{e_1, \dots, e_q\}$. Let $N \subseteq F$ be a submodule, $\rho \subseteq F \times F$ be a Gröbner reduction. Then,

$$F_r = N_r + I_r, \qquad r \in \mathbb{N}^p. \tag{2.4}$$

Consequently

$$F = N \oplus I, \qquad F_r = N_r \oplus I_r, \qquad r \in \mathbb{N}^p. \tag{2.5}$$

Proof. Let $f \in F_r$. Reduce f to normal form $f \longrightarrow^\star h = \mathrm{NF}(f)$. By axiom 3. we have that $f \equiv h \pmod N$, therefore $f - h = n \in N$. By axiom 5. $h \in F_r$. Thus, $h \in I \cap F_r = I_r$. As both, f and h are in F_r, so is n. Therefore, $f = n + r \in N_r + I_r$. □

Remark. Equation (2.4) corresponds to 'division with remainder' in the classical theory. Similar, equation (2.5) describes 'uniqueness of normal forms' in Gröbner basis computation.

Lemma 25 (Monomial submodules and Gröbner reduction)**.**
Assume that the set of monomials $\mathbb{M}$ in R is a monoid. Let $N \subseteq F$ be a monomial submodule (i.e N is generated by a subset of $\mathbb{M}E$). Choose a monomial $\mathbb{K}$-linear complement I of N in F (e.g., $I = \mathbb{K} \cdot S$ where $S = \{t \in \mathbb{M}E \colon t \notin N\}$). Let p_I denote projection $N \oplus I \longrightarrow I$ and let $\rho \subset F \times F$ be the relation

$$\rho = p_I|_{F \setminus I}.$$

Then, with arbitrary monomial filtration, ρ is a Gröbner reduction for N.

Proof. Let N be generated by $X \subseteq \mathbb{M}E$. The general element of N is $n = \sum_{x \in X} a_x x$. The elements $a_x \in R$ are

$$a_x = \sum_{\mathfrak{m} \in \mathbb{M}} a_x^{\mathfrak{m}} \cdot \mathfrak{m}, \quad a_x^{\mathfrak{m}} \in \mathbb{K} \qquad \text{whence } n = \sum_{x \in X} \sum_{\mathfrak{m} \in \mathbb{M}} a_x^{\mathfrak{m}} \cdot \mathfrak{m}x. \tag{2.6}$$

Since $\mathbb{M}\mathbb{M} \subseteq \mathbb{M}$, the expressions $\mathfrak{m}x$ are monomials in $\mathbb{M}E$. After (possibly) some cancellations, equation (2.6) results in the unique representation of n as $\mathbb{K}$-linear combination of $\mathbb{M}E$. Since each surviving term is a (monomial) multiple of a generator monomial of N, it is in N, this means, N is a monomial module.

Let $S = \{t \in \mathbb{M}E \colon t \notin N\}$, and let $I = \mathbb{K} \cdot S$, the vector space generated by elements from S. By construction, I is a *monomial* subspace of F.

Evidently $N \cap I = 0$.

Write $f \in F$ as $\mathbb{K}$-linear combination of elements of $\mathbb{M}E$. We may split this expression as

$$f = \sum_{t \in S} f_t t + \sum_{t \notin S} f_t t, \qquad f_t \in \mathbb{K}$$

which shows that $f \in N + I$. Consequently $F = N \oplus I$. The relation ρ results in

$$\rho \colon f \longrightarrow h \Leftrightarrow f \in F \setminus I \wedge h = p_I(f)$$

Thus, with exception of elements in I, every $f \in F$ reduces to normal form in one step. If $f \longrightarrow h$ then $f \in F \setminus I$ and $h = p_I(f) = p_I(n + r) = r$; thus, $f - h = n \in N$, i.e., $f \equiv h \mod N$. Consequently ρ is a strong reduction for N.

Let $f \in F_r$ and $f \longrightarrow h$. By monomiality of F_r, $\mathrm{T}(f) \subseteq F_r$. Because $f = n + h$ is a direct decomposition, it follows that $\mathrm{T}(h) \subseteq F_r$. Consequently $h \in F_r$ and ρ is a Gröbner reduction. □

2.3. Computation of Dimension in Finitely Generated Modules

The task of computing the dimension in finitely generated modules of differential and difference-differential operators has been considered by many researchers.

But already the first legitimate question in this context "What Dimension is actually meant?" is not obviously answered. While in general topological spaces X the *Krull-Dimension* appears (the supremum over all lengths of chains $(X_0, X_1, \ldots, X_n)$, where $X_k \subseteq X$, ordered by inclusion), in multi-filtered rings the *Gelfand-Kirillov Dimension* [Tor99] is an important quantity.

Let $\mathbb{K}$ be a field of characteristic zero. Previously, we have considered filter-spaces F_r which are subspaces of F and form a $\mathbb{K}$-vector space. As a $\mathbb{K}$-vector space, we have a basis, the number of $\mathbb{K}$-linear independent basis elements is called the vector space dimension. Vector space dimension is described by the *Hilbert-Function*. When talking about dimension in this section we mean vector-space dimension over the field $\mathbb{K}$, and restrict our attention to finite-dimensional spaces. For an algebra over a field, the dimension as vector space is finite if and only if its Krull dimension is 0. We will denote dimension of the vector space V over $\mathbb{K}$ by the symbol $\dim_{\mathbb{K}}(V)$.

In differential algebra, the notion of differential dimension polynomial was introduced by Kolchin [Kol64]. This dimension polynomial carries certain invariants, and its computation has been addressed for the last 50 years. Methods for its computation include characteristic sets as well as Gröbner basis techniques. After all, the differential dimension polynomial is obtained from a differential ring equipped with an univariate filtration, namely the order of the differential operator. This was the starting point of considerations in different base rings (that we've considered in the introduction), for a survey on related techniques see [KLAV98, Lev07, ZW08a, ZW08b, Lev12].

In this section, we let the commutative subring $K \subseteq R$ contained in R be a field $\mathbb{K}$ of characteristic zero.

Theorem 12 (Main Theorem on Dimension)**.**
Let $M = Rm_1 + \cdots + Rm_q$ be a finite R-module with free presentation

$$0 \longrightarrow N \longrightarrow F \xrightarrow{\pi} M \longrightarrow 0$$

where $F = R^q$. Assume given a strong reduction for N with set of irreducibles I. Let $V \subseteq F$ be a monomial $\mathbb{K}$-linear subspace that is ρ-stable and let U be the set of irreducible monomials in V. Then $\pi(U)$ is a $\mathbb{K}$-vector space bases for $\pi(V)$. In particular we obtain that

$$\dim_{\mathbb{K}} \pi(V) = |\pi(U)| = |U|.$$

Proof. Let $f, h \in I$. Then $\pi(f) = \pi(h)$ implies that $f - h \in N \cap I = 0$ whence $\pi|I$ is injective. Since $U = I \cap \mathbb{M}E \cap V \subseteq I$ it is plain that $\pi|U$ is injective, whence $|\pi(U)| = |U|$. Let

$$\sum_j c_j \pi(\mu_j) = 0, \qquad c_j \in \mathbb{K},\, \mu_j \in I \cap \mathbb{M}E.$$

Then $\sum_j c_j \mu_j \in N \cap I = 0$. Therefore $c_j = 0\ \forall j$. This demonstrates that $\pi(I \cap \mathbb{M}E)$ is $\mathbb{K}$-linearly independent. Thus $\pi(U) \subseteq \pi(I \cap \mathbb{M}E)$ is linearly independent. Now we may reduce elements $f \in F$ until an irreducible r is reached. Doing this for elements $f \in V$ and taking into account that the reduction stays inside V we obtain an irreducible $r \in V$. Thus

$$\forall f \in V\, \exists r \in I \cap V \text{ with } \pi(r) = \pi(f).$$

Now take $m \in \pi(V)$. $\exists f \in V$ with $m = \pi(f)$. Choose $r \in I \cap V$ with $\pi(r) = \pi(f)$,

$$r = \sum_j c_j \mu_j, \qquad c_j \in \mathbb{K}, \mu_j \in \mathbb{M}E.$$

Since V is monomial, all μ_j are in V and because $r \in I$, all terms of r must be in I. Therefore

$$\mu_j \in V \cap \mathbb{M}E \cap I = U \qquad \forall j.$$

Consequently

$$m = \pi(r) = \sum_j c_j \pi(\mu_j) \in \mathbb{K} \cdot \pi(U).$$

So $\langle \pi(U) \rangle_{\mathbb{K}} = V$ and $\pi(U)$ is a $\mathbb{K}$-basis. $\square$

In the general situation consider a finite module M over an arbitrary monomially filtered ring R

$$R = \bigcup_{r \in \mathbb{N}^p} R_r \quad M = Rm_1 + \cdots + Rm_q.$$

Choose a free presentation

$$0 \longrightarrow N \longrightarrow F \xrightarrow{\pi} M \longrightarrow 0 \tag{2.7}$$

with $F = R^q$. We get the following corollary.

Corollary 6 (Dimension in Filterspaces).
Let F be equipped with extended filtration from R and consider M with the filtration $M_r = \pi(F_r)$. For $r \in \mathbb{N}^p$ let U_r be the set of irreducible monomials in the filter space F_r. Assume given a Gröbner reduction for N. Then the sets $\pi(U_r)$ provide $\mathbb{K}$-vector space bases for the spaces M_r. In particular

$$\dim_{\mathbb{K}} M_r = |\pi(U_r)| = |U_r| \qquad r \in \mathbb{N}^p.$$

Proof. Apply Theorem 12 with $V = F_r$. □

Combining this corollary with Lemma 25 gives:

Corollary 7 (Dimension in Monomial Submodules).
Assume that the monomials in R form a multiplicative monoid, let $N \subseteq F$ be a monomial submodule. Let $S = \{t \in \mathbb{M}E \colon t \notin N\}$. Then, for arbitrary monomial filtration R_r and extended filtration F_r, we have

$$\dim_{\mathbb{K}} \left(F/N\right)_r = |S_r|.$$

Proof. Let $I = \mathbb{K} \cdot S$. Then $I \cap \mathbb{M}E = S$ and thus $I \cap \mathbb{M}E \cap F_r = S \cap F_r = S_r$. Using Corollary 6 proves the assertion. □

Let us consider a degenerate case, where the filtration F_r is chosen trivial by setting for all $r \in \mathbb{N}^p$ the filter space $F_r = F$. Careful specialization of Corollary 6, gives us the following interesting result.

Corollary 8 (Irreducible Monomials provide Basis).
Let F be the free module $F = \mathbb{K}^{(\mathbb{M}E)}$, and let I denote the irreducible monomials in F. Choose the free presentation (2.7). Then, the monomials $\pi(I \cap \mathbb{M}E)$ provide a linear independent $\mathbb{K}$-basis of the R-module M.

Proof. To prove $\pi(I \cap \mathbb{M}E)$ is $\mathbb{K}$-linear independent let (for $\lambda_k \in I \cap \mathbb{M}E$)

$$\sum_{k=0}^{n} c_k\, \pi(\lambda_k) = \pi\left(\sum_{k=0}^{n} c_k\, \lambda_k\right) = 0 \Rightarrow \sum_{k=0}^{n} c_k\, \lambda_k \in N \cap I \Rightarrow \sum_{k=0}^{n} c_k\, \lambda_k = 0,$$

and therefore $c_k = 0$ for all k.

To show that $\pi(I \cap \mathbb{M}E)$ spans M consider $y \in M$, hence there exists a sequence of coefficients $c_k \in \mathbb{K}$ and $\lambda_k \in I \cap \mathbb{M}E$ such that

$$y \in M \Rightarrow y = \pi\left(\sum_{k=0}^{n} c_k \lambda_k\right) = \sum_{k=0}^{n} c_k \pi(\lambda_k).$$

The claim follows. □

We now consider the situation as in Theorem 12, that is, we are concerned with the free presentation (2.7). To compute $\dim_{\mathbb{K}} \pi(V)$ with $V \subseteq F$ where V is a monomial set, we can now use the following algorithm (formulated by Günter Landsmann). The algorithm presumes that the monomials contained in V, namely $V \cap \mathbb{M}E$, can be finitely enumerated, i.e. there exist finitely many monomials $\mathfrak{m} \in V \cap \mathbb{M}E$.

Algorithm 1 Compute $\dim_{\mathbb{K}} \pi(V)$

Require:
  $R = \mathbb{K}^{(\mathbb{M})}$;
  F a free R-module generated by $E := \{e_1, \ldots, e_q\}$;
  $V \subseteq F$ a monomial subset;
Ensure: $\dim_{\mathbb{K}} \pi(V)$
  $d \leftarrow \dim_{\mathbb{K}}(V)$;
  $B \leftarrow \emptyset$;
  order the set $V \cap \mathbb{M}E : V \cap \mathbb{M}E = \{s_j : 1 \leq j \leq d\}$;
  for $1 \leq j \leq d$ **do**
    compute $\mathrm{NF}(s_j)$ and write it as $\mathbb{K}$-linear combination of monomials

$$\mathrm{NF}(s_j) = \sum_{t \in \mathbb{M}E} s_t^j \cdot t$$

    $B \leftarrow B \cup \mathrm{T}(\mathrm{NF}(s_j))$ where $\mathrm{T}(\mathrm{NF}(s_j)) = \{t : s_t^j \neq 0\}$;
  $N \leftarrow |B|$;
  order the set $B : B = \{t_i : 1 \leq i \leq N\}$;
  construct the $d \times N$ matrix $M := (M_{i,j})$ with $M_{i,j} = s_{t_i}^j$;
    ▷ The j-th row of M is the coordinate vector of s_j in the basis B
  return $\mathrm{rk}(M)$

Theorem 13. *Algorithm 1 is correct.*

Proof. Since V is monomial, we have $V = \mathbb{K} \cdot (V \cap \mathbb{M}E)$. Therefore

$$\pi(V) = \pi(\mathbb{K} \cdot (V \cap \mathbb{M}E)) = \mathbb{K} \cdot \pi(V \cap \mathbb{M}E),$$

so the set of all $\pi(t)$ with $t \in V \cap \mathbb{M}E$ generates $\pi(V)$ as $\mathbb{K}$-module. Obviously, we may replace each such t by its normal-form $\mathrm{NF}(t)$, i.e. $\pi(V)$ is $\mathbb{K}$-generated by the set

$$\{\pi(\mathrm{NF}(t)) : t \in V \cap \mathbb{M}E\},$$

and this set must contain a $\mathbb{K}$-basis as a subset. The rank is now the desired dimension. □

2.4. Example from Physics

2.4.1. The Wave-Equation

The wave-equation is an example of a hyperbolic linear partial differential equation of second order. It can be used, to model physical phenomena such as Hooke's Law, appearing at the modelling of the stiffness of a spring. The homogeneous wave-equation in d-dimensional space is given by

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \nabla_x^2 u = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \sum_{i=1}^{d}\left(\frac{\partial^2 u}{\partial x_i^2}\right) = 0, \qquad c > 0. \tag{2.8}$$

We let ∇_x^2 denote the *Laplace-Operator* applied to the spatial coordinates $x_1, \dots, x_d$, which is usually denoted by Δ. However, we've reserved Δ for the set of derivations of a (difference-) differential ring, that's why we've decided to use this notion. Without loss of generality, we assume that $c = 1$, which can be achieved by substituting $t' = c \cdot t$.

Let's model this equation in terms of a module of differential operators over a differential field. To that end, we consider the differential field $\mathbb{K}(x_1, \dots, x_d, t)$ with set of derivations $\Delta := \{\delta_{x_1}, \dots, \delta_{x_d}, \delta_t\}$, where δ_k acts by derivation w.r.t. k, and consider the cyclic module that is generated by the wave-equation, i.e. we consider the cyclic free module generated by

$$g_d := \delta_t^2 - \sum_{i=1}^{d}\delta_{x_i}^2, \tag{2.9}$$

that corresponds to (2.8). Let Θ denote the commutative semigroup generated by Δ, we define on Θ the map

$$\mathrm{ord}_\Delta : \Theta \to \mathbb{N}, \qquad \delta_{x_1}^{k_1}\dots\delta_{x_d}^{k_d}\delta_t^{k_t} \mapsto \mathrm{ord}_\Delta(\delta_{x_1}^{k_1}\dots\delta_{x_d}^{k_d}\delta_t^{k_t}) := k_1 + \ldots + k_d + k_t, \tag{2.10}$$

and extend to the ring of differential operators R, as in Definition 12.

Lemma 26. *The family of subsets*

$$R_k := \{f \in R : \mathrm{ord}_\Delta(f) \le k\} \tag{2.11}$$

forms a filtration on R, i.e. $\mathrm{ord}_\Delta(\cdot)$ is a filter valuation in the sense of Definition 12.

Proof. As it is easily seen (for example at (1.19) with $\sigma = \mathrm{id}$), we have for the choice $\theta_1 = \delta^m, \theta_2 = \delta^n$ where $m, n \in \mathbb{N}^{d+1}$, and for $c \in \mathbb{K}\backslash\{0\}$ that

$$\theta_1 \cdot c \cdot \theta_2 = c \cdot \theta_1\theta_2 + \sum_{l \lneq_\pi m} c_l \cdot \delta^l \cdot \theta_2,$$

the coefficients c_l might zero. In particular we have

$$\mathrm{ord}_\Delta(\theta_1 \cdot c \cdot \theta_2) = \mathrm{ord}_\Delta(\theta_1) + \mathrm{ord}_\Delta(\theta_2).$$

From that, we obtain by Lemma 15 that for all $a, b \in R$ we have

$$\mathrm{ord}_\Delta(ab) \leq \mathrm{ord}_\Delta(a) + \mathrm{ord}_\Delta(b) \Rightarrow R_r R_s \subseteq R_{r+s}.$$

The remaining properties are obvious. □

The same reasoning can be applied to other settings.

Example 14 (Example for Filter-valuations)**.** *The following maps are examples for filter valuations:*

- *The total degree* $\deg(\cdot)$ *at the ring of commutative polynomials;*
- *The degree* $|\cdot|_{\mathcal{O}_i}$ *w.r.t. a partition of Ore-variables* $\mathcal{O} := \{\partial_1, \ldots, \partial_n\}$ *into* p *disjoint subsets* $\mathcal{O}_1, \ldots, \mathcal{O}_p$*.*

Computation of the Univariate Hilbert Function

Suppose, we want to find the univariate dimension polynomial for the cyclic module generated by g_d as in (2.9). If R denotes the ring of differential operators of the differential field $(\mathbb{K}(x_1, \ldots, x_d, t), \Delta)$, we have the exact sequence

$$0 \longrightarrow Rg_d \longrightarrow R \xrightarrow{\pi} R/Rg_d \longrightarrow 0, \tag{2.12}$$

and we consider the reduction relation

$$f \longrightarrow h :\Longleftrightarrow h = f - c \cdot \delta_t^{k_t} \delta_{x_1}^{k_1} \ldots \delta_{x_d}^{k_d} \cdot g_d \;\wedge\; \mathrm{LT}(f) = \mathrm{LT}(\delta_t^{k_t} \delta_{x_1}^{k_1} \ldots \delta_{x_d}^{k_d} \cdot g_d). \tag{2.13}$$

By Corollary 6 we need to find a closed form representation for the irreducible monomials. We fix an order on the indeterminates by ranking

$$\delta_t > \delta_{x_1} > \ldots > \delta_{x_d}. \tag{2.14}$$

An element is reducible with respect to the wave-equation if the order in the first ranked variable δ_t is ≥ 2, therefore the irreducibles are those which have order in δ_t equal to zero or one. Hence, a possible description of the irreducible monomials contained in $\Theta \cap R_r$ is given by

$$|I \cap \Theta \cap R_r| = \left\{ (0, k_1, \ldots, k_d) : \sum_{i=1}^{d} k_i \leq r \right\} \cup \left\{ (1, k_1, \ldots, k_d) : 1 + \sum_{i=1}^{d} k_i \leq r \right\}.$$

But the cardinality of this sets can be given explicit as univariate polynomial in r, by using Lemma 4. With that, we find that

$$\phi(r) = \binom{r+d}{d} + \binom{r+d-1}{d}, \qquad r \geq 0, d \geq 1. \tag{2.15}$$

In particular, for $d = 3$ we obtain the result

$$\phi(r) = \binom{r+3}{3} + \binom{r+3-1}{3} = \binom{r+3}{3} + \binom{r+2}{3},$$

which happens to coincide with [KLAV98, Example 5.7.5.], the minimal differential dimension polynomial of the wave-equation.

Computation of the Multivariate Hilbert Function

In this paragraph, we will consider a multi-filtered version and therefore a multivariate generalization of result (2.15).

If we make a full partition of the set Δ, i.e. we decompose the set Δ with $d+1$ elements $\{\delta_{x_1}, \ldots, \delta_{x_d}, \delta_t\}$ into exactly $d+1$ sets, each containing one δ_{x_i} (respectively one containing δ_t), we get for g_d a $(d+1)$-variate Hilbert function, that is, we consider the $(d+1)$-fold filtration

$$R_{r_1,\ldots,r_{d+1}} := \{f \in R : \operatorname{ord}_{\delta_t}(f) \leq r_{d+1} \wedge \forall i : \operatorname{ord}_{\delta_{x_i}}(f) \leq r_i\}. \tag{2.16}$$

Let us consider the case $d = 1$, i.e. we consider $g_1 = \delta_t^2 - \delta_x^2$, that is, we have

$$(\mathbb{K}(x,t), \Delta), \qquad \Delta := \{\delta_t, \delta_x\}, \qquad R_{r_1,r_2} := \{f \in R : \operatorname{ord}_{\delta_t}(f) \leq r_1 \wedge \operatorname{ord}_{\delta_x}(f) \leq r_2\}.$$

The irreducibles in R are obviously given by

$$I_t = \{f \in R : \nexists h : f \longrightarrow h\} = \{f \in R : \operatorname{ord}_{\delta_t}(f) \leq 1\},$$

hence $I_t \cap Rg = 0$, because any multiple h of g satisfies $\operatorname{ord}_{\delta_t}(h) \geq 2$. When we now reduce by reduction relation (2.13), it is possible to give explicit the normal form. In particular, if we reduce $\delta_t^k \delta_x^l$ by g_1, we obtain

$$\delta_t^k \delta_x^l \longrightarrow \delta_t^k \delta_x^l - \delta_t^{k-2} \delta_x^l \cdot g_1 = \delta_t^{k-2} \delta_x^{l+2} \longrightarrow^\star \begin{cases} \delta_x^{k+l} & k \text{ even}, \\ \delta_t \delta_x^{k+l-1} & k \text{ odd}. \end{cases}$$

For the last step, we keep in mind that in $N = Rg_1$, we have $\delta_t^2 - \delta_x^2 = 0$, in particular $\delta_t^2 = \delta_x^2 \pmod{N}$.

Now we can consider the monomials $\delta_t^i \delta_x^j$ in the (i,j)-plane, and count the irreducible monomials contained in R_{r_1,r_2}. Suppose, we know that the irreducibles can be described by a polynomial. Interpolating this data, the bivariate dimension polynomial can be written explicit as

$$\phi(r_1, r_2) := \begin{cases} r_1 + 1, & r_2 = 0, \\ r_2 + 1, & r_1 = 0, \\ 2(r_1 + r_2), & r_1, r_2 \geq 1. \end{cases}$$

At the border $r_1 = 0$ or $r_2 = 0$, the result $2r_1 + 2r_2$ is not valid, which is (as in the case of univariate Hilbert polynomials over the graded polynomial ring) a matter of regularity.

For two variables we obtain by reduction the following

$$\delta_t^k \delta_x^l \delta_y^m \longrightarrow \delta_t^{k-2} \delta_x^{l+2} \delta_y^m + \delta_t^{k-2} \delta_x^l \delta_y^{m+2} =: h, \qquad k \geq 2.$$

It now might happen, that $k - 2 = \operatorname{ord}_{\delta_t}(h) \geq 2$. In that case, we can reduce the monomials $\delta_t^{k-2} \delta_x^{l+2} \delta_y^m$ and $\delta_t^{k-2} \delta_x^l \delta_y^{m+2}$ once more by g_2, until we arrive a linear combination of monomials $\mathfrak{m}_i$ where $\operatorname{ord}_{\delta_t}(\mathfrak{m}_i) \leq 1$.

Example 15. *For the monomial $\delta_t^5 \delta_x^4 \delta_y^3$ we have that*

$$\begin{tikzcd}[column sep=small]
 & \delta_t^5 \delta_x^4 \delta_y^3 \longrightarrow \delta_t^3 \delta_x^6 \delta_y^3 + \delta_t^3 \delta_x^4 \delta_y^5 \arrow[dl] \arrow[dr] & \\
\delta_t \delta_x^8 \delta_y^3 + \delta_t \delta_x^6 \delta_y^5 & & \delta_t \delta_x^6 \delta_y^5 + \delta_t \delta_x^4 \delta_y^7,
\end{tikzcd} \tag{2.17}$$

so we have

$$\delta_t^5 \delta_x^4 \delta_y^3 \longrightarrow^{\star} \mathrm{NF}(\delta_t^5 \delta_x^4 \delta_y^3) = \delta_t \delta_x^8 \delta_y^3 + 2 \cdot \delta_t \delta_x^6 \delta_y^5 + \delta_t \delta_x^4 \delta_y^7 \in I_t.$$

We apply this algorithm to the cyclic module generated by (2.9). Suppose, from some additional insight, we knew that we can express the Hilbert function by a multivariate polynomial. Let ϕ_k denote the multivariate dimension polynomial associated to g_k in $k + 1$ variables (i.e. we consider the cyclic R-module $N = Rg_k$). With Algorithm 1, we derive by interpolation of small data that for $r_i \geq 1$

$$\begin{aligned}
\phi_1(r_1, r_2) &= 2(r_1 + r_2), \\
\phi_2(r_1, r_2, r_3) &= 2(1 + r_1 r_2 + r_1 r_3 + r_2 r_3), \\
\phi_3(r_1, r_2, r_3, r_4) &= 2(r_1 + r_2 + r_3 + r_4 + r_1 r_2 r_3 + r_1 r_2 r_4 + r_1 r_3 r_4 + r_2 r_3 r_4).
\end{aligned}$$

Next, we will stay with (2.9), but split the indeterminates into the sets

$$\Delta_1 := \{\delta_t\}, \qquad \Delta_2 := \{\delta_{x_1}, \ldots, \delta_{x_d}\}, \tag{2.18}$$

where

$$\operatorname{ord}_{\Delta_1}(f) := \operatorname{ord}_{\delta_t}(f), \qquad \operatorname{ord}_{\Delta_2}(f) := \sum_{i=1}^{d} \operatorname{ord}_{\delta_{x_i}}(f), \qquad f \in R.$$

Motivation to consider this setting might be given by some physical interpretation, where time plays an exposed role compared to geometry. The partition of the set Δ gives rise the bivariate filtration:

$$R_{r_1, r_2} := \{f \in R : \operatorname{ord}_{\Delta_1}(f) \leq r_1 \wedge \operatorname{ord}_{\Delta_2}(f) \leq r_2\}. \tag{2.19}$$

Let now ψ_d denote the Hilbert function associated to filtration (2.19) with respect to $N = Rg_d$. Obviously, $\psi_1 = \phi_1$. Again, we apply Algorithm 1, do an interpolation of the resulting data, and obtain the following sequence of bivariate polynomials (we assume $r_i \geq 1$):

$$\begin{aligned}
\psi_1(r_1, r_2) &= 2(r_1 + r_2) \\
\psi_2(r_1, r_2) &= 1 + r_1 + r_2 + 2r_1r_2 + r_2^2 \\
\psi_3(r_1, r_2) &= 1 + (5/3)r_2 + r_2^2 + r_1 + (1/3)r_2^3 + 2r_1r_2 + r_1r_2^2
\end{aligned}$$

Let us identify the d-dimensional wave equation (2.9) by the vector

$$g_d \mapsto w(g_d) := (\mathrm{ord}_{\delta_t}(g_d), \mathrm{ord}_{\delta x_1}(g_d), \ldots, \mathrm{ord}_{\delta_{x_d}}(g_d)) = \underbrace{(2, \ldots, 2)}_{(d+1)-copies} .$$

Clearly, the reducible monomials $\mathfrak{m}$ w.r.t. the wave-equation satisfy $w(\mathfrak{m}) \geq_\pi w(g_d)$. We quote now the result [KLAV98, Proposition 2.2.11], that provides an explicit formula for the number of points in $\mathbb{N}^n$ less than $w(g_d)$ is given.

Theorem 14 ([KLAV98, Proposition 2.2.11]). *Let a partition*

$$\{1, \ldots, m\} = v_1 \cup \ldots \cup v_q, \qquad v_i \cap v_j = \emptyset \text{ for } i \neq j$$

be fixed, such that $m_i = |v_i|$ and $m = m_1 + \ldots + m_q$. For any subset

$$A := \{a_1, \ldots, a_n\} \subseteq \mathbb{N}^m, \qquad a_k = (a_{k1}, \ldots, a_{km}) \quad 1 \leq k \leq n,$$

such that $n \in \mathbb{N}^+$ let

$$A_r := \{(a_1, \ldots, a_m) \in A : \sum_{i \in v_k} a_i \leq r_k \ \forall k : 1 \leq k \leq q\}, \qquad r = (r_1, \ldots, r_q) \in \mathbb{N}^q.$$

Define the set of those elements not greater than or equal to any m-tuple from A w.r.t. product order, that is

$$U_A(r) := \{(u_1, \ldots, u_m) \in \mathbb{N}^m : \forall (a_1, \ldots, a_m) \in A_r \, \exists \, 1 \leq i \leq m : a_i > u_i\}$$

where $r = (r_1, \ldots, r_q) \in \mathbb{N}^q$, and denote

- *$S(\ell, n) :=$ the set of all ℓ-element subsets of $\{1, \ldots, n\}$;*
- *$\bar{a}_{\emptyset j} := 0$ and $\bar{a}_{\sigma j} := \max\{a_{vj} : v \in \sigma\}$ for $\sigma \in S(\ell, n)$ and $1 \leq j \leq m$;*
- *$b_{\sigma i} = \sum_{h \in v_i} \bar{a}_{\sigma h}$;*

Then, $|U_A(r_1, \ldots, r_q)|$ can be given as polynomial p in q variables by

$$|U_A(r_1, \ldots, r_q)| = p(r_1, \ldots, r_q) = \sum_{\ell=0}^{n} (-1)^\ell \sum_{\sigma \in S(\ell, n)} \prod_{i=1}^{q} \binom{t_i + m_i - b_{\sigma i}}{m_i},$$

hence, we have $\deg_{x_i}(p(r_1, \ldots, r_q)) \leq m_i$ *for $1 \leq i \leq q$, in particular* $\deg(p) \leq m$.

Example 16. *Let us apply Theorem 14 to g_d. As already mentioned before, the wave-equation can be identified by its exponent $g_d \mapsto w(g_d) = (2, \ldots, 2) \in \mathbb{N}^{d+1}$. If we partition $\Delta := \{\delta_t, \delta_{x_1}, \ldots, \delta_{x_d}\}$ into $(d+1)$-subsets as before, we get with the notation of Theorem 14 that $m_i = 1$ for $1 \leq k \leq d+1$ and*

$$a = (a_{11}, \ldots, a_{1(d+1)}) = (2, \ldots, 2) \in \mathbb{N}^{d+1}$$

and therefore

$$\bar{a}_{\emptyset k} = 0, \qquad b_{\sigma k} = \bar{a}_{\sigma k} = 2, \qquad 1 \leq k \leq d+1.$$

Hence, Theorem 14 provides an explicit closed formula for the dimension polynomial by

$$\phi_d(r_1, \ldots, r_{d+1}) = \prod_{i=1}^{d+1} \binom{r_i + 1 - 0}{1} - \prod_{i=1}^{d+1} \binom{r_i + 1 - 2}{1} = \prod_{i=1}^{d+1} (r_i + 1) - \prod_{i=1}^{d+1} (r_i - 1).$$

In the same manner, we can partition the set Δ as in (2.18), derive $m_1 = 1$ and $m_2 = d$,

$$g_d \mapsto w(g_d) = (\operatorname{ord}_{\Delta_1}(g_d), \operatorname{ord}_{\Delta_2}(g_d)) = (2, 2) = (a_{11}, a_{12}) = a \in \mathbb{N}^2,$$

and therefore

$$\bar{a}_{\emptyset 1} = \bar{a}_{\emptyset 2} = 0, \qquad b_{\sigma 1} = \bar{a}_{\sigma 1} = b_{\sigma 2} = \bar{a}_{\sigma 2} = 2.$$

and compute the explicit closed form as

$$\psi_d(r_1, r_2) = (r_1 + 1)\binom{r_2 + d}{d} - (r_1 - 1)\binom{r_2 + d - 2}{d}.$$

If we specialize $d = 1, 2, 3$, we obtain the same results as with Algorithm 1, hence, we've derived a verification of the results.

2.4.2. The Heat-Equation

As a second example for a physical system that can be described in an operator-algebra, we consider the heat equation. The heat equation is derived from Fourier's law (stating that the time rate of heat transfer through material is proportional to the negative gradient in temperature and to the area), and from the conversation law of energy. The equation derived for the wave-propagation of heat in an isotropic and homogenous medium in d-dimensional space is

$$\frac{\partial u}{\partial t} = \alpha \cdot \nabla_x^2 u = \alpha \cdot \sum_{i=1}^{d} \left(\frac{\partial^2 u}{\partial x_i^2} \right), \qquad \alpha \in \mathbb{R},$$

the parameter α denotes the thermal diffusitivity, which depends on the used material. The heat equation is an example of a parabolic partial differential equation. We now consider the cyclic free module generated by

$$h_d := \delta_t - \alpha \cdot \sum_{i=1}^{d} \delta_{x_i}^2, \qquad \alpha \in \mathbb{R}. \tag{2.20}$$

Without loss of generality, we assume $\alpha = 1$.

Computation of the Univariate Hilbert Function

We keep our setting from before, that is, we consider

- the differential field $(\mathbb{K}, \Delta)$ with $\Delta := \{\delta_t, \delta_{x_1}, \ldots, \delta_{x_d}\}$;
- the filter valuation (2.10);
- the filtration on R given by (2.11);
- the exact sequence (2.12);
- the reduction relation (2.13) (with g_d replaced by h_d);
- and the order (2.14).

The essential condition $I_t \cap (Rh_d) = 0$ is equivalent to the statement, that irreducible monomials $\mathfrak{m}$ satisfy $\mathrm{ord}_{\delta_t}(\mathfrak{m}) = 0$, for every monomial with $\mathrm{ord}_{\delta_t}(\mathfrak{m}) \geq 1$ can be reduced further by h_d. This allows us, by Corollary 7, and Lemma 4 to find the univariate dimension polynomial by

$$\phi(r) = |I \cap \Theta \cap R_r| := \left\{(0, k_1, \ldots, k_d) : \sum_{i=1}^{d} k_i \leq r\right\} = \binom{r+d}{d}.$$

Computation of the Multivariate Hilbert Function

Similiar to the univariate case, we can now compute the result of reduction w.r.t. h_d as

$$\delta_t^k \delta_{x_1}^{l_1} \ldots \delta_{x_d}^{l_d} \longrightarrow \delta_t^k \delta_{x_1}^{l_1} \ldots \delta_{x_d}^{l_d} - \delta_t^{k-1} \cdot \delta_{x_1}^{l_1} \ldots \delta_{x_d}^{l_d} \cdot h_d = \delta_t^{k-1} \cdot (\sum_{i=1}^{d} \delta_{x_i}^{l_i+2} \cdot \prod_{\substack{j=1 \\ j \neq i}}^{d} \delta_{x_j}^{l_j}).$$

For example, for $d = 3$, we have

$$\mathfrak{m} := \delta_t^k \delta_{x_1}^{l_1} \delta_{x_2}^{l_2} \delta_{x_3}^{l_3} \longrightarrow \delta_t^{k-1}(\delta_{x_1}^{l_1+2} \delta_{x_2}^{l_2} \delta_{x_3}^{l_3} + \delta_{x_1}^{l_1} \delta_{x_2}^{l_2+2} \delta_{x_3}^{l_3} + \delta_{x_1}^{l_1} \delta_{x_2}^{l_2} \delta_{x_3}^{l_3+2})$$

This reduction can now be iterated as in (2.17), until $\mathrm{ord}_{\delta_t}(\mathfrak{m}) = 0$.

In the following, ϕ_k denotes the dimension polynomial associated to the cyclic free module generated by h_k in $k+1$ variables, associated to the filtration (2.16). The interpolation for values $r_i \geq 1$ again gives polynomials:

$$\begin{aligned}
\phi_1(r_1, r_2) &= 2r_1 + r_2 + 1, \\
\phi_2(r_1, r_2, r_3) &= 2r_1r_2 + 2r_1r_3 + r_2r_3 + r_2 + r_3 + 1, \\
\phi_3(r_1, r_2, r_3, r_4) &= 2r_1r_2r_3 + 2r_1r_2r_4 + 2r_1r_3r_4 + r_2r_3r_4 + \\
&\quad r_2r_3 + r_2r_4 + r_3r_4 + 2r_1 + r_2 + r_3 + r_4 + 1.
\end{aligned}$$

Next, we do again the split (2.18), i.e. we consider the bivariate filtration (2.19), to derive the following bivariate dimension polynomials ($r_i \geq 1$):

$$\begin{aligned}
\psi_1(r_1, r_2) &= 2r_1 + r_2 + 1,\\
\psi_2(r_1, r_2) &= 2r_1r_2 + (1/2)r_2^2 + r_1 + (3/2)r_2 + 1,\\
\psi_3(r_1, r_2) &= 1 + (11/6)r_2 + r_2^2 + (1/6)r_2^3 + r_1 + 2r_1r_2 + r_1r_2^2.
\end{aligned}$$

Example 17. *Let us once more apply Theorem 14 to the heat equation h_d given by* (2.20)*. The full partition of Δ into $(d+1)$-sets corresponds to the exponent vector*

$$a = (\mathrm{ord}_{\delta_t}(h_d), \mathrm{ord}_{\delta_{x_1}}(h_d), \ldots, \mathrm{ord}_{\delta_{x_d}}(h_d)) = (a_{11}, \ldots, a_{1(d+1)}) = (1, 2, \ldots, 2) \in \mathbb{N}^{d+1},$$

$m_1 = 1$, $m_2 = d$, and therefore

$$\forall k : \bar{a}_{\emptyset k} = 0, \qquad b_{\sigma 1} = \bar{a}_{\sigma 1} = 1 \wedge b_{\sigma j} = \bar{a}_{\sigma j} = 2, \qquad 1 < j \leq n.$$

The Theorem implies

$$\phi_d(r_1, \ldots, r_{d+1}) = \prod_{i=1}^{d+1}(r_i + 1) - r_1 \cdot \prod_{i=2}^{d+1}(r_i - 1).$$

For the split (2.18) *we get $a = (1, 2)$ such that*

$$\bar{a}_{\emptyset 1} = \bar{a}_{\emptyset 2} = 0, \qquad b_{\sigma 1} = \bar{a}_{\sigma 1} = 1 \wedge b_{\sigma 2} = \bar{a}_{\sigma 2} = 2,$$

and finally

$$\psi_d(r_1, r_2) = (r_1 + 1)\binom{r_2 + d}{d} - r_1\binom{r_2 + d - 2}{d}.$$

Once more, if we specialize $d = 1, 2, 3$, we obtain the same results as with Algorithm 1, hence, we've derived a verification of the results.

2.5. Homomorphic Image of Gröbner Reduction

We consider a ring R containing a commutative ring K, and a set of monomials $\Lambda \subseteq R$, such that $R = K^{(\Lambda)}$ (recall notation (1.25)) and the free R-module F

$$F = Re_1 \oplus \cdots \oplus Re_q$$

as well as a submodule $N \subseteq F$. Let $S = K^{(\Omega)}$ be another such ring (with set of monomials $\Omega \subseteq S$) and let $\varphi : S \to R$ denote a surjective homomorphism of rings such that $\varphi(K) = K$ and $\varphi(\Omega) = \Lambda$. Let

$$G = Se_1 \oplus \cdots \oplus Se_q$$

denote the free S-module of same rank as R. We extend the map φ to a homomorphism of S-modules (denoted by the same symbol)

$$\varphi\colon G \to F, \qquad \sum_{i=1}^{q} r_i e_i \mapsto \varphi\left(\sum_{i=1}^{q} r_i e_i\right) := \sum_{i=1}^{q} \varphi(r_i) e_i. \tag{2.21}$$

Suppose now, we are given a strong reduction σ for G, and we want to apply the knowledge from the S-module G in the R-module F, where G and F are connected by (2.21). This is, we want the following diagram to commute:

$$G \qquad \begin{tikzcd} g \arrow[r, mapsto, "\varphi"] \arrow[d, "\sigma"', "\star"] & \varphi(g) = f \arrow[d, "\rho", "\star"'] \\ \mathrm{NF}_\sigma(g) \arrow[r, mapsto, "\varphi"] & \varphi(\mathrm{NF}_\sigma(g)) = \mathrm{NF}_\rho(\varphi(g)) \end{tikzcd} \qquad N = \varphi(G) \subseteq F \tag{2.22}$$

In this picture, σ is a reduction in G, ρ is a reduction in F. In particular, the elements g and $\mathrm{NF}_\sigma(g)$ are in G, while $\varphi(g) = f$ and $\varphi(\mathrm{NF}_\sigma(g)) = \mathrm{NF}_\rho(\varphi(g))$ are in N. We reproduce [FL15a, Proposition 6].

Theorem 15 (Homomorphic Image of Gröbner Reduction)**.** *If $\sigma \subseteq G \times G$ is a strong reduction for $\varphi^{-1}(N)$ then there is a strong reduction $\rho \subseteq F \times F$ such that*

$$\varphi(\underset{\sigma}{\mathrm{NF}}(g)) = \underset{\rho}{\mathrm{NF}}(\varphi(g)),$$

that is, in diagram (2.22), when σ is a strong reduction, the reduction ρ can also be chosen as strong reduction. Further, if σ is a Gröbner reduction for $\varphi^{-1}(N)$ with respect to a monomial filtration $S = \bigcup_{r\in\mathbb{N}^p} S_r$ then ρ is a Gröbner reduction for N with respect to the filtration

$$R = \bigcup_{r\in\mathbb{N}^p} \varphi(S_r).$$

Proof. Let

$$I_\sigma = \{g \in G : \nexists z \text{ with } g \longrightarrow z\}$$

denote the monomial subspace of irreducibles in G. By Lemma 24 we have that

$$G = \varphi^{-1}(N) \oplus I_\sigma \Rightarrow F = N \oplus \varphi(I_\sigma).$$

Let $p\colon F \to \varphi(I_\sigma)$ denote projection. We define the relation $\rho \subseteq F \times F$ by

$$f \longrightarrow_\rho h \Leftrightarrow f \notin \varphi(I_\sigma) \wedge h = p(f).$$

It is clear, that ρ is noetherian (i.e. Axiom 1. is fulfilled). $\varphi(I_\sigma)$ is the K-space of ρ-irreducibles, and $\varphi(I_\sigma)$ is monomial. Indeed, if

$$f = \varphi(i) \in \varphi(I_\sigma) \wedge i = \sum_{t\in\Omega E} i_t t \Rightarrow f = \sum_{t\in\Omega E} \varphi(i_t)\varphi(t). \tag{2.23}$$

By monomiality of I_σ we know that all monomials t occurring in this sum are in I_σ and so the corresponding $\varphi(t)$ are in $\varphi(I_\sigma)$. Since $\varphi(\Omega E) = \Lambda E$, i.e., φ maps monomials in G onto monomials in F, by collecting terms in (2.23) we see, that $\mathrm{T}(f) \subseteq \varphi(I_\sigma)$ demonstrating Axiom 2. Axiom 3. and 4. are obvious.

Take $g \in G$ and let $i = \mathrm{NF}_\sigma(g)$. Then $g \longrightarrow_\sigma^\star i$ and

$$g - i = \nu \in \varphi^{-1}(N)$$

according to Axiom 3. for σ. Now we have

$$\varphi(g) = \varphi(\nu) + \varphi(i) \in N \oplus \varphi(I_\sigma).$$

If $\nu \in \ker\varphi$ then $\varphi(g) = \varphi(i)$ equals its own normal form. Further,

$$\nu \notin \ker\varphi \Rightarrow \varphi(g) \longrightarrow_\rho \varphi(i).$$

In both cases we derive $\varphi(i) = \mathrm{NF}_\rho(\varphi(g))$.

Now assume that $S = \bigcup_{r\in\mathbb{N}^p} S_r$ is a filtration and that σ is a Gröbner reduction with respect to the extended filtration

$$G_r = S_r e_1 \oplus \cdots \oplus S_r e_q.$$

Then Lemma 24 assures that

$$G_r = \varphi^{-1}(N)_r \oplus (I_\sigma)_r \qquad \forall r \in \mathbb{N}^p.$$

By Lemma 20, we find that $R_r = \varphi(S_r)$ is a filtration on R and $F_r = \varphi(G_r)$ yields the extended filtration $F = \bigcup_{r\in\mathbb{N}^p} F_r$.

Let $f \longrightarrow_\rho h$ and $f \in F_r$. There is a $g \in G_r$ with $\varphi(g) = f$. Let $i = \mathrm{NF}_\sigma(g)$. Then $i \in G_r$ and so $\varphi(i) \in \varphi(G_r) = F_r$. But $\varphi(i) = \mathrm{NF}_\rho(f)$ and therefore we see that $h \in F_r$. Consequently ρ is a Gröbner reduction. $\square$

Applying the last theorem to the ring D provides an alternative method for constructing a Gröbner reduction in free D-modules. By Corollary 4, we've encountered such a homomorphism, and hence find a possible way of handling the negative exponents that occur in monomials of difference-differential operators, by designing a Gröbner reduction for positive exponents exclusively.

3. Classic Examples and their Relation to Gröbner Reduction

In this chapter we are going to present some classic examples appearing in literature, and their relation to Gröbner Reduction. Part of this work was presented in the report [FL15b], developed in joint work with Günter Landsmann.

While we've formulated the algebraic setting on a general (non-commutative) ring, in literature more concrete examples are considered. At concrete examples, a reduction relation is presented and a corresponding theory of Gröbner bases for computation of the multivariate Hilbert function is introduced.

As a matter of fact, most of this principles can be viewn under the aspect of Gröbner Reduction. Surprisingly, the only major requirement on the underlying ring R is the existence of monomials $\mathbb{M}$ contained in R, i.e. $\mathbb{M} \subseteq R$, and a commutative ring K such that $K \subseteq R$.

Throughout this chapter R denotes an arbitrary (possibly non-commutative) ring containing a commutative ring K in such a way that R is a free K-module. All rings that will occur are of this type. There are situations where the ring K contains a field $\mathbb{K}$ that is central in R. Then, there may be two different monomial concepts:

1. $R = K^{(\mathbb{M}_1)}$ (R is the free K-module with basis $\mathbb{M}_1$);
2. $R = \mathbb{K}^{(\mathbb{M}_2)}$ (R is a vector space over a field $\mathbb{K}$ with basis $\mathbb{M}_2$).

In certain instances we will need the assumption that K is a field. This will be emphasized at occurrence.

Further, we will consider several rings R and investigate reduction relations for submodules of free modules over them. Always, F will denote the free R-module with basis $E := \{e_1, \dots, e_q\}$.

While the rings are equipped with certain filtrations there is always present a well-ordering $\prec$ of the monomials $\mathbb{M}E$ that distinguishes for all $F\backslash\{0\}$ a leading term $\mathrm{LT}(f)$ and a leading coefficient $\mathrm{LC}(f)$.

In each of the examples below we are now concerned with two reduction relations: For the free module F let $f, g, h \in F$, $g \neq 0$. Then we have

1. full reduction ρ:

$$f \longrightarrow_g^\rho h \quad :\Longleftrightarrow \quad \exists \mathfrak{m} \in \mathbb{M}: \ \mathrm{LT}(\mathfrak{m}g) \in \mathrm{T}(f) \wedge h = f - \frac{f_{\mathrm{LC}(\mathfrak{m}g)}}{\mathrm{LC}(\mathfrak{m}g)}\mathfrak{m}g \ \wedge \ P; \tag{3.1}$$

2. leading term reduction σ:

$$f \longrightarrow_g^\sigma h \quad :\Longleftrightarrow \quad \exists \mathfrak{m} \in \mathbb{M}: \ \mathrm{LT}(\mathfrak{m}g) = \mathrm{LT}(f) \wedge h = f - \frac{f_{\mathrm{LC}(\mathfrak{m}g)}}{\mathrm{LC}(\mathfrak{m}g)}\mathfrak{m}g \ \wedge \ P; \tag{3.2}$$

The symbol 'P' denotes a predicate $P = P(f, g, \mathfrak{m}, h)$ depending on the actual situation. For a subset $G \subseteq F$ one has then in both cases

$$f \longrightarrow_G h \quad :\Longleftrightarrow \quad \exists g \in G : f \longrightarrow_g h.$$

These reduction concepts are the core of Gröbner bases.

In literature, paradigms related to what we've encountered in Chapter 2 are discussed. However, the concepts appearing in current papers do not satisfy the second axiom of Definition 17, i.e. the irreducibles form in general not a $\mathbb{K}$-vector space, but only a monomial subset of F, i.e. we have for all $f \in F : f \in I \Rightarrow \mathrm{T}(f) \subseteq I$, where I is as in (2.2). From that point of view, we replace axiom 2. by

2'. I is a monomial subset of F, that is

$$\forall f \in F \, (f \in I \Rightarrow \mathrm{T}(f) \subseteq I) \, ;$$

We call a reduction relation $\rho \subseteq F \times F$ a *weak reduction*, if it satisfies axioms 1., 3. and 4. from Definition 17 and axiom 2'.

If it additionally satisfies axiom 5. from Definition 17, the reduction relation is called a *weak Gröbner reduction*.

Definition 18. A *weak reduction* $\rho \subseteq F \times F$ satisfies

1. ρ is noetherian, i.e. every sequence of reduction steps terminates;

2'. I is a monomial subset of F, i.e. $\forall f \in F \, (f \in I \Rightarrow \mathrm{T}(f) \subseteq I)$;

3. $f \longrightarrow h \Rightarrow f \equiv h \pmod N$;

4. $I \cap N = 0$;

If additionally

5. $f \longrightarrow h \wedge f \in F_r \Rightarrow h \in F_r$;

it is called a *weak Gröbner reduction.*

Even when G is a Gröbner basis, ρ is in general not a strong reduction as shown at Example 19. Plainly every (strong) Gröbner reduction in the sense of Definition 17 is also a weak Gröbner reduction.

Definition 19 (Gröbner Basis in a Free Module)**.**
Consider a submodule N of a free module $F = K^{(\mathbb{M}E)}$. Assume given

- a well-order $\prec$ on $\mathbb{M}E$;
- a predicate $P = P(f, g, \mathfrak{m}, h)$;
- ρ the full reduction defined by these data.

A subset $G \subseteq N$ is a *Gröbner basis* for N if and only if ρ is a weak reduction for N.

Given a filtered ring and the obvious necessary data, we need to check the defining axioms in order to reveal the relation ρ as a (weak) Gröbner reduction for $N = RG$. The set G is then exposed as a Gröbner basis for N and the filter groups are ρ-stable.

So we fix a set $G \subseteq F$ and write ρ and σ for the relations $f \longrightarrow_g^\rho h$ and $f \longrightarrow_g^\sigma h$ respectively, i.e. $f \longrightarrow_\rho h$ means $f \longrightarrow_g^\rho h$ and similar for σ. All our examples follow the pattern along the following lines.

Termination

Fix a positive integer q and design an injection $\varphi : \mathbb{M}E \to \mathbb{N}^q$. The set $\mathbb{N}^q$ is ordered lexicographically, i.e.

$$a < b :\Leftrightarrow a_{\min\{i:\ a_i \neq b_i\}} < b_{\min\{i:\ a_i \neq b_i\}}, \qquad a, b \in \mathbb{N}^q$$

The set of monomials $\mathbb{M}E$ inherits a well-ordering $\prec$ by means of this injection.

We call such an order *induced by the injection* φ. The well order $\prec$ extends to a well order on the set of all finite subsets of $\mathbb{M}E$ (this is $\{\mathrm{T}(f) : f \in F\}$):

$$\mathrm{T}(f) \prec \mathrm{T}(g) :\Leftrightarrow \max\{\mathrm{T}(f) \bigtriangleup \mathrm{T}(g)\} \in \mathrm{T}(g). \tag{3.3}$$

where $\bigtriangleup$ is the symmetric difference (consider e.g. [BWK93]).

Lemma 27. *For $f, h \in F$, and full reduction $\longrightarrow_\rho \subseteq F \times F$ defined by* (3.1)*, we have:*

$$f \longrightarrow_\rho h \Rightarrow \mathrm{T}(h) \prec \mathrm{T}(f).$$

Proof. For arbitrary g, from $f \longrightarrow_g^\rho h$ we see that $\mathrm{LT}(\mathfrak{m}g) \in \mathrm{T}(f)\backslash\mathrm{T}(h)$ - where $\mathfrak{m}$ is a term as mentioned in (3.1) - whereas for all terms t with $t \succ \mathrm{LT}(\mathfrak{m}g)$ we have that $h_t = f_t$. This demonstrates that $\mathrm{T}(f) \succ \mathrm{T}(h)$. □

Since $\sigma \subseteq \rho$ it is clear that $I_\rho \subseteq I_\sigma$ and σ terminates if ρ terminates. Consequently both relations terminate there.

The set of irreducible monomials should be a monomial subset of F

A reformulation of the predicate 'the set of irreducible monomials form a monomial subset of F' is given by the simple formula

$$f \in I \Rightarrow \mathrm{T}(f) \subseteq I.$$

The relation ρ has the property

$$\begin{aligned} f \text{ is } \rho\text{-reducible} &\Longleftrightarrow \exists h : f \longrightarrow_\rho h \\ &\Longleftrightarrow \exists g \in G \exists \mathfrak{m} \in \mathbb{M} \left(\mathrm{LT}(\mathfrak{m}g) \in \mathrm{T}(f) \wedge P(f, g, \mathfrak{m}, f - \frac{f_{\mathrm{LT}(\mathfrak{m}g)}}{\mathrm{LC}(\mathfrak{m}g)} \mathfrak{m}g) \right). \end{aligned}$$

In case that P does not involve h, i.e., $P = P(f, g, \mathfrak{m})$, we obtain

$$f \text{ is } \rho\text{-reducible} \Leftrightarrow \exists g \in G \exists \mathfrak{m} \in \mathbb{M} \left(\mathrm{LT}(\mathfrak{m}g) \in \mathrm{T}(f) \wedge \mathrm{LC}(\mathfrak{m}g) | f_{\mathrm{LT}(\mathfrak{m}g)} \wedge P(f, g, \mathfrak{m}) \right).$$

For I_ρ to be monomial it is then enough to verify the *monomial irreducibility condition*

$$\begin{aligned} \exists g \in G \exists \mathfrak{m} \in \mathbb{M} \left(\mathrm{LT}(\mathfrak{m}g) \in \mathrm{T}(f) \wedge \mathrm{LC}(\mathfrak{m}g) \in K^\times \wedge P(\mathrm{LT}(\mathfrak{m}g), g, \mathfrak{m}) \right) \Rightarrow \\ \exists g \in G \exists \mathfrak{m} \in \mathbb{M} \left(\mathrm{LT}(\mathfrak{m}g) \in \mathrm{T}(f) \wedge \mathrm{LC}(\mathfrak{m}g) | f_{\mathrm{LT}(\mathfrak{m}g)} \wedge P(f, g, \mathfrak{m}) \right). \end{aligned} \tag{3.4}$$

The monomial irreducibility condition for σ under the assumption $P = P(f, g, \mathfrak{m})$ is the same as as for ρ, except that on the right hand side of the implication we have $\mathrm{LT}(\mathfrak{m}g) = \mathrm{LT}(f)$. For details see [FL15b].

It is clear that this condition is hard to satisfy. Indeed, I_σ is not monomial in general.

Compatibility of reduction with congruence modulo $N = RG$

This is always obvious from the general pattern (3.1) and (3.2).

$I \cap N = 0$, i.e. each non-zero element in N is reducible

The validity of this condition must be guaranteed by an appropriate choice of the generator set G which is achieved by the usual Buchberger completion procedure.

Each filter should be ρ-stable

In our examples we will consider univariate filtrations $(F_r)_{r \in \mathbb{N}}$ of F that are constructed due to the following scheme:

We start with an 'order-function' $\nu : \mathbb{M} \to \mathbb{N}$ where $\nu(\mathfrak{m})$ can be read off from $\mathfrak{m} \in \mathbb{M}$, i.e. $\nu(\mathfrak{m})$ is the sum of certain exponents that are present in $\mathfrak{m}$. The function $\mathfrak{m}$ extends to $\mathbb{M}E$ by setting

$$\nu(\mathfrak{m}e) := \nu(\mathfrak{m}), \qquad \mathfrak{m} \in \mathbb{M}, e \in E.$$

and further to entire F (we always use the same symbol)

$$\nu(f) := \begin{cases} \max\{\nu(t) : t \in \mathrm{T}(f)\}, & f \in F\backslash\{0\}, \\ -\infty & f = 0. \end{cases}$$

Then, for $f, g \in F, c \in K$, we have

$$\nu(f \pm g) \leq \max\{\nu(f), \nu(g)\}, \qquad \nu(c \cdot f) \leq \nu(f).$$

The *univariate filtration induced by* ν is then

$$F_r^{(\nu)} := \{f \in F : \nu(f) \leq r\}, \qquad r \in \mathbb{N}.$$

This is closely related to what has been discussed at Definition 12, respectively its properties that have been examined at Lemma 15. Remark that this defines implicitly the sets $R_r^{(\nu)}$, where $r \in \mathbb{N}$, since $R = R^1$. From the properties of ν it is plain that

- the sets $F_r^{(\nu)}$ are monomial K-modules;
- $r \leq s \Rightarrow F_r^{(\nu)} \subseteq F_s^{(\nu)}$;
- $\bigcup\limits_{r=0}^{\infty} F_r^{(\nu)} = F$.

It remains to check that $R_r \cdot F_s \subseteq F_{r+s}$ (for $r, s \in \mathbb{N}$), which is then the only property of filtrations that depends on the actual ring structure of R. The multivariate filtrations that we consider are constructed from univariate ones by means of intersection: Given order functions $\nu_1, \ldots, \nu_p$ and $r = (r_1, \ldots, r_p) \in \mathbb{N}^p$ we set

$$F_r^{(\nu_1,\ldots,\nu_p)} = F_{r_1}^{(\nu_1)} \cap \ldots \cap F_{r_p}^{(\nu_p)} = \{f \in F : \nu_1(f) \leq r_1 \wedge \ldots \wedge \nu_p(f) \leq r_p\}.$$

The next theorem condenses the preceding discussion.

Theorem 16. *Let G e a subset of the free R-module $F = K^{(\mathbb{M}E)}$. Assume that*

- *$\prec$ is a well order on $\mathbb{M}E$;*
- *$P = P(f, g, \mathfrak{m}, h)$ is a predicate $F \times G \times \mathbb{M} \times F \to \{0, 1\}$;*
- *ρ is the full reduction defined by $(\prec, P, G)$;*
- *$\left(R_r^{(j)}\right)_{r\in\mathbb{N}}$ is defined by an order function $\nu_j : \mathbb{M} \to \mathbb{N}$, $\qquad 1 \leq j \leq p$;*
- *$R_r = R_{r_1}^{(1)} \cap \ldots \cap R_{r_p}^{(p)}$, $\qquad r \in \mathbb{N}^p$;*
- *$F_r = \bigoplus_{e\in E} R_r e$.*

Under these assumptions, if

1. $R_r^{(j)} \cdot R_s^{(j)} \subseteq R_{r+s}^{(j)}, \qquad r, s \in \mathbb{N}, j = 1, \ldots, p;$

2. $P = P(f, g, \mathfrak{m})$ *and the monomial irreducibility condition* (3.4) *holds;*

3. $f \longrightarrow_g^{\rho} h \wedge f \in F_r \Rightarrow h \in F_r$ *for all* $g \in G$*;*

4. $N \cap I_\rho = 0$*;*

then $(F_r)_{r\in\mathbb{N}^p}$ *is a monomial filtration on* F *w.r.t. the monomial filtration* $(R_r)_{r\in\mathbb{N}^p}$ *and the full reduction* ρ *is a weak Gröbner reduction for* $N = RG$*.*

The Difficulties arising at the considered Rings

All the theories that are developed in current literature, take care of the differences at the concrete setting. But what exactly are the differences?

The most obvious difference is *the commutation rule of ring*. The non-commutativity might even has influence on the support (i.e. the set $\mathrm{T}(f)$) of the considered element f in the free module. For example, while in a difference-ring we have $\sigma^l \cdot a = \sigma^l(a)\sigma^l$, the situation is different in a differential ring, where lower order terms are introduced (compare (1.28)). We've encountered this behaviour already at Lemma 6 (reformulated at (1.19)), Lemma 10 or for the Weyl-algebra at (1.33).

But more can be said. At the Weyl algebra we have the situation, that the monomials xd do not form a multiplicative monoid. This is the major difference to all other considered examples, and causes slight difficulties in the computation. However, as we will find out, the non-commutativity is in some sense well-behaved, that the leading terms are preserved, the details are carried out in the upcoming sections, in particular at Lemma 31, or for the Ore-algebra at Lemma 42 and Lemma 43.

We note also, that Gröbner bases in [DL12a], are defined by Definition 21, which differs from our setting Definition 19. Still, we will show how this connects to our concepts.

In difference-differential rings, we still have non-commutativity, but the monomials form a multiplicative monoid. Most technical part is here to ensure the condition

$$f \in F_r \wedge f \longrightarrow h \Rightarrow h \in F_r, \qquad r \in \mathbb{N}^p.$$

For the bivariate case, that can be covered by an appropriate choice of $\prec_1$ and $\prec_2$, by restricting the reduction as in Definition 7. This is demonstrated at Theorem 20. A similar situation is at the Ore-algebra. For the Ore-algebra, this condition achieved by using multiple term orderings, see Definition 24.

The Weyl algebra produces an involved theory, and the same is true for the Ore-algebra. Having in mind section 1.4, it is not too surprising that the commutative case can re-use ideas from this rings, and theory can be simplified.

3.1. Bernstein Polynomials over the Weyl Algebra $\mathrm{A}_n(\mathbb{K})$

The theory of the Weyl algebra $\mathrm{A}_n(\mathbb{K})$ in n variables, is the study of modules over rings of differential operators with polynomial coefficients over the field $\mathbb{K}$. There exists a vast account on literature, most noteworthy the book by Coutinho [Cou95] that provides a readable introduction to the topic. In this section, we review work appearing in [DL12a] under the scope of Gröbner reduction.

In the difference-differential ring, or in the ring of Ore-polynomials, the monomials $\mathbb{M}$ form a multiplicative monoid, i.e. $\mathbb{M}\mathbb{M} \subseteq \mathbb{M}$. For the Weyl-Algebra, this is not longer the case. As indicated in section 1.4, there are several possible viewpoints on the Weyl algebra, with two different concepts of monomials.

Let $\mathbb{K}$ be a field of characteristic zero and let d_i denote the i-th partial derivative of the polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$. The Weyl algebra $\mathrm{A}_n(\mathbb{K})$ is the $\mathbb{K}$-algebra generated by $\mathbb{K}[x_1, \ldots, x_n] \cup \{d_1, \ldots, d_n\}$ as a subalgebra of $\mathrm{End}_{\mathbb{K}}(\mathbb{K}[x_1, \ldots, x_n])$. The multiplication in this ring obeys the rules (1.33).

Let A denote the ring $\mathrm{A}_n(\mathbb{K})$. We may consider A as a free $\mathbb{K}[x_1, \ldots, x_n]$-module with basis $\mathbb{M}_1 = \{d^l : l \in \mathbb{N}^n\}$. Then $\mathbb{M}_1$ is a monoid isomorphic to $\mathbb{N}^n$ and, according to our notational convention K and R specialize to $K = \mathbb{K}[x_1, \ldots, x_n]$ and $R = K^{(\mathbb{M}_1)}$.

We will here stress the second approach: A as a free $\mathbb{K}$-vector space with distinguished set of monomials $\mathbb{M}_2 = \{x^k d^l : (k, l) \in \mathbb{N}^n \times \mathbb{N}^n\}$. In the following we write Λ for $\mathbb{M}_2$.

Explicitly, the product of two monomials in $\mathrm{A}_n(\mathbb{K})$ is

$$x^k d^l \cdot x^p d^q = \sum_{v \in \mathbb{N}^n} \binom{l}{v} x^k d^v (x^p) d^{l+q-v}, \qquad k, l, p, q \in \mathbb{N}^n, \binom{l}{v} = \prod_{i=1}^{n} \binom{l_i}{v_i} \tag{3.5}$$

To visualize the scope of the sum we may write

$$x^k d^l \cdot x^p d^q = \sum_{v \leq_\pi \inf\{l,p\}} \binom{l}{v} \frac{p!}{(p-v)!} x^{k+p-v} d^{l+q-v}.$$

Filtrations in the Weyl algebra

Let $F = A^{(E)}$ the free A-module with basis $E := \{e_1, \ldots, e_q\}$.

Lemma 28 (Cancellation laws in A). *Let $\lambda, \mu \in \Lambda$ and $t_1, t_2 \in \Lambda E$. Then*

1. $\lambda \cdot t_1 = \lambda \cdot t_2 \Rightarrow t_1 = t_2$;
2. $\lambda \cdot t_1 = \mu \cdot t_1 \Rightarrow \lambda = \mu$;

Proof. $\lambda = x^k d^l$, $t_1 = x^p d^q e_1$ and $t_2 = x^r d^s e_2$.

1. If $\lambda t_1 = \lambda t_2$ then $x^k d^l \cdot x^p d^q e_1 = x^k d^l \cdot x^r d^s e_2$, therefore $e_1 = e_2$. We get

$$\sum_{u \leq_\pi \inf\{l,p\}} b_u x^{k+p-u} d^{l+q-u} = \sum_{v \leq_\pi \inf\{l,p\}} c_v x^{k+r-v} d^{l+s-v}, \qquad b_u, c_v \in \mathbb{N}.$$

Therefore there exists $u, v \in \mathbb{N}^n$ such that

$$x^{k+p} d^{l+q} = c_v x^{k+r-v} d^{l+s-v} \qquad x^{k+r} d^{l+s} = b_u x^{k+p-u} d^{l+q-u}.$$

It follows that

$$p = r - v, \quad q = s - v, \quad r = p - u, \quad s = q - u$$

from which we derive $u = v = 0$. Consequently $t_1 = t_2$.

2. Is proven in the same style. $\square$

For $\lambda \in \Lambda$ we define the three order functions $\Lambda \to \mathbb{N}$. For $\lambda = x^k d^l \in \Lambda$

$$\nu_1(\lambda) = k_1 + \cdots + k_n, \qquad \nu_2(\lambda) = l_1 + \cdots + l_n, \qquad \nu_3(\lambda) = \nu_1(\lambda) + \nu_2(\lambda).$$

Exactly like in (3.11) these order functions extend to functions $\nu_j : F \to \mathbb{N}$:

$$\nu_j(\lambda e) := \nu_j(\lambda), \qquad \nu_j(f) := \begin{cases} \max\{\nu_j(t) : t \in \mathrm{T}(f)\}, & f \in F \backslash \{0\}; \\ -\infty, & f = 0, \end{cases}$$

where $\lambda \in \Lambda$ and $e \in E$.

From the definition it is clear that for $f, g \in F$ and $c \in \mathbb{K}\backslash\{0\}$

$$\nu_j(f \pm g) \leq \max\{\nu_j(f), \nu_j(g)\}, \qquad \nu_j(c \cdot f) = \nu_j(f), \qquad j = 1, 2, 3. \tag{3.6}$$

Note that the extensions of ν_3 to F is not the sum of the extensions to F from ν_1 and ν_2.

We define three well-orders $\prec_x, \prec_d$ and $\prec_{xd}$ on ΛE:

The first $\prec_x$ comes from the injection

$$\Lambda E \to \mathbb{N}^{2n+3}, \; t = x^k d^l e \mapsto (\nu_1(t), \nu_2(t), k_1, \ldots, k_n, l_1, \ldots, l_n, e)$$

$\prec_d$ comes from

$$\Lambda E \to \mathbb{N}^{2n+3}, \; t = x^k d^l e \mapsto (\nu_2(t), \nu_1(t), l_1, \ldots, l_n, k_1, \ldots, k_n, e)$$

and finally $\prec_{xd}$ by the map

$$\Lambda E \to \mathbb{N}^{2n+2}, \; t = x^k d^l e \mapsto (\nu_3(t), k_1, \ldots, k_n, l_1, \ldots, l_n, e)$$

where E is ordered naturally and $\mathbb{N}^p$ is ordered lexicographically. Each one determines leading term and leading coefficient written LT_k, LC_k where $k \in \{x, d, xd\}$.

Lemma 29. *Let*

$$\lambda = x^k d^l \in \Lambda, \qquad t = x^r d^s e \in \Lambda E, \qquad f \in F \backslash \{0\}.$$

We have $k, l, r, s \in \mathbb{N}^n$ and set $|k| = k_1 + \ldots + k_n$ and similar for l, r, s. Obviously

$$|k + l| = |k| + |l|.$$

Then:

1. $\nu_1(\lambda t) = |k| + |r|$, $\nu_2(\lambda t) = |l| + |s|$, $\nu_3(\lambda t) = |k| + |r| + |l| + |s| = \nu_1(\lambda t) + \nu_2(\lambda t)$
2. $\mathrm{LT}_x(\lambda t) = \mathrm{LT}_d(\lambda t) = \mathrm{LT}_{xd}(\lambda t) = x^{k+r} d^{l+s} e$
3. $\nu_1(f) = \nu_1(\mathrm{LT}_x(f))$, $\nu_2(f) = \nu_2(\mathrm{LT}_d(f))$, $\nu_3(f) = \nu_3(\mathrm{LT}_{xd}(f))$,
4. $\nu_j(\lambda t) = \nu_j(\lambda) + \nu_j(t), \qquad j = 1, 2, 3$

Proof. Points 1. and 2. follow from (3.5). For 3. take $s \in \mathrm{T}(f)$. Obvious

$$s \preccurlyeq_k \mathrm{LT}_k(f), \qquad k \in \{x, d, xd\},$$

and by the choice of $\prec_k$ have

$$\nu_1(s) \leq \nu_1(\mathrm{LT}_x(f)), \quad \nu_2(s) \leq \nu_2(\mathrm{LT}_d(f)), \quad \nu_3(s) \leq \nu_3(\mathrm{LT}_{xd}(f))$$

Hence, we obtain

$$\begin{aligned} \nu_1(f) &= \max\{\nu_1(s) : s \in \mathrm{T}(f)\} = \nu_1(\mathrm{LT}_x(f)); \\ \nu_2(f) &= \max\{\nu_2(s) : s \in \mathrm{T}(f)\} = \nu_2(\mathrm{LT}_d(f)); \\ \nu_3(f) &= \max\{\nu_3(s) : s \in \mathrm{T}(f)\} = \nu_3(\mathrm{LT}_{xd}(f)); \end{aligned}$$

4. is obvious from 1. □

Lemma 30. *Let $\lambda, \mu \in \Lambda$ and $s, t \in \Lambda E$. Then, for $k \in \{x, d, xd\}$:*

- $\lambda \prec_k \mu \Rightarrow \mathrm{LT}_k(\lambda t) \prec_k \mathrm{LT}_k(\mu t)$;
- $s \prec_k t \Rightarrow \mathrm{LT}_k(\lambda s) \prec_k \mathrm{LT}_k(\lambda t)$.

Proof. First, let $\lambda = x^k d^l$, $\mu = x^r d^s$, $t = x^\alpha d^\beta e$ and $\lambda \prec_x \mu$. Then

$$\mathrm{LT}_x(\lambda t) = x^{k+\alpha} d^{l+\beta} e \qquad \mathrm{LT}_x(\mu t) = x^{r+\alpha} d^{s+\beta} e.$$

If $\nu_1(\lambda) < \nu_1(\mu)$ then $|k| < |r|$, so

$$\nu_1(\mathrm{LT}_x(\lambda t)) = |k + \alpha| = |k| + |\alpha| < |r| + |\alpha| = |r + \alpha| = \nu_1(\mathrm{LT}_x(\mu t)),$$

and thus $\mathrm{LT}_x(\lambda t) \prec_x \mathrm{LT}_x(\mu t)$.

If $\nu_1(\lambda) = \nu_1(\mu)$ and $\nu_2(\lambda) < \nu_2(\mu)$ then $|k| = |r|$ and $|l| < |s|$ so

$$\nu_1(\mathrm{LT}_x(\lambda t)) = |k| + |\alpha| = |r| + |\alpha| = \nu(\mathrm{LT}_x(\mu t)),$$

and

$$\nu_2(\mathrm{LT}_x(\lambda t)) = |l| + |\beta| < |s| + |\beta| = \nu_2(\mathrm{LT}_x(\mu t)),$$

which also means $\mathrm{LT}_x(\lambda t) \prec_x \mathrm{LT}_x(\mu t)$.

If $\nu_1(\lambda) = \nu_1(\mu)$ and $\nu_2(\lambda) = \nu_2(\mu)$ (i.e. $|k| = |r|$ and $|l| = |s|$) and $k \neq r$ then let $j := \min\{i : k_i \neq r_i\}$. We obtain

$$(|k+\alpha|, |l+\beta|, k+\alpha, l+\beta, e) <_{\mathrm{lex}} (|r+\alpha|, |s+\beta|, r+\alpha, s+\beta, e).$$

Once again this means that $\mathrm{LT}_x(\lambda t) \prec_x \mathrm{LT}_x(\mu t)$.

The same argument can be used for the last remaining case $\nu_1(\lambda) = \nu_1(\mu)$ and $\nu_2(\lambda) = \nu_2(\mu)$ and $k = r$. Then we must have $l \neq s$, let $j := \min\{i : l_i \neq s_i\}$, then $l_j < s_j$ which results again in $\mathrm{LT}_x(\lambda t) = \mathrm{LT}_x(\mu t)$.

Second, let $\lambda = x^\alpha d^\beta$, $s = x^k d^l e_m$, $t = x^p d^q e_n$, such that $s \prec_x t$. The proof works similar as before. The only difference is one more case: When $x^k d^l = x^p d^q$ then e_m must be smaller e_n and the statement follows.

The proof of the statements for $k \in \{d, xd\}$ is the same with the pairs

$$(\nu_2(\cdot), \mathrm{LT}_d, \prec_d), \qquad (\nu_3(\cdot), \mathrm{LT}_{xd}, \prec_{xd}).$$

$\square$

Lemma 31. *Let $\mathbb{K}$ be a field of characteristic zero, $A = \mathrm{A}_n(\mathbb{K})$, $F = A^{(E)}$ the free A-module on the set E. Let $a \in A\backslash\{0\}$, $f \in F\backslash\{0\}$. Then, for $k \in \{x, d, xd\}$:*

- $\mathrm{LT}_k(a \cdot f) = \mathrm{LT}_k(\mathrm{LT}_k(a) \cdot \mathrm{LT}_k(f))$;
- $\mathrm{LC}_k(a \cdot f) = \mathrm{LC}_k(a) \cdot \mathrm{LC}_k(f)$.

Proof. We show the statement for $k = x$, the cases $k = d, xd$ are handled analogously. Let $\lambda_0 = \mathrm{LT}_x(a)$, $a_0 = \mathrm{LC}_x(a)$ and $t_0 = \mathrm{LT}_x(f)$, $f_0 = \mathrm{LC}_x(f)$. Thus, we have

$$a = a_0\lambda_0 + \sum_{\lambda \prec_x \lambda_0} a_\lambda \lambda \qquad f = f_0 t_0 + \sum_{t \prec_x t_0} f_t t.$$

Then, we have

$$a \cdot f = \underbrace{a_0 f_0 \lambda_0 t_0}_{(S1)} + \underbrace{\sum_{t \prec_x t_0} a_0 f_t \lambda_0 t}_{(S2)} + \underbrace{\sum_{\lambda \prec_x \lambda_0} a_\lambda f_0 \lambda t_0}_{(S3)} + \underbrace{\sum_{\lambda \prec_x \lambda_0} \sum_{t \prec_x t_0} a f_t \lambda t}_{(S4)}$$

Pick out a term $\lambda_0 t$ of sum $(S2)$. Then, from Lemma 30 we immediately derive that

$$\mathrm{LT}_x(\lambda_0 t) \prec_x \mathrm{LT}_x(\lambda_0 t_0).$$

When choosing a term from sum $(S3)$ we obtain

$$\mathrm{LT}_x(\lambda t_0) \prec_x \mathrm{LT}_x(\lambda_0 t_0),$$

or at $(S4)$ we have

$$\mathrm{LT}_x(\lambda t) \prec_x \mathrm{LT}_x(\lambda_0 t) \prec_x \mathrm{LT}_x(\lambda_0 t_0).$$

Now, let $s \in \mathrm{T}(a \cdot f)$. Then there exists $\lambda \in \Lambda, t \in \Lambda E$ with $\lambda \in \mathrm{T}(a)$ and $t \in \mathrm{T}(f)$ and $s \in \mathrm{T}(\lambda t)$. If follows that $s \preccurlyeq_x \mathrm{LT}_x(\lambda t)$.

$$\begin{aligned}
&\lambda = \lambda_0 \wedge t = t_0 \Rightarrow s \preccurlyeq_x \mathrm{LT}_x(\lambda_0 t_0);\\
&\lambda = \lambda_0 \wedge t \neq t_0 \Rightarrow s \preccurlyeq_x \mathrm{LT}_x(\lambda_0 t) \prec_x \mathrm{LT}_x(\lambda_0 t_0);\\
&\lambda \neq \lambda_0 \wedge t = t_0 \Rightarrow s \preccurlyeq_x \mathrm{LT}_x(\lambda t_0) \prec_x \mathrm{LT}_x(\lambda_0 t_0);\\
&\lambda \neq \lambda_0 \wedge t \neq t_0 \Rightarrow s \prec_x \mathrm{LT}_x(\lambda_0 t_0).
\end{aligned}$$

Consequently

$$\mathrm{LT}_x(a \cdot f) = \mathrm{LT}_x(\lambda_0 t_0) = \mathrm{LT}_x(\mathrm{LT}_x(a) \cdot \mathrm{LT}_x(f)),$$

and

$$\mathrm{LC}_x(a \cdot f) = a_0 f_0 = \mathrm{LC}_x(a) \cdot \mathrm{LC}_x(f).$$

□

Corollary 9. *Let $\lambda \in \Lambda \backslash \{0\}$ and $f \in F \backslash \{0\}$. Then*

$$\mathrm{LT}_k(\lambda f) = \mathrm{LT}_k(\lambda \cdot \mathrm{LT}_k(f)), \qquad k \in \{x, d, xd\}.$$

Corollary 10. *Let $a \in A$ and $f \in F$. Then:*

$$\nu_k(a \cdot f) = \nu_k(a) + \nu_k(f), \qquad k = 1, 2, 3.$$

Proof. If $a = 0$ or $f = 0$ the statements are obvious. So assume $a \neq 0$ and $f \neq 0$. Set

$$\begin{aligned}
\lambda_1 &= x^k d^l = \mathrm{LT}_x(a) & t_1 &= x^\alpha d^\beta e_1 = \mathrm{LT}_x(f)\\
\lambda_2 &= x^p d^q = \mathrm{LT}_d(a) & t_2 &= x^\gamma d^\delta e_2 = \mathrm{LT}_d(f)\\
\lambda_3 &= x^r d^s = \mathrm{LT}_{xd}(a) & t_3 &= x^\varepsilon d^\zeta e_3 = \mathrm{LT}_{xd}(f)
\end{aligned}$$

From Lemma 29 we get

$$\mathrm{LT}_x(\lambda_1 t_1) = x^{k+\alpha} d^{l+\beta} e_1,\ \ \mathrm{LT}_d(\lambda_2 t_2) = x^{p+\gamma} d^{q+\delta} e_2,\ \ \mathrm{LT}_{xd}(\lambda_3 t_3) = x^{r+\varepsilon} d^{s+\zeta} e_3.$$

Hence, by the preceding discussion we have

$$\nu_1(a \cdot f) = \nu_1(\mathrm{LT}_x(a \cdot f)) = \nu_1(\mathrm{LT}_x(\mathrm{LT}_x(a) \cdot \mathrm{LT}_x(f))) = \nu_1(\mathrm{LT}_x(\lambda_1 \cdot t_1))$$

$$
\begin{aligned}
&= \nu_1(x^{k+\alpha}d^{l+\beta}e_1) = |k+\alpha| = |k| + |\alpha| = \nu_1(\mathrm{LT}_x(a)) + \nu_1(\mathrm{LT}_x(f)) \\
&= \nu_1(a) + \nu_1(f).
\end{aligned}
$$

$$
\begin{aligned}
\nu_2(a \cdot f) &= \nu_2(\mathrm{LT}_d(a \cdot f)) = \nu_2(\mathrm{LT}_d(\mathrm{LT}_d(a) \cdot \mathrm{LT}_d(f))) = \nu_2(\mathrm{LT}_d(\lambda_2 \cdot t_2)) \\
&= \nu_2(x^{p+\gamma}d^{q+\delta}e_2) = |q+\delta| = |q| + |\delta| = \nu_2(\mathrm{LT}_d(a)) + \nu_2(\mathrm{LT}_d(f)) \\
&= \nu_2(a) + \nu_2(f).
\end{aligned}
$$

$$
\begin{aligned}
\nu_3(a \cdot f) &= \nu_3(\mathrm{LT}_{xd}(a \cdot f)) = \nu_3(\mathrm{LT}_{xd}(\mathrm{LT}_{xd}(a) \cdot \mathrm{LT}_{xd}(f))) = \nu_3(\mathrm{LT}_{xd}(\lambda_3 \cdot t_3)) \\
&= \nu_3(x^{r+\varepsilon}d^{s+\zeta}e_1) = |r+\varepsilon+s+\zeta| = |r+s| + |\varepsilon+\zeta| \\
&= \nu_3(\mathrm{LT}_{xd}(a)) + \nu_3(\mathrm{LT}_{xd}(f)) = \nu_3(a) + \nu_3(f).
\end{aligned}
$$

□

Lemma 28 generalizes to the statement that $\mathrm{A}_n(\mathbb{K})$ is a domain.

Corollary 11. *Let $a \in A$, $f \in F$. Then $a \cdot f = 0 \Rightarrow a = 0 \vee f = 0$.*

Proof. Assume $a \neq 0$ and $f \neq 0$. Let ν denote any of ν_1, ν_2, ν_3. Then $\nu(a) \geq 0$ and $\nu(f) \geq 0$. It follows that $\nu(a \cdot f) = \nu(a) + \nu(f) \geq 0$. Consequently $a \cdot f \neq 0$. □

Definition 20 (Filtration on $\mathrm{A}_n(\mathbb{K})$)**.** Let $A = \mathrm{A}_n(\mathbb{K})$, $F = A^{(E)}$. For $r, s \in \mathbb{N}$ we set

$$
F_r^{(k)} := \{f \in F : \ \nu_k(f) \leq r\}, \qquad F_{r,s} := F_r^{(1)} \cap F_s^{(2)}, \qquad k = 1, 2, 3.
$$

We will show that these sets define filtrations on F. Remark that we have defined implicitly $A_r^{(i)}$ and $A_{r,s}$ since we may consider A as the free module A^1.

Lemma 32. *Let $F_r^{(k)}$ be defined as in Definition 20.*

1. $\left(F_r^{(i)}\right)_{r \in \mathbb{N}}$ *defines a univariate filtration on F ($1 \leq i \leq 3$).*
2. $(F_{r,s})_{r,s \in \mathbb{N}}$ *defines a bivariate filtration on F.*

Proof. From (3.6) it is clear that all the sets $F_r^{(i)}$ - hence also the $F_{r,s}$ are monomial $\mathbb{K}$-vector spaces, that is, vector spaces with the property

$$
f \in F_r^{(i)} \Leftrightarrow \mathrm{T}(f) \subseteq F_r^{(i)}, \qquad 1 \leq i \leq 3.
$$

Immediate from Corollary 10 we obtain

$$
A_r^{(i)} \cdot F_s^{(i)} \subseteq F_{r+s}^{(i)}, \qquad 1 \leq i \leq 3.
$$

Therefore, also $A_{r,s}F_{t,u} \subseteq F_{r+t,s+u}$. □

Corollary 12. *For $r \in \mathbb{N}$ we have that $F_r^{(3)} \subseteq F_{r,r} \subseteq F_{2r}^{(3)}$.*

Proof. By monomiality, if $f \in F_r^{(3)}$ then $\mathrm{T}(f) \subseteq F^{(3)}$. Thus, for arbitrary $t \in \mathrm{T}(f)$, $\nu_1(t) + \nu_2(t) = \nu_3(t) \leq r$. Therefore, also $\nu_1(t) \leq r$ and $\nu_2(t) \leq r$, i.e. $t \in F_r^{(1)} \cap F_r^{(2)} = F_{r,r}$. Thus $\mathrm{T}(f) \subseteq F_{r,r}$ and so $f \in F_{r,r}$.
Now assume that $f \in F_{r,r}$. Then $\mathrm{T}(f) \subseteq F_{r,r}$. Therefore, if $t \in \mathrm{T}(f)$ then $\nu_3(t) = \nu_1(t) + \nu_2(t) \leq r + r$. Consequently $t \in F_{2r}^{(3)}$. This shows that $f \in F_{2r}^{(3)}$. □

3.1.1. (x, ∂)-Gröbner Bases

Dönch and Levin [DL12a] introduced the notion of (x, ∂)-Gröbner basis for free modules over $\mathrm{A}_n(\mathbb{K})$. We present their concepts here and show the relation to Gröbner reduction.

Remark. In section 1.4 we have used the symbols x_i and d_i as generators for the free algebra whose generators satisfy the commutation rules (1.33). In [DL12a] the Weyl algebra $\mathrm{A}_n(\mathbb{K})$ over $\mathbb{K}$ is considered, where $\mathrm{A}_n(\mathbb{K})$ is generated by $x_1, \dots, x_n$ and $\partial_1, \dots, \partial_n$ such that ∂_i plays the role of d_i. This explains, why their reduction is called (x, ∂)-reduction. Consequently, we should name the concept (x, d)-reduction. However, for the better reference to literature, we choose to stay along with the name (x, ∂)-reduction, although actually no ∂ appears in our considerations (as we have reserved it for Ore-operators).

If not explicitly mentioned differently, we consider for a monomial always an n-tuple

$$x^k d^l = x_1^{k_1} \dots x_n^{k_n} d_1^{l_1} \dots d_n^{l_n}, \qquad k = (k_1, \dots, k_n), l = (l_1, \dots, l_n) \in \mathbb{N}^n.$$

Lets consider [DL12a]. First the authors defined a divisibility notation in a non-standard way mimicking commutative monomials

$$x^k d^l | x^r d^s :\Leftrightarrow (k, l) \leq_\pi (r, s), \qquad k, l, r, s \in \mathbb{N}^n. \tag{3.7}$$

This notion extends to divisibility of monomials ΛE by defining $t_1 := x^k d^l e_1$, $t_2 := x^r d^s e_2$ of the free A-module $F = A^{(E)}$ by setting

$$t_1 | t_2 :\Leftrightarrow x^k d^l | x^r d^s \wedge e_1 = e_2.$$

In this case the quotient t_2/t_1 is the element $x^{r-k} d^{s-l} \in \Lambda$.

Lemma 33. *Let $t_1, t_2 \in \Lambda E$. Then*

$$t_1 | t_2 \Rightarrow \frac{t_2}{t_1} \cdot t_1 = t_2 + \sum_j n_j s_j, \qquad n_j \in \mathbb{N}^+, s_j \in \Lambda E$$

such that

$$\forall j : \nu_1(s_j) < \nu_1(t_2) \wedge \nu_2(s_j) < \nu_2(t_2).$$

Proof. $t_1 = x^k d^l e$, $t_2 = x^r d^s e$, $k \leq_\pi r$ and $l \leq_\pi s$. Using formula (3.5) gives

$$\begin{aligned}
\frac{t_2}{t_1} \cdot t_1 &= x^{r-k} d^{s-l} \cdot x^k d^l e = \sum_{v \leq_\pi s-l} \binom{s-l}{v} x^{r-k} d^v(x^k) d^{s-l+l-v} e \\
&= \binom{s-l}{0} x^{r-k} d^0(x^k) d^{s-0} e + \sum_{0 \neq v \leq_\pi s-l} \binom{s-l}{v} x^{r-k} d^v(x^k) d^{s-l+l-v} e \\
&= x^r d^s e + \sum_{0 \neq v \leq_\pi s-l} \binom{s-l}{v} x^{r-k} d^v(x^k) d^{s-v} e = t_2 + \sum_j n_j s_j
\end{aligned}$$

Since the index v in the previous line is in $\mathbb{N}^n \backslash \{0\}$ the conditions on the s_j are obvious. $\square$

Example 18. *For $n = 1$, i.e. in $\mathrm{A}_1(\mathbb{K})$, we have that x^2d^3 divides x^4d^4 in the sense of (3.7), and we can write*

$$\frac{x^4d^4}{x^2d^3} \cdot x^2d^3 = x^2d \cdot x^2d^3 = x^4d^4 + 2x^3d^3.$$

We have

$$3 = \nu_1(x^3d^3) < \nu_1(x^4d^4) = 4 \wedge 3 = \nu_2(x^3d^3) < \nu_2(x^4d^4) = 4.$$

Lemma 34. *Let $t_1, t_2, w \in \Lambda E$. Then*

$$t_1 \prec_x t_2 \wedge t_2|w \Rightarrow \left(\frac{w}{t_2} \cdot t_1\right)_w = 0.$$

Proof. Set $t_1 = x^\alpha d^\beta e_1$, $t_2 = x^\gamma d^\delta e_2$, $w = x^r d^s e_2$ such that $\gamma \leq_\pi r$ and $\delta \leq_\pi s$. From $t_1 \prec_x t_2$ we get $\nu_1(t_1) \leq \nu_1(t_2)$ whence $|\alpha| \leq |\gamma|$. From (3.5) we obtain

$$\begin{aligned}\frac{w}{t_2} \cdot t_1 = x^{r-\gamma}d^{s-\delta} \cdot x^\alpha d^\beta e_1 &= \sum_{u \leq_\pi s-\delta} \binom{s-\delta}{u} x^{r-\gamma} d^u (x^\alpha) d^{s-\delta+\beta-u} e_1 \\ &= \sum_{u \leq_\pi s-\delta} \binom{s-\delta}{u} \frac{\alpha!}{(\alpha-u)!} x^{r-\gamma+\alpha-u} d^{s-\delta+\beta-u} e_1\end{aligned}$$

To derive a contradiction assume that $\left(\frac{w}{t_2} \cdot t_1\right)_w \neq 0$. Then

$$\exists u \left(0 \leq_\pi u \leq_\pi s-\delta \wedge x^{r-\gamma+\alpha-u} d^{s-\delta+\beta-u} \cdot e_1 = x^r d^s \cdot e_2\right)$$

that is

$$e_1 = e_2, \qquad \alpha = \gamma + u \wedge \beta = \delta + u \leq_\pi s,$$

i.e. $u = \beta - \delta \geq_\pi 0$, $\delta \leq_\pi \beta$.

If $0 <_\pi u$ then $|\alpha| = |\gamma| + |u| > |\gamma|$, a contradiction. Therefore

$$u = 0, \qquad \alpha = \gamma \wedge \beta = \delta \wedge e_1 = e_2,$$

i.e. $t_1 = t_2$. This contradicts the assumption $t_1 \prec_x t_2$.

Consequently $\left(\frac{w}{t_2} \cdot t_1\right)_w = 0$. ∎

Let now $f, g, h \in F$ such that $g \neq 0$. (x, ∂)-reduction, defined in [DL12a], amounts to the following

$$\begin{aligned}& f \xrightarrow{(x,\partial)}_g h \iff \exists w \in \mathrm{T}(f): \\ & \left(\mathrm{LT}_x(g) \mid w \wedge h = f - \frac{f_w}{\mathrm{LC}_x(g)} \frac{w}{\mathrm{LT}_x(g)} g \ \wedge\ \nu_2\left(\frac{w}{\mathrm{LT}_x(g)} \mathrm{LT}_d(g)\right) \leq \nu_2(\mathrm{LT}_d(f))\right).\end{aligned} \tag{3.8}$$

Lemma 35. *Assume that* $f \xrightarrow{(x,\partial)}_g h$ *and let* w *be the term mentioned in* (3.8)*. Then* $w \notin \mathrm{T}(h)$*.*

Proof. Isolating the x-leader of g gives

$$\begin{aligned}
h &= f - \frac{f_w}{\mathrm{LC}_x(g)} \frac{w}{\mathrm{LT}_x(g)} \left(\mathrm{LC}_x(g)\mathrm{LT}_x(g) + \sum_{t \prec_x \mathrm{LT}_x(g)} g_t t \right) \\
&= f - \frac{f_w}{\mathrm{LC}_x(g)} \frac{w}{\mathrm{LT}_x(g)} \mathrm{LC}_x(g)\mathrm{LT}_x(g) - \frac{f_w}{\mathrm{LC}_x(g)} \frac{w}{\mathrm{LT}_x(g)} \sum_{t \prec_x \mathrm{LT}_x(g)} g_t t \\
&= f - f_w \frac{w}{\mathrm{LT}_x(g)} \mathrm{LT}_x(g) - \sum_{t \prec_x \mathrm{LT}_x(g)} \frac{f_w}{\mathrm{LC}_x(g)} \frac{w}{\mathrm{LT}_x(g)} g_t t
\end{aligned}$$

Application of Lemma 33 gives

$$\begin{aligned}
h &= f - f_w \left(w + \sum_j n_j s_j \right) - \sum_{t \prec_x \mathrm{LT}_x(g)} \frac{f_w}{\mathrm{LC}_x(g)} \frac{w}{\mathrm{LT}_x(g)} g_t t \\
&= f - f_w w - \sum_j n_j f_w s_j - \sum_{t \prec_x \mathrm{LT}_x(g)} \frac{f_w g_t}{\mathrm{LC}_x(g)} \frac{w}{\mathrm{LT}_x(g)} t
\end{aligned} \tag{3.9}$$

where all $n_j > 0$ and $\nu_1(s_j) < \nu_1(w)$ and $\nu_2(s_j) < \nu_2(w)$.

Considering (3.9), the coefficient of w in h is

$$h_w = f_w - f_w - 0 - \sum_{t \prec_x \mathrm{LT}_x(g)} \frac{f_w g_t}{\mathrm{LC}_x(g)} \left(\frac{w}{\mathrm{LT}_x(g)} \cdot t \right)_w .$$

Lemma 34 now immediately provides $h_w = 0$. □

It is now possible to relate (x, ∂)-reduction to Gröbner reduction.

Theorem 17. *Let* P *denote the predicate*

$$P(f, g, \lambda) :\Leftrightarrow \nu_2(\lambda \cdot g) \leq \nu_2(f)$$

Then,

$$f \xrightarrow{(x,\partial)}_g h \Leftrightarrow \exists \lambda \in \Lambda \left(\mathrm{LT}_x(\lambda g) \in \mathrm{T}(f) \wedge h = f - \frac{f_{\mathrm{LT}_x(\lambda g)}}{\mathrm{LC}_x(\lambda g)} \lambda \wedge P(f, g, \lambda) \right).$$

Consequently, using notation equation (3.1)*, we have that*

$$f \xrightarrow{(x,\partial)}_g h \Leftrightarrow f \longrightarrow_g^\rho h.$$

Proof. Observe that

$$P(f,g,\lambda) \Leftrightarrow \nu_2(\lambda \cdot \mathrm{LT}_d(g)) \leq \nu_2(\mathrm{LT}_d(f)), \qquad \text{cf. [DL12a]}$$

Let $f \overset{(x,\partial)}{\longrightarrow}_g h$ and set $\lambda = w/\mathrm{LT}_x(g)$, where w is the term mentioned in (3.8). Write

$$\mathrm{LT}_x(g) = x^k d^l e \quad w = x^{k+r} d^{l+s} e \Rightarrow \lambda = x^r d^s.$$

Then

$$\mathrm{LT}_x(\lambda g) = \mathrm{LT}_x(\lambda \cdot \mathrm{LT}_x(g)) = \mathrm{LT}_x(x^r d^s \cdot x^k d^l e) = x^{r+k} d^{s+l} e = w,$$

and $\mathrm{LC}_x(\lambda g) = \mathrm{LC}_x(g)$. It follows that

$$h = f - \frac{f_w}{\mathrm{LC}_x(g)} \frac{w}{\mathrm{LT}_x(g)} g = f - \frac{f_{\mathrm{LT}_x(\lambda g)}}{\mathrm{LC}_x(\lambda g)} \lambda g \qquad \wedge \qquad \mathrm{LT}_x(\lambda g) = w \in \mathrm{T}(f).$$

Since $f \overset{(x,\partial)}{\longrightarrow}_g h$ holds, the predicate $P(f,g,\lambda)$ is true. Consequently $f \longrightarrow_g^\rho h$.

Conversely assume that $f \longrightarrow_g^\rho h$. Let $\lambda \in \Lambda$ be such that $\mathrm{LT}_x(\lambda g) \in \mathrm{T}(f)$ and

$$h = f - \frac{f_{\mathrm{LT}_x}(\lambda g)}{\mathrm{LC}_x(\lambda g)} \lambda g \wedge P(f,g,\lambda).$$

Set $w = \mathrm{LT}_x(\lambda g)$. Then $w \in \mathrm{T}(f)$. Write λ as $\lambda = x^u d^v$ and $\mathrm{LT}_x(g) = x^k d^l e$. Then

$$w = \mathrm{LT}_x(\lambda \cdot \mathrm{LT}_x(g)) = \mathrm{LT}_x(x^u d^v \cdot x^k d^l e) = x^{u+k} d^{v+l} e.$$

Thus $\mathrm{LT}_x(g) | w$ and $w/\mathrm{LT}_x(g) = x^u d^v = \lambda$. Since $\mathrm{LC}_x(\lambda g) = 1 \cdot \mathrm{LC}_x(g)$ we obtain

$$h = f - \frac{f_w}{\mathrm{LC}_x(g)} \frac{w}{\mathrm{LT}_x(g)} g \wedge \nu_2 \left(\frac{w}{\mathrm{LT}_x(g)} \mathrm{LT}_d(g) \right) \leq \nu_2(\mathrm{LT}_d(f)).$$

Consequently $f \overset{(x,\partial)}{\longrightarrow}_g h$. □

Lemma 36. *I_ρ is monomial.*

Proof. Let $\mathrm{LT}_x(\lambda g) \in \mathrm{T}(f)$ and $P(\mathrm{LT}_x(\lambda g), g, \lambda)$ hold. This means that

$$\nu_2(\lambda \cdot g) \leq \nu_2(\mathrm{LT}_x(\lambda \cdot g)).$$

Since $\mathrm{LT}_x(\lambda g) \in \mathrm{T}(f)$ it follows that $\mathrm{LT}_x(\lambda g) \preccurlyeq_d \mathrm{LT}_d(f)$, and therefore

$$\nu_2(\mathrm{LT}_x(\lambda g)) \leq \nu_2(\mathrm{LT}_d(f)) = \nu_2(f).$$

Thus, $\nu_2(\lambda g) \leq \nu_2(f)$. Consequently $\mathrm{LT}_x(\lambda g) \in \mathrm{T}(f) \wedge P(f,g,\lambda)$. This demonstrates that the monomial irreducibility condition (3.4) is satisfied. □

In [DL12a] the authors define Gröbner bases differently. Formulated in our notation:

Definition 21. Let N be a submodule of $F = \mathbb{K}^{(\Lambda E)}$ and $G \subseteq N\backslash\{0\}$. G is a (x,∂)-Gröbner basis for N if and only if

$$\forall f \in N\backslash\{0\}\ \exists g \in G\,(\mathrm{LT}_x(g)|\mathrm{LT}_x(f) \wedge \nu_2(g) - \nu_2(\mathrm{LT}_x(g)) \leq \nu_2(f) - \nu_2(\mathrm{LT}_x(f)))\,.$$

We position this notion into the frame of our concepts.

Theorem 18. *Given $G \subseteq N$, let σ denote the leading term reduction corresponding to the relation $f \overset{(x,\partial)}{\longrightarrow}_G h$, i.e.*

$$f \longrightarrow_g^\sigma h :\Leftrightarrow \exists \lambda \in \Lambda \left(\mathrm{LT}_x(\lambda g) = \mathrm{LT}_x(f) \wedge h = f - \frac{f_{\mathrm{LC}_x(\lambda g)}}{\mathrm{LC}_x(\lambda g)} \lambda g \wedge \nu_2(\lambda g) \leq \nu_2(f) \right);$$

Then G is an (x,∂)-Gröbner basis for N if and only if $I_\sigma \cap N = 0$.

Proof. We fix some notation. Elements $f, g \in F\backslash\{0\}$ are written

$$f = f_0 t_0 + \sum_{t \prec_x t_0} f_t t = f_0' t_0' + \sum_{t \prec_d t_0'} f_t t$$
$$g = g_1 t_1 + \sum_{t \prec_x t_1} g_t t = g_1' t_1' + \sum_{t \prec_d t_1'} g_t t$$

with coefficients $f_0, f_t, f_0', g_1, g_t, g_1' \in \mathbb{K}$, i.e. we've got the leading terms

$$\mathrm{LT}_x(f) = t_0 = x^{k_0} d^{l_0} e_0, \qquad \mathrm{LT}_d(f) = t_0' = x^{k_0'} d^{l_0'} e_0'$$
$$\mathrm{LT}_x(g) = t_1 = x^{r_1} d^{s_1} e_1, \qquad \mathrm{LT}_d(g) = t_1' = x^{r_1'} d^{s_1'} e_1'.$$

Assume that G is an (x,∂)-Gröbner basis for N and $f \in N\backslash\{0\}$. $\exists g \in G$ such that

$$\mathrm{LT}_x(g)|\mathrm{LT}_x(f) \wedge \nu_2(g) - \nu_2(\mathrm{LT}_x(g)) \leq \nu_2(f) - \nu_2(\mathrm{LT}_x(f))$$

Using the notation from above we get

$$x^{r_1} d^{s_1} e_1 | x^{k_0} d^{l_0} e_0 \wedge s_1' - s_1 \leq_\pi l_0' - l_0,$$

hence $r_1 \leq_\pi k_0 \wedge s_1 \leq_\pi l_0 \wedge e_1 = e_0$. Set $\lambda = x^{k_0 - r_1} d^{l_0 - s_1}$. Then

$$\mathrm{LT}_x(\lambda \cdot g) = \mathrm{LT}_x(x^{k_0 - r_1} d^{l_0 - s_1} \cdot x^{r_1} d^{s_1} e_1) = x^{k_0} d^{l_0} e_0 = \mathrm{LT}_x(f).$$

Further, we get

$$\nu_2(\lambda \cdot g) = \nu_2(\mathrm{LT}_d(x^{k_0 - r_1} d^{l_0 - s_1} \cdot x^{r_1'} d^{s_1'} e_1')) = |l_0 - s_1 + s_1'| \leq |l_0'| = \nu_2(f),$$

(note that $|\cdot|$ implies summation of the entries of the multi-index, and not absolute values). This shows that

$$f \longrightarrow_g^\sigma f - \frac{\mathrm{LC}_x(f)}{\mathrm{LC}_x(\lambda g)} \lambda g,$$

that means, f is σ-reducible. Consequently, each $f \in N\backslash\{0\}$ is σ-reducible whence $N \cap I_\sigma = 0$.

Conversely, assume that $N \cap I_\sigma = 0$ and let $f \in N\backslash\{0\}$ is σ-reducible, $\exists g \in G \exists \lambda \in \Lambda$ such that

$$\mathrm{LT}_x(\lambda \cdot g) = \mathrm{LT}_x(f) \wedge h = f - \frac{\mathrm{LC}_x(f)}{\mathrm{LC}_x(\lambda g)}\lambda g \wedge \nu_2(\lambda \cdot g) \leq \nu_2(f). \tag{3.10}$$

Write $\lambda = x^a d^b$. From (3.10) we get

$$x^{k_0} d^{l_0} e_0 = \mathrm{LT}_x(f) = \mathrm{LT}_x(\lambda g) = \mathrm{LT}_x(x^a d^b \cdot x^{r_1} d^{s_1} e_1),$$

hence

$$k_0 = a + r_1 \wedge l_0 = b + s_1 \wedge e_0 = e_1 \Rightarrow r_1 \leq_\pi k_0 \wedge s_1 \leq_\pi l_0,$$

i.e. $\mathrm{LT}_x(g) | \mathrm{LT}_x(f)$. Moreover

$$\nu_2(\lambda g) = \nu_2(\mathrm{LT}_d(\lambda g)) = \nu_2(\mathrm{LT}_d(x^a d^b \cdot x^{r_1'} d^{s_1'} e_1')) = b + s_1'.$$

From (3.10) we obtain $b + s_1' \leq l_0'$ and so

$$l_0 + s_1' = b + s_1 + s_1' \leq l_0' + s_1,$$

Therefore,

$$\begin{aligned}\nu_2(g) - \nu_2(\mathrm{LT}_x(g)) &= \nu_2(x^{r_1'} d^{s_1'} e_1') - \nu_2(x^{r_1} d^{s_1} e_1) = s_1' - s_1 \\ &\leq l_0' - l_0 = \nu_2(f) - \nu_2(\mathrm{LT}_x(f)).\end{aligned}$$

Therefore G is an (x, ∂)-Gröbner basis for N. □

Corollary 13. *Let ρ denote (x, ∂)-reduction* (3.8) *for N. If G is an (x, ∂)-Gröbner basis for N then ρ is a weak reduction for N. Thus, an (x, ∂)-Gröbner basis for N is a Gröbner basis for N w.r.t. ρ.*

Proof. Consider an (x, ∂)-Gröbner basis for N. Since $I_\rho \subseteq I_\sigma$ we obtain that $I_\rho \cap N = 0$. Together with Lemma 36 this says that ρ is a weak reduction for N. □

Even when G is a Gröbner basis, the corresponding reduction relation ρ is in general not a strong reduction.

Example 19. *Consider $A = \mathrm{A}_1(\mathbb{K})$, $g = xd + d^2 \in A$. Let ρ be the (x, ∂)-reduction defined by $G = \{g\}$ and $N = Ag$.*

We show that $N \cap I_\rho = 0$. Let $a \in A$

$$a = a_0 x^{k_0} d^{l_0} + \sum_{\mu \prec_x x^{k_0} d^{l_0}} a_\mu \mu \qquad a_0 \neq 0.$$

Then,

$$\begin{aligned} a \cdot g &= a_0 x^{k_0} d^{l_0}(xd + d^2) + \sum_{\mu \prec_x x^{k_0} d^{l_0}} a_\mu \mu (xd + d^2) \\ &= a_0 x^{k_0} d^{l_0} \cdot xd + a_0 x^{k_0} d^{l_0} \cdot d^2 + \sum_{\mu \prec_x x^{k_0} d^{l_0}} (a_\mu \mu xd + a_\mu \mu d^2) \end{aligned}$$

Set $\lambda = x^{k_0} d^{l_0}$. *Then* $\mathrm{LT}_x(\lambda \cdot g) = \mathrm{LT}_x(x^{k_0} d^{l_0} \cdot (xd + d^2)) = x^{k_0+1} d^{l_0+1}$. *Hence,*

$$\mathrm{LT}_x(a \cdot g) = \mathrm{LT}_x(\mathrm{LT}_x(a) \cdot \mathrm{LT}_x(g)) = \mathrm{LT}_x(x^{k_0} d^{l_0} \cdot xd) = x^{k_0+1} d^{l_0+1}$$

and thus $\mathrm{LT}_x(\lambda g) = \mathrm{LT}_x(a \cdot g) \in \mathrm{T}(a \cdot g)$. *On the other hand, we have*

$$\nu_2(\lambda g) = \nu_2(\mathrm{LT}_d(\lambda g)) = \nu_2(\mathrm{LT}_d(\lambda \cdot \mathrm{LT}_d(g))) = \nu_2(\mathrm{LT}_d(x^{k_0} d^{l_0} \cdot d^2)) = l_0 + 2,$$

hence

$$\nu_2(a \cdot g) = \nu_2(\mathrm{LT}_d(a \cdot g)) = \nu_2(\mathrm{LT}_d(\mathrm{LT}_d(a) \cdot \mathrm{LT}_d(g))) = \nu_2(\mathrm{LT}_d(a) \cdot d^2).$$

Now, $x^{k_0} d^{l_0} \preccurlyeq_d \mathrm{LT}_d(a)$. *Applying Lemma 30 gives*

$$x^{k_0} d^{l_0+2} = \mathrm{LT}_d(x^{k_0} d^{l_0} \cdot d^2) \preccurlyeq_d \mathrm{LT}_d(\mathrm{LT}_d(a) d^2).$$

Therefore,

$$l_0 + 2 = \nu_2(x^{k_0} d^{l_0+2}) \leq \nu_2(\mathrm{LT}_d(\mathrm{LT}_d(a) \cdot d^2)) = \nu_2(\mathrm{LT}_d(a) \cdot d^2),$$

and so $\nu_2(\lambda g) \leq \nu_2(a \cdot g)$. *All in all,*

$$\exists \lambda \in \Lambda \left(\mathrm{LT}_x(\lambda g) = \mathrm{LT}_x(ag) \wedge \nu(\lambda g) \leq \nu_2(ag) \right),$$

and choosing h *appropriately we see that* $ag \longrightarrow_g^\sigma h$, *hence* ag *is* σ*-reducible. Therefore* $N \cap I_\sigma = N \cap I_\rho = 0$. *Consequently* ρ *is a weak reduction for* $N = Ag$ *and* $\{g\}$ *a Gröbner basis.*

Now consider $f_1, f_2, g \in A$

$$f_1 = xd, \qquad f_2 = d^2.$$

Then it is obvious that $f_1, f_2 \in I_\rho$. *But* $f_1 + f_2$ *is not:*

$$\mathrm{LT}_x(1 \cdot g) = xd \in \mathrm{T}(f) \wedge \nu_2(1 \cdot g) = 2 \leq \nu_2(f_1 + f_2),$$

i.e. $f_1 + f_2$ *reduces to zero (w.r.t.* ρ*), so* $f_1 + f_2$ *is* $\notin I_\rho$. *This shows that* I_ρ *is not closed under addition and therefore* ρ *is not a strong reduction for* N.

Theorem 19. *Let $f, g, h \in F$ such that $g \neq 0$. Assume that $f \xrightarrow{(x,\partial)}_g h$. Then, for arbitrary $r, s \in \mathbb{N}$*

1. $f \in F_r^{(1)} \Rightarrow h \in F_r^{(1)}$;
2. $f \in F_r^{(2)} \Rightarrow h \in F_r^{(2)}$;
3. $f \in F_{r,s} \Rightarrow h \in F_{r,s}$;
4. $f \in F_r^{(3)} \Rightarrow h \in F_{2r}^{(3)}$;

Consequently the full reduction corresponding to an (x,∂)-Gröbner basis for a submodule $N \subseteq F$ is a weak Gröbner reduction for N with respect to these filtrations.

Proof. By Lemma 31 we may assume that $f \longrightarrow_g^\rho h$, i.e.

$$\exists \lambda \in \Lambda \left(\mathrm{LT}_x(\lambda g) \in \mathrm{T}(f) \wedge h = f - \frac{f_{\mathrm{LT}_x(\lambda g)}}{\mathrm{LC}_x(\lambda g)} \lambda g \wedge P(f, g, \lambda) \right).$$

1. Assume that $f \in F_r^{(1)}$. Then $\nu_1(f) \leq r$, whence $\forall t \in \mathrm{T}(f) : \nu_1(t) \leq r$. Take $t \in \mathrm{T}(h)$. If $t \in \mathrm{T}(f)$ then $\nu_1(t) \leq r$. If $t \notin \mathrm{T}(f)$ then

   $$0 \neq h_t = -\frac{f_{\mathrm{LT}_x(\lambda g)}}{\mathrm{LC}_x(\lambda g)} (\lambda g)_t$$

   Thus

   $$(\lambda g)_t \neq 0 \Rightarrow t \in \mathrm{T}(\lambda g) \Rightarrow t \preccurlyeq_x \mathrm{LT}_x(\lambda g) \in \mathrm{T}(f).$$

   Therefore

   $$\nu_1(t) \leq \nu_1(\mathrm{LT}_x(\lambda g)) \leq r \Rightarrow \mathrm{T}(h) \subseteq F_r^{(1)} \Rightarrow h \in F_r^{(1)}.$$

2. Let $f \in F_r^{(2)}$. Then $\mathrm{T}(f) \subseteq F_r^{(2)}$. Writing out the predicate P we obtain

   $$\nu_2(\lambda \cdot \mathrm{LT}_d(g)) \leq \nu_2(\mathrm{LT}_d(f)).$$

   Take $t \in \mathrm{T}(h)$. If $t \in \mathrm{T}(f)$ then $t \in F_r^{(2)}$. If $t \notin \mathrm{T}(f)$ then, with the same argument as in the previous case, we obtain $t \in \mathrm{T}(\lambda g)$. Therefore $t \preccurlyeq_d \mathrm{LT}_d(\lambda g) = \mathrm{LT}_d(\lambda \cdot \mathrm{LT}_d(g))$. Then we derive

   $$\nu_2(t) \leq \nu_2(\mathrm{LT}_d(\lambda \cdot \mathrm{LT}_d(g)) = \nu_2(\lambda \cdot \mathrm{LT}_d(g)) \leq \nu_2(\mathrm{LT}_d(f)),$$

   whence $\nu_2(t) \leq r$, that is, $t \in F_r^{(2)}$. This shows that $\mathrm{T}(h) \subseteq F_r^{(2)}$. Since the filter-sets are vector spaces we arrive at $h \in F_r^{(2)}$.

3. If $f \in F_{r,s}$ then $f \in F_r^{(1)} \cap F_s^{(2)}$. Therefore also $h \in F_r^{(1)} \cap F_s^{(2)} = F_{r,s}$.

4. This follows from Corollary 12 and the previous point.

□

3.2. The Ring of Difference-Differential Operators

At the introductory chapter 1, we have encountered the ring of difference-differential operators over a field $\mathbb{K}$. In this section, we consider the situation where we have a commutative ring K contained in the considered ring D, and we are given a tuple $\delta = (\delta_1, \ldots, \delta_m)$ of derivations and a tuple of automorphisms $\sigma = (\sigma_1, \ldots, \sigma_n)$ of K. All these maps are assumed to commute with each other. The ring D is then constructed as the free K-module on the set of formal expressions

$$\Lambda_{m,n} := \{\delta^k \sigma^l = \delta_1^{k_1} \cdots \delta_m^{k_m} \sigma_1^{l_1} \cdots \sigma_n^{l_n}, \quad (k_i \in \mathbb{N}, l_i \in \mathbb{Z})\}$$

and a product that reflects the properties of derivations and automorphisms. We consider the elements of the set $\Lambda_{m,n}$ as the distinguished monomials, and write Λ for $\Lambda_{m,n}$. Consequently elements of D are finite K-linear combinations

$$\sum_{(k,l) \in \mathbb{N}^m \times \mathbb{Z}^n} a_{k,l} \delta^k \sigma^l, \qquad a_{k,l} \in K,$$

and the product is driven by the rules

$$\delta_i \cdot c = c \cdot \delta_i + \delta_i(c) \qquad \sigma_j \cdot c = \sigma_j(c) \sigma_j, \qquad c \in K.$$

A left module over D is also called a *difference-differential module*, or $\Delta\Sigma$-module over K.[1] The concept covers difference modules ($\Delta = \emptyset$) as well as differential modules ($\Sigma = \emptyset$) as special instances. But we can also make a link to the Weyl-algebra, considered in the previous section.

Lemma 37. *Consider a field* $\mathbb{K}$ *with* $\operatorname{char}(\mathbb{K}) = 0$*. Let*

1. $K = \mathbb{K}[x_1, \ldots, x_n]$;
2. $\Delta = \left\{ \frac{\mathrm{d}}{\mathrm{d}x_1}, \ldots, \frac{\mathrm{d}}{\mathrm{d}x_n} \right\}$;
3. $\Sigma = \emptyset$.

Then the resulting $\Delta\Sigma$*-ring is the Weyl-algebra* $\mathrm{A}_n(\mathbb{K})$*.*

Proof. This is due to the fact that partial derivatives have no relations among each other. Precisely: Let $\Delta^\star$ be the monoid generated (in $\operatorname{End}_{\mathbb{K}}(K)$) by Δ. Then $\Delta^\star \cong \mathbb{N}^n$ and $\mathrm{A}_n(\mathbb{K})$ is a free K-module with basis $\Delta^\star$. □

We continue our consideration with the choice $K = \mathbb{K}$. We use the notation

$$y^k = \delta^k(y) \text{ and } y_s = \sigma^s(y), \qquad k \in \mathbb{N}^m,\ s \in \mathbb{Z}^n.$$

[1] In the literature the tuples δ and σ are denoted informally as the sets Δ and Σ, whence the name. Note though, that the mappings δ_i need not be distinct. The same is the case with the σ_j.

The product in D can then be written explicitely

$$\delta^k\sigma^l \cdot y \cdot \delta^r\sigma^s e = \sum_{u \leq_\pi k} \binom{k}{u} y_l^{k-u} \cdot \delta^{u+r}\sigma^{l+s} e, \qquad y \in K.$$

For $\lambda = \delta^k\sigma^l \in \Lambda$ we set

$$\nu_1(\lambda) = k_1 + \ldots + k_m, \qquad \nu_2(\lambda) = |l_1| + \ldots + |l_n|, \qquad \nu_0(\lambda) = \nu_1(\lambda) + \nu_2(\lambda). \quad (3.11)$$

The extensions of these functions to D, given by

$$\nu_j : D \to \mathbb{N}, \qquad a \mapsto \max\{\nu_j(\lambda) : \lambda \in \mathrm{T}(a)\}$$

induce the univariate filtrations

$$D_r^{(j)} = \{a \in D : \nu_j(a) \leq r\} \qquad j = 0, 1, 2. \quad (3.12)$$

Lemma 38. *Let D be filtered as in* (3.12), *F a free D-module.*

- *The family $\left(D_r^{(j)}\right)_{r \in \mathbb{N}}$ is a monomial filtration on D,* $\qquad j = 0, 1, 2.$
- $\left(F_r^{(j)}\right)_{r \in \mathbb{N}}$ *is a monomial filtration on F,* $\qquad j = 0, 1, 2.$

Proof. The proof is a variation of Example 7. □

Fix an enumeration of the set E and set

$$t = \delta^k\sigma^l e_i \mapsto (\nu_j(t), i, k_1, \ldots, k_m, |l_1|, \ldots, |l_n|, \mathrm{sgn}(l_1) + 1, \ldots, \mathrm{sgn}(l_n) + 1) \qquad j = 0, 1, 2.$$

The corresponding well-orders for monomials $s = \delta^k\sigma^l e_{i_1}$, $t = \delta^r\sigma^s e_{i_2}$ in ΛE are now

$$\begin{aligned} s \prec_j t \quad &:\Leftrightarrow \\ &(\nu_j(s), i_1, k_1, \ldots, k_m, |l_1|, \ldots, |l_n|, \mathrm{sgn}(l_1) + 1, \ldots, \mathrm{sgn}(l_n) + 1) \\ &<_{\mathrm{lex}} \\ &(\nu_j(t), i_2, r_1, \ldots, r_m, |s_1|, \ldots, |s_n|, \mathrm{sgn}(s_1) + 1, \ldots, \mathrm{sgn}(s_n) + 1) \end{aligned}$$

Then

$$s \preccurlyeq_j t \Rightarrow \nu_j(s) \leq \nu_j(t), \qquad j = 0, 1, 2.$$

These orders single out $\mathrm{LT}_j(f)$ and $\mathrm{LC}_j(f)$ for each $f \in F \backslash \{0\}$. According to (3.1) we get

$$f \longrightarrow_g^{\rho_j} h \Leftrightarrow \exists \lambda \in \Lambda \Big(\mathrm{LT}_j(\lambda g) \in \mathrm{T}(f) \wedge h = f - \frac{f_{\mathrm{LT}_j(f)}}{\mathrm{LC}_j(\lambda g)} \lambda g \Big)$$

and for $G \subseteq F$ the reduction ρ_j is

$$f \longrightarrow_G^{\rho_j} h \Leftrightarrow \exists g \in G \text{ such that } f \longrightarrow_g^{\rho_j} h.$$

Note that the predicate 'P' mentioned in (3.1) is empty here, i.e. we may set $P = \mathrm{TRUE}$.

Lemma 39. *If* $f \longrightarrow_G^{\rho_j} h = f - \lambda g$ *and* $f \in F_r^{(j)}$ *then* $h \in F_r^{(j)}$.

Proof. There exists $g \in G$ and $\lambda \in \Lambda$ such that $\mathrm{LT}_j(\lambda g) \in \mathrm{T}(f)$. Therefore, from monomiality of the filtration, we get $\mathrm{LT}_j(\lambda g) \in F_r^{(j)}$. Let $b \in \mathrm{T}(\lambda g)$ be an arbitrary monomial. Then from $b \preccurlyeq_j \mathrm{LT}_j(\lambda g)$ we obtain $\nu_j(b) \leq \nu_j(\mathrm{LT}_j(\lambda g)) \leq r$, that is, $b \in F_r^{(j)}$. Consequently $\lambda g \in F_r$, and so is $h = f - c \cdot \lambda g$. □

Together with the previous remarks, the last Lemma exhibits the relations ρ_j as Gröbner reductions.

3.2.1. Relative Reduction

In [ZW07a] the filtration $\left(F_r^{(0)}\right)_{r \in \mathbb{N}}$ is treated by using a variant of the term order $\prec_0$ and its corresponding reduction. In [ZW08a] the bivariate filtration $D_{r,s} = D_r^{(1)} \cap D_s^{(2)}$ occurs. For the purpose of reduction the following two term orders have been used. For monomials $u = \delta^k \sigma^l e_i$ and $v = \delta^r \sigma^s e_j$ in ΛE, set

$$\begin{aligned} u \prec_1 v \quad &\Leftrightarrow \\ &(\nu_2(u), \nu_1(u), i, k_1, \ldots, k_m, |l_1|, \ldots, |l_n|, \mathrm{sgn}(l_1) + 1, \ldots, \mathrm{sgn}(l_n) + 1) \\ &<_{\mathrm{lex}} \\ &(\nu_2(v), \nu_1(v), j, r_1, \ldots, r_m, |s_1|, \ldots, |s_n|, \mathrm{sgn}(s_1) + 1, \ldots, \mathrm{sgn}(s_n) + 1) \end{aligned} \tag{3.13}$$

respectively

$$\begin{aligned} u \prec_2 v \quad &\Leftrightarrow \\ &(\nu_1(u), \nu_2(u), i, k_1, \ldots, k_m, |l_1|, \ldots, |l_n|, \mathrm{sgn}(l_1) + 1, \ldots, \mathrm{sgn}(l_n) + 1) \\ &<_{\mathrm{lex}} \\ &(\nu_1(v), \nu_2(v), j, r_1, \ldots, r_m, |s_1|, \ldots, |s_n|, \mathrm{sgn}(s_1) + 1, \ldots, \mathrm{sgn}(s_n) + 1) \end{aligned} \tag{3.14}$$

The appropriate reduction concept - called *relative reduction* in [ZW08a] - takes into account both of these orders. Let $f, g, h \in F$. Then $f \xrightarrow{\mathrm{rel}}_g h$ if and only if

$$\exists \lambda \in \Lambda \Big(\mathrm{LT}_1(\lambda g) = \mathrm{LT}_1(f) \wedge \mathrm{LT}_2(\lambda g) \preccurlyeq_2 \mathrm{LT}_2(f) \wedge h = f - \frac{\mathrm{LC}_1(f)}{\mathrm{LC}_1(\lambda g)} \lambda g \Big).$$

Here we meet leading term reduction (3.2) involving the predicate

$$P(f, g, \lambda) \Leftrightarrow \mathrm{LT}_2(\lambda g) \preccurlyeq_2 \mathrm{LT}_2(f).$$

Again, for $G \subseteq F$ relative reduction is

$$f \xrightarrow{\mathrm{rel}}_G h \Leftrightarrow \exists g \in G \text{ with } f \xrightarrow{\mathrm{rel}}_g h.$$

Theorem 20. *Let the orders $\prec_1$ and $\prec_2$ be given by* (3.13) *resp.* (3.14)*, consider the bivariate filtration $F_{r,s}$ of F, given by*

$$F_{r,s} = \bigoplus_{e \in E} D_{r,s} e,$$

induced by the filtration $(D_{r,s})_{(r,s)\in\mathbb{N}^2}$ on D, the set $D_{r,s}$ as in Example 7. Then

$$f \xrightarrow{\text{rel}}_g h \wedge f \in F_{r,s} \ \Rightarrow h \in F_{r,s}.$$

Consequently, the full reduction associated to $\xrightarrow{\text{rel}}_G$ is a Gröbner reduction.

Proof. Assume $f \xrightarrow{\text{rel}}_g h$ and $f \in F_{r,s}$. Thus $|f|_1 \leq r$ and $|f|_2 \leq s$. We set

$$u := \mathrm{LT}_1(f) = \mathrm{LT}_1(\lambda g), \quad u' := \mathrm{LT}_2(f), \quad c = \mathrm{LC}_1(\lambda g).$$

Thus we may write

$$\begin{array}{rcl} f &=& f_u u + \varphi = f_{u'} u' + \varphi' \\ \lambda g &=& cu + \psi. \end{array}$$

From the assumption we obtain that $\lambda g \preccurlyeq_2 u'$ and

$$h = f - \frac{\mathrm{LC}_1(f)}{\mathrm{LC}_1(\lambda g)} \lambda g = f_u u + \varphi - \frac{f_u}{c}(cu + \psi) = \varphi - \frac{f_u}{c}\psi.$$

Therefore

$$\mathrm{T}(h) \subseteq \mathrm{T}(\varphi) \cup \mathrm{T}(\psi) = \big(\mathrm{T}(f) \cup \mathrm{T}(\lambda g)\big) \setminus \{u\}.$$

Take $\mu \in \mathrm{T}(h)$. If $\mu \in \mathrm{T}(f)$ then $|\mu|_1 \leq r \wedge |\mu|_2 \leq s$. If $\mu \in \mathrm{T}(\lambda g)$ then, since $\lambda g \preccurlyeq_2 u'$, we obtain $\mu \preccurlyeq_2 u'$ and therefore $|\mu|_1 \leq |u'|_1 \leq r$. Because $u = \mathrm{LT}_1(\lambda g)$ we obtain $\mu \prec_1 u$ and thus $|\mu|_2 \leq |u|_2 \leq s$. So in any case we obtain $|\mu|_1 \leq r \wedge |\mu|_2 \leq s$, that is, $|h|_1 \leq r \wedge |h|_2 \leq s$. Therefore $h \in F_{r,s}$. Obviously $f \xrightarrow{\text{rel}}_g h$ implies that $\mathrm{LT}_1(h) \prec_1 \mathrm{LT}_1(f)$. Consequently $\xrightarrow{\text{rel}}_G$ is a noetherian reduction compatible with the filtration. □

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3.2.2. Computation of Multivariate Δ-Σ Dimension Polynomials

In the general case, Corollary 6 applies. We will generalize the dimension polynomial computed in [ZW07a, ZW08a, Lev12, Lev13]. To that end, we will set up a refined filtration of the ring D, controlled by a partition of the basic operators in the difference-differential field (K, Δ, Σ). Again, K is actually the field $\mathbb{K}$. After designing a Gröbner reduction for a submodule, Corollary 6 will give us an improved picture of the filter spaces in the quotient.

Consider the sets

$$\Delta = \{\delta_1, \ldots, \delta_m\}, \qquad\qquad \Sigma = \{\sigma_1, \ldots, \sigma_n\}$$

of the difference-differential field (K, Δ, Σ). We divide Δ and Σ into p respectively q pairwise disjoint subsets

$$\Delta = \Delta_1 \cup \cdots \cup \Delta_p, \qquad \Sigma = \Sigma_1 \cup \cdots \cup \Sigma_q \tag{3.15}$$

where

$$\Delta_1 = \{\delta_1, \ldots, \delta_{m_1}\} \qquad \Delta_k = \{\delta_{m_1+\cdots+m_{k-1}+1}, \ldots, \delta_{m_1+\cdots+m_k}\}, \qquad 2 \le k \le p,$$

and $m_1 + \cdots + m_p = m$. Similar for Σ

$$\Sigma_1 = \{\sigma_1, \ldots, \sigma_{n_1}\} \qquad \Sigma_k = \{\sigma_{n_1+\cdots+n_{k-1}+1}, \ldots, \sigma_{n_1+\cdots+n_k}\}, \qquad 2 \le k \le q$$

where $n_1 + \cdots + n_q = n$.

Definition 22. For a monomial $\lambda = \delta_1^{k_1} \cdots \delta_m^{k_m} \sigma_1^{l_1} \cdots \sigma_n^{l_n} \in \Lambda_{m,n}$ we define

$$|\lambda|_{\Delta_r} = \sum_{\delta_i \in \Delta_r} k_i, \qquad 1 \le r \le p, \qquad |\lambda|_{\Sigma_s} = \sum_{\sigma_i \in \Sigma_s} |l_i|, \qquad 1 \le s \le q.$$

For a general difference-differential operator we set

$$|a|_\Phi := \max\{|\lambda|_\Phi \colon \lambda \in \mathrm{T}(a)\}, \qquad \Phi \in \{\Delta_1, \ldots, \Delta_p, \Sigma_1, \ldots, \Sigma_q\}.$$

The following device defines a $p+q$-variate filtration on D. For $r \in \mathbb{N}^{p+q}$ set

$$D_r = \{u \in D \colon \forall i: \ 1 \le i \le p: \ |u|_{\Delta_i} \le r_i \wedge \forall j: \ 1 \le j \le q: \ |u|_{\Sigma_j} \le r_{p+j}\}. \tag{3.16}$$

We reproduce now the result obtained in [FL15a, Theorem 2].

Theorem 21. *Let K be a $\Delta\Sigma$-field and M a finitely generated difference-differential module. Produce a partition of the sets Δ, Σ as described in (3.15) and equip the operator ring D with the filtration described in (3.16). Extend the filtration to the finite free presentation*

$$0 \longrightarrow N \longrightarrow F \xrightarrow{\pi} M \longrightarrow 0$$

where F has K-basis E, and let $\prec$ be a generalized term order on ΛE.

If G is a Gröbner basis of N then the cardinality of the set

$$U_r = \{t \in \Lambda E \cap F_r \colon \forall g \in G \, \forall \lambda \in \Lambda$$
$$(t = \mathrm{LT}_\prec(\lambda g) \Rightarrow \exists i : |\lambda g|_{\Delta_i} > r_i \vee \exists j : |\lambda g|_{\Sigma_j} > r_{p+j})\}$$

provides the values of the Hilbert function of M, i.e.,

$$\dim_K M_r = |U_r| \quad \forall r \in \mathbb{N}^{p+q}.$$

Proof. The relation

$$\begin{aligned} f \longrightarrow h \Leftrightarrow \exists g \in G\ \exists \lambda \in \Lambda\ \mathrm{LT}_\prec(\lambda g) = \mathrm{LT}_\prec(f) \qquad & \text{and} \\ \forall i:\ 1 \le i \le p\, |\lambda g|_{\Delta_i} \le |f|_{\Delta_i} \wedge \forall j:\ 1 \le j \le q\, |\lambda g|_{\Sigma_j} \le |f|_{\Sigma_j} \qquad & \text{and} \\ h = f - \frac{\mathrm{LC}_\prec(f)}{\mathrm{LC}_\prec(\lambda g)} \lambda g & \end{aligned}$$

defines a Gröbner reduction for N and Corollary 6 is applicable. □

3.3. The Ring of Ore Polynomials

As we've indicated in the introduction, we continue now to consider a structure over the commutative ring K. Given a K-endomorphism $\sigma\colon K \to K$, a σ-skew derivation is an additive map $\delta\colon K \to K$ satisfying

$$\delta(ab) = \sigma(a)\delta(b) + \delta(a)b, \qquad a, b \in K.$$

An Ore-variable over K is a pair $\partial = (\sigma, \delta)$ where σ is an endomorphism and δ is a σ-skew derivation.

Let $\partial_i = (\sigma_i, \delta_i)$ be Ore-variables $(1 \le i \le n)$ such that all mappings σ_i, δ_j commute with each other. Then the Ore algebra $\mathbb{O}$ defined by $\mathcal{O} = \{\partial_1, \ldots, \partial_n\}$ is the set of K-linear combinations on the set of formal expressions $\partial^k = \partial_1^{k_1} \cdots \partial_n^{k_n}$ with multiplication determined by the rules

$$\partial_i \cdot \partial_j = \partial_j \cdot \partial_i \text{ and } \partial_i \cdot x = \sigma_i(x)\partial_i + \delta_i(x) \qquad x \in K.$$

We set $\Lambda := \{\partial^k\colon k \in \mathbb{N}^n\} \cong \mathbb{N}^n$, as usual its elements are called monomials.

With the convenient notation

$$x_k^l = (\delta^l \circ \sigma^k)(x) \qquad k, l \in \mathbb{N}^n,\ x \in K.$$

the product in the ring $\mathbb{O}$ may be written explicitely

$$x\partial^l \cdot y\partial^q = \sum_{v \in \mathbb{N}^n} \binom{l}{v} x y_v^{l-v} \cdot \partial^{q+v} = \sum_{v \le_\pi l} \binom{l}{v} x y_{l-v}^{v} \cdot \partial^{l+q-v}$$

where $x, y \in K$ and $l, q \in \mathbb{N}^n$. In particular

$$x\partial^0 \cdot y\partial^q = xy\partial^q$$

demonstrating that K is naturally a subring of $\mathbb{O}$.

Remark. With the notion from Lemma 10 we have $S_{k,l}(x) := x_{k+l}^l$, which is valid for the univariate case. Here, we want to develop a theory of the n-fold case, thats why we have chosen to change notation.

In [Lev07], the set $\mathcal{O}$ of variables is splitted into disjoint subsets $\mathcal{O} = \mathcal{O}_1 \cup \cdots \cup \mathcal{O}_p$. This gives order functions

$$\nu_j \colon \Lambda \to \mathbb{N}, \qquad \partial^k \mapsto \nu_j(\partial^k) := \sum_{\partial_i \in \mathcal{O}_j} k_i \qquad 1 \le j \le p,$$

and the total degree function $\nu_0(\partial^k) = \nu_1(\partial^k) + \cdots + \nu_p(\partial^k)$, that extends to the free module $F = \mathbb{D}^{(E)}$ where $E = \{e_1, \ldots, e_q\}$. In the notion of Chapter 2, Example 8, we have $\nu_j(\cdot) = |\cdot|_{\mathcal{O}_j}$.

As usual

$$\nu_j(\partial^k e) = \nu_j(\partial^k), \qquad\qquad \nu_j(f) = \begin{cases} \max_{t \in \mathrm{T}(f)} \nu_j(t) & \ldots\ f \in F \setminus \{0\} \\ -\infty & \ldots\ f = 0 \end{cases}$$

for all $j = 0, 1, \ldots, p$.

For the remaining part of this section we assume that K is a field $\mathbb{K}$.

Lemma 40. *Let $x, y \in K \backslash \{0\}$, $k, l \in \mathbb{N}^n$. Then, we have*

$$\nu_j(x\partial^k \cdot y\partial^l e) = \nu_j(\partial^k) + \nu_j(\partial^l), \qquad j = 0, \ldots, p.$$

Proof. Take a term $t \in \mathrm{T}(x\partial^k \cdot y\partial^l)$.

$$x\partial^k \cdot y\partial^l = \sum_{v \le_\pi k} \binom{k}{v} x y^v_{k-v} \cdot \partial^{l+k-v} \Rightarrow \exists v \le_\pi k : t = \partial^{l+v},$$

thus, if $1 \le j \le p$ then

$$\nu_j(t) = \sum_{\partial_i \in \mathcal{O}_i} (l_i + v_i) \le \sum_{\partial_i \in \mathcal{O}_i} (l_i + k_i) = \sum_{\partial_i \in \mathcal{O}_i} l_i + \sum_{\partial_i \in \mathcal{O}_i} k_i = \nu_j(\partial^l) + \nu_j(\partial^k) = \nu_j(xy_k \cdot \partial^{k+l}).$$

Since $xy_k \neq 0$ the assertion follows. The statement for $j = 0$ follows by summing up all $j = 1, \ldots, p$. $\square$

The p orders on ΛE considered in [Lev07] are defined by the p injections

$$\begin{aligned} &\tau_j : \Lambda \to \mathbb{N}^{n+p+1} \\ &\quad \partial^k \mapsto \tau_j(\partial^k) := (\nu_j(\lambda), \nu_0(\lambda), \nu_1(\lambda), \ldots, \widehat{\nu_j(\lambda)}, \ldots, \nu_p(\lambda), k^j, k^1, \ldots, \widehat{k^j}, \ldots, k^p) \end{aligned} \tag{3.17}$$

with notation

$$k = (k_1, \ldots, k_n) = (k^1, \ldots, k^p), \qquad k_i \in \mathbb{N}, k^j \in \mathbb{N}^{\mathrm{Card}(\mathcal{O}_j)},$$

the elements designated by $\widehat{\nu_j(\lambda)}$ and $\widehat{k^j}$ left out. The extensions of τ_j to ΛE are given by

$$\varphi_j \colon \Lambda E \to \mathbb{N}^{n+p+2}, \qquad \partial^k e_i \mapsto \varphi_j(\partial^k e_i) := (\tau(\partial^k), i).$$

Thus, for terms $t_1, t_2 \in \Lambda E$,

$$t_1 \prec_j t_2 \Leftrightarrow \varphi_j(t_1) <_{\text{lex}} \varphi_j(t_2).$$

Note that

$$\nu_j(\partial^k) = |k^j| = \sum_{\partial_i \in \mathcal{O}_j} k_i, \qquad 1 \le j \le p.$$

Lemma 41. *Let $k, l \in \mathbb{N}^n$, τ_j as in* (3.17)*. Then,*

$$\tau_j(\partial^{k+l}) = \tau_j(\partial^k) + \tau_j(\partial^l), \qquad 1 \le j \le p.$$

Proof. Using the notation from above it is plain that $(k+l)^j = k^j + l^j$. Therefore

$$\nu_j(\partial^{k+l}) = |(k+l)^j| = |k^j + l^j| = |k^j| + |l^j| = \nu_j(\partial^k) + \nu_j(\partial^l).$$

From this observation the statement is obvious. □

Leading term and leading coefficient functions are written $\mathrm{LT}_j(\cdot)$, $\mathrm{LC}_j(\cdot)$ for $1 \le j \le p$. As before it is plain that

$$\nu_j(f) = \nu_j(\mathrm{LT}_j(f)) \qquad \forall j.$$

Lemma 42. *Let $x, y \in K$, $k, l \in \mathbb{N}^n$, $e \in E$. Then,*

$$\mathrm{LT}_j(x\partial^k \cdot y\partial^l e) = \partial^{k+l} e, \qquad \forall j : 1 \le j \le p.$$

Proof. Take a term $t \in \mathrm{T}(x\partial^k \cdot y\partial^l e)$ with $t \neq \partial^{k+l}e$. Then $\exists v <_\pi k$ with $t = \partial^{v+l}e$. Let $i_0 = \min\{i \colon v_i \neq k_i\}$. Then

$$v_{i_0} < k_{i_0} \wedge \forall i < i_0 \colon v_i = k_i \Rightarrow (l+v)_{i_0} < (k+l)_{i_0} \wedge \forall i < i_0 : (l+v)_i = (k+l)_i.$$

Thus

$$i_0 = \min\{i \colon (l+v)_i \neq (k+l)_i\} \wedge (l+v)_{i_0} < (k+l)_{i_0} \wedge \forall i \colon (l+v)_i \le (k+l)_i.$$

Therefore

$$\nu_0(t) = |l+v| = |l| + |v| < |l| + |k| = |l+k| = \nu_0(\partial^{l+k}e)$$

If $\partial_{i_0} \in \mathcal{O}_j$ then

$$\nu_j(t) < \nu_j(\partial^{k+l}e) \Rightarrow t = \partial^{l+v}e \prec_j \partial^{k+l}e.$$

On the other hand, if $\partial_{i_0} \notin \mathcal{O}_j$ then

$$\nu_j(t) \le \nu_j(\partial^{k+l}e) \wedge \nu_0(t) < \nu_0(\partial^{k+l}e)$$

and again $t \prec_j \partial^{k+l}e$. □

Corollary 14. *Let $a, b, x, y \in K\backslash\{0\}$, $k, l, r \in \mathbb{N}^n$, $e_u, e_v \in E$ such that $u < v$. Then,*

$$\begin{aligned} \partial^k \prec_j \partial^l \Rightarrow \quad & \mathrm{LT}_j(x\partial^k \cdot a\partial^r e_u) \prec_j \mathrm{LT}_j(y\partial^l \cdot b\partial^r e_v) \\ \wedge \quad & \mathrm{LT}_j(a\partial^r \cdot x\partial^k e_u) \prec_j \mathrm{LT}_j(b\partial^r \cdot y\partial^l e_v). \end{aligned} \qquad 1 \leq j \leq p.$$

Proof. We have to show that $\partial^{k+r} e_u \prec_j \partial^{l+r} e_v$. From the hypothesis we have

$$\tau_j(\partial^k) <_{\mathrm{lex}} \tau_j(\partial^l).$$

Thus, using Lemma 41

$$\tau_j(\partial^{k+r}) = \tau_j(\partial^k) + \tau_j(\partial^r) <_{\mathrm{lex}} \tau_j(\partial^l) + \tau_j(\partial^r) = \tau_j(\partial^{l+r}).$$

Therefore also

$$(\tau_j(\partial^{k+r}), u) <_{\mathrm{lex}} (\tau_j(\partial^{l+r}), v).$$

□

Lemma 43. *Let an Ore-operator $a \in \mathbb{O} \setminus \{0\}$ be given, such that $\mathrm{LT}_j(a) = \partial^{k_0}$ and let $f \in F \setminus \{0\}$. Then, for all j*

- $\mathrm{LT}_j(a \cdot f) = \mathrm{LT}_j(a) \cdot \mathrm{LT}_j(f)$;
- $\mathrm{LC}_j(a \cdot f) = \mathrm{LC}_j(a) \cdot \sigma^{k_0}(\mathrm{LC}_j(f))$.

Proof. Set

$$\mathrm{LC}_j(a) = a_0, \qquad \mathrm{LT}_j(f) = \partial^{l_0} e_0, \qquad \mathrm{LC}_j(f) = f_0.$$

Thus

$$a = a_0\partial^{k_0} + \sum_{\partial^k \prec_j \partial^{k_0}} a_k \partial^k \text{ and } f = f_0 \partial^{l_0} e_0 + \sum_{\partial^l e \prec_j \partial^{l_0} e_0} f_{l,e} \partial^l e.$$

$$\begin{aligned} a \cdot f = \underbrace{a_0\partial^{k_0} \cdot f_0 \partial^{l_0} e_0}_{(S1)} + \underbrace{\sum_{\partial^l e \prec_j \partial^{l_0} e_0} a_0 \partial^{k_0} \cdot f_{l,e} \partial^l e}_{(S2)} + \underbrace{\sum_{\partial^k \prec_j \partial^{k_0}} a_k \partial^k \cdot f_0 \partial^{l_0} e_0}_{(S3)} \\ + \underbrace{\sum_{\partial^k \prec_j \partial^{k_0}} \sum_{\partial^l e \prec_j \partial^{l_0} e_0} a_k \partial^k \cdot f_{l,e} \partial^l e}_{(S4)} \end{aligned}$$

Pick out a summand of sum $(S2)$. If $\partial^l \prec_j \partial^{l_0}$ then

$$\mathrm{LT}_j(a_o \partial^{k_0} \cdot f_{l,e} \partial^l e) \prec_j \mathrm{LT}_j(a_0 \partial^{k_0} \cdot f_0 \partial^{l_0} e_0).$$

If $\partial^l = \partial^{l_0}$ then $e < e_0$ and

$$\mathrm{LT}_j(a_o \partial^{k_0} \cdot f_{l,e} \partial^l e) = \partial^{k_0 + l} e \prec_j \partial^{k_0 + l_0} e_0 = \mathrm{LT}_j(a_k \partial^k \cdot f_0 \partial^{l_0} e_0).$$

For a summand of sum $(S3)$ we obtain

$$\mathrm{LT}_j(a_k\partial^k \cdot f_0\partial^{l_0}e_0) \prec_j \mathrm{LT}_j(a_0\partial^{k_0} \cdot f_0\partial^{l_0}e_0).$$

As to sum $(S4)$, from the scope of the first sum we derive

$$\mathrm{LT}_j(a_k\partial^k \cdot f_{l,e}\partial^l e) \prec_j \mathrm{LT}_j(a_0\partial^{k_0} \cdot f_0\partial^l e_0).$$

If $\partial^l \prec_j \partial^{l_0}$ then

$$\mathrm{LT}_j(a_0\partial^{k_0} \cdot f_0\partial^l e_0) \prec_j \mathrm{LT}_j(a_0\partial^{k_0} \cdot f_0\partial^{l_0}e_0).$$

If $\partial^l = \partial^{l_0}$ then

$$\mathrm{LT}_j(a_0\partial^{k_0} \cdot f_0\partial^l e_0) = \partial^{k_0+l} = \partial^{k_0+l_0} = \mathrm{LT}_j(a_0\partial^{k_0} \cdot f_0\partial^{l_0}e_0).$$

So, in any case,

$$\mathrm{LT}_j(a_k\partial^k \cdot f_{l,e}\partial^l e) \prec_j \mathrm{LT}_j(a_0\partial^{k_0} \cdot f_0\partial^{l_0}e_0).$$

Let $t \in \mathrm{T}(a \cdot f)$. Then t must be a term (surviving after cancellation) of one of the sum expressions $(S1)$-$(S4)$. Consequently

$$t \preccurlyeq_j \mathrm{LT}_j(a_0\partial^{k_0} \cdot f_0\partial^{l_0}e_0) = \partial^{k_0+l_0}e_0 = \partial^{k_0} \cdot \partial^{l_0}e_0 = \mathrm{LT}_j(a \cdot \mathrm{LT}_j(f)).$$

Moreover we see that the expression $\partial^{k_0} \cdot \partial^{l_0}e_0$ does not cancel out. It follows that $\mathrm{LT}_j(a \cdot f) = \mathrm{LT}_j(a) \cdot \mathrm{LT}_j(f)$.

From the expansion of expression $(S1)$

$$a_0\partial^{k_0} \cdot f_0\partial^{l_0}e_0 = \sum_{v \le \pi} \binom{k_0}{v} a_0(f_0)_v^{k_0-v}\partial^{l_0+v}e_0.$$

we derive

$$\mathrm{LC}_j(a \cdot f) = a_0(f_0)_{k_0} = \mathrm{LC}_j(a) \cdot \mathrm{LC}_j(f)_{k_0} = \mathrm{LC}_j(a) \cdot \sigma^{k_0}(\mathrm{LC}_j(f))$$

□

Corollary 15. *Let $\lambda = \partial^k \in \Lambda$, $f \in F \setminus \{0\}$. Assume that $\partial = (\sigma, \delta)$, i.e.*

$$\partial \cdot x = \sigma(x) \cdot \partial + \delta(x).$$

Then for all $j = 1, \ldots, p$, we have

- $\mathrm{LT}_j(\lambda \cdot f) = \lambda \cdot \mathrm{LT}_j(f)$;
- $\mathrm{LC}_j(\lambda \cdot f) = \sigma^k(\mathrm{LC}_j(f))$.

Corollary 16. *Let $a \in \mathbb{D}\setminus\{0\}$, $f \in F\setminus\{0\}$. Then, for all $j = 0, \ldots, p$*

$$\nu_j(a \cdot f) = \nu_j(a) + \nu_j(f).$$

Proof. Let $1 \le j \le p$ and set $\mathrm{LT}_j(a) = \partial^{k_0}$, $\mathrm{LT}_j(f) = \partial^{l_0} e_0$. Then

$$\begin{aligned}\nu_j(a \cdot f) &= \nu_j(\mathrm{LT}_j(a \cdot f)) = \nu_j(\mathrm{LT}_j(a) \cdot \mathrm{LT}_j(f)) = \nu_j(\partial^{k_0} \cdot \partial^{l_0} e_0) = \nu_j(\partial^{k_0}) + \nu_j(\partial^{l_0}) \\ &= \nu_j(\mathrm{LT}_j(a)) + \nu_j(\mathrm{LT}_j(f)) = \nu_j(a) + \nu_j(f).\end{aligned}$$

The statement for $j = 0$ follows by summation. $\square$

Corollary 17. *Let $a \in \mathbb{O}$, $f \in F$. Then $a \cdot f = 0 \Rightarrow a = 0 \vee f = 0$. Consequently, $\mathbb{O}$ is a domain.*

Proof. Let $a \neq 0 \wedge f \neq 0$. Then $\nu(a) > 0 \wedge \nu(f) > 0$. Therefore

$$\nu(a \cdot f) = \nu(a) + \nu(f) > 0$$

hence $a \cdot f \neq 0$. $\square$

The order functions ν_j propose a natural filtration concept.

$$F_r^{(j)} = \{f \in F \colon \nu_j(f) \le r\} \qquad r \in \mathbb{N},\ 0 \le j \le p.$$

For $r \in \mathbb{N}^p$ we set

$$F_r = \bigcap_{j=1}^{p} F_{r_j}^{(j)} = \{f \in F \colon \nu_1(f) \le r_1 \wedge \cdots \wedge \nu_p(f) \le r_p\}.$$

Again the sets $\mathbb{O}_t^{(j)}$ and $\mathbb{O}_r$ are implicitely defined ($F = \mathbb{O}^1$).

Lemma 44. *Let $r \in \mathbb{N}$. Tor all $1 \le j \le p$, the sets $F_r^{(j)}$ defines a univariate filtration on F with respect to the univariate filtration $\mathbb{O}_r^{(j)}$ in $\mathbb{O}$. For $r = (r_1, \ldots, r_p) \in \mathbb{N}^p$, we have that*

$$F_r = \bigcap_{j=1}^{p} F_{r_j}^{(j)}$$

is a p-fold filtration with respect to $\mathbb{O}_r$.

Proof.

- If $f, g \in F_r^{(j)}$ then $\nu_j(f+g) \le \max\{\nu_j(f), \nu_j(g)\} \le r$, thus the sets $F_r^{(j)}$ are abelian groups;
- $r \le s$ in $\mathbb{N}$ implies $F_r^{(j)} \subseteq F_s^{(j)}$;
- $\bigcup_{r=0}^{\infty} F_r = F$;
- If $a \in \mathbb{O}_r^{(j)} \wedge f \in \mathbb{O}_s^{(j)}$ then $\nu_j(a \cdot f) = \nu_j(a) + \nu_j(f) \le r + s$, hence $\mathbb{O}_r^{(j)} \cdot F_s^{(j)} \subseteq F_{s+t}^{(j)}$.

$\square$

3.3.1. Reduction with Respect to Several Term Orders

In [Lev07] the following theory is developed.

Definition 23. Let $f, g \in F$, $g \neq 0$. Let $k, i_1, \ldots, i_r$ be distinct elements in $\{1, \ldots, p\}$, $\mathcal{I} = \{i_1, \ldots, i_r\}$ and $\mathcal{L} = (k, \mathcal{I})$. Then f is $\mathcal{L}$-reduced w.r.t. g if and only if

$$\nexists \lambda \in \Lambda : \Big(\lambda \cdot \mathrm{LT}_k(g) \in \mathrm{T}(f) \wedge \forall i \in \mathcal{I} \colon \nu_i(\lambda \cdot \mathrm{LT}_i(g)) \leq \nu_i(\mathrm{LT}_i(f))\Big)$$

f is $\mathcal{L}$-reduced w.r.t. $G \subseteq F$ if and only if f is $\mathcal{L}$-reduced w.r.t. g for all $g \in G$.

The corresponding reduction concept in [Lev07] is given in the next definition.

Definition 24. Let $f, g, h \in F$, $g \neq 0$. $\mathcal{I}$ and $\mathcal{L} = (k, \mathcal{I})$ as before. Then we say that f $\mathcal{L}$-reduces to h if and only if

$$\begin{aligned} f \xrightarrow{\mathcal{L}}_g h \iff\ & \exists w \in \mathrm{T}(f) : \mathrm{LT}_k(g) \mid w \wedge \\ & h = f - \frac{f_w}{\tau_{(w/\mathrm{LT}_k(g))}(\mathrm{LC}_k(g))} \frac{w}{\mathrm{LT}_k(g)} g \wedge \\ & \forall i \in \mathcal{I} : \nu_i\left(\frac{w}{\mathrm{LT}_k(g)} \mathrm{LT}_i(g)\right) \leq \nu_i(\mathrm{LT}_i(f)) \end{aligned} \tag{3.18}$$

Here for $\lambda \in \Lambda$ the symbol τ_k denotes the exponent of λ, as power of ∂, considered as the corresponding endomorphism of K, precisely

$$\lambda = \partial^k, x \in K \Rightarrow \tau_\lambda(x) = \sigma^k(x).$$

Theorem 22. *Let $f, g, h \in F$, $g \neq 0$ and $\mathcal{L} = (k, \mathcal{I})$ as before. Let P denote the predicate*

$$P(f, g, \lambda) \Leftrightarrow \forall i \in \mathcal{I} : \nu_i(\lambda \cdot g) \leq \nu_i(f).$$

Let ρ denote the reduction relation

$$f \longrightarrow_g^\rho h \Leftrightarrow \exists \lambda \in \Lambda \Big(\mathrm{LT}_k(\lambda \cdot g) \in \mathrm{T}(f) \wedge h = f - \frac{f_{\mathrm{LT}_k(\lambda \cdot g)}}{\mathrm{LC}_k(\lambda \cdot g)} \lambda \cdot g \wedge P(f, g, \lambda)\Big).$$

Then

$$f \longrightarrow_g^\rho h \Leftrightarrow f \xrightarrow{\mathcal{L}}_g h$$

Proof. Assume that $f \xrightarrow{\mathcal{L}}_g h$ and let $w \in \mathrm{T}(f)$ as mentioned in (3.18). Let $\mathrm{LT}_k(g) = \partial^l e$. Since $\mathrm{LT}_k(g) | w$ we may write $w = \partial^{p+l} e$. Set $\lambda = (w/\mathrm{LT}_k(g)) = \partial^p$. Then, by Corollary 15,

$$\mathrm{LT}_k(\lambda \cdot g) = \lambda \cdot \mathrm{LT}_k(g) = \partial^p \cdot \partial^l e = \partial^{p+l} e = w$$

In terms of the τ-notation we obtain

$$\tau_{(w/\mathrm{LT}_k(g))}\big(\mathrm{LC}_k(g)\big) = \tau_\lambda(\mathrm{LC}_k(g)) = \sigma^p(\mathrm{LC}_k(g)) = \mathrm{LC}_k(\lambda \cdot g).$$

Consequently $\mathrm{LT}_k(\lambda\cdot g)\in \mathrm{T}(f)$ and $h = f - \frac{f_{\mathrm{LT}_k(\lambda\cdot g)}}{\mathrm{LC}_k(\lambda\cdot g)}\lambda\cdot g$. Since

$$\nu_i(\lambda\cdot g) = \nu_i(\mathrm{LT}_i(\lambda\cdot g)) = \nu_i(\lambda\cdot \mathrm{LT}_i(g)) = \nu_i\Big(\frac{w}{\mathrm{LT}_k(g)}\cdot \mathrm{LT}_i(g)\Big)$$

the formula $P(f,g,\lambda)$ is exactly the additional condition in (3.18), which means that $f \longrightarrow_g^{\rho} h$.

Conversely assume that $f \longrightarrow_g^{\rho} h$. Let $\lambda = \partial^p$ as mentioned in the formula, $\mathrm{LT}_k(g) = \partial^l e$ and set

$$w = \mathrm{LT}_k(\lambda\cdot g) = \lambda\cdot \mathrm{LT}_k(g) = \partial^p\cdot\partial^l e = \partial^{p+l}e.$$

Then $\mathrm{LT}_k(g)|w$, and

$$\frac{w}{\mathrm{LT}_k(g)} = \partial^p = \lambda \Rightarrow \mathrm{LC}_k(\lambda\cdot g) = \sigma^p(\mathrm{LC}_k(g)).$$

In τ-notation:

$$\tau_{(w/\mathrm{LT}_k(g))}\big(\mathrm{LC}_k(g)\big) = \tau_{\partial^p}\big(\mathrm{LC}_k(g)\big) = \sigma^p(\mathrm{LC}_k(g)) = \mathrm{LC}_k(\lambda\cdot g).$$

Therefore $w\in \mathrm{T}(f)\wedge \mathrm{LT}_k(g)|w$ and

$$h = f - \frac{f_w}{\tau_{(w/\mathrm{LT}_k(g))}(\mathrm{LC}_k(g))}\frac{w}{\mathrm{LT}_k(g)}g \wedge P(f,g,\lambda)$$

and this means that $f \xrightarrow{\mathcal{L}}_g h$. $\square$

Lemma 45. *P satisfies the monomial irreducibility condition* (3.4)*. Therefore*

$$f\in I_\rho \Rightarrow \mathrm{T}(f)\subseteq I_\rho.$$

Proof. Assume

$$\exists g\in G\,\exists\lambda\in\Lambda\colon \mathrm{LT}_k(\lambda\cdot g)\in \mathrm{T}(f)\wedge \mathrm{LC}_k(\lambda\cdot g)\in K^{\times}\wedge \forall i\in\mathcal{I}: \nu_i(\lambda\cdot g)\le \nu_i(\mathrm{LT}_k(\lambda\cdot g)).$$

Then $\nu_i(\lambda\cdot g) = \nu_i(\mathrm{LT}_k(\lambda\cdot g))$. As $\mathrm{LT}_k(\lambda\cdot g)\in \mathrm{T}(f)$ it follows

$$\nu_i(\mathrm{LT}_k(\lambda\cdot g))\le \nu_i(f) \Rightarrow \nu_i(\lambda\cdot g)\le\nu_i(f).$$

Consequently

$$\exists g\in G\,\exists\lambda\in\Lambda\colon \mathrm{LT}_k(\lambda\cdot g)\in \mathrm{T}(f)\wedge \mathrm{LC}_k(\lambda\cdot g)|f_{\mathrm{LT}_k(\lambda\cdot g)}\wedge\forall i\in\mathcal{I}:\nu_i(\lambda\cdot g)\le\nu_i(f).$$

$\square$

3.4. Computation of Multivariate Hilbert-Polynomials for Polynomial Ideals

The concept of dimension polynomial for the multivariate polynomial ring was introduced by Hilbert already in the 19^{th} century. Consider the multivariate polynomial ring $R = \mathbb{K}[x_1, \dots, x_n]$ in n unknowns over a field $\mathbb{K}$. Then R is a direct sum $R = \bigoplus_{k=0}^{\infty} R_k$, where R_k is generated by

$$\{x_1^{k_1} \dots x_n^{k_n} : \; k_1 + \dots + k_n = k\}.$$

Clearly, $\dim_{\mathbb{K}} R_k$, is equal to the number of such monomials, as they form a basis of R_k. By Lemma 5, the number of monomials in n variables generating R_k is given by

$$\dim_{\mathbb{K}} R_k = \binom{n+k-1}{k}, \qquad k = k_1 + \dots + k_n, \quad k, k_1, \dots, k_n \in \mathbb{N}, k \geq 0, n \geq 1.$$

Theorem 23 (Hilbert)**.**
Let $R = \mathbb{K}[x_1, \dots, x_n] = \bigoplus_{k=0}^{\infty} R_k$ be a graded polynomial ring over a field $\mathbb{K}$, and $M = \bigoplus_{k=0}^{\infty} M_k$ a graded R-module. Then, there exists a polynomial $\phi(t)$ with rational coefficients such that

$$\dim_{\mathbb{K}} M_k = \phi(k), \qquad k \text{ large enough.}$$

This polynomial ϕ is called the *Hilbert polynomial* of the graded module M. Obviously we obtain bounds for $\phi(k)$ by

$$0 \leq \phi(k) \leq \sum_{i=0}^{k} \binom{n+i-1}{i} = \binom{n+k}{k}.$$

As we have proven a theorem on the dimension of a filtered module, we cover the Hilbert polynomial as a special case of our considerations. Indeed, in the general case we can apply Corollary 7 to actually compute a multivariate generalization of the Hilbert polynomial.

The computation of Hilbert polynomials in this setting is addressed in [Sta78, KW88, BS92, Eis95, CLO97]. A generalization towards bivariate Hilbert polynomials was made in [Lev99]. With the help of Gröbner reduction we are in position to reason about general multivariate Hilbert polynomials in this setting.

The basis of one algorithm, for the computation of Hilbert polynomials, is to use polynomial Gröbner bases. From that point of view, it is reasonable to first make the connection to our concept of Gröbner reduction, i.e. we show how to achieve a filtration that is compatible to polynomial reduction.

Lemma 46. *Let $\mathfrak{a} \trianglelefteq R := \mathbb{K}[x_1, \dots, x_n]$ be an ideal and G a Gröbner of $\mathfrak{a}$ w.r.t. any term order $\prec$, then for $r \in \mathbb{N}^n$,*

$$R_r := \{f \in \mathbb{K}[x_1, \dots, x_n] : \mathrm{LT}(f) \preceq x_1^{r_1} \dots x_n^{r_n}\}$$

defines a monomial filtration with the additional property

$$m \in R_r \wedge n \preccurlyeq m \Rightarrow n \in R_r.$$

Consequently $\longrightarrow_G$ *is a Gröbner reduction w.r.t.* $(R_r)_{r\in\mathbb{N}^n}$.

Example 20. *If we consider the ideal* $\mathfrak{a} \trianglelefteq R := \mathbb{K}[x, y]$, *where*

$$\mathfrak{a} := {}_R\langle f_1 := x^4y^3 + xy^6, f_2 := xy^5, f_3 := 2x^5y^2 - 4x^3y^5\rangle \trianglelefteq R,$$

we obtain as a Gröbner basis

$$G := \{g_1 := x^4y^3, g_2 := xy^5, g_3 := x^5y^2\}.$$

with respect to the lexicographic order (where $x > y > z$*). Graphically, this ideal corresponds to*

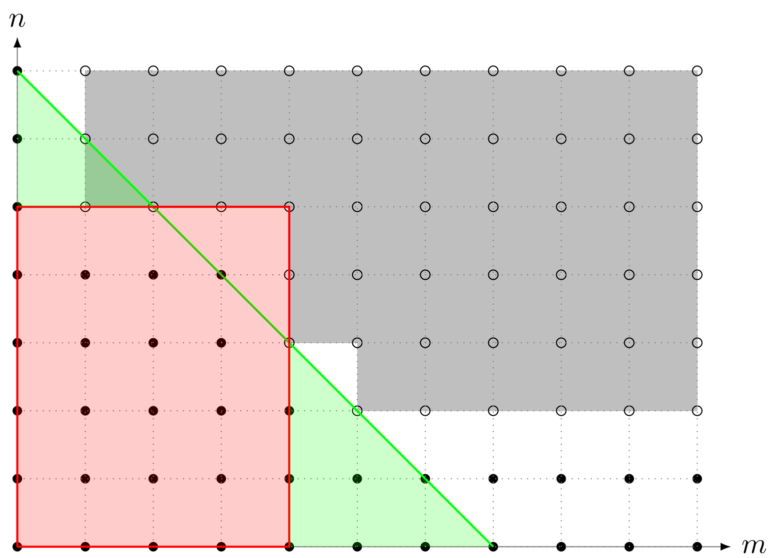

Figure 3.1.: $(m, n) \mapsto x^m y^n$

Let us consider the bivariate dimension polynomial associated to the filtration

$$R_{r,s} = \{f \in R : \ \deg_x(f) \le r \ \wedge \ \deg_y(f) \le s\}, \qquad r, s \in \mathbb{N}.$$

Plugging in values $(r, s) \gg (0, 0)$, *we can count the number of irreducible monomials. This value can be interpolated as bivariate polynomial by:*

$$\#\ \textit{of irred. monomials in } R_{r,s} : \ p_2(r, s) = 2r + s + 10 \ \in \mathbb{K}[r, s], \qquad (r, s) \ge_\pi (5, 4).$$

From that, we see that the growth of elements is linear *by increasing the degree in one direction. For the univariate filtration*

$$R_k := \{f \in R : \deg(f) \le k\},$$

we count as irreducible monomials

k	*6*	*7*	*8*	*9*	*10*	*11*
no. of irred. terms	*27*	*31*	*34*	*37*	*40*	*43*

hence, we obtain $p_1(k) = 3k+10$ *for* $k \geq 7$*. For example, there are* $\binom{8+2}{2} = 45$ *monomials in 2 variables of total degree 8. From this 45 monomials, there are 34 irreducible, leaving 11 reducible elements w.r.t. polynomial reduction with lexicographic order. They are given by*

$$R_8 \backslash I = \{xy^7, xy^6, xy^5, x^2y^6, x^2y^5, x^3y^5, x^4y^4, x^4y^3, x^5y^3, x^5y^2, x^6y^2\}$$

From this 11 monomials

- *One monomial* (xy^5) *has degree 6, hence* $\binom{6+2}{2} - 1 = 27$ *irreducible monomials of degree 6;*
- *Five monomials* ($xy^5, xy^6, x^2y^5, x^4y^3, x^5y^2$) *have degree* ≤ 7*, hence* $\binom{7+2}{2} - 5 = 31$ *irreducible monomials of degree 7.*

Further, we observe, that for all $k \in \mathbb{N}$ *we have*

$$R_k \subseteq R_{k,k} \subseteq R_{2k} \quad \wedge \quad p_2(k,k) = p_1(k).$$

We now consider the general case of a p-fold filtration. Obviously, we could take the theory developed for the ring of Ore polynomials. However, due to the behaviour of monomials in $\mathbb{K}[x_1, \ldots, x_n]$, we can greatly simplify this theory.

As in section 3.3 we split the unkowns into p disjoint subsets $X_1, \ldots, X_p$, such that their union equals X and each x_i corresponds to exactly one X_j. The degree w.r.t. this partition is then given by

$$\deg_{X_j} : T^n(X) \to \mathbb{N}, \qquad x^k \mapsto \sum_{x_i \in X_j} k_i, \qquad 1 \leq j \leq p.$$

The free module F is constructed by the generator $E := \{e := 1\}$, i.e. $F = R^1$. With this setting, the functional $\deg_{X_j}(\cdot)$ extends to R and F respectively by choosing

$$\deg_{X_j}(x^k e) := \deg_{X_j}(x^k), \qquad \deg_{X_j}(f) = \begin{cases} \max_{t \in \mathrm{T}(f)} \deg_{X_j}(t) & \ldots f \in F \backslash \{0\}; \\ -\infty & \ldots f = 0, \end{cases}$$

for all $1 \leq j \leq p$.

Let us now formulate the basic characteristics of $\deg_{X_j}(\cdot)$.

Lemma 47. *Let* $R := \mathbb{K}[x_1, \ldots, x_n]$*, a partition* $X := \bigcup_{i=1}^{p} X_i$ *be fixed, Then, for* $k, l \in \mathbb{N}^n$*,* $a, b \in \mathbb{K} \backslash \{0\}$*,* $f, g \in R \backslash \{0\}$ *we have:*

1. $\deg_{X_j}(af \pm bg) \leq \max\{\deg_{X_j}(f), \deg_{X_j}(g)\}$;

2. $\deg_{X_j}(fg) = \deg_{X_j}(f) + \deg_{X_j}(g)$*;*

In particular, this points can be specialized to

3. $\deg_{X_j}(ax^k \cdot bx^l) = \deg_{X_j}(x^k) + \deg_{X_j}(x^l) = \sum_{x_i \in X_j}(k_i + l_i)$*;*

4. $\deg_{X_j}(ax^k \cdot f) = \sum_{x_i \in X_j} k_i + \deg_{X_j}(f)$*;*

We can now apply the considerations from before, and designate the leading term w.r.t. the partition X_j, this is, identify each monomial by the integer-vector

$$\tau_j : T^n(X) \to \mathbb{N}^{n+1}, \qquad x^k \mapsto (\deg_{X_j}(x^k), k_1, \ldots, k_n),$$

giving a one-to-one correspondence between $T^n(X)$ and $\mathbb{N}^{n+1}$. By ordering the terms λ appearing in $f \in R$ compared in lexicographic order, this gives $\mathrm{LT}_j(f)$ and $\mathrm{LC}_j(f)$.

However, it might happens, that $\tau_j(\lambda) = \tau_k(\lambda)$ for $j \neq k$. If, for example, we have $X := \{x_1, x_2, x_3\}$, we can consider the partition $X_1 := \{x_1\}, X_2 := \{x_2, x_3\}$. Then,

$$\tau_1(x_1^3 x_2 x_3^2) = (3, 3, 1, 2) = \tau_2(x_1^3 x_2 x_3^2).$$

By Lemma 47, we immediately obtain

$$\begin{aligned}\tau_j(x^k \cdot x^l) &= \tau_j(x^{k+l}) = (\deg_{X_j}(x^{k+l}), k_1 + l_1, \ldots, k_n + l_n) \\ &= (\deg_{X_j}(x^k), k_1, \ldots, k_n) + (\deg_{X_j}(x^l), l_1, \ldots, l_n) = \tau_j(x^k) + \tau_j(x^l).\end{aligned}$$

Similar as in the ring $\mathbb{O}$, we plainly have for $f \in R$ that

$$\mathrm{LT}_j(f) = \deg_{X_j}(\mathrm{LT}_j(f)), \qquad 1 \leq j \leq p.$$

Lemma 48. *Let* $1 \leq j \leq p$*. Given* $f, g \in R \backslash \{0\}$*, we have:*

- $\mathrm{LT}_j(f \cdot g) = \mathrm{LT}_j(f) \cdot \mathrm{LT}_j(g)$*;*
- $\mathrm{LC}_j(f \cdot g) = \mathrm{LC}_j(f) \cdot \mathrm{LC}_j(g)$*.*

This applies in particular to $f \in T^n(X) \subseteq R$*, i.e.*

$$\mathrm{LT}_j(x^k \cdot g) = \left(\prod_{x_i \in X_j} x_i^{k_i}\right) \cdot \mathrm{LT}_j(g), \qquad \mathrm{LC}_j(ax^k \cdot g) = a \cdot \mathrm{LC}_j(g) \qquad a \in \mathbb{K}.$$

The degree functionals $\deg_{X_j}$ induce a natural filtration of $F = R^1$, by setting

$$F_r^{(j)} := \{f \in F : \deg_{X_j}(f) \leq r\}, \qquad r \in \mathbb{N}, 1 \leq j \leq p,$$

respectively, its p-fold counterpart defined as the intersection $F_r = F_{r_1}^{(1)} \cap \ldots \cap F_{r_p}^{(p)}$.

As before, we show that each $F_r^{(j)}$ defines an univariate filtration on F w.r.t. the univariate filtration on R_r on R. Therefore, by Lemma 16, F_r is a p-fold filtration w.r.t. R_r.

We want now to characterize the full reduction that is obtained by this setting. This is a straight-forward specialization of Theorem 22. We state it explicit here.

Definition 25 (Full reduction in $\mathbb{K}[X]$)**.**
Let $f, g \in F = R^1$, s.t. $g \neq 0$, denote the predicate P by

$$P(f, g, x^k) \Leftrightarrow \deg_{X_j}(x^k \cdot g) \leq \deg_{X_j}(f), \qquad 2 \leq j \leq p. \tag{3.19}$$

Then, ρ is the full reduction

$$f \longrightarrow_g^\rho h \Leftrightarrow \exists x^k \in T^n(X) \left(\mathrm{LT}_1(x^k g) \in \mathrm{T}(f) \wedge h = f - \frac{f_{\mathrm{LT}_1(x^k \cdot g)}}{\mathrm{LC}_1(x^k \cdot g)} x^k g \wedge P(f, g, x^k) \right).$$

Let now

$$r_i := \deg_{X_i}(f) = \deg_{X_i}(\mathrm{LT}_i(f)), \qquad 1 \leq i \leq p.$$

The predicate (3.19) fits in full reduction in the sense that

$$x^k g \in F_{r_1}^{(1)} \text{ because } \mathrm{LT}_1(x^k g) \in \mathrm{T}(f) \subseteq F_r^{(1)},$$

and $x^k g \in F_{r_2}^{(2)} \cap \ldots \cap F_{r_n}^{(n)}$ because of P. From that, one obtains that an arbitrary permutation of the indices results in the same result. Namely, we could replace the argument in Definition 25 by

$$\exists x^k \in T^n(X) \left(\exists \ell :\ 1 \leq \ell \leq p : \mathrm{LT}_\ell(x^k \cdot g) \in \mathrm{T}(f) \wedge h = f - \frac{f_{\mathrm{LT}_\ell(x^k \cdot g)}}{\mathrm{LC}_\ell(x^k \cdot g)} x^k g \wedge P(f, g, x^k, \ell) \right); \tag{3.20}$$

where

$$P(f, g, x^k, \ell) :\Leftrightarrow \deg_{X_j}(x^k \cdot g) \leq \deg_{X_j}(f), \qquad 1 \leq j \leq p, j \neq \ell.$$

Lemma 49. *The predicate P given by* (3.19) *satisfies the monomial irreducibility condition* (3.4)*, i.e. $f \in I_\rho \Rightarrow \mathrm{T}(f) \subseteq I_\rho$.*

Proof. See the proof of Lemma 45. $\square$

4. Relative Reduction and Buchberger's Algorithm in Filtered Free Modules

While in the previous section, we've considered (and extended) the existing approaches for the computation of a Gröbner reduction, we now want to propose a Buchberger procedure, for certain type of filtrations. We are interested in multivariate filtrations that are built as intersection of univariate filtrations. In particular, we reflect the discussions of the author and Prof. Alexander Levin, and propose a method for computing a generating set, such that every non-zero element in the considered submodule reduces with respect to this generating set to zero. The author has, together with Prof. Levin submitted a pre-print to the proceedings of the ACA (Applications of Computer Algebra), see [FL16].

In [ZW08a], a method was presented, that provided the method terminates, produces a Gröbner basis with respect to relative reduction. In fact, this is a variant of Buchberger's algorithm (i.e. adding non-zero remainders with respect to multivariate division), by taking into account relative reduction. However, in [Dön13], an example was presented where the method does not terminate. This termination property has been examined in [HZ15].

We reproduce the discussion from [Dön13, HZ15]. Let

$$\prec = \operatorname{lex}(x_3 > x_1 > x_2), \qquad \prec' = \operatorname{grevlex}(x_3, x_2, x_1).$$

on $\{x_1, x_2, x_3\}$ and let G_i be defined as

$$\begin{aligned} G_i := \{&f_0 := x_1^3 x_2^2 + x_1^4 x_2, f_1 := x_2^3 x_3^2 + x_1 x_2^2 x_3^2\} \\ &\cup \{g_j := x_1^{3+4j} x_2 x_3 + x_2^{2+4j} x_3^2 \mid j = 0 \dots i\}. \end{aligned}$$

Claim 1. G_0 is a Gröbner basis w.r.t. $\prec'$ for $\langle f_0, g_0 \rangle$. To that end, we observe that the S-polynomials can be reduced to zero. Indeed,

$$\begin{aligned} S_{\prec'}(f_0, g_0) &= x_1^4 x_2 x_3 - x_2^3 x_3^2 = x_1 \cdot g_0 - \underbrace{(x_2^3 x_3^2 + x_1 x_2^2 x_3^2)}_{=:f_1} \\ S_{\prec'}(f_0, f_1) &= x_2 x_3^2 \cdot f_0 - x_1^3 \cdot f_1 = 0 \\ S_{\prec'}(f_1, g_0) &= x_1^4 x_2^2 x_3^2 - x_2^4 x_3^3 = x_2 x_3^2 \cdot f_0 - x_2^2 x_3 \cdot g_0 + 0. \end{aligned}$$

For the second part of the loop, we $\prec$-reduce the S-polynomial $S(f_0, g_i)$ relative to $\prec'$.

Claim 2. Let $H_i := \{f_0, g_0, \ldots, g_i\}$. The S-polynomial

$$S_\prec(f_0, g_i) = x_1^3 x_2^{3+4i} x_3^2 - x_1^{7+4i} x_2 x_3 \text{ is not } \prec\text{-reducible modulo } H_i \text{ relative to } \prec' .$$

For the g_i we observe for $0 \leq j \leq i$ that

$$\begin{aligned}
\mathrm{LT}_\prec(x_1^3 x_2^{1+4(i-j)} g_j) &= x_1^3 x_2^{3+4i} x_3^2 = \mathrm{LT}_\prec(S_\prec(f_0, g_i)) \\
\mathrm{LT}_{\prec'}(S_\prec(f_0, g_i)) &= x_1^{7+4i} x_2 x_3 \prec' x_1^{6+4j} x_2^{2+4(i-j)} x_3 = \mathrm{LT}_{\prec'}(x_1^3 x_2^{1+4(i-j)} g_j),
\end{aligned}$$

hence (by the second line) reduction with one of $\{g_0, \ldots, g_i\}$ is prohibited. For f_0 we observe $\deg_{x_3}(x_1^{a_1} x_2^{a_2} x_3^{a_3} \cdot f_0)$ is the same for both monomials occurring in f_0, hence we make a tie break, and the monomial with higher degree w.r.t. x_1 is the dominant part. But this is always the monomial $x_1^4 x_2$ which has already higher degree than $S(f_0, g_i)$, hence $S_\prec(f_0, g_i)$ can not be reduced with f_0 either.

It turns out, that $S_\prec(f_0, g_i)$ can be reduced by f_1, in particular

$$\begin{aligned}
\mathrm{LT}_\prec(x_1^2 x_2^{1+4i} \cdot f_1) &= x_1^3 x_2^{3+4i} x_3^2 = \mathrm{LT}_\prec(S_\prec(f_0, g_i)) \\
\mathrm{LT}_{\prec'}(x_1^2 x_2^{1+4i} \cdot f_1) &= x_1^2 x_2^{4+4i} x_3^2 \prec' x_1^{7+4i} x_2 x_3 = \mathrm{LT}_{\prec'}(S_\prec(f_0, g_i)),
\end{aligned}$$

hence, it is possible to achieve the following chain of reductions:

$$\begin{aligned}
S_\prec(f_0, g_i) &= x_1^3 x_2^{3+4i} x_3^2 - x_1^{7+4i} x_2 x_3 \longrightarrow_{f_1} S_\prec(f_0, g_i) - x_1^2 x_2^{1+4i} \cdot f_1 \\
&= -x_1^{7+4i} x_2 x_3 - x_1^2 x_2^{4+4i} x_3^2 =: h_1 \longrightarrow_{f_1} h_1 + x_1 x_2^{2+4i} \cdot f_1 \\
&= -x_1^{7+4i} x_2 x_3 + x_1 x_2^{5+4i} x_3^2 =: h_2 \longrightarrow_{f_1} h_2 - x_2^{3+4i} \cdot f_1 \\
&= -x_1^{7+4i} x_2 x_3 + x_2^{6+4i} x_3^2 = -g_{i+1}.
\end{aligned}$$

At each intermediate step, we observe, that the elements h_1, h_2 and g_{i+1} are irreducible w.r.t. H_i. But obviously

$$H_{i+1} = H_i \cup \{g_{i+1}\},$$

hence, we add infinitely many elements g_{i+1}, which causes the algorithm to fail.

Although relative reduction is formulated for modules over the ring of difference-differential operators, the example above is formulated for usual commutative polynomials. We've already seen, that the interplay of rings allows to pass from one setting to the other, hence the same or similar examples could have been formulated for other rings (where this type of Buchberger algorithm is formulated).

We want to generalize the algorithm proposed in [ZW08a], and view it under the aspect of Gröbner reduction. First, let us fix the setup, that we assume in this chapter.

General Assumption 2. We consider a ring R generated by a set of monomials $\Lambda \subseteq R$. Further, we assume that R contains a commutative subring $K \subseteq R$, that serves as coefficient domain i.e. we have $R = K^{(\Lambda)}$. The letter F shall denote the free R-module generated by $E := \{e_1, \ldots, e_q\}$ such that $F = R^{(E)} = K^{(\Lambda E)}$.

We assume, that R is p-fold filtered by

$$R_r := \{f \in R : \nu_1(f) \leq r_1 \wedge \ldots \wedge \nu_p(f) \leq r_p\}, \qquad r = (r_1, \ldots, r_p) \in \mathbb{N}^p, \tag{4.1}$$

each $\nu_i(\cdot)$ is assumed to be a filter valuation (in the sense of Definition 12). For monomials $t_1, t_2 \in \Lambda E \subseteq F$, define

$$t_1 \prec_m^n t_2 :\Longleftrightarrow (\nu_m(t_1), \nu_n(t_1), \varphi(t_1)) <_{\text{lex}} (\nu_m(t_2), \nu_n(t_2), \varphi(t_2)), \qquad 1 \leq m, n \leq p,$$

and

$$t_1 \prec_m t_2 :\Longleftrightarrow (\nu_m(t_1), \nu(t_1), \varphi(t_1)) <_{\text{lex}} (\nu_m(t_2), \nu(t_2), \varphi(t_2)), \qquad 1 \leq m \leq p,$$

where $\varphi : \Lambda E \to \mathbb{N}^s$ (s a positive integer) uniquely identify t_i, such as the exponent vector in a power product. We define for $t = \lambda e \in \Lambda E$, that $\nu_i(\lambda e) := \nu_i(\lambda)$ and $\nu(t) := \nu_1(t) + \ldots + \nu_p(t)$ and extend this to F by

$$\nu_k(f) := \max_{\prec_k}\{\nu_k(t) : t \in \mathrm{T}(f)\}, \qquad \nu(f) := \nu_1(f) + \ldots + \nu_p(f), \qquad f \in F \backslash \{0\}.$$

We get the following facts:

- $t_1 \prec_m t_2 \Rightarrow \nu_m(t_1) \leq \nu_m(t_2)$;
- $t_1 \prec_m^n t_2 \Rightarrow ((\nu_m(t_1) < \nu_m(t_2)) \vee (\nu_m(t_1) = \nu_m(t_2) \wedge \nu_n(t_1) \leq \nu_n(t_2)))$;

In [HZ15] for the ring R the ring of difference-differential operators D is considered. For the termination of Buchberger's algorithm, the notion of *difference-differential degree compatibility* is introduced. In fact, if the pair of generalized term orders $\prec_1$ and $\prec_2$ are difference-differential degree compatible, the Buchberger's algorithm as introduced in [ZW08a] terminates. We want to lift their approach to our general setting, and generalize relative reduction to *set-relative reduction*. To that end, we introduce so called *admissible orders*.

Definition 26. Let R be a bifiltered ring with filtration

$$R_{r,s} = \{f \in R : \nu_1(f) \leq r \wedge \nu_2(f) \leq s\}, \qquad (r, s) \in \mathbb{N}^2,$$

where ν_i are filter valuations. Let F be a free R-module with basis E and let F be the induced filtration given by $\{F_{r,s} = R_{r,s}E : (r, s) \in \mathbb{N}^2\}$. Let $\prec_1$ and $\prec_2$ be two term orders on $\Lambda E \subseteq F$. The term orders $\prec_1$ and $\prec_2$ are called admissible if and only if for $t_1, t_2 \in \Lambda E$:

- if $t_1 \prec_1 t_2$ then $\nu_1(t_1) < \nu_1(t_2)$ or $(\nu_1(t_1) = \nu_1(t_2) \wedge \nu_2(t_1) < \nu_2(t_2))$;
- if $t_1 \prec_2 t_2$ then $\nu_2(t_1) < \nu_2(t_2)$ or $(\nu_2(t_1) = \nu_2(t_2) \wedge \nu_1(t_1) < \nu_1(t_2))$;
- $t_1 \prec_1 t_2 \Longleftrightarrow t_1 \prec_2 t_2$ when $\nu_1(t_1) = \nu_1(t_2)$ and $\nu_2(t_1) = \nu_2(t_2)$.

Example 21. *If* $\varphi : \Lambda E \to \mathbb{N}^n$ *(n is some positive integer) uniquely identifies a monomial, then the pair*

$$t_1 \prec t_2 :\Leftrightarrow (u_1(t_1), u_2(t_1), \varphi(t_1)) <_{lex} (u_1(t_2), u_2(t_2), \varphi(t_2)),$$
$$t_1 \prec' t_2 :\Leftrightarrow (u_2(t_1), u_1(t_1), \varphi(t_1)) <_{lex} (u_2(t_2), u_1(t_2), \varphi(t_2))$$

is an example for a pair of admissible orders on the monomials ΛE *in* F.

The following Lemma is proven in [Lev00, Lemma 4.1].

Theorem 24. *Consider the ring* R *with set of monomials* $\Lambda \subseteq R$. *Let the monomials* Λ *be built as* $\Lambda = A^k \cdot B^l$ *where* $A := \{a_1, \dots, a_m\}$, $B := \{b_1, \dots, b_n\}$ *and*

$$A^k \cdot B^l := a_1^{k_1} \dots a_m^{k_m} \cdot b_1^{l_1} \dots b_n^{l_n}, \qquad k = (k_1, \dots, k_m) \in \mathbb{N}^m, l = (l_1, \dots, l_n) \in \mathbb{Z}^n.$$

Let S *be an infinite sequence of terms* ΛE *(where* $E := \{e_1, \dots, e_q\}$*). Then, there exists an index* $1 \leq j \leq q$, *and an infinite subsequence*

$$\{\lambda_1 e_j, \lambda_2 e_j, \dots, \lambda_k e_j, \dots\} \subseteq S$$

such that λ_k *divides* λ_{k+1} *for all* $k = 1, 2, \dots$.

Example 22. *The monomials in the rings of our interested are covered by Theorem 24:*

- *commutative polynomials (with* $A = \{x_1, \dots, x_m\}$ *and* $n = 0$*);*
- *difference operators (with* $m = 0$ *and* $B = \{\sigma_1, \dots, \sigma_n\}$*);*
- *differential operators (with* $A = \{\delta_1, \dots, \delta_m\}$ *and* $n = 0$*);*
- *difference-differential operators (with* $A = \{\delta_1, \dots, \delta_m\}$ *and* $B = \{\sigma_1, \dots, \sigma_n\}$*);*
- *Ore-operators (with* $A = \{\partial_1, \dots, \partial_m\}$ *and* $n = 0$*).*

As a next step, we generalize [HZ15, Lemma 3.2] to monomially filtered rings with that particular kind of monomials.

Lemma 50. *Let* F *be a free* R*-module,* $\prec$ *and* $\prec'$ *be a pair of admissible term orders on* ΛE, *and* $G_i := \{g_1, \dots, g_s, r_1, \dots, r_i\} \subseteq F \backslash \{0\}$. *If* r_{i+1} *is* $\prec$*-reduced modulo* G_i *relative to* $\prec'$, *and if*

$$\text{for any } \lambda \in \Lambda, h \in G_i : \mathrm{LT}_\prec(r_{i+1}) \neq \mathrm{LT}_\prec(\lambda \cdot h),$$

then the ascending chain $G_1 \subseteq G_2 \subseteq \dots$ *stabilizes.*

Proof. Since for all $\lambda \in \Lambda$ and $h \in G_i$ we have $\mathrm{LT}_\prec(r_{i+1}) \neq \mathrm{LT}_\prec(\lambda \cdot h)$, the element r_{i+1} is irreducible w.r.t. G_i. Condition (1.22) is involving the term order $\prec'$ would apply only if r_{i+1} would be reducible, hence, we are considering the usual notion of reducedness w.r.t. G_i for the order $\prec$. If the chain $G_1 \subseteq G_2 \subseteq \dots$ does not stabilize, then there would be an infinite subsequence of monomials $\{\lambda_1 e, \lambda_2 e, \dots\} \subseteq \Lambda E$, such that $\lambda_k | \lambda_{k+1}$ contrary to the statement of Theorem 24. $\square$

Theorem 25. *If R denotes an arbitrary monomially bi-filtered ring, where R is built from monomials of the form $A^k \cdot B^l$ with $(k, l) \in \mathbb{N}^m \times \mathbb{Z}^n$, and the orders $\prec$ and $\prec'$ are chosen to be admissible, then Buchberger's algorithm as formulated in [ZW08a] terminates in a finite number of steps.*

Proof. We start with $G := \{g_1, \dots, g_q\}$ and already assume that it is a Gröbner basis with respect to $\prec'$ (which can be ensured by Lemma 50). Suppose that the relative reduction proceeds by generating the sequence of sets $G_i := \{g_1, \dots, g_q, r_1, \dots, r_i\}$ for $i \geq 1$ and let r_{i+1} be reduced with respect to G_i. Then, either

- $\mathrm{LT}_{\prec}(r_{i+1}) \neq \mathrm{LT}_{\prec}(\lambda h)$ for any $\lambda \in \Lambda$ and $h \in G_i$, or
- $\mathrm{LT}_{\prec}(r_{i+1}) = \mathrm{LT}_{\prec}(\lambda h)$ for some $\lambda \in \Lambda, h \in G_i$ such that $\mathrm{LT}_{\prec'}(r_{i+1}) \prec' \mathrm{LT}_{\prec'}(\lambda h)$.

By Lemma 50, the first case cannot occur infinitely often many times (that is, if all G_i are obtained from G_{i-1} via a transition of the first type, then the ascending chain $G_1 \subseteq G_2 \subseteq \dots$ stabilizes). For the second case, we have

$$\begin{aligned}&\mathrm{LT}_{\prec}(r_{i+1}) = \mathrm{LT}_{\prec}(\lambda h),\\ &\mathrm{LT}_{\prec'}(r_{i+1}) \prec' \mathrm{LT}_{\prec'}(\lambda h) \Leftrightarrow \nu_2(\mathrm{LT}_{\prec'}(r_{i+1})) < \nu_2(\mathrm{LT}_{\prec'}(\lambda h)).\end{aligned}$$

If the algorithm does not terminate, then $(G_i)_{i \geq 1}$ is a strictly increasing sequence. Therefore, we can assume that there are infinitely many pairs $(i, j) \in \mathbb{N}^2$ with $i > j$ such that

$$\mathrm{LT}_{\prec}(r_i) = \mathrm{LT}_{\prec}(\lambda r_j) \wedge \mathrm{LT}_{\prec'}(r_i) \prec' \mathrm{LT}_{\prec'}(\lambda r_j).$$

We obtain a strictly descending (w.r.t. the order $\prec'$) infinite sequence of monomials in ΛE that contradicts the fact that ΛE is well-ordered w.r.t. $\prec'$. $\square$

While relative reduction is concerned with bivariate filtrations, we consider the situation of a multivariate filtration. To that end, we introduce the concept of *set-relative reduction.*

Theorem 26. *Let F be the free R-module, where R is a (not necessarily commutative) noetherian ring and the fixed commutative subring K of R is a field. Let $f \in F$ and $G = \{g_1, \dots, g_q\} \subseteq F \backslash \{0\}$. Let $\mathcal{A}$ be a subset of $\{\prec_1, \dots, \prec_p\}$, and an order $\prec_m^n$ $(1 \leq m, n \leq p)$ defined above be fixed. Then, there exists $h_1, \dots, h_q \in R$ and $r \in F$ such that*

$$f = h_1 g_1 + \dots + h_q g_q + r$$

and

- *$h_i = 0$ or for all $\prec$ in $\mathcal{A}$ we have*
$$\mathrm{LT}_{\prec}(h_i g_i) \preceq \mathrm{LT}_{\prec}(f), \qquad 1 \leq i \leq q;$$
- *$r = 0$ or for all $\prec$ in $\mathcal{A}$ with $\mathrm{LT}_{\prec}(r) \preceq \mathrm{LT}_{\prec}(f)$ we have that*
$$\mathrm{LT}_{\prec_m^n}(r) \notin \{\mathrm{LT}_{\prec_m^n}(\lambda g_i) : \mathrm{LT}_{\prec_n^m}(\lambda g_i) \preceq_n^m \mathrm{LT}_{\prec_n^m}(r) : \lambda \in \Lambda\}, \qquad 1 \leq m, n \leq p.$$

Proof. We give a constructive proof, along the lines of [ZW08a, Theorem 3.1]. First, we initialize $r = f$ and $h_i = 0$ for $1 \leq i \leq q$. The next steps are repeated until $r = 0$ or there exists no λ and g_i that satisfy the conditions of the theorem. If there exists $\lambda \in \Lambda$ such that for all $\prec$ in A we have $\mathrm{LT}_\prec(r) \preceq \mathrm{LT}_\prec(f)$ and

$$\mathrm{LT}_{\prec_m^n}(r) = \mathrm{LT}_{\prec_m^n}(\lambda g_i) \qquad \wedge \qquad \mathrm{LT}_{\prec_n^m}(\lambda g_i) \preceq_n^m \mathrm{LT}_{\prec_n^m}(r),$$

we are allowed to perform the reduction step, and update the quantities r to r' resp. h_i to h_i' as

$$r' = r - \frac{\mathrm{LC}_{\prec_m^n}(r)}{\mathrm{LC}_{\prec_m^n}(\lambda g_i)}\lambda g_i \qquad\qquad h_i' = h_i + \frac{\mathrm{LC}_{\prec_m^n}(r)}{\mathrm{LC}_{\prec_m^n}(\lambda g_i)} \cdot \lambda.$$

Obviously, we have that

$$\mathrm{LT}_{\prec_m^n}(r') \prec_m^n \mathrm{LT}_{\prec_m^n}(r), \qquad \text{while for all } \prec \text{ in } \mathcal{A} \text{ we have } \mathrm{LT}_\prec(\lambda g_i) \preceq \mathrm{LT}_\prec(f).$$

Since the monomials $\Lambda E \subseteq F$ are well-ordered, this can only be repeated finitely often. Summing up the λ_i to h_i we obtain for all $\prec$ in $\mathcal{A}$ that $\mathrm{LT}_\prec(h_i g_i) \preceq \mathrm{LT}_\prec(f)$. □

Definition 27. Let $\mathcal{A} \subseteq \{\prec_1, \ldots, \prec_p\}$, $\prec_m^n \in \{\prec_i^k \colon 1 \leq i, k \leq p\}$, $f \in F$, the elements $\{g_1, \ldots, g_q\} \subseteq F\backslash\{0\}$ and $r \in F$. We say that f *$\mathcal{A}$-reduces relative to* $\prec_m^n$ to r if only if the conditions of Theorem 26 apply, and we call this process *set-relative reduction.* The reduction is fully characterized by the tuple

$$(\mathcal{A}, \prec) \in \mathcal{P}(\{\prec_1, \ldots, \prec_p\}) \times \{\prec_i^k \colon 1 \leq i, k \leq p\}.$$

Consider now a set $V := \{f_1, \ldots, f_p\} \subseteq F\backslash\{0\}$. A set $G := \{g_1, \ldots, g_q\} \subseteq F\backslash\{0\}$ is a Gröbner basis for V if and only if every element $f \in {}_R\langle V\rangle$ can be reduced modulo G to zero in finitely many steps. For the reduction, we consider set relative reduction. As easily observed, this definition is equivalent to $(RG) \cap I = 0$.

Based on set-relative reduction, we can now consider a p-step procedure for computing Gröbner bases in this setting. We restrict our attention to rings R where the monomials are of the form $A^k \cdot B^l$ with $(k, l) \in \mathbb{N}^m \times \mathbb{Z}^n$., i.e. we can presume that Theorem 24 and Theorem 25 holds.

We also assume that the fixed orthant decomposition of $\mathbb{N}^m \times \mathbb{Z}^n$ consists of k orthants and $\Lambda^{(j)}$ $(1 \leq j \leq k)$ is a subset of Λ (we use notation (1.24) from Theorem 8, that although formulated for difference-differential operators, can be generalized to monomials of the form $A^k \cdot B^l$ easily), whose exponent vectors lie in the j-th orthant. Furthermore, $K[\Lambda^{(j)}]$ will denote the subring of R generated (as a K-vector space - as in Theorem 26, we assume that K is a field) by the set $\Lambda^{(j)}$.

Exactly in the fashion of Theorem 8, we consider elements $f, g \in F$, where $\prec$ is a monomial order of ΛE, and $V(j, f, g)$ will denote a finite system of generators of the $K[\Lambda^{(j)}]$-module

$${}_{K[\Lambda^{(j)}]}\langle \mathrm{LT}_\prec(\lambda f) \in \Lambda^{(j)}E : \ \lambda \in \Lambda\rangle \ \cap \ {}_{K[\Lambda^{(j)}]}\langle \mathrm{LT}_\prec(\eta g) \in \Lambda^{(j)}E : \ \eta \in \Lambda\rangle$$

With that notation, for every generator $v \in V(j, f, g)$, the element

$$S_{\prec}(j, f, g, v) = \frac{v}{\mathrm{LT}_{j,\prec}(f)} \frac{f}{\mathrm{LC}_{j,\prec}(f)} - \frac{v}{\mathrm{LT}_{j,\prec}(g)} \frac{f}{\mathrm{LC}_{j,\prec}(g)}$$

is said to be an S-polynomial of f and g w.r.t. j, v and $\prec$.

Based on the preceding results, we can now present Buchberger's algorithm in this setting.

Algorithm 2 Given V, compute a set G such that ${}_R\langle V\rangle = {}_R\langle G\rangle$ and $(RG) \cap I = 0$.

Require: F is a free R-module, $V := \{f_1, \ldots, f_r\} \subseteq F\backslash\{0\}$;
  The ring R is p-fold filtered as in (4.1).
Ensure: $G := \{g_1, \ldots, g_s\} \subseteq F\backslash\{0\}$ where ${}_R\langle G\rangle = {}_R\langle V\rangle$ such that $I \cap (RG) = 0$.
  (I understood as the irreducible elements w.r.t. set-relative reduction)
  $\mathcal{A} \leftarrow \{\prec_p\}$;
  $G^{(0)} \leftarrow \{f_1, \ldots, f_r\}$;
  while there exist $j \in \{1, \ldots, k\}$, $f, g \in G^{(0)}$ and $v \in V(j, f, g)$ such that $S_{\prec_p}(j, f, g, v)$ $\mathcal{A}$-reduces to $r \neq 0$ relative to $\prec_p$ by $G^{(0)}$ **do**
    $G^{(0)} \leftarrow G^{(0)} \cup \{r\}$;
  $G^{(1)} \leftarrow G^{(0)}$;
  for $\ell = p - 1, \ldots, 1$ **do**
    $(A, \prec) \leftarrow (\{\prec_p, \ldots, \prec_{\ell+1}\}, \prec_\ell^{\ell+1})$;
    while there exist $j \in \{1, \ldots, k\}$, $f, g \in G^{(p-\ell)}$ and $v \in V(j, f, g)$ such that $S_{\prec_\ell}(j, f, g, v)$-$\mathcal{A}$-reduces to $r \neq 0$ relative to $\prec$ by $G^{(p-\ell)}$ **do**
      $G^{(p-\ell)} \leftarrow G^{(p-\ell)} \cup \{r\}$;
    $G^{(p-\ell+1)} \leftarrow G^{(p-\ell)}$;
  return $G^{(p)}$

Theorem 27. *Consider $V := \{f_1, \ldots, f_s\} \subseteq F\backslash\{0\}$, let $G := \{g_1, \ldots, g_q\} \subseteq F\backslash\{0\}$ be a $\{\prec_1, \ldots, \prec_p\}\backslash\{\prec_i\}$-Gröbner basis of V relative to $\prec_i$. Then, G is a Gröbner basis of V with respect to $\prec_j$ for all $1 \leq j \leq p$.*

Proof. If $f \in F$ can be $\{\prec_1, \ldots, \prec_p\}\backslash\{\prec_i\}$-reduced modulo G relative to $\prec_i$ to zero, then f can be reduced to zero modulo G with respect $\{\prec_1, \ldots, \prec_p\}\backslash\{\prec_i\}$ in the classic way. To see, that G is a Gröbner basis w.r.t. $\prec_i$ we take $f \in {}_R\langle V\rangle$, which we represent as in Theorem 26 as

$$f = h_1 g_1 + \ldots + h_q g_q.$$

In every step of the set-relative reduction, we have

$$\mathrm{LT}_{\prec_i}(\lambda_j g_j) \preceq_i \mathrm{LT}_{\prec_i}(r), \qquad \lambda_j \in \mathrm{T}(h_j),\ 1 \leq j \leq q.$$

We now conclude that

$$r = c_j \lambda_j g_j + r_j \Rightarrow \mathrm{LT}_{\prec_i}(r_j) \preceq_i \mathrm{LT}_{\prec_i}(r), \qquad c_j \in K.$$

If now $\mathrm{LT}_{\prec_i}(r) \preceq_i \mathrm{LT}_{\prec_i}(f)$ then $\mathrm{LT}_{\prec_i}(r_j) \preceq_i \mathrm{LT}_{\prec_i}(f)$. This is in particular fulfilled for the first step, where $r = f$. So in every step, we have

$$\mathrm{LT}_{\prec_i}(r_j) \preceq_i \mathrm{LT}_{\prec_i}(f).$$

Further, if

$$\begin{aligned}\mathrm{LT}_{\prec_i}(h_j g_j) \preceq_i \mathrm{LT}_{\prec_i}(\lambda_j g_j) &\Rightarrow \mathrm{LT}_{\prec_i}((h_j + c_j \lambda_j) g_j) \preceq_i \mathrm{LT}_{\prec_i}(\lambda_j g_j),\\ &\Rightarrow \mathrm{LT}_{\prec_i}(h_j g_j) \preceq_i \mathrm{LT}_{\prec_i}(r_j) \preceq_i \mathrm{LT}_{\prec_i}(f).\end{aligned}$$

But this obviously fulfilled, since in the first step, we have $h_j = 0$ for all j. $\square$

5. Outlook

In this thesis, we've presented the concept of Gröbner reduction, for a common treatment of modules over certain (non-commutative) rings of operators. While this approach covers quite some rings of interest in the mathematical landscape, there still might be extensions to a wider class of rings, that would be subject to further study. The book [RM87] is devoted to non-commutative noetherian rings, and gives a lot of different perspectives on related constructions. There might be also some applications to physical systems, that are described by operator algebras, where Gröbner reduction could be applied.

Another research perspective might be formulated as follows: Consider the ring $R = K^{(\mathbb{M})}$, as well as an R-module M, and a binary reduction relation $\rho \subseteq M \times M$. Let W be a well-founded relation, $\mathrm{rk} : M \to W$ a function, and $X \subseteq M$ a set. Then, we define

$$u \longrightarrow v :\Longleftrightarrow \exists x \in X : v = u - x \wedge \mathrm{rk}(v) < \mathrm{rk}(u).$$

For example, one possible choice would be $W = \mathbb{M}$, the function rk is the leading term $\mathrm{LT} : R \to \mathbb{M}$. Hence, leading term reduction can be viewed under that 'model-reduction'. It turns out, that a great account of Theorems appearing in classic textbooks on Gröbner bases (such as [AL94, BWK93, Win96, CLO97]), can be proven under the aspect of this reduction relation with a view towards Gröbner reduction. A further choice of rk would be as sketched at the beginning of Chapter 3, namely (3.3), in particular $\mathrm{rk} : F \to \mathcal{P}_{fin}(\Lambda E)$ (the set of finite subsets of monomials $\mathbb{M}E$), that assigns to $f \in F$ its support $\mathrm{T}(f)$.

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Scientific Work

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Gröbner bases, Resultants and Linear Algebra (GBRELA) 2013
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Linz Algebra Research Day (LARD) 2013 X-Mas Edition
Dimension in Difference-Differential Rings

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Linz Algebra Research Day (LARD) 2015
Recent Results on Gröbner Reduction on Free Modules

Applications of Computer Algebra (ACA) 2016
Bases for Modules of Difference-Operators by Gröbner Reduction

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Reduction Concepts in Free Modules