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Eidesstattliche Erklärung

Ich erkläre an Eides statt, dass ich die vorliegende Dissertation selbstständig und ohne fremde Hilfe verfasst, andere als die angegebenen Quellen und Hilfsmittel nicht benutzt bzw. die wörtlich oder sinngemäss entnommenen Stellen als solche kenntlich gemacht habe. Die vorliegende Dissertation ist mit dem elektronisch übermittelten Textdokument identisch.

Linz, October 2016

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Kurzfassung

Das Hauptziel der vorliegenden Dissertation ist das Studium neuer Algorithmen zur Bestimmung von polynomialen, rationalen und algebraischen Lösungen von algebraischen gewöhnlichen Differentialgleichungen (AODEs) erster Ordnung. Das Problem der Bestimmung von Lösungen von AODEs erster Ordnung in geschlossener Form hat eine lange Geschichte, und spielt nach wie vor in vielen Bereichen der Mathematik eine Rolle. Es gibt einige Lösungsmethoden für spezielle Klassen solcher ODEs. Jedoch gibt es noch immer keinen Entscheidungsalgorithmus für allgemeine AODEs erster Ordnung, selbst wenn man nur nach bestimmten Arten von Lösungen sucht, z. B. polynomialen, rationalen oder algebraischen Funktionen. Unser Interesse gilt algebraischen allgemeinen Lösungen, rationalen allgemeinen Lösungen, speziellen rationalen Lösungen sowie Polynomlösungen. Verschiedene Algorithmen zur Bestimmung solcher Arten von Lösungen von AODEs erster Ordnung werden präsentiert.

Wir nähern uns AODEs erster Ordnung aus verschiedenen Richtungen. Wenn die Ableitung der gesuchten Lösung als neue Unbekannte behandelt wird, kann die AODE erster Ordnung als Hyperfläche über dem Grundkörper betrachtet werden. Hierfür sind Werkzeuge aus der algebraischen Geometrie anwendbar. Insbesondere nutzen wir birationale Transformationen von algebraischen Hyperflächen, um die gegebene Differentialgleichung in eine andere zu transformieren, die im Idealfall einfacher zu lösen ist. Dieser geometrische Ansatz führt uns zu einer Prozedur zur Bestimmung einer algebraischen allgemeinen Lösung einer parametrisierbaren AODE erster Ordnung. Eine allgemeine Lösung enthält eine beliebige Konstante. Für das Problem der Bestimmung einer rationalen allgemeinen Lösung, in welcher die Konstante rational auftritt, schlagen wir einen Entscheidungsalgorithmus für die gesamte Klasse von AODEs erster Ordnung vor.

Die geometrische Methode ist nicht anwendbar, um spezielle rationale Lösungen zu erhalten. Stattdessen studieren wir diese Art von Lösungen unter kombinatorischen und algebraischen Gesichtspunkten. In der kombinatorischen Betrachtung spielen die Pole der Koeffizienten der Differentialgleichung eine wichtige Rolle, nämlich für die Abschätzung von Kandidaten für Pole der rationalen Lösung und deren Vielfachheiten. Wir schlagen eine algebraische Methode, basierend auf der Theorie algebraischer Funktionenkörper, vor, um den Grad einer rationalen Lösung abzuschätzen. Eine Kombination dieser Methoden führt uns zu einem Algorithmus zur Bestimmung aller rationalen allgemeinen Lösungen für eine generische Klasse von AODEs erster Ordnung, die jede AODE erster Ordnung aus der Sammlung von Kamke einschließt. Für Polynomlösungen funktioniert der Algorithmus in der gesamten Klasse von AODEs erster Ordnung.

Abstract

The main aim of this thesis is to study new algorithms for determining polynomial, rational and algebraic solutions of first-order algebraic ordinary differential equations (AODEs). The problem of determining closed form solutions of first-order AODEs has a long history, and it still plays a role in many branches of mathematics. There is a bunch of solution methods for specific classes of such ODEs. However still no decision algorithm for general first-order AODEs exists, even for seeking specific kinds of solutions such as polynomial, rational or algebraic functions. Our interests are algebraic general solutions, rational general solutions, particular rational solutions and polynomial solutions. Several algorithms for determining these kinds of solutions for first-order AODEs are presented.

We approach first-order AODEs from several aspects. By considering the derivative as a new indeterminate, a first-order AODE can be viewed as a hypersurface over the ground field. Therefore tools from algebraic geometry are applicable. In particular, we use birational transformations of algebraic hypersurfaces to transform the differential equation to another one for which we hope that it is easier to solve. This geometric approach leads us to a procedure for determining an algebraic general solution of a parametrizable first-order AODE. A general solution contains an arbitrary constant. For the problem of determining a rational general solution in which the constant appears rationally, we propose a decision algorithm for the general class of first-order AODEs.

The geometric method is not applicable for studying particular rational solutions. Instead, we study this kind of solutions from combinatorial and algebraic aspects. In the combinatorial consideration, poles of the coefficients of the differential equation play an important role in the estimation of candidates for poles of a rational solution and their multiplicities. An algebraic method based on algebraic function field theory is proposed to globally estimate the degree of a rational solution. A combination of these methods leads us to an algorithm for determining all rational solutions for a generic class of first-order AODEs, which covers every first-order AODEs from Kamke's collection. For polynomial solutions, the algorithm works for the general class of first-order AODEs.

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List of Algorithms

Chapter 1

Introduction

A first-order AODE is a differential equation of the form $F(x, y, y') = 0$, where F is a polynomial in three variables with coefficients in an algebraically closed field, for instance $\overline{\mathbb{Q}}$, the field of algebraic numbers. Solving the differential equation is the problem of determining differentiable functions $y = y(x)$ such that $F(x, y(x), y'(x)) = 0$. If $y(x)$ is an algebraic (rational, polynomial) functions, then it is called an algebraic (rational, polynomial, respectively) solution. A solution may contain an arbitrary constant. Such a solution is called a general solution. For example, $y(x) = x^2 + c$ is a general solution of the differential equation $y' - 2x = 0$. Solving first-order AODEs is a fundamental problem in the theory of (non-linear) algebraic differential equations.

First-order AODEs have been studied a lot and there are many solution methods for special classes of such ODEs. The study of these ODEs can be dated back to the work of Fuchs [14], and later by Poincaré [32]. In [27], Malmquist studied the class of first-order AODEs having transcendental meromorphic solutions, and Eremenko revisited later in [11]. In the 1970s, Matsuda classified differential function fields having no movable critical points up to isomorphism of differential fields [28]. The theory by Matsuda brings modern wind to the algebraic theory of first-order AODEs. Following this direction, Eremenko presented a theoretical consideration on a degree bound for rational solutions [12].

The problem of finding closed form solutions of first-order AODEs has been considered widely in the literature. Among non-linear first-order AODEs, Riccati equations can be considered as the simplest ones. In [25], Kovacic solved completely the problem of computing Liouvilian solutions of a second order linear ODE with rational function coefficients. In the process, Kovacic also proposed an algorithm for determining all rational solutions of a Riccati equation. Solving a first-order first-degree AODE is a much harder problem. The problem of determining an algebraic general solution for a first-order first-degree AODE is one of an equivalent version of the Poincaré problem. This problem is still open. In [6], Carnicer investigated a degree bound for algebraic solutions for first-order first-degree AODEs in non-dicritical cases. Hubert [21] found implicit solutions by computing Gröbner bases.

The problem of studying symbolic solutions for first-order AODEs from an algebro-geometric approach has received much attention in the last decade. The first algorithm for the class of first-order autonomous AODEs has been proposed by Feng and Gao [13, 1]. The algorithm is based on the fact that by considering the derivative as a new indeterminate, the differential equation can be viewed an algebraic curve. Applying this idea to the general class of first-order AODEs, and combining it with Fuchs' theorem on first-order AODEs without movable critical points, Chen and Ma [9] presented an algorithm for determining a special class of rational general solutions. However, their algorithm is incomplete due to two reasons: the necessary condition for the existence of the solution is not proven to be algorithmically checkable, and a good rational parametrization is required in advance. Ngô and Winkler [29, 31, 30] applied the algebro-geometric approach to general non-autonomous first-order AODEs. Using parametrization of algebraic surfaces, they associate to the given parametrizable AODE an associated system of algebraic equations in the parameters. This associated system is a planar rational system. In order to complete the algorithm, a degree bound for irreducible invariant algebraic curves of the planar rational system is required. The problem of finding a uniform bound for the degree of invariant algebraic curves for planar rational systems is known as the Poincaré problem. This difficult problem has been solved by Carnicer [6], but only generically for the non-dicritical case. So the algorithm of Ngô and Winkler, although producing general rational solutions in almost all situations where such a solution exists, is still no complete decision algorithm. Following this direction, a generalization to the class of higher order AODEs [20], and even to algebraic partial differential equations [15] is presented. So far no general algorithm for deciding the existence and, in the positive case, computing an algebraic/rational general solution, and all particular rational solutions exists.

In this thesis, we present:

1. A procedure for determining an algebraic general solution of a parametrizable first-order AODE (see Algorithm 2).
2. A full algorithm for determining a rational general solution, in which the constant appears rationally, for a general first-order AODE (see Algorithm 5).
3. Algorithms for computing all rational solutions for a generic class of first-order AODEs (see Algorithm 7, 10, 12).
4. An algorithm for computing all polynomial solutions for an arbitrary first-order AODE (see Algorithm 8).

This generalizes the works by Feng and Gao [13], Chen and Ma [9], Ngô and Winkler [29, 31, 30], Behloul and Cheng [3].

In Chapter 2, we recall basic notations from differential algebra and algebraic geometry. In Chapter 3, we approach first-order AODEs from an algebraic geometric aspect. By considering the derivative as a new indeterminate, a given first-order AODE can be seen as an algebraic equation. This algebraic equation defines an algebraic surface over the ground field. An algebraic solution of the differential equation corresponds to an algebraic curve on the surface which satisfies certain condition. Therefore tools from algebraic geometry are applicable. In particular, birational transformation of algebraic surfaces is used to transform the differential equation to a planar rational system. The key point is that there is a faithful relation between algebraic general solutions of the given differential equation and algebraic general solutions of the planar rational system. Solving a general planar rational system is still a very hard problem. But in many cases, the obtained planar rational system is easy to solve. This algebraic geometric approach leads to a procedure for determining an algebraic general solution for first-order AODEs.

A similar method is presented in Chapter 4. By considering the derivation as a new indeterminate, we can also view the differential equation as an algebraic equation which defines an algebraic curve over the field of rational functions over the ground field. With a similar process, birational transformation of algebraic curves is used to transform the given differential equation to a first-order first-degree AODE. We prove that optimal parametrization of algebraic curves over the field of rational functions can be achieved within the field of rational functions. This guarantees us to do the process in a controllable way. Consequently, a decision algorithm for determining a rational general solution for which the constant appears rationally of a first-order AODE is established.

In Chapter 5, we study particular rational and polynomial solutions. The problem of computing all rational solutions which are not necessary general solutions requires more algebraic techniques. Two other methods are introduced. A combinatorial approach is given to estimate possible pole positions for a rational solution, and bound the order of these poles. An algebraic method based on algebraic function field theory is proposed to globally estimate the degree of a rational solution. A combination of these methods is applicable for a generic class of first-order AODEs which covers all parametrizable first-order AODEs. A long the way, a general algorithm for determining all polynomial solutions of first-order AODEs is obtained.

Chapter 2

Preliminaries

In this chapter, we briefly recall some basic notions in differential algebra and algebraic geometry. This chapter should not be considered as an introduction to differential algebra and parametrization of hypersurface. For further detail, we refer the reader to [24, 34] for differential algebra, and to [38, 35] for parametrization of algebraic curves and surfaces.

2.1 Differential algebra

2.1.1 Ring of differential polynomials

Definition 2.1.1. A differential ring is a ring R equipped with a derivation σ, which is a map from R to itself, such that:

i. σ is a group homomophism, i.e. $\forall a, b \in R, \sigma(a+b) = \sigma(a) + \sigma(b)$.

ii. σ satisfies Leibniz's rule: $\forall a, b \in R, \sigma(ab) = \sigma(a)b + a\sigma(b)$.

If furthermore R is a field, it is called a differential field.

Every field $\mathbb{K}$ can be seen as a differential field with the trivial derivation which maps all elements of the field to zero. Let $\mathbb{K}(x)$ be the field of rational functions in x with coefficients in $\mathbb{K}$. The trivial derivation on $\mathbb{K}$ can be extended to a derivation $\frac{d}{dx}$ in such a way that $\frac{d}{dx}(x) = 1$.

Let $\mathbb{K}$ be a differential field with a derivation σ, and let $y_1, y_2, ..., y_n$ be n independent indeterminates over $\mathbb{K}$. Elements of the differential polynomial ring $R := \mathbb{K}\{y_1, y_2, ..., y_n\}$ are polynomials in $y_1, y_2, ..., y_n$ and their derivatives with coefficients in $\mathbb{K}$. The derivation on R is extended naturally from σ. By abuse of notation, we denote the derivation on R by σ . The j-th derivative of y_i will be also denoted by y_{ij}. We shall call y_i its own derivative of order zero and shall be sometimes written by y_{i0}. The first derivative of y_i is usually written by y_i' instead of y_{i1}. In the remain of this chapter, if there is no caution, R denotes the polynomial differential ring with the derivation σ. R is a standard example for a differential ring.

Definition 2.1.2. An ideal I in R is called a differential ideal if I is closed under the derivation, i.e. $\sigma(I) \subseteq I$. If furthermore I is a radical (resp. prime) ideal of R, it is called a radical (resp. prime) differential ideal.

Let Σ be a subset of R. The differential ideal generated by Σ, denoted by $[\Sigma]$, is the ideal generated by elements of Σ and their derivatives. It is in fact equal to the intersection of all differential ideals containing Σ. The radical of $[\Sigma]$, denoted by $\{\Sigma\}$, is a differential ideal. It is called the radical ideal generated by Σ. Therefore $\{\Sigma\}$ is the intersection of all radical differential ideals containing Σ.

In commutative algebra, a radical ideal of a ring of polynomials in finitely many variables over a field can be always factored as the intersection of finitely many prime ideals. This fact is still true on a polynomial differential ring (or in general a radical Noetherian differential ring). In [34], Ritt proved that every radical differential ideal in R is the radical of a finitely generated differential ideal. As a consequence, the class of radical differential ideals in R satisfies the ascending chain condition. Note that it is not true for the class of all differential ideals. Therefore every radical differential ideal in R is the intersection of finitely many prime differential ideals.

Theorem 2.1.3 (see [34])**.** *Let I be a radical differential ideal in R. There exist uniquely, up to a permutation of indexes, prime differential ideals $P_1, P_2, ..., P_r$ for some $r \in \mathbb{N}$ such that:*

i. $I = P_1 \cap P_2 \cap ... \cap P_r$ *and*

ii. The intersection is irredundant, i.e. $\forall i,\ P_i \not\subseteq \bigcap_{j \neq i} P_j$.

Such prime differential ideals are called the essential components of the ideal I.

Definition 2.1.4 (see [34])**.** Let Σ be a subset of R, and L a differential field extended from $\mathbb{K}$ which respect to which the y_is are indeterminates. An element $\xi := (\xi_1, \xi_2, ..., \xi_n) \in L^n$ is called a zero of Σ if ξ vanishes all elements of Σ.

It is clear that if ξ vanishes a differential polynomial F, then it also vanishes all derivatives of F. Therefore Σ, $[\Sigma]$, and $\{\Sigma\}$ agree the set of zeros.

Definition 2.1.5. Let I be a prime differential ideal in R. A zero ξ of I, which lies in a differential field L extended from $\mathbb{K}$, is called a generic zero if every differential polynomial in R vanished at ξ must be in I.

In the other words, ξ is a generic zero of the prime differential I if and only if the set

$$\mathbb{I}(\xi) := \{F \in R \,|\, F(\xi) = 0\}$$

is exactly I. It is well-known that every prime differential ideal has a generic zero.

2.1.2 Ritt's reduction

Given a prime differential ideal I, and a solution ξ. To check whether ξ is a general solution of I, we usually face the ideal membership problem. In particular, we need to know when a given differential polynomial is belong to the ideal I. It can be done systematically by using Ritt's reduction.

Definition 2.1.6. Let $\Delta y := \{y_{ij} \,|\, i = 1, ..., n, j \in \mathbb{N}\}$ be the set of indeterminates and their derivatives in R. A ranking " $<$ " of $y_1, ..., y_n$ is a total ordering on Δy such that:

i. $\forall i = 1, ...n,\ \forall j_1, j_2 \in \mathbb{N},\ j_1 < j_2 \Rightarrow y_{ij_1} < y_{ij_2}$, and

ii. $\forall u, v \in \Delta y,\ u < v \Rightarrow u' < v'$.

The ranking is called orderly if it furthermore satisfies:

iii. $\forall i_1, i_2 = 1, ..., n,\ \forall j_1, j_2 \in \mathbb{N},\ j_1 \leq j_2 \Rightarrow y_{i_1 j_1} \leq y_{i_2 j_2}$.

Example 2.1.7. In the ring $K\{y\}$ of differential polynomials in one variables, there is only one ranking and it satisfies the order $y < y' < y'' < \ldots < y^{(n)} < \ldots$.

Example 2.1.8. In the ring $K\{s, t\}$ of differential polynomials in two variables, the ranking defined by $s^{(i)} < t^{(j)}$ for every $i, j \in \mathbb{N}$ is called a lexicographic order.

Now fix a ranking "$<$" on Δy in the polynomial differential ring R, and $F \in R - \mathbb{K}$ a differential polynomial. Since the ranking orders totally the set Δy, there exists an element in Δy appearing in the normal expression of F with the highest ranking. It is called the leader of F, denoted by u_F. Rewrite F as a polynomial in u_F:

$$F = I_0 + I_1 u_F + ... + I_d u_F^d$$

where d is the degree of F with respect to u_F, and $I_0, I_1, ..., I_d \in R$. Then I_d, the leading coefficient of F with respect to u_F, is called the initial of F, and is denoted by I_F. The derivative of F with respect to u_F

$$\frac{\partial F}{\partial u_F} = I_1 + 2I_2 u_F + ... + dI_{d-1} u_F^{d-1}$$

is called the seperant of F, and is denoted by S_F. A simple computation gives:

Lemma 2.1.9. *For every $k \geq 1$, we have*

$$F^{(k)} = S_F u_F^{(k)} + G$$

where $G \in R$ is a polynomial with coefficients in $\mathbb{K}$ and variables the elements of Δy having ranking not exceed $u_F^{(k-1)}$.

In order to define a reduction on R, we need to extend the ranking "$<$" up to an order on the whole ring R.

Definition 2.1.10. Let "$<$" be a ranking on Δy. We define a ranking on R, which is recalled by "$<$", as follow: for any F and G differential polynomials in R,

i. If $F \in \mathbb{K}$ and $G \in R \setminus \mathbb{K}$, then $F < G$.

ii. If $F, G \in R \setminus \mathbb{K}$, then

$$F < G :\Leftrightarrow (u_F < u_G) \vee ((u_F = u_G) \wedge (\deg_{u_F} F < \deg_{u_G} G))$$

Otherwise, they have the same rank.

Now let us fix a ranking on R. Let G be differential polynomials in R such that $G \notin \mathbb{K}$. A differential polynomial F is said to be *lower* than G if $\deg_{u_F} F < \deg_{u_G} G$. The polynomial F is said to be *reduced* with respect to G if F is lower than G and derivatives of G. If the ranking is orderly, then F is lower than G if and only if F is reduced with respect to G.

We are going to recall the Ritt's reduction on polynomial differential rings. Ritt's reduction can be seen as a generalization of Euclidean algorithm. It is well-known that if f and g are two polynomials in one variable whose coefficients lie in a field, then one can divide f by g by using Euclidean algorithm. It remains a unique pair of polynomials (q, r) with $\deg r < \deg g$ such that $f = gp + r$. This fact is no longer true for general multivariate differential polynomials, even only for ordinary multivariate polynomials. However, in polynomial differential rings, we have a so call pseudo-remainder algorithm to divide a differential polynomial to an auto-reduced set of differential polynomials.

A subset Σ of R is called an *auto-reduced set* if $\Sigma \cap \mathbb{K} = \emptyset$ and every element of Σ is reduced with respect to the others.

Definition 2.1.11 (Ritt's reduction)**.** Let Σ be an auto-reduced set of R with respect to a given ranking " $<$ ", and let F be a differential polynomial in R. A reduction of F with respect to Σ is a differential polynomial $F_0 \in R$ such that:

i. F_0 is reduced with respect to elements of Σ, and

ii. $F_0 \leq F$, and

iii. For each $G \in \Sigma$, there are $i_G, s_G \in \mathbb{N}$ such that

$$\left(\prod_{G \in \Sigma} I_G^{i_G} S_G^{s_G} \right) F - F_0$$

can be written as a linear combination over R of all derivatives $\sigma^j G$ where $G \in \Sigma$ and $\sigma^j G \leq u_F$.

In this case, F_0 is called the differential pseudo remainder of F with respect to Σ, and denoted by $\text{prem}(F, \Sigma)$.

The following proposition is a criteria to check whether a solution of a prime differential ideal which is generated by a given auto-reduced set is general. It is extremely useful in practice. For further details we refer the reader to [34].

Proposition 2.1.12. *Let I be a differential prime ideal in R which is generated by an auto-reduced set Σ. Then a zero ξ of I is a generic zero if and only if*

$$\forall F \in R, \;\; F(\xi) = 0 \Rightarrow prem(F, \Sigma) = 0$$

2.2 General solutions of AODEs

From now on by $\mathbb{K}$ we denote a computational algebraically closed field of characteristic zero with the trivial derivation. In practice, we might choose $\mathbb{K} = \overline{\mathbb{Q}}$ the field of algebraic numbers. All derivatives are understood as the usual ones.

An algebraic ordinary differential equation is a differential equation of the form

$$F(x, y, y', ..., y^{(n)}) = 0 \tag{2.1}$$

where $F \in \mathbb{K}[x]\{y\} \setminus \mathbb{K}[x][y]$. Without lost on generality, we can always assume that F is an irreducible polynomial in $\mathbb{K}[x, y, y', ..., y^{(n)}]$. Otherwise F can be factored as the product of irreducible factors. In this case, the set of solutions of the given differential equation is equal to the union of the sets of solutions of AODEs which are defined by the irreducible factors of F.

The notion "general solution" of an AODEs can be described from differential algebra context. In general a general solution of an AODE is defined as a generic zero of a certain associated prime differential ideal. In our situation such ideal much be established from F. It is well-known that, in a polynomial ring in finitely many variables over a field, a principle ideal generated by an irreducible polynomial is prime. It is no longer true in the case of differential polynomial rings. In particular, neither $[F]$ nor $\{F\}$ is a prime differential ideal of $\mathbb{K}(x)\{y\}$, even if F is an irreducible polynomial. Fortunately, Ritt proved that:

Lemma 2.2.1 (see [34]). *Let $F \in \mathbb{K}(x)\{y\}$ such that F is an irreducible polynomial in $\mathbb{K}[x, y, y', ..., y^{(n)}]$. Then the ideal $\{F\}$ can be factored as:*

$$\{F\} = (\{F\} : S_F) \cap \{F, S_F\}$$

where $(\{F\} : S_F) := \{G \in \mathbb{K}(x)\{y\} \,|\, G.S_F \in \{F\}\}$ is a prime differential ideal.

The lemma shows that the ideal $(\{F\} : S_F)$ is the unique essential component (among finitely many essential components of $\{F\}$) that does not contain the separant S_F of F. On the other hand the second component $\{F, S_F\}$ is the intersection of the other essential components of $\{F\}$. It leads us to the definition of general solutions of an AODE.

Definition 2.2.2. Consider the differential equation $F(x, y, y', \ldots, y^{(n)}) = 0$.

i. A zero of the radical ideal $\{F\}$ is called a solution of the differential equation.

ii. A generic zero of the differential ideal $(\{F\} : S_F)$ is called a general solution.

iii. A zero of the ideal $\{F, S_F\}$ is called a singular solution.

Definition 2.2.3. Consider the differential equation $F(x, y, y', \ldots, y^{(n)}) = 0$, let ξ be a solution which is contained in a differential field L extended from $\mathbb{K}(x)$. We denote by K the field of constants of L.

i. ξ is called an algebraic solution if there is a non-zero polynomial $G \in K[x, y]$ such that $G(x, \xi) = 0$.
In this case, G is called an annihilating polynomial of ξ.

ii. If furthermore $\deg_y G = 1$, then ξ is called a rational solution.

iii. ξ is called an algebraic (resp. rational, polynomial) general solution if it is a general solution and algebraic (resp. rational, polynomial).

Given an algebraic solution ξ of the differential equation $F(x, y, y', \ldots, y^{(n)}) = 0$, there are infinitely many corresponding annihilating polynomials. If we ask for irreducible polynomials among them, then there is only one up to multiplying by a non-zero constant. If $G(x, y)$ is an irreducible annihilating polynomial of ξ, then all root $y = y(x)$ of the algebraic equation $G(x, y) = 0$ are solutions of the differential equation, (see [1], Lemma 2.4). Therefore, by abuse of notation, G is sometimes called a solution.

The following lemma is concluded from Prposition 2.1.12.

Lemma 2.2.4. *A solution ξ of the differential equation*

$$F(x, y, y', \ldots, y^{(n)}) = 0$$

is a general solution if and only if

$$\forall H \in k(x)\{y\},\ H(\xi) = 0 \Rightarrow prem(H, F) = 0$$

Proposition 2.2.5. *Let ξ be an algebraic solution of the differential equation*

$$F(x, y, y', \ldots, y^{(n)}) = 0$$

with the irreducible annihilating polynomial G. If ξ is a general solution, then at least one of the coefficients of G contain a constant which is transcendental over $\mathbb{K}$.

Proof. By contradiction, if $G(x, y) \in \mathbb{K}[x, y]$, then $G = \operatorname{prem}(G, F) \neq 0$. It is contradiction with the fact that $G(x, \xi) = 0$. Thus at least one of the coefficients of G is a constant which is not in $\mathbb{K}$. Since $\mathbb{K}$ is algebraically closed, such constant is transcendental over $\mathbb{K}$. □

From the previous proposition, an algebraic general solution can be viewed as a class of algebraic solutions which is parametrized by a certain number of parameters. In particular if $y^{(n)}$ is the highest derivation appearing on F, then the number of independent parameters needed to paramatrized a general solution is exactly n (see [24, Thm. 6, Sec. 12, Chp. 2]).

2.3 Parametrization of algebraic curves and surfaces

In Chapter 3 and 4, we intrinsically use an algebraic geometric approach for solving first-order AODEs. Consider a first-order AODE, $F(x, y, y') = 0$, for an irreducible non-constant polynomial F. We view the equation to be an algebraic one by replacing the derivative by an independent variable, i. e. $F(x, y, z) = 0$. Depending on the ground field the zero set of such an equation defines an algebraic curve or an algebraic surface.

$$\mathcal{C} = \left\{(a_1, a_2) \in \mathbb{A}^2(\overline{\mathbb{K}(x)}) \mid F(x, a_1, a_2) = 0\right\},$$
$$\mathcal{S} = \left\{(a_0, a_1, a_2) \in \mathbb{A}^3(\mathbb{K}) \mid F(a_0, a_1, a_2) = 0\right\}.$$

For higher dimensional spaces such zero sets of single polynomials are called hypersurfaces.

Definition 2.3.1. The algebraic curve $\mathcal{C}$ is called the *corresponding curve*. The algebraic surface $\mathcal{S}$ is called the *corresponding surface*.

By definition the algebraic curves and surfaces are given implicitly by the defining equation. Very often it is useful to have a parametric expression for the points on the curve or surface. Let $\mathbb{F}$ be some algebraically closed field.

Definition 2.3.2. A *rational parametrization*, or briefly, a parametrization of a curve $\mathcal{C}$ over $\mathbb{A}^2(\mathbb{K})$ is a rational map $\mathcal{P} : \mathbb{A}^1(\mathbb{K}) \to \mathcal{C} \subseteq \mathbb{A}^2(\mathbb{K})$ such that the image of $\mathcal{P}$ is dense in $\mathcal{C}$ (with respect to the Zariski topology).

Similarly a (rational) parametrization of a surface $\mathcal{S}$ over $\mathbb{A}^3(\mathbb{K})$ is a rational map $\mathcal{P} : \mathbb{A}^2(\mathbb{K}) \to \mathcal{S} \subseteq \mathbb{A}^3(\mathbb{K})$ such that the image of $\mathcal{P}$ is dense in $\mathcal{S}$.

If, furthermore, $\mathcal{P}$ is a birational equivalence, $\mathcal{P}$ is called a *proper* parametrization.

A parametrization is called optimal, if the degree of its coefficient field is minimal (see [38] for further details).

Let $\mathcal{P}_{\mathcal{C}}(t) = (p_1(x,t), p_2(x,t))$ be a parametrization over $\overline{\mathbb{K}(x)}$ of the corresponding curve of an AODE. Then $\mathcal{P}_{\mathcal{S}}(s_1, s_2) = (s_1, p_1(s_1, s_2), p_2(s_1, s_2))$ is an algebraic parametrization of the corresponding surface. If $\mathcal{P}_{\mathcal{C}}$ is rational in x then $\mathcal{P}_{\mathcal{S}}$ is a rational parametrization. However, there are first-order AODEs which admit a rational parametrization of the corresponding surface but not of the corresponding curve. Consider for instance the AODE, $F(x, y, y') = y'^2 - y^3 - x^2 = 0$. The corresponding curve has genus 1, whereas the corresponding surface can be parametrized by $\left(\frac{s(1-s^2)}{t^3}, \frac{1-s^2}{t^2}, \frac{1-s^2}{t^3}\right)$.

It is well-known that if an algebraic curve or surface admits a rational parametrization, then it admits a proper parametrization. In the affirmative case, for curves one can compute such a proper parametrization with optimal coefficient field. For more details on rationality we refer to [38] and [43, 39, 35] for curves and surfaces respectively.

Theorem 2.3.3 (Rationality Criterion)**.** *An algebraic curve admits a rational parametrization if and only if its genus is equal to zero.*

An algebraic surface admits a rational parametrization if and only if both its arithmetic genus and the second plurigenus are equal to zero.

Furthermore, there is a relation between different proper parametrizations of curves and surfaces respectively.

Lemma 2.3.4. *Let $\mathcal{P}$ and $\mathcal{Q}$ be two proper parametrizations of some algebraic hypersurface. Then there exists a rational function R such that $\mathcal{Q} = \mathcal{P}(R)$.*

- *In case of curves, R is a Möbius transformation, i. e. a linear rational function $R(s_1) = \frac{a_0 + a_1 s_1}{b_0 + b_1 s_1}$ with $a_0 b_1 - a_1 b_0 \neq 0$.*

- *In case of surfaces, R is a Cremona transformation, i. e. a birational map of the plane to itself, and hence by the Theorem of Castelnuovo-Noether a finite composition of quadratic transformations and projective linear transformations (c.f. [39, 43]).*

Definition 2.3.5. A point A on the corresponding curve $\mathcal{C}$ is called an *algebraic solution point* if its coordinates have the form $(y(x), y'(x))$ for some $y(x) \in \overline{\mathbb{K}(x)}$. If furthermore $y(x) \in \mathbb{K}(x)$, A is called a *rational solution point*.

Finding an algebraic/rational general solution of $F(x, y, y') = 0$ is reduced to looking for a class of algebraic/rational solution points $(y(x), y'(x))$ which depend on a parameter c.

Chapter 3

Algebraic general solutions of first-order AODEs

This chapter is based on the author's works in [41]. In this chapter we present a procedure for determining an algebraic general solution of a first-order AODE. In order to use the technique of rational parametrization, we add an additional assumption to the initial differential equation, that the algebraic equation obtained when we replace the derivation y' by a new indeterminate defines a rational surface. A first-order AODE satisfying this additional assumption is called surface-parametrizable. The general schedule for determining an algebraic general solution of a surface-parametrizable first-order AODE is as follows. We associate for each surface-parametrizable first-order AODE a planar rational system, which is so called the associated differential system. The key observation is that algebraic general solutions of the initial differential equation can be determined faithfully from an algebraic general solution of the associated differential system (see Section 3.1). This step is inherited from the work by Ngô and Winkler in [29].

The problem of determining an algebraic general solution of a surface parametrizable first-order AODE is now reduced to the problem of computing an algebraic general solution of a planar rational system. The latter problem is hard in general. But in case a rational first integral is provided, or even only a degree bound for a rational first integral is given, we propose an algorithm to determine an algebraic general solution (see Section 3.2). Finally, if a surface-parametrizable first-order AODE is given together with a degree bound for an algebraic general solution, we can compute an algebraic general solution explicitly (see Section 3.3).

3.1 Associated Differential System

In this section, we construct for each surface-parametrizable first-order AODE a planar rational system. Although the construction is as similar as the one described in Ngô and Winkler [29], it is briefly summarized here for self-containedness. Several facts relating to their algebraic general solutions are investigated.

Let us first give a formal definition for surface-parametrizable first-order AODE.

Definition 3.1.1. A first-order AODE $F(x, y, y') = 0$ is called surface parametrizable if its corresponding surface, say $\mathcal{S}$, in $\mathbb{A}^2(\mathbb{K})$ defined by $F(x, y, z) = 0$ is rational.

In the other words, there is a rational map $\mathcal{P} : \mathbb{A}^2_{\mathbb{K}} \to \mathcal{S} \subset \mathbb{A}^3_{\mathbb{K}}$ defined by $\mathcal{P}(s,t) := (\chi_1(s,t), \chi_2(s,t), \chi_3(s,t))$ for some rational functions $\chi_1, \chi_2, \chi_3 \in \mathbb{K}(s,t)$ such that $F(\mathcal{P}(s,t)) = 0$, and $\mathcal{P}$ is invertible. Such $\mathcal{P}$ is called a *proper parametrization* of the surface $\mathcal{S}$. Algorithm for determining a parametrization of a rational surface is investigated, for instance, there is one in [35]. During this section and so on, we always assume that a surface parametrizable first-order AODE is equipped with a proper parametrization $\mathcal{P}$.

Now let us fix an algebraic general solution $\xi = \xi(x)$ of the surface parametrizable first-order AODE $F(x,y,y') = 0$. Then $F(x, \xi(x), \xi'(x)) = 0$. Denote $(s(x), t(x)) := \mathcal{P}^{-1}(x, \xi(x), \xi'(x))$, a representation of the inverse of $(x, \xi(x), \xi'(x))$ via $\mathcal{P}$. Since $\mathcal{P}$ is proper, $(s(x), t(x))$ is a pair of algebraic functions satisfying $\mathcal{P}(s(x), t(x)) = (x, \xi(x), \xi'(x))$. Therefore

$$\begin{cases} \chi_1(s(x), t(x)) = x \\ \chi_2'(s(x), t(x)) = \chi_3(s(x), t(x)) \end{cases}$$

Differentiating both sides of the first equation, and expanding the second one gives us a linear system on $s'(x)$ and $t'(x)$.

$$\begin{cases} s'(x)\dfrac{\partial}{\partial s}\chi_1(s(x), t(x)) + t'(x)\dfrac{\partial}{\partial t}\chi_1(s(x), t(x)) = 1 \\ s'(x)\dfrac{\partial}{\partial s}\chi_2(s(x), t(x)) + t'(x)\dfrac{\partial}{\partial t}\chi_2(s(x), t(x)) = \chi_3(s(x), t(x)) \end{cases}$$

Since $\mathcal{P}$ is a birational equivalent, the Jacobian matrix

$$\begin{bmatrix} \frac{\partial \chi_1}{\partial s} & \frac{\partial \chi_2}{\partial s} & \frac{\partial \chi_3}{\partial s} \\ \frac{\partial \chi_1}{\partial t} & \frac{\partial \chi_2}{\partial t} & \frac{\partial \chi_3}{\partial t} \end{bmatrix}$$

has generic rank 2. Without lost on general, we can always assume that the determinant

$$\begin{vmatrix} \frac{\partial \chi_1}{\partial s} & \frac{\partial \chi_2}{\partial s} \\ \frac{\partial \chi_1}{\partial t} & \frac{\partial \chi_2}{\partial t} \end{vmatrix}$$

is non-zero. Furthermore, we claim that $g(s(x), t(x)) \neq 0$. It will be asserted by the following lemma.

Lemma 3.1.2. *With notation as above. Then*

$$\forall R \in \mathbb{K}(s,t),\ R(s(x), t(x)) = 0 \Rightarrow R = 0$$

Now $s'(x)$ and $t'(x)$ can be solved by Cramer's rule from the linear system. Thus $(s(x), t(x))$ is an algebraic solution of the planar rational system:

$$\begin{cases} s' = \dfrac{\chi_3(s,t)\frac{\partial}{\partial t}\chi_1(s,t) - \frac{\partial}{\partial t}\chi_2(s,t)}{g(s,t)} \\ t' = \dfrac{\frac{\partial}{\partial s}\chi_2(s,t) - \chi_3(s,t)\frac{\partial}{\partial s}\chi_1(s,t)}{g(s,t)} \end{cases} \tag{3.1}$$

Definition 3.1.3. System (3.1) is called the associated differential system of the differential equation $F(x, y, y') = 0$ with respect to the proper parametrization $\mathcal{P}$.

We summary here the result of the construction of the associated system above.

Theorem 3.1.4. *Let $F(x, y, y') = 0$ be a surface parametrizable first-order AODE and consider its associated differential system (3.1) with respect to a given proper parametrization $\mathcal{P}$. If $y = y(x)$ is an algebraic general solution of the differential equation $F(x, y, y') = 0$, then*

$$(s(x), t(x)) := \mathcal{P}^{-1}(x, y(x), y'(x))$$

is an algebraic general solution of the associated system.

Proof. $(s(x), t(x))$ is an algebraic solution of the associated system as we established. Lemma 3.1.2 asserts that it is in fact a general solution. □

Theorem 3.1.5. *Let $F(x, y, y') = 0$ be a surface parametrizable first-order AODE and consider its associated system (3.1) with respect to a given proper parametrization $\mathcal{P}$. If $(s(x), t(x))$ is an algebraic general solution of the associated system, then*

$$y(x) := \chi_2\left(s\left(2x - \chi_1(s(x), t(x))\right), t\left(2x - \chi_1(s(x), t(x))\right)\right)$$

is an algebraic general solution of the differential equation $F(x, y, y') = 0$.

Proof. As in the construction, $s(x), t(x)$ must satisfies the following system:

$$\begin{cases} \chi_1'(s(x), t(x)) = 1 \\ \chi_2'(s(x), t(x)) = \chi_3(s(x), t(x)) \end{cases}$$

The first relation yields $c := \chi_1(s(x), t(x)) - x$ is an arbitrary constant. Thus we have

$$\begin{cases} \chi_1(s(x-c), t(x-c)) = x \\ \chi_2(s(x-c), t(x-c)) = y(x) \\ \chi_3(s(x-c), t(x-c)) = y'(x) \end{cases}$$

Therefore $y(x)$ is an algebraic general solution of $F(x, y, y') = 0$.

It remains to prove that $y(x)$ is a general solution. To this end, let arbitrary $G \in \mathbb{K}(x)\{y\}$ such that $G(y(x)) = 0$. Since F is of order 1, $\mathrm{prem}(G, F) \in \mathbb{K}(x)[y, y']$. Let $R \in \mathbb{K}, y, y']$ be the numerator of $\mathrm{prem}(G, F)$. Then $R(x, y(x), y'(x)) = 0$. It implies $R(\mathcal{P}(s(x-c), t(x-c))) = 0$. Since c can be chosen arbitrary, we have $R(\mathcal{P}(s(x), t(x))) = 0$. Now, applying the lemma 3.1.2 yields $R(\mathcal{P}(s, t)) = 0$. So that $R(x, y, z) = R(\mathcal{P}(\mathcal{P}^{-1}(x, y, z))) = 0$. It follows $\mathrm{prem}(G, F) = 0$. Hence $y(x)$ is a general solution. □

The previous two theorems establish a one-to-one correspondence between algebraic general solutions of a paramatrizable first-order AODE and algebraic general solutions of its associated system which is a planar rational system. Furthermore the correspondence is formulated explicitly. Once an algebraic general solution of its associated system is known, the corresponding algebraic general solution of the given surface parametrizable first-order AODE can be determined immediately. The problem of finding an algebraic general solution of paramatrizable first-order AODEs can be reduced to the problem of determining an algebraic general solution of a planar rational system.

Is is important to notice that the one-to-one correspondence holds not only for the class of algebraic general solutions, but also for the general class of general solutions (which do not necessary satisfy the property of being algebraic functions). By just repeating the above process without assumption that the general solutions are algebraic, we obtain:

Theorem 3.1.6. *Let $F(x, y, y') = 0$ be a surface parametrizable first-order AODE and consider its associated system (3.1) with respect to a given proper parametrization $\mathcal{P}$.*

i. If $y(x)$ is a general solution of the differential equation $F(x, y, y') = 0$, then

$$(s(x), t(x)) := \mathcal{P}^{-1}(x, y(x), y'(x))$$

is a general solution of the associated system.

ii. If $(s(x), t(x))$ is a general solution of the associated system, then

$$y(x) := \chi_2(s(2x - \chi_1(s(x), t(x))), t(2x - \chi_1(s(x), t(x))))$$

is a general solution of the given differential equation.

3.2 Planar rational system and its algebraic general solutions

This section is devoted to the problem of computing explicitly an algebraic general solution of the planar rational system. Whereas the problem of finding explicit algebraic solutions of planar rational systems has received only little attention in the literature, the problem of finding implicit algebraic solutions, or in the other words, finding irreducible invariant algebraic curves and rational first integral, has been heavily studying. Some historical details and recent results which are helpful for our proofs will be recalled. By combining these results and the idea for finding algebraic general solutions of autonomous first-order AODEs of Aroca et. al. (see [1]), we will present an algorithm for determining an algebraic general solution of a planar rational system with a given rational first integral.

3.2.1 Planar rational system

Definition 3.2.1. A planar rational system is a differential system of order 1 of the form:

$$\begin{cases} s' = M(s, t) \\ t' = N(s, t) \end{cases} \tag{3.2}$$

where M, N are rational functions on s, t with coefficients in $\mathbb{K}$.

If M, N are polynomials, it is called a planar polynomial system.

Given a planar rational system, we are interested in its algebraic solutions. The key objects to investigate information about algebraic solutions of a planar rational system are invariant algebraic curves and rational first integrals. Our plan is first using known algorithms to find a rational first integral of the system, and then its irreducible algebraic curves. Secondly, each irreducible algebraic curve will derive the system into two autonomous first-order AODEs which can be solved explicitly by the procedure of Aroca et. al. (see [1]).

Let us now explain why irreducible invariant algebraic curves are good candidates for implicit algebraic solutions. Assume that $(s(x), t(x))$ is an algebraic solution of planar rational system (3.2), where $s(x), t(x)$ are algebraic functions whose coefficients lie in a differential field K extended from $\mathbb{K}$ by constants. Let $G(s,t) \in K[s,t]$ be an irreducible polynomial such that $G(s(x), t(x)) = 0$. Then we also have $\frac{d}{dx}G(s(x), t(x)) = 0$. Expanding the left hand side gives

$$s'(x)\frac{\partial}{\partial s}G(s(x), t(x)) + t'(x)\frac{\partial}{\partial t}G(s(x), t(x)) = 0$$

Since $(s(x), t(x))$ is a solution of the system (3.2), we imply that

$$M(s(x), t(x))\frac{\partial}{\partial s}G(s(x), t(x)) + N(s(x), t(x))\frac{\partial}{\partial t}G(s(x), t(x)) = 0$$

Now let us rewrite M, N as reduced rational functions, say $\frac{M_1}{M_2}, \frac{N_1}{N_2}$ respectively, where M_1, M_2, N_1, N_2 are polynomials in s and t with coefficients in $\mathbb{K}$ and such that $\gcd(M_1, M_2) = \gcd(N_1, N_2) = 1$. Then we obtain

$$\left(M_1 N_2 \frac{\partial G}{\partial s} + M_2 N_1 \frac{\partial G}{\partial t}\right)(s(x), t(x)) = 0$$

If both $s(x)$ and $t(x)$ are constants, we call $(s(x), t(x))$ a constant solution. Constant solutions of the system (3.2) are exact solutions of the algebraic system $M_1(s,t) = N_1(s,t) = 0$. Thus it can be computed easily by solving the algebraic system. It is clear that the system (3.2) has a constant solution as a general solution if and only if $M = N = 0$. From now on, we only consider non-constant solutions, i.e. solution $(s(x), t(x))$ such that not both coordinates are constants.

The last equality says that the polynomial function $M_1 N_2 \frac{\partial G}{\partial s} + M_2 N_1 \frac{\partial G}{\partial t}$ is vanished along the irreducible curve defined by $G(s,t) = 0$. Therefore, by Hilbert Nullstellensatz, $M_1 N_2 \frac{\partial G}{\partial s} + M_2 N_1 \frac{\partial G}{\partial t}$ must be divisible by G. This fact leads us to the definition of invariant algebraic curve.

Definition 3.2.2. An algebraic curve defined by $G(s,t) = 0$ is called an *invariant algebraic curve* of the planar rational system

$$\begin{cases} s' = M(s,t) \\ t' = N(s,t) \end{cases} \tag{3.3}$$

where M, N are rational functions on s, t with coefficients in $\mathbb{K}$, if

$$M_1 N_2 \frac{\partial G}{\partial s} + M_2 N_1 \frac{\partial G}{\partial t} = GH$$

for some $H \in \mathbb{K}[s,t]$. In this case, H is called the cofactor of G.

Definition 3.2.3. A differentiable function $W(s,t)$ on two variables s,t with coefficients in $\mathbb{K}$ is a first integral of the planar rational system

$$\begin{cases} s' = M(s,t) \\ t' = N(s,t) \end{cases} \tag{3.4}$$

where M,N are rational functions on s,t with coefficients in $\mathbb{K}$, if it is not a constant function and

$$M\frac{\partial G}{\partial s} + N\frac{\partial G}{\partial t} = 0$$

If furthermore W is a rational function, it is called a *rational first integral*.

It is not hard to see that the set of all first integrals of a planar rational system together with constant functions has an algebraic structure as a field. The intersection of such field and $\mathbb{K}(s,t)$ is the set of all rational first integrals with constants in $\mathbb{K}$. If the planar differential system has a rational first integrals, there is a non-composite reduced rational function, say F, such that every rational first integral has the form $u(F(s,t))$ for some univariate rational function u with coefficients in $\mathbb{K}$ (see [4]). In the other words, the set of all rational first integrals of the planar rational system is either an empty set or $\mathbb{K}(F)\setminus\mathbb{K}$, where $\mathbb{K}(F)$ is the field extended from $\mathbb{K}$ by F. Such the F is unique up to a composition with a homography. In particular, instead of finding all rational first integrals, looking for a non-composite one is enough.

On the other hand, the set of rational first integrals, and all invariant algebraic curves of a planar rational system does not change if we multiply the right hand side of the two differential equations of the system by the same non-zero rational function in $\mathbb{K}(s,t)$. Therefore it is suffices to consider planar polynomial systems for studying invariant algebraic curves and rational first integrals. Furthermore, by multiplying the right hand side of the differential equations in the system (3.2) by $\frac{M_2N_2}{gcd(M_1N_2,M_2N_1)}$, one can always assume that M,N are polynomials such that $gcd(M,N)=1$.

Problem of finding invariant algebraic curves and rational first integrals of a planar polynomial system dates back from the work of Darboux in the 1870s and Poincaré in the 1890s. Darboux showed that if a planar polynomial system has a large enough number of invariant algebraic curves, then it has a rational first integral (see [10]). Once a non-composite rational first integral is found, for instance $\frac{P}{Q}$ which is a reduced rational function, the algebraic curves defined by irreducible factors of $P-cQ$ are all but finitely many irreducible invariant algebraic curves of the planar rational system (see [4]). In [32], Poincaré pointed out that in order to find a non-composite rational first integral, it is sufficient to find an upper bound for the degree of irreducible invariant algebraic curves. In 1979, Jouanolou proved that such upper bound exists (see [22]). However there is still no effective way to determine such bound in general. The problem of determining such a bound is called as *Poincaré problem* which is a well-known difficult problem. Although *Poincaré problem* is still open, it is already solved in several specific cases. Partial results can be found, for instance, in [6, 8, 7, 32].

With a given degree bound, there are already some algorithms computing a rational first integral and invariant algebraic curves whose degree does not exceed the bound. In [33], Prelle and Singer proposed a procedure, which is usually called as Prelle-Singer procedure, for computing invariant algebraic curves and a rational first integral of a planar polynomial system. They first compute invariant algebraic curves with a given setting degree by using undeterminate coefficient method. The degree bound is setting to be increasing one by one up to obtaining large enough number of invariant algebraic curves. Once enough number of invariant algebraic curves are found, there is a certain formula to determine a rational first integral.

Recently, Bostan et. al. presented an efficient algorithm for computing a non-composite rational first integral with a degree bound of a planar polynomial system (see [4]). The idea is based on the fact that the planar polynomial system (3.2) has a rational first integral if and only if all power series solutions of the corresponding differential equation

$$y' = \frac{M(x,y)}{N(x,y)}$$

are algebraic. In this case, the irreducible annihilating polynomials of these power series solutions lead to rational first integrals. Once a degree bound of the rational first integral is fixed, only finitely many first coefficients of the power series solutions are necessary. Also in the paper, the authors proposed an algorithm for computing all irreducible invariant algebraic curves with a degree bound. The algorithm has been implementing in Maple package *RationalFirstIntegrals*. Later, we will use their package to determine a non-composite rational first integral of the associated system of a given first-order AODE.

The following theorem is a classical result on relation between irreducible invariant algebraic curves and rational first integrals of a planar rational system. We recall here for technique purpose. For further detail, we prefer to many classical literatures about rational first integrals, for instance, see [33].

Theorem 3.2.4. *There is a natural number N such that a given planar rational system has a rational first integral if and only if the system has more than N irreducible invariant algebraic curves. Furthermore, if $W = \frac{P}{Q}$ is a reduced rational first integral then every irreducible invariant algebraic curves is defined by an irreducible factor of $c_1 P - c_2 Q$, where c_1, c_2 are arbitrary constants.*

3.2.2 Algebraic general solutions of planar rational systems

As a preparation step for the next sections, we present in this section the formal definition of an algebraic general solution of a planar rational system from differential algebra context. Some first properties which will be used later are recalled.

Let us recall the planar rational system (3.2)

$$\begin{cases} s' = M(s,t) \\ t' = N(s,t) \end{cases}$$

where $M, N \in \mathbb{K}(s,t)$ are rational functions. In order to give a formal definition for a general solution of a planar rational system, we need to investigate a differential prime ideal in $\mathbb{K}(x)\{s,t\}$ constructed from the two differential equations of the system.

Let us consider the differential polynomial ring $\mathbb{K}(x)\{s,t\}$ equipped with the order lex ranking " $<$ " such that $s < t$. Rewrite M, N as reduced fractions $\frac{M_1}{M_2}, \frac{N_1}{N_2}$ respectively and denote:

$$\tilde{M} := M_2(s,t)s' - M_1(s,t)$$
$$\tilde{N} := N_2(s,t)t' - N_1(s,t)$$

Then the leader of $\tilde{M}$ and $\tilde{N}$ are s' and t' respectively which are of degree 1. Their initials, and also the separant, are equal to $M_2(s,t)$ and $N_2(s,t)$. Therefore $\{M, N\}$ is an auto-reduced set of $\mathbb{K}(x)\{s,t\}$. Let denote

$$\Sigma := \{M_2^i N_2^j \,|\, i, j \in \mathbb{N}\}$$

Then we have:

Proposition 3.2.5. *The ideal* $I := \left([\tilde{M}, \tilde{N}] : \Sigma\right) \subset \mathbb{K}(x)\{s,t\}$ *is a prime differential ideal.*

Proof. Consider the homomorphism $\phi : \mathbb{K}(x)\{s,t\} \to \mathbb{K}(x)(s,t)$ defined by $\phi(s) = s$, $\phi(t) = t$, $\phi(s') = M(s,t)$ and $\phi(t') = N(s,t)$. We claim that $I = \ker \phi$.

Since $\phi(s' - M(s,t)) = \phi(t' - N(s,t)) = 0$, $I \subseteq \ker \phi$. Conversely, let any $H \in \ker \phi$. By using Ritt's reduction, there are $i_M, i_N \in \mathbb{N}$ such that the differential polynomial $M_2^{i_M} N_2^{i_N} H - \text{prem}(H, \{\tilde{M}, \tilde{N}\})$ can be written as a linear combination of $\tilde{M}, \tilde{N}$ and their derivatives with coefficients in $k(x)\{s,t\}$. Therefore,

$$\phi\left(M_2^{i_M} N_2^{i_N} H - \text{prem}(H, \{\tilde{M}, \tilde{N}\})\right) = 0$$

Thus

$$\phi\left(\text{prem}(H, \{\tilde{M}, \tilde{N}\})\right) = M_2^{i_M} N_2^{i_N} \phi(H) = 0$$

Moreover, $\text{prem}(H, \{\tilde{M}, \tilde{N}\})$ must lie in $\mathbb{K}(x)[s,t]$ since it is reduced by $\tilde{M}$ and $\tilde{N}$. So that $\text{prem}(H, \{\tilde{M}, \tilde{N}\}) = 0$. It implies $H \in I$.

We have proved the claim that $I = \ker \phi$. Thus the factor ring $\mathbb{K}(x)\{s,t\}/I$ is isomorphic with a subring of $\mathbb{K}(s,t)$, which is an integral domain. Hence, I is a differential prime ideal. ☐

Definition 3.2.6. With notations as above.

i. A zero of the set $\{s' - M(s,t), t' - N(s,t)\}$ is called a solution of the planar rational system (3.2).

ii. A generic zero of the ideal I is called a general solution of the planar rational system (3.2).

Definition 3.2.7. Let $(\xi_1, \xi_2) \in L^2$ be a solution of the planar rational system (3.2), where L is a differential field extended from $\mathbb{K}(x)$. Denote by K the field of constants of L.

i. The solution (ξ_1, ξ_2) is called an algebraic general solution if there are non-zero polynomials $G_1, G_2 \in K[x,y]$ such that $G_1(x, \xi_1) = G_2(x, \xi_2) = 0$.

ii. If furthermore $\deg_y G_1 = \deg_y G_2 = 1$, then (ξ_1, ξ_2) is called a rational solution.

iii. (ξ_1, ξ_2) is called an algebraic (resp. rational) general solution if it is a general solution and is algebraic (resp. rational).

The following lemma is an immediate consequence of Proposition 2.1.12.

Lemma 3.2.8. *A solution (ξ_1, ξ_2) of the planar rational system (3.2) is a general solution if and only if*

$$\forall H \in k(x)\{s,t\},\ H(\xi_1,\xi_2) = 0 \Rightarrow prem\left(H, \{\tilde{M}, \tilde{N}\}\right) = 0$$

Similar to Proposition 2.2.5, a general solution of a planar rational system must contain an arbitrary constant. It is stated as the following proposition:

Proposition 3.2.9. *Assume $(\xi_1, \xi_2) \in L^2$ is an algebraic general solution of a planar rational system, where L is a differential field extended from $\mathbb{K}(x)$. Denote by K the field of constants of L. Let $G \in K[s,t]$ be an irreducible polynomial such that $G(\xi_1, \xi_2) = 0$. Then at least one of the coefficients of G contains a constant which is transcendental over $\mathbb{K}$.*

Proof. By contradiction, assume that G contains no such constant, then $G \in \mathbb{K}[s,t]$. Since G is vanished at the general solution, $G = \text{prem}\left(G, \{\tilde{M}, \tilde{N}\}\right) = 0$. It is a contradiction. □

3.2.3 Algorithm and Examples

Next we will consider the problem of finding an explicit algebraic general solution of a planar rational system with a given irreducible invariant algebraic curve. The following property is a motivation.

Proposition 3.2.10. *If the parametriazable first-order AODE $F(x, y, y') = 0$ has an algebraic general solution, then its associated differential system with respect to a proper parametrization has a rational first integral.*

Proof. If the differential equation $F(x, y, y') = 0$ has an algebraic general solution, then so is its associated system. By applying proposition 3.2.9, the associated system must have an irreducible invariant algebraic curve $G(s,t) = 0$ such that G is monic and at least one of the coefficients of G contains a constant which is transcendental over $\mathbb{K}$. In the other words, the associated system has infinitely many irreducible invariant algebraic curves. Thus it has a rational first integral. □

Theorem 3.2.11. *Assume that $W = \frac{P}{Q}$ is a reduced rational first integral of the planar rational system*

$$\begin{cases} s' = \dfrac{M_1(s,t)}{M_2(s,t)} \\ t' = \dfrac{N_1(s,t)}{N_2(s,t)} \end{cases}$$

where $M_1, M_2, N_1, N_2 \in \mathbb{K}[s,t]$, and that $(s(x), t(x))$ is an algebraic solution in which not both $s(x)$ and $t(x)$ are constants. Then $(s(x), t(x))$ is an algebraic general solution if and only if $W(s(x), t(x))$ is a constant which is transcendental over $\mathbb{K}$.

Proof. Assume that $(s(x), t(x))$ is an algebraic general solution of the planar rational system, then

$$\begin{aligned} W'(s(x), t(x)) &= s'(x)\frac{\partial W}{\partial s}(s(x), t(x)) + t'(x)\frac{\partial W}{\partial t}(s(x), t(x)) \\ &= \left(M\frac{\partial W}{\partial s} + N\frac{\partial W}{\partial t}\right)(s(x), t(x)) = 0 \end{aligned}$$

Therefore $W(s(x), t(x)) = c$ is an arbitrary constant. If $c \in \mathbb{K}$, then $P - cQ \in \mathbb{K}[s, t]$ has an irreducible factor in $\mathbb{K}[s, t]$ vanished at $(s(x), t(x))$. It can not happen duo to proposition 3.2.9. Hence $c \notin \mathbb{K}$. Since $\mathbb{K}$ is algebraically closed, c is transcendental over $\mathbb{K}$.

Conversely, assume that $(s(x), t(x))$ is a non-constant algebraic solution of the given planar rational system such that $W(s(x), t(x)) = c$, where c is a constant being transcendental over $\mathbb{K}$. Let G be an irreducible polynomial such that $G(s(x), t(x)) = 0$. Since $P - cQ$ is also vanished a long $(s(x), t(x))$, G must be an irreducible factor of $P - cQ$. As in [36, Ch. 3, Thm. 3.6], G has the form $A + \alpha B$ for some $A, B \in \mathbb{K}[s, t]$, $B \neq 0$, and $\alpha \in \overline{\mathbb{K}(c)}$ which is still transcendental over $\mathbb{K}$.

Now let $H \in \mathbb{K}(x)\{s, t\}$ be a differential polynomial such that $H(s(x), t(x)) = 0$. We denote $\tilde{H} := \text{prem}(H, \{\tilde{M}, \tilde{N}\})$ where $\tilde{M} := M_2 s' - M_1$ and $\tilde{N} := N_2 t' - N_1$. To finish the proof, we need to show that $\tilde{H} = 0$. It is clear that $\tilde{H} \in \mathbb{K}(x)\{s, t\}$ and satisfies $\tilde{H}(s(x), t(x)) = 0$. Let consider both $G = A + \alpha B$ and $\tilde{H}$ as polynomials in s, t with coefficient in $\mathbb{K}(\alpha, x)$. Then they are both vanished along $(s(x), t(x))$, and G is, again, irreducible. Thus $\tilde{H}$ must be divisible by G. It is only possible in the case $\tilde{H} = 0$, because α is transcendental not only on $\mathbb{K}$ but also on $\mathbb{K}(x)$. Hence $(s(x), t(x))$ is a general solution. □

The following corollary is an immediately consequence of the above theorem. It help us to split a planar rational system into two autonomous first-order AODEs, which lead us to the algorithm for determining explicit algebraic general solution of a planar rational system.

Corollary 3.2.12. *Assume that* $W = \frac{P}{Q}$ *is a reduced rational first integral of the system*

$$\begin{cases} s' = \dfrac{M_1(s, t)}{M_2(s, t)} \\ t' = \dfrac{N_1(s, t)}{N_2(s, t)} \end{cases}$$

where $M_1, M_2, N_1, N_2 \in \mathbb{K}[s, t]$ *and that* $(s(x), t(x))$ *is an algebraic general solution. Then*

i. $s(x)$ *is an algebraic general solution over* $\mathbb{K}(c)$ *of the autonomous first-order AODE* $F_1(s', s) = 0$*, where*

$$F_1 := Res_t(P - cQ, M_2 s' - M_1)$$

ii. $t(x)$ *is an algebraic general solution over* $\mathbb{K}(c)$ *of the autonomous first-order AODE* $F_2(s', s) = 0$*, where*

$$F_2 := Res_s(P - cQ, N_2 s' - N_1)$$

Fortunately the problem of finding algebraic general solutions of autonomous first-order AODEs is investigated. In [1], Aroca et. al. proposed a criteria to decide whether an autonomous first-order AODE having an algebraic general solution and compute such solution in affirmative case. Combining the previous theorem and the corollary, together with the result of Aroca et. al., an algorithm for computing explicit algebraic general solutions of planar rational systems with a given rational first integral will be proposed next. For determining a rational first integral, one can use the package *RationalFirstIntegrals* which have been implementing by A. Bostan et. al. [4].

Algorithm 1 Algebraic general solutions of a planar rational system

Require: The planar rational system

$$\begin{cases} s' = \dfrac{M_1(s,t)}{M_2(s,t)} \\ t' = \dfrac{N_1(s,t)}{N_2(s,t)} \end{cases}$$

and $W = \frac{P}{Q}$ a reduced rational first integral.

Ensure: An algebraic general solution $(s(x), t(x))$.

If $M = 0$, then $s(x) = c$ and $t(x)$ is an algebraic general solution of $t' = N(c, t)$
If $N = 0$, then $t(x) = c$ and $s(x)$ is an algebraic general solution of $s' = M(s, c_1)$
Compute $F_1 := \mathrm{Res}_t(P - c_1 Q, M_2(s,t)s' - M_1(s,t))$
$\mathcal{S}$:= the set of all irreducible factors of F_1 in $\overline{\mathbb{K}(c)}[s', s]$ containing s'
for all $H \in \mathcal{S}$ **do**
    If $H(s', s) = 0$ has no algebraic solution, then return "No algebraic general solution"
    $s(x)$:= an algebraic solution of $H(s', s) = 0$
    $t(x)$:= a solution of the equation $W(s(x), t) = c_1$
    If $s'(x) - M(s(x), t(x)) = t'(x) - N(s(x), t(x)) = 0$, then return "$(s(x + c_2), t(x + c_2))$"
end for
Return "No algebraic general solution"

Example 3.2.13. Consider the palanar rational system

$$\begin{cases} s' = t \\ t' = \dfrac{t^2}{2s} \end{cases} \tag{3.5}$$

By multiplying the right hand sides of the two differential equations of the system with $\frac{2s}{t}$, we obtain a new system which shares the same set of rational first integrals and invariant algebraic curves:

$$\begin{cases} s' = 2s \\ t' = t \end{cases} \tag{3.6}$$

Using the package *RationalFirstIntegrals* of A. Bostan et. al. (see [4]), we can evaluate a non-composite rational first integral of the last system, for instance

$W = \frac{-64s}{100s-t^2}$. W is also a non-composite rational first integral of the system (3.5). Now we can use the algorithm 1 to find an algebraic general solution of the system (3.5). First we set

$$F_1(s, s') := \text{Res}_t(s' - t, -64s - c_1(100s - t^2)) = (64 + 100c_1)s - c_1 s'$$

which is an irreducible polynomial in $\overline{k(c_1)}[s, s']$. Solving the differential equation $F_1(s, s') = 0$ (by using Aroca's et. al. algorithm, or just by integrating) yields an algebraic solution:

$$s(x) = \frac{1}{c_1}(16 + 25c_1)x^2$$

Next, we find $t(x)$ by solving the algebraic equation $W(s(x), t) = c_1$. It gives two candidates $\frac{2}{c_1}(16+25c_1)x$ and $-\frac{2}{c_1}(16+25c_1)x$. By substituting them to the system (3.5), we see that

$$(s(x), t(x)) := \left(\frac{1}{c_1}(16 + 25c_1)x^2, \frac{2}{c_1}(16 + 25c_1)x\right)$$

is an algebraic solution. Since the system is autonomous, $(s(x + c_2), t(x + c_2))$ is an algebraic general solution.

Example 3.2.14. Consider the planar rational system

$$\begin{cases} s' = \dfrac{t^2}{2} \\ t' = \dfrac{t^3}{2s^2 - 1} \end{cases} \tag{3.7}$$

A rational first integral, for instance $W = \frac{s^2-1}{t^4}$, can be found by a process similar to the one in the previous example. Let

$$F_1(s, r) := \text{Res}_t\left(sr - t^2, s^2 - 1 - ct^4\right) = \left(cs^2r^2 - s^2 + 1\right)^2$$

By solving the autonomous differential equation $F(s, s') = 0$ we obtain an algebraic solution, for instance,

$$s(x) = \pm\sqrt{\frac{x^2}{c} + 1}$$

Next we find $t(x)$ by solving the algebraic equation $W(s(x), t) = c$. Therefore, $t(x) = \pm\sqrt{\frac{x}{c}}$. Finally,

$$\left(\pm\sqrt{\frac{(x + d)^2}{c} + 1}, \pm\sqrt{\frac{x + d}{c}}\right)$$

are algebraic general solutions of the given planar rational system, where c and d are arbitrary constants.

3.3 Algebraic general solutions with degree bound

In this section, we will combine previous results in this chapter to study further the problem of finding an algebraic general solution of a surface parametrizable first-order AODEs. In particular, given a surface parametrizable first-order AODE and a positive integer n, we will present an algorithm for finding an algebraic general solution whose irreducible annihilating polynomial has total degree less than or equal to n.

Consider the surface parametrizable first-order AODE $F(x, y, y') = 0$ with a given proper parametrization

$$\mathcal{P}(s,t) := (\chi_1(s,t), \chi_2(s,t), \chi_3(s,t))$$

for some $\chi_1, \chi_2, \chi_3 \in \mathbb{K}(s,t)$. Assume that $y = y(x) \in \overline{K(x)}$ is an algebraic general solution of the differential equation $F(x, y, y') = 0$, where the field K is extended from $\mathbb{K}$ by transcendence constants. Let $Y(x, y) \in K[x, y]$ be an irreducible annihilating polynomial of $y(x)$. We sometimes call $Y(x, y) = 0$ an algebraic general solution instead of $y(x)$.

We denote $\deg Y, \deg_x Y$ as the total degree of $Y(x, y)$ and the degree of x in Y, respectively.

Theorem 3.3.1. *With notation as above, let $(\sigma_1(x,y,z), \sigma_2(x,y,z)) := \mathcal{P}^{-1}(x,y,z)$ be the inverse map of $\mathcal{P}$. If the differential equation $F(x,y,y') = 0$ has an algebraic general solution $Y(x,y) = 0$ with $\deg Y \leq n$, then the associated system has a rational first integral whose total degree is less than or equal to*

$$m := n^3 \left(\deg_x \sigma_1 + \deg_y \sigma_1 + \deg_x \sigma_2 + \deg_y \sigma_2\right) + 2n^2 \left(\deg_z \sigma_1 + \deg_z \sigma_2\right).$$

Proof. Denote $s(x) := \sigma_1(x, y(x), y'(x))$ and $t(x) := \sigma_2(x, y(x), y'(x))$, then the pair $(s(x), t(x))$ is an algebraic general solution of the associated system. Let $G(s,t) \in K[s,t]$ be an irreducible polynomial such that $G(s(x), t(x)) = 0$. $G(s,t) = 0$ is in fact an irreducible invariant algebraic curve of the associated system with coefficients in K. We will first claim that $\deg G \leq m$.

Denote

$$Q_1(x,y) := \sigma_1\left(x, y, -\frac{\frac{\partial}{\partial x} Y(x,y)}{\frac{\partial}{\partial y} Y(x,y)}\right)$$

and

$$Q_2(x,y) := \sigma_2\left(x, y, -\frac{\frac{\partial}{\partial x} Y(x,y)}{\frac{\partial}{\partial y} Y(x,y)}\right)$$

which are rational functions in $\mathbb{K}(x,y)$. Then $s(x) = Q_1(x, y(x))$ and $t(x) = Q_2(x, y(x))$. The degree of x and y on Q_1 and Q_2 can be estimated in terms of σ_1, σ_2 and Y as follow:

$$\deg_x Q_1 \leq n. \deg_x \sigma_1 + \deg_z \sigma_1 \tag{3.8}$$

$$\deg_y Q_1 \leq n. \deg_y \sigma_1 + \deg_z \sigma_1 \tag{3.9}$$

$$\deg_x Q_2 \leq n. \deg_x \sigma_2 + \deg_z \sigma_2 \tag{3.10}$$

$$\deg_y Q_2 \leq n. \deg_y \sigma_2 + \deg_z \sigma_2 \tag{3.11}$$

Now in order to get annihilating polynomials of $s(x), t(x)$, using the resultant is a fast way. In particular, the polynomials

$$R_1(s,x) := \mathrm{Res}_y(numer(Q_1) - s.denom(Q_1), Y(x,y))$$
$$R_2(s,x) := \mathrm{Res}_y(numer(Q_2) - t.denom(Q_2), Y(x,y))$$

are annihilating polynomials of $s(x)$ and $t(x)$ respectively, where $numer(Q_1)$ is the numerator of Q_1 and $denom(Q_1)$ the denominator one. Therefore $H(s,t) := \mathrm{Res}_x(R_1(s,x), R_2(t,x))$ is a polynomial in $K[s,t]$ satisfying $H(s(x),t(x)) = 0$. It implies that G must be divide H. From the definition of the resultant, one can determine immediately an upper bound for the total degree of H, and thus of G. In fact,

$$\deg_s H \le \deg_s R_1 . \deg_x R_2 \le N^2(\deg_x Q_2 + \deg_y Q_2)$$

Equivalently, we also have

$$\deg_t H \le N^2(\deg_x Q_1 + \deg_y Q_1)$$

Combining with (3.8), (3.9),(3.10) and (3.11) yields $\deg G \le m$.

Moreover, since $(s(x), t(x))$ is an algebraic general solution, $G(s, t = 0)$ can be seen as the class of all irreducible invariant algebraic curves of the associated system. Therefore its degree bound is also a degree bound for the non-composite rational first integral. □

As an immediate consequence, the theorem leads us to the following algorithm for finding an algebraic general solution $Y(x,y) = 0$ with $\deg Y \le n$ of the differential equation $F(x,y,y') = 0$.

Example 3.3.2. Consider the differential equation

$$y'^3 - 4xyy' + 8y^2 = 0 \tag{3.12}$$

The solution surface is rational, because it admits the proper parametrization

$$\mathcal{P}(s,t) := \left(\frac{t^3 + 8s^2}{4st}, s, t\right)$$

The inverse map of the parametrization is $(\sigma_1(x,y,z), \sigma_2(x,y,z)) := (y,z)$. The associated system of the given differential equation with respect to $\mathcal{P}$ is

$$\begin{cases} s' = t \\ t' = \dfrac{t^2}{2s} \end{cases}$$

If we look for an algebraic general solution $Y(x,y) = 0$ with $\deg Y \le 2$, then we need to find a rational first integral of total degree at most 16 of the associated system. As we have seen in previous example, the associated system has the rational first integral $W = \frac{64s}{100s - t^2}$ of total degree 2, and the algebraic general solution $(s(x), t(x)) := \left(\frac{1}{c_1}(16 + 25c_1)(x + c_2)^2, \frac{2}{c_1}(16 + 25c_1)(x + c_2)\right)$. By apply the theorem 3.1.5, we have

$$y(x) = \frac{1}{c_1^3}(c_1 x - 25c_1 - 16)(16c_1 x + 25c_1^2 x - 625c_1^2 - 800c_1 - 256)$$

is an algebraic general solution of the given differential equation.

Algorithm 2 Algebraic general solutions of a first-order AODE with degree bound

Require: Differential equation $F(x,y,y') = 0$ with a proper parametrization $\mathcal{P}$, and a positive integer n.

Ensure: An algebraic general solution $Y(x,y) = 0$ such that $\deg Y \leq n$.

1: $(\sigma_1, \sigma_2) := \mathcal{P}^{-1}$

2: Determine a degree bound for a rational first integral for the associated differential system

$$m := n^3 \left(\deg_x \sigma_1 + \deg_y \sigma_1 + \deg_x \sigma_2 + \deg_y \sigma_2\right) + 2n^2 \left(\deg_z \sigma_1 + \deg_z \sigma_2\right)$$

3: Determine the associated differential system $\{s' = M, t' = N\}$, where

$$M(s,t) := \frac{\chi_3(s,t)\frac{\partial}{\partial t}\chi_1(s,t) - \frac{\partial}{\partial t}\chi_2(s,t)}{\frac{\partial}{\partial s}\chi_1(s,t)\frac{\partial}{\partial t}\chi_2 1(s,t) - \frac{\partial}{\partial t}\chi_1(s,t)\frac{\partial}{\partial s}\chi_2(s,t)}$$

$$N(s,t) := \frac{\frac{\partial}{\partial s}\chi_2(s,t) - \chi_3(s,t)\frac{\partial}{\partial s}\chi_1(s,t)}{\frac{\partial}{\partial s}\chi_1(s,t)\frac{\partial}{\partial t}\chi_2 1(s,t) - \frac{\partial}{\partial t}\chi_1(s,t)\frac{\partial}{\partial s}\chi_2(s,t)}$$

4: If the associated differential system has no rational first integral of total degree at most m, then return "No algebraic general solution of total degree at most n". Otherwise, go to next step.

5: $W :=$ a rational first integral of degree at most m of the system, and solving the system by using the algorithm 1

6: If the system has no algebraic general solution, then return "No algebraic general solution of total degree at most n"

7: $(s(x), t(x)) :=$ an algebraic general solution of the system

8: Compute $y(x) := \chi_2\left(s(2x - \chi_1(s(x), t(x))), t(2x - \chi_1(s(x), t(x)))\right)$

9: $Y(x,y) :=$ an irreducible annihilating polynomial of $y(x)$

10: If $\deg Y > n$, then return "No algebraic general solution of total order at most n"

11: Return "$Y(x,y) = 0$".

Chapter 4

Rational general solutions of first-order AODEs

This chapter is devoted for studying rational general solutions of first-order AODEs. A general solution contains an arbitrary constant. A rational general solution in which the constant appears rationally is called strong. In this chapter, we present a full algorithm for determining a strong rational general solution for a first-order AODE.

In order to obtain the algorithm, we also approach the differential equation from a geometric point of view. However, different from the previous chapter, we are going to view first-order AODEs as algebraic curves over the field of algebraic functions. We intrinsically use parametrization of algebraic curves to transform the differential equation to a first-order first-degree AODE (see Section 4.3). Parametrizations to be used must be "good" enough to make sure that every coefficient appears during the transformation is a rational function. In order to do that, we study some properties of optimal parametrizations for rational curves over the field of rational functions (see Section 4.2). Among first-order first-degree AODEs, only Riccati and linear differential equations potentially admit a rational general solution. This leads us to a decision algorithm for determining a strong rational general solution of a first-order AODE (see Section 4.5). Further detail can be found in [42, 16].

4.1 Strong rational general solution

In this section, we give a necessary condition for a first-order AODE to admit a rational general solution of the form $y(x,c) \in \mathbb{K}(x,c) \setminus \mathbb{K}(x)$, where c is a transcendental constant. Consider a first-order AODE, $F(x,y,y') = 0$, for an irreducible polynomial F. We view the equation to be an algebraic one by replacing the derivative by an independent variable, i. e. $F(x,y,z) = 0$.

Definition 4.1.1. The algebraic curve $\mathcal{C}_F$ over $\overline{\mathbb{K}(x)}$ defined by $F(x,y,z) = 0$ is called the *corresponding curve* of the differential equation $F(x,y,y') = 0$.

The following theorem is a slightly different version of Theorem 2.4 in [9]. Note, that we assume irreducibility in $\mathbb{K}[x,y,z]$.

Theorem 4.1.2. *Let F be an irreducible polynomial in $\mathbb{K}[x,y,z] \setminus \mathbb{K}[x,y]$. If the differential equation $F(x,y,y') = 0$ has a rational solution of the form $y(x,c) \in$*

$\mathbb{K}(x,c) \setminus \mathbb{K}(x)$ *for an arbitrary constant* c*, then its corresponding curve in* $\mathbb{A}^2(\overline{\mathbb{K}(x)})$ *is rational, and admits a parametrization with coefficients in* $\mathbb{K}(x)$.

Proof. First, we need to prove that F is still irreducible as a polynomial in $\overline{\mathbb{K}(x)}[y,z]$. In order to do that, let us consider the ideal

$$I := \{H \in \overline{\mathbb{K}(x)}[y,z] \,|\, H(x, y(x,c), y'(x,c)) = 0\}$$

in the polynomial ring $\overline{\mathbb{K}(x)}[y,z]$. We claim that I is a principle prime ideal. Consider the ring homomorphism $\phi : \overline{\mathbb{K}(x)}[y,z] \to \overline{\mathbb{K}(x)}(c)$, defined by $\phi(H) := H(x, y(x,c), y'(x,c))$ for $H \in \overline{\mathbb{K}(x)}[y,z]$. The kernel of ϕ is exactly I. Therefore ϕ induces an embedding from the quotient ring $\overline{\mathbb{K}(x)}[y,z]/I$ to $\overline{\mathbb{K}(x)}(c)$. Thus $\overline{\mathbb{K}(x)}[y,z]/I$ is a domain, and then I is a prime ideal. Since $\overline{\mathbb{K}(x)}[y,z]$ is a noetherian unique factorization domain, we know from [18, Prop. 1.12A, p. 7] that every prime ideal of height one is principle. Hence, I is principle.

Next we prove that I can be generated by an irreducible polynomial G in $\mathbb{K}[x,y,z]$. We construct such a generator by the method of Gröbner bases. Let $y(x,c) = \frac{P_1(x,c)}{P_2(x,c)}$ and $y'(x,c) = \frac{Q_1(x,c)}{Q_2(x,c)}$ be in reduced form, i. e. $P_1, P_2, Q_1, Q_2 \in \mathbb{K}[x,c]$ such that $\gcd(P_1,P_2) = \gcd(Q_1,Q_2) = 1$. From the definition of the ideal I, we know by implizitation that

$$I = \langle yP_2 - P_1,\, zQ_2 - Q_1, 1 - P_2t_1, 1 - Q_2t_2\rangle \cap \overline{\mathbb{K}(x)}[y,z]\,.$$

In which the first component of the right hand side is an ideal in $\overline{\mathbb{K}(x)}[c,t_1,t_2,y,z]$ generated by the polynomials $yP_2 - P_1, zQ_2 - Q_1, 1 - P_2t_2$ and $1 - Q_2t_2$. We fix the lexicographic ordering on $\overline{\mathbb{K}(x)}[c,t_1,t_2,y,z]$ with $c > t_1 > t_2 > y > z$. Using this ordering we compute a reduced Gröbner basis of I by first computing a reduced Gröbner basis for the first component of the right hand side, and then eliminating all elements containing c, t_1, t_2. Buchberger's algorithm and reduction of the obtained basis yields a list of polynomials in the variables c, t_1, t_2, y, z with coefficients in $\mathbb{K}(x)$. Therefore, after eliminating polynomials containing c, t_1, t_2, we obtain a reduced Gröbner basis of I which contains only polynomials in $\mathbb{K}(x)[y,z]$. Since I is principle, the reduced Gröbner basis of I contains only one element, say $G_1 \in \mathbb{K}(x)[y,z]$. Moreover, since I is a prime ideal, G_1 must be irreducible over $\overline{\mathbb{K}(x)}[y,z]$ and hence also in $\mathbb{K}(x)[y,z]$. Let $G \in \mathbb{K}[x,y,z]$ such that $G_1 = \frac{a(x)}{b(x)}G$ for some $a(x), b(x) \in \mathbb{K}[x]$ and G is primitive over $\mathbb{K}[x]$. Hence, G is irreducible over $\mathbb{K}(x)[y,z]$ (since G_1 is irreducible). Then we have $I = \langle G_1\rangle = \langle G\rangle$ over $\overline{\mathbb{K}(x)}[y,z]$. Therefore, G is irreducible over $\overline{\mathbb{K}(x)}[y,z]$.

Since F is an irreducible element in the ideal I, F differs from G by multiplication with a non-zero constant factor in $\mathbb{K}$. Therefore, F is also irreducible over $\overline{\mathbb{K}(x)}[y,z]$.

By now, the corresponding curve $\mathcal{C}_F$ is irreducible. Since $F(x, y(x,c), y'(x,c)) = 0$, $\mathcal{C}_F$ can be parametrized by a pair of rational functions $\mathcal{P}(t) := (y(x,t), \frac{\partial}{\partial x}y(x,t))$. Hence $\mathcal{C}$ is rational. □

Theorem 4.1.2 motivates the following definitions.

Definition 4.1.3. The first-order AODE, $F(x,y,y') = 0$, is called *parametrizable* if its corresponding curve is rational.

A parametrizable first-order AODE is surface parametrizable. But the converse direction is not always true. In fact, we will see in Section 4.2 that if a first-order AODE is parametrizable, then its corresponding curve can be parametrized by a pair $(p_1(x,t), p_2(x,t))$ of rational functions in x and t. In this case, $(x, p_1(x,t), p_2(x,t))$ is a rational parametrization for the corresponding surface. However, it is easy to check that the differential equation

$$y'^2 - y^3 - x = 0$$

is surface parametrizable but not parametrizable.

All differential equations of the form $y'F_1(x,y) = F_0(x,y)$, where $F_0, F_1 \in \mathbb{K}[x,y]$, are parametrizable. As a consequence, we might also say that all quasi-linear differential equations of the form $y' = \frac{F_0(x,y)}{F_1(x,y)}$ are parametrizable.

Note, that almost all of the first-order AODEs listed in the collection of Kamke [23] are parametrizable. In fact 89 percent are parametrizable AODEs. The remaining ones consist of two classes. One part contains the reducible AODEs, hence, parametrizability of the factors can be considered. Around one half of the reducible AODEs have parametrizable factors. The other part consists of AODEs or which the corresponding curve has genus greater than 0.

The class of first-order AODEs covers around 64 percent of the entire collection of first-order ODEs in Kamke. Some of the remaining ODEs contain arbitrary functions. For certain choices of these functions, the ODEs might be algebraic. For further details on statistical investigations of Kamke's list we refer to [16].

A rational general solution of a first-order AODE is not necessary of the form $y(x,c) \in \mathbb{K}(x,c) \setminus \mathbb{K}(x)$ for some transcendental constant c. However, if the $y(x,c)$ is a solution of a first-order AODE, then it is a general solution in the sense of Ritt. In fact, let assume that $H \in K(x)\{y\}$ be an arbitrary differential polynomial such that $H(y(x,c)) = 0$, and that $G := \text{prem}(H, F)$. Then $G \in \mathbb{K}(x)[y, y']$. From the definition of pseudo differential remainder, we know that there are natural numbers m, n such that $S_F^m I_F^n G - H$ is a linear combination of F and its derivatives with coefficients in $\mathbb{K}(x)\{y\}$, where S_F and I_F are separant and initial of F respectively. S_F and I_F are not vanished at $y = y(x,c)$. Otherwise, as we have seen in the proof of Theorem 4.1.2, that S_F and I_F are different from F by multiplying a rational function in $\mathbb{K}(x)$, which is not possible. Therefore G is vanished at $y = y(x,c)$. It implies that G is different from F by multiplying a rational function in $K(x)$. This implies $G = 0$. Hence $y(x,c)$ is a general solution.

Definition 4.1.4. A solution y of the differential equation $F(x, y, y') = 0$ is called a *strong rational general solution* if $y = y(x,c) \in \mathbb{K}(x,c) \setminus \mathbb{K}(x)$, where c is a transcendental constant over $\mathbb{K}(x)$.

Theorem 4.1.2 is not true if the given rational general solution is not strong. For instance, the differential equation

$$x^3y'^3 - (3x^2y - 1)y'^2 + 3xy^2y' - y^3 + 1 = 0$$

has a rational general solution

$$y(x) = cx + (c^2 + 1)^{\frac{1}{3}},$$

which is not strong. The corresponding curve has genus 1. Therefore, the differential equation has no strong rational general solution. However, as we will see later, if a parametrizable first-order AODE has a rational general solution, then it has a strong rational general solution.

4.2 Optimal parametrization of rational curves

We have seen that the corresponding curve of a first-order AODE having a strong rational general solution is rational. Moreover, by Theorem 4.1.2 the corresponding curve admits a parametrization with coefficients in $\mathbb{K}(x)$. In case we have a parametrization with coefficients in $\mathbb{K}(x)$ we can decide the existence of a strong rational general solution and compute it. Indeed, as we show in this section, such a parametrizations always exists.

Optimal parametrization is a key notion to answer the question. Several algorithms for determining an optimal parametrization of a rational curve were provided. In [38], Sendra, Winkler and Pérez-Díaz proposed an algorithm for computing an optimal parametrization of a rational curve over the field $\mathbb{Q}$ of rational numbers. Similar result for the class of rational curves over the field $\mathbb{Q}(x)$ of rational functions is presented in [19]. From a different method, Beck and Schicho studied the optimal parametrization problem for rational curves over perfect fields [2]. Since $\mathbb{K}(x)$ is a perfect field, the algorithm of Beck and Schicho is applicable over $\mathbb{K}(x)$. Below, we follows the idea by Hilgarter and Winkler [19] to describe the field of an optimal parametrization of a rational curve over $\mathbb{K}(x)$.

Let us fix a rational curve $\mathcal{C}$ in $\mathbb{A}^2(\overline{\mathbb{K}(x)})$ defined by $G(x, y, z) = 0$ for some irreducible polynomial $G \in \mathbb{K}(x)[y, z]$. As a consequence of Hilbert-Hurwitz theorem [38, Ch. 5, p. 152], $\mathcal{C}$ can be rationally transformed down to a line or a conic over $\mathbb{K}(x)$, depending on whether the total degree of G is odd or even, respectively. The transformation was described in [38] by using the notion of adjoint curves. The line is always parametrizable over $\mathbb{K}(x)$. To parametrize the conic, it is sufficient to search for a $\mathbb{K}(x)$-rational point on it.

In the following we show, along the lines of [19, 38], that indeed there always exists such a $\mathbb{K}(x)$-rational point. Let us consider the projective conic $\mathcal{E} \in \mathbb{P}^2(\overline{\mathbb{K}(x)})$ defined by $G(y, z, w) = 0$, where

$$G(y, z, w) := A_1 y^2 + A_2 yz + A_3 z^2 + A_4 yw + A_5 zw + A_6 w^2$$

is a polynomial in $\mathbb{K}[x][y, z, w]$ such that $(A_1, A_2, A_3) \neq (0, 0, 0)$. Our next goal is to determine a $\mathbb{K}(x)-$rational point of $\mathcal{E}$.

Without loss of generality, we may assume that $A_1 \neq 0$. Otherwise, we just swap y with z or w. Then G can be written as

$$G(y, z, w) = A_1 \left(y + \frac{A_2}{2A_1} z + \frac{A_4}{2A_1} w \right)^2 + \left(\frac{4A_1A_3 - A_2^2}{4A_1} \right) z^2 + \\ + \left(\frac{2A_1A_5 - A_2A_4}{2A_1} \right) zw + \left(\frac{4A_1A_6 - A_4^2}{4A_1} \right) w^2$$

If $4A_1A_3 - A_2^2 = 0$, we see immediately that $G\left(\frac{A_2}{2A_1}, -1, 0\right) = 0$. Therefore $\left(\frac{A_2}{2A_1}, -1, 0\right) \in \mathbb{P}^2(\mathbb{K}(x))$ is a $\mathbb{K}(x)-$rational point of $\mathcal{E}$. In general, if $4A_1A_3 - A_2^2 = 0$ or $4A_1A_6 - A_4^2 = 0$ or $4A_3A_6 - A_5^2 = 0$, the conic $\mathcal{E}$ is called a parabola. However, the condition for a conic to be a parabola does not invariant under linear projective transformations. In other words, a parabola can be transformed to a conic which is not a parabola by using a suitable linear projective map.

Let us assume that $4A_1A_3 - A_2^2 \neq 0$. We rewrite G as follow:

$$G(y,z,w) = \overline{A}_1\left(y + \frac{A_2}{2A_1}z + \frac{A_4}{2A_1}w\right)^2 + \overline{A}_2\left(z + \frac{2A_1A_5 - A_2A_4}{4A_1A_3 - A_2^2}\cdot w\right)^2 + \overline{A}_3w^2$$

where

$$\begin{aligned}
\overline{A}_1 &= A_1\\
\overline{A}_2 &= \frac{4A_1A_3 - A_2^2}{4A_1}\\
\overline{A}_3 &= \frac{4A_1A_6 - A_4^2}{4A_1} - \frac{(2A_1A_5 - A_2A_4)^2}{4A_1\left(4A_1A_3 - A_2^2\right)}
\end{aligned}$$

Therefore, by using the linear transformation

$$\begin{bmatrix}\overline{y}\\ \overline{z}\\ \overline{w}\end{bmatrix} = \begin{bmatrix}1 & \frac{A_2}{A_1} & \frac{A_4}{A_1}\\ 0 & 1 & \frac{2A_1A_5 - A_2A_4}{4A_1A_3 - A_2^2}\\ 0 & 0 & 1\end{bmatrix}\cdot\begin{bmatrix}y\\ z\\ w\end{bmatrix}$$

the conic $\mathcal{E}$ can be transformed to a projective conic which is defined by

$$\overline{A}_1\overline{y}^2 + \overline{A}_2\overline{z}^2 + \overline{A}_3\overline{w}^2 = 0$$

Moreover, by multiplying both side of the equation by the common denominator, we may assume that $\overline{A}_1, \overline{A}_2, \overline{A}_3$ are polynomials.

Next, let $\overline{A}_1\overline{A}_3 = AP^2$ and $\overline{A}_2\overline{A}_3 = BQ^2$ for some $A, B, P, Q \in \mathbb{K}[x]$ such that A and B are square-free polynomials. We transform the previous conic one more time by using the following linear transformation:

$$\begin{bmatrix}Y\\ Z\\ W\end{bmatrix} = \begin{bmatrix}P & 0 & 0\\ 0 & Q & 0\\ 0 & 0 & \overline{A}_3\sqrt{-1}\end{bmatrix}\cdot\begin{bmatrix}\overline{y}\\ \overline{z}\\ \overline{w}\end{bmatrix}$$

The obtained conic is the one defined by $AY^2 + BZ^2 - W^2 = 0$. By abuse of notation, we rename this conic by $\mathcal{E}$. Note that, the above transformations are bijective if $\overline{A}_3 \neq 0$ and easy to computer the inverse maps. (The case when $\overline{A}_3 = 0$ is trivial.)

Proposition 4.2.1. *For every square-free polynomials $A, B \in \mathbb{K}[x]$, the projective conic defined by $AY^2 + BZ^2 - W^2 = 0$ always has a $\mathbb{K}(x)-$rational point.*

Before giving a proof for this proposition, we need the following lemma.

Lemma 4.2.2. *Let A, B be polynomials in $\mathbb{K}[x]$ such that A is square-free and $\deg A \geq \deg B \geq 1$. Then there exists $a, b, m \in \mathbb{K}[x]$ such that a is square-free, $\deg a < \deg A$, and $b^2 - B = am^2A$.*

Proof. Denote by n the degree of A and let $x_1, \dots, x_n \in \mathbb{K}$ be roots of A. There exists a polynomial $b \in \mathbb{K}[x]$ of degree at most $n-1$ such that $b(x_i) = \sqrt{B(x_i)}$ for every $i = 1, \dots, n$, where $\sqrt{B(x_i)}$ is a square root of $B(x_i)$. We see that $B(x) \equiv b(x)^2 \mod (x - x_i)$ for every $i = 1, \dots, n$. Since A is square-free, Chinese Remainder Theorem yields $B(x) \equiv b(x)^2 \mod A(x)$.

Now let $a, m \in \mathbb{K}[x]$ such that a is square-free and $\frac{b^2-B}{A} = a.m^2$. Note that such a pair (a, m) is always exist. It remains to prove that $\deg a < \deg A$. Indeed, we have

$$\begin{aligned}\deg a &= \deg(b^2 - B) - \deg(Am^2)\\ &\le \deg(b^2 - B) - \deg A\\ &\le \max\{2(\deg A - 1), \deg B\} - \deg A\\ &< \deg A\,.\end{aligned}$$

□

From the proof, we see that $\deg b \le \deg A - 1$. This fact leads us to an algorithmic way to determine the triple (a, b, m) by using indeterminate coefficient method. In particular, we first set b a polynomial of degree $\deg A - 1$ in x with indeterminate coefficients. The remainder of the division $b^2 - B$ by A can be computed by using Euclidean algorithm. Since A divides $b^2 - B$, the remainder must be equal to zero. This yields an algebraic system on the indeterminate coefficients. By solving the obtained algebraic system, we can find all possible choices for b, and hence for a and m.

Proof of Proposition 4.2.1. This proof follows the lines of [19].

Let $A, B \in \mathbb{K}[x]$ be square-free polynomials, and consider the projective conic $\mathcal{E}$ defined by $AY^2 + BZ^2 - W^2 = 0$. Denote $d(\mathcal{E}) := \min(\deg A, \deg B)$. We prove the existence of a $\mathbb{K}(x)-$rational point on $\mathcal{E}$ by induction on $d(\mathcal{E})$. In the induction base case, i. e. $d(\mathcal{E}) = 0$, for instance $\deg A = 0$, then $(1 : 0 : \sqrt{A}) \in \mathbb{P}^2(\mathbb{K}(x))$ is a $\mathbb{K}(x)-$rational point of the conic.

Let $m \ge 1$ be an arbitrary natural number, and assume that for every projective conic $\tilde{\mathcal{E}}$ defined by $\tilde{A}Y^2 + \tilde{B}Z^2 - W^2 = 0$ for some square-free polynomials $\tilde{A}, \tilde{B} \in \mathbb{K}[x]$, if $d(\tilde{\mathcal{E}}) < m$ then $\tilde{\mathcal{E}}$ admits a $\mathbb{K}(x)-$rational point. We need to prove that if $d(\mathcal{E}) = m$, then $\mathcal{E}$ also admits a $\mathbb{K}(x)-$rational point.

In case $d(\mathcal{E}) = m$, we process as follows. We may assume further that $\deg A \ge \deg B = m$, otherwise we just swap Y and Z. By Lemma 4.2.2, there exists $A_1, b, m \in \mathbb{K}[x]$ such that A_1 is square-free, $\deg A_1 < \deg A$, and $b^2 - B = A_1 m^2 A$. We transform the coordinate system (Y, Z, W) to the new one $(\overline{Y}, \overline{Z}, \overline{W})$ by the linear transformation

$$\begin{bmatrix}\overline{Y}\\ \overline{Z}\\ \overline{W}\end{bmatrix} = \begin{bmatrix}Am & 0 & 0\\ 0 & b & 1\\ 0 & B & b\end{bmatrix}\begin{bmatrix}Y\\ Z\\ W\end{bmatrix}.$$

Then we see that

$$A_1\overline{Y}^2 + B\overline{Z}^2 - \overline{W}^2 = (b^2 - B)(AY^2 + BZ^2 - W^2)\,.$$

Since B is square-free, $b^2 - B \neq 0$. Thus the conic $\mathcal{E}$ has a $\mathbb{K}(x)-$rational point if and only if the projective conic $\mathcal{E}_1$ defined by $A_1\overline{Y}^2 + B\overline{Z}^2 - \overline{W}^2 = 0$ has a $\mathbb{K}(x)-$rational point.

If $\deg A_1 < \deg B$, then $d(\mathcal{E}_1) = \deg A_1 < \deg B = m$. Therefore $\mathcal{E}_1$ satisfies the induction hypothesis. This show that $\mathcal{E}_1$ admits a $\mathbb{K}(x)-$rational point. Then so is $\mathcal{E}$.

In case $\deg A_1 \geq \deg B$, we can repeat the above process recursively until we get a projective conic $\mathcal{E}_k$ defined by $A_kY^2 + BZ^2 - W^2 = 0$, where A_k is square-free and $\deg A_k < \deg B$. Note that, the polynomial B does not change via these transformations. At this step we have $d(\mathcal{E}_k) = \deg A_k < \deg B = m$. In other words, $\mathcal{E}_k$ satisfies the induction hypothesis. Therefore $\mathcal{E}_k$ contains a $\mathbb{K}(x)-$rational point. Then so is $\mathcal{E}$. □

The proof is constructive. We now conclude the above discussion by the following theorem.

Theorem 4.2.3. *A rational curve defined over $\mathbb{K}(x)$, i. e. a curve which can be parametrized over $\overline{\mathbb{K}(x)}$, can actually be parametrized over $\mathbb{K}(x)$. Therefore optimal parametrizations of a rational curve over $\mathbb{K}(x)$ always have coefficients in $\mathbb{K}(x)$.*

Furthermore, an algorithm for determining such an optimal parametrization can be provided by following the process of Hillgarter and Winkler [19]. We summarize the discussion by a short description for the algorithm.

Algorithm 3 OptimalPara (Optimal Parametrization)

Require: A rational curve $\mathcal{C}$ over $\mathbb{K}(x)$

Ensure: An optimal parametrization for $\mathcal{C}$

1: Determine a birational transformation, say $\mathcal{G}$, to transform the curve down to a conic, say $\mathcal{E}$, or a line by algorithm derived from the theorem of Hilbert-Hurwitz (see Theorem 5.8 and Algorithm HLBERT-HURWITZ in [38]). If it is a line, go to step 2. Otherwise, go to step 3.

2: Determine an optimal parametrization for the line, say $\mathcal{P}(t)$, and then return $\mathcal{G}^{-1}(\mathcal{P}(t))$.

3: Linearly transform the conic $\mathcal{E}$ to a projective conic of the form $AY^2 + BZ^2 - W^2 = 0$ for some $A, B \in \mathbb{K}[x]$ square-free polynomials.

4: Construct a $\mathbb{K}(x)-$rational point for the latter conic as the method described in this section.

5: Determine the corresponding $\mathbb{K}(x)-$rational point point in $\mathcal{E}$, say M.

6: Determine a parametrization, say $\mathcal{P}(t)$, for $\mathcal{E}$ by using the point M (see Algorithm CONIC-PARAMETRIZATION [38]).

7: Return $\mathcal{G}^{-1}(\mathcal{P}(t))$

4.3 Associated differential equation

In this section, we only work with the class of parametrizable first-order AODEs. Based on optimal parametrizations of the corresponding curves, we construct for each parametrizable first-order AODE an associated differential equation, which is a quasi-linear ordinary differential equation. Several facts about connections between rational general solutions of a parametrizable first-order AODE and its associated differential equation will be presented. The problem which remains is looking for rational general solutions of quasi-linear differential equations. This problem is discussed at the end of this section.

Consider a parametrizable first-order AODE $F(x, y, y') = 0$ and assume that an optimal parametrization $\mathcal{P} = p_1, p_2 \in (\mathbb{K}(x)(t))^2$ of the corresponding curve is given, where we write $p_i(t) = p_i(x, t)$ to indicate the dependence on x. Let $y(x) \in \overline{\mathbb{K}(x)}$ be an algebraic solution. Then the pair of two algebraic functions $(y(x), y'(x))$ can be seen as an algebraic solution point on the corresponding curve $\mathcal{C}$. Two cases arise.

(i) $(y(x), y'(x)) \notin \mathrm{im}(\mathcal{P})$, where $\mathrm{im}(\mathcal{P})$ is the image of $\mathcal{P}$. Then $(y(x), y'(x))$ can be determined from the finite set $\mathcal{C} \setminus \mathrm{im}(\mathcal{P})$.

(ii) $(y(x), y'(x)) = \mathcal{P}(\omega(x))$ for some $\omega(x) \in \overline{\mathbb{K}(x)}$. In this case we identify the algebraic function $\omega(x)$ with a point on the affine line $\mathbb{A}^1(\overline{\mathbb{K}(x)})$.

Let us take a look at the algebraic function $\omega(x)$. It satisfies the system

$$\begin{cases} p_1(x, \omega(x)) = y(x)\,, \\ p_2(x, \omega(x)) = y'(x)\,. \end{cases}$$

Therefore,

$$\frac{d}{dx} p_1(x, \omega(x)) = p_2(x, \omega(x))\,.$$

By expanding the left hand side, we have

$$\omega'(x) \cdot \frac{\partial p_1}{\partial t}(x, \omega(x)) + \frac{\partial p_1}{\partial x}(x, \omega(x)) = p_2(x, \omega(x))$$

Thus $\omega(x)$ either satisfies the algebraic relation

$$\begin{cases} \dfrac{\partial p_1}{\partial t}(x, \omega(x)) = 0\,, \\ \dfrac{\partial p_1}{\partial x}(x, \omega(x)) = p_2(x, \omega(x))\,, \end{cases}$$

or it is an algebraic solution of the quasi-linear differential equation

$$\omega' = \frac{p_2(x, \omega) - \frac{\partial p_1}{\partial x}(x, \omega)}{\frac{\partial p_1}{\partial t}(x, \omega)}\,. \tag{4.1}$$

The ODE in (4.1) will be of further importance.

Definition 4.3.1. Let $F(x, y, y') = 0$ be an AODE and let $\mathcal{P}(t) = (p_q(x, t), p_2(x, t))$ be a proper rational parametrization of the corresponding curve. Then the ODE (4.1) is called the *associated differential equation*.

In the above, we have proven the following lemma.

Lemma 4.3.2. *With notations as above, if $y = y(x) \in \overline{\mathbb{K}(x)}$ is an algebraic solution of the differential equation $F(x, y, y') = 0$, then one of the following holds:*

(i) The algebraic solution point $(y(x), y'(x))$ *lies in the finite set* $\mathcal{C} \setminus \mathrm{im}(\mathcal{P})$.

(ii) $y(x) = p_1(x, \omega(x))$ *for some algebraic solution* $\omega(x)$ *of the algebraic system:*

$$\begin{cases} \dfrac{\partial p_1}{\partial t}(x, \omega) = 0\,, \\ \dfrac{\partial p_1}{\partial x}(x, \omega) = p_2(x, \omega)\,. \end{cases}$$

(iii) $y(x) = p_1(x, \omega(x))$ *for some algebraic solution* $\omega(x)$ *of the associated quasi-linear differential equation* (4.1).

Theorem 4.3.3. *We use the notation from above and assume that the parametrization* $\mathcal{P}$ *is proper. Then there is a one-to-one correspondence between rational general solutions of the differential equation* $F(x, y, y') = 0$ *and rational general solutions of its associated differential equation* (4.1).

In particular, if $\omega(x)$ *is a rational general solution of the associated equation* (4.1)*, then* $y(x) = p_1(x, \omega(x))$ *is a rational general solution of given differential equation.*

Conversely, if $y(x)$ *is a rational general solution of the given differential equation, then* $\omega(x) = \mathcal{P}^{-1}(y(x), y'(x))$ *is a rational general solution of the associated equation* (4.1)*, where* $\mathcal{P}^{-1}$ *is a rational representation of the inverse of* $\mathcal{P}$.

Proof. Assume that $\omega(x)$ is a rational general solution of the associated differential equation (4.1), and denote $y(x) := p_1(x, \omega(x))$. From the construction above, it is clear that $y(x)$ is a rational solution of the differential equation $F(x, y, y') = 0$.

It remains to show that $y(x)$ is a general solution. Let $G \in \mathbb{K}(x)\{y\}$ be a differential polynomial such that $G(y(x)) = 0$, and let $H := \mathrm{prem}(G, F)$. We need to show that $H = 0$. Since y' is the highest derivative occurring in F, we know that $H \in \mathbb{K}(x)[y, y']$. Both G and F vanish at $y(x)$, hence so does H regarded as a differential polynomial. Therefore, $H(\mathcal{P}(\omega(x))) = H(y(x), y'(x)) = 0$ regarding H as a polynomial. Note, that $(H \circ \mathcal{P})(\omega) = H(f_1(x, \omega), f_2(x, \omega)) \in k(x, \omega)$. In order to fulfill $(H \circ \mathcal{P})(\omega) = 0$, ω has to be in $\overline{\mathbb{K}(x)}$. Since $\omega(x)$ is a general solution of the associated differential equation, it contains an arbitrary constant and hence, $H \circ \mathcal{P} = 0$. Therefore, $H = (H \circ \mathcal{P}) \circ \mathcal{P}^{-1} = 0$.

Equivalently, if $y(x)$ is a rational general solution of the given differential equation, then, by the construction of the associated equation, $\omega(x) := \mathcal{P}^{-1}(y(x), y'(x))$ is a rational solution of (4.1). By a similar argument as above ω is a rational general solution of the associated differential equation (4.1). □

Lemma 4.3.2 tells us that for finding rational solutions of a parametrizable first-order AODE, working with the class of quasi-linear first-order ODEs is essentially enough. If we look for rational general solutions, the situation is even much stricter. In fact, in [3], Behloul and Cheng proved that if a quasi-linear differential equation has infinitely many rational solutions, then it must be either a linear differential equation or a Riccati equation. The following theorem is a combination of Theorem 4.3.3 and the result of Behloul and Cheng.

Theorem 4.3.4. *Let* $F(x, y, y') = 0$ *be a first-order AODE.*

(i) If $F = 0$ has a strong rational general solution, then it is parametrizable and its associated differential equation is of the form

$$\omega' = a_0(x) + a_1(x)\omega + a_2(x)\omega^2 \,, \tag{4.2}$$

for some $a_0, a_1, a_2 \in \mathbb{K}(x)$.

(ii) If $F = 0$ is parametrizable and has a rational general solution, then its associated quasi-linear differential equation is of the form (4.2).

Proof. If a parametrizable first-order AODE has a rational general solution, then so does its associated differential equation. In this case the associated differential equation has infinitely many rational solutions. Then (ii) follows from the result of Behloul and Cheng in [3]. Finally, (i) follows immediately from Theorem 4.1.2 and (ii). □

Corollary 4.3.5. *If a parametrizable first-order AODE has a rational general solution, then it has a strong rational general solution.*

Proof. It is a consequence of the previous theorem and [37, Cor. 2.1, p. 18] □

We are looking for rational general solutions of first-order AODEs. The problem remained now is computing a rational general solution of the differential equation (4.2). In the case $a_2 = 0$, it is a linear differential equation of degree 1 which can be easily solved by integrating. In the case $a_2 \neq 0$, it is a classical Riccati equation.

For the problem of computing rational general solution, or even all rational solutions, of a Riccati equation, readers can refer [25] for a completed algorithm. In [25], Kovacic proposes an algorithm for computing Liouvilian solutions of a linear second order ODE. As a special case, Section 3.1 in that paper leads to a full algorithm for determining all rational solutions of a Riccati equation. Note that for a Riccati equation, the notion of rational general solutions and strong rational general solutions are coincide. In [9], Chen and Ma do a slight modification of the algorithm by Kovacic to seek for only strong rational general solution. We will discuss about rational solutions of Riccati equations further in Section 4.4 and 5.3.3.

4.4 Rational general solutions of Riccati equations

We now restrict our work to the class of Riccati equations over the field $\mathbb{K} = \mathbb{C}$. A Riccati equation is a differential equation of the form

$$\omega' = a_0(x) + a_1(x)\omega + a_2(x)\omega^2 \,, \tag{4.3}$$

where $a_0, a_1, a_2 \in \mathbb{K}(x)$, $a_2 \neq 0$. In this section we provide an algorithm for deciding and finding a rational general solution of (4.3). The problem of finding rational solutions of a Riccati equation has been intensively studied. Algorithms for finding all rational solutions of a Riccati equation can be found for instance in [5, 40] or [37, Alg. 2.2, p. 21] or [25]. We adapt ideas of Bronstein [5], Yuan [44], and Kovacic [25] and propose an algorithm for specifically determining rational general solutions of a Riccati equation if there is any.

We know that by the transformation

$$\omega = \frac{y}{a_2(x)} - \frac{a_1(x)}{2a_2(x)} - \frac{a_2'(x)}{2a_2(x)^2}$$

the differential equation (4.3) can be transformed into the form

$$y' - y^2 = a(x)\,, \tag{4.4}$$

where

$$a = a_0a_2 - \frac{a_1^2}{4} + \frac{a_1'}{2} - \frac{3a_2'^2}{4a_2^2} - \frac{a_1a_2'}{2a_2} + \frac{a_2''}{2a_2}$$

is again a rational function over $\mathbb{K}$.

Definition 4.4.1. The differential equation (4.4) is called the *normal form* of the Riccati equation (4.3).

If a is a constant, then the differential equation (4.4) has a rational general solution if and only if $a = 0$. This follows from the algorithm of Feng and Gao [1]. In this case, the rational general solution is given by $y(x) = \frac{1}{x-c}$ for an arbitrary constant c. From now on, we assume that a is a non-constant rational function over k.

Definition 4.4.2. We say that the AODE (4.4) has *suitable poles* if the following conditions on $a = \frac{P}{Q}$, with $P, Q \in \mathbb{K}[x]$ and $\gcd(P,Q) = 1$, hold:

(i) $a(x)$ has only double poles.

(ii) $\deg P \leq \deg Q - 2$. Consequently, $a(x)$ has no pole at infinity.

(iii) If $x_0 \in \mathbb{C}$ is a pole of $a(x)$, then $a(x)(x-x_0)^2|_{x=x_0}$ is of the form $\frac{1}{4}(1-n^2)$ for some positive integer $n \geq 2$.

By abuse of notation we also say, that $a(x)$ has suitable poles.

Due to [44], having suitable poles is a necessary condition for the existence of a rational general solution for a normal Riccati equation (4.4) with $a \neq 0$.

Proposition 4.4.3 (Yuan [44])**.** *If the differential equation (4.4), with $a \neq 0$, has a rational general solution, then the equation has suitable poles.*

Proposition 4.4.4. *Assume that the normal Riccati equation (4.4) has suitable poles, and $y(x) = \frac{S(x)}{T(x)} \in \mathbb{K}(x)$ is a rational solution, where $S, T \in \mathbb{K}[x]$. Then*

(i) $\deg S < \deg T$*, and*

(ii) $y(x)$ *has only simple poles.*

Proof. We assume w. l. o. g. that $\gcd(S,T) = 1$.

(i) If otherwise $\deg S \geq \deg T$, then $y(x)$ can be expanded as a formal series

$$y(x) = \sum_{n=-\infty}^{N} c_n x^n ,$$

where $N = \deg S - \deg T \geq 0$, $c_n \in \mathbb{K}$ for all n, $c_N \neq 0$. By substituting in (4.4), we have

$$\left(\ldots + N c_N x^{N-1}\right) + \left(\ldots + c_N^2 x^{2N}\right) = a(x) .$$

Since $N - 1 < 2N$, the term $c_N^2 x^{2N}$ on the left hand side must be killed by a term on the series expansion of $a(x)$ at infinity. This contradicts the property (ii) in Definition 4.4.2.

(ii) Now let $x_0 \in \mathbb{C}$ be a pole of $y(x)$ of order $M \geq 1$. The formal Laurent series expansion of $y(x)$ around the pole x_0 is

$$y(x) = \frac{c_{-M}}{(x - x_0)^M} + \frac{c_{-M+1}}{(x - x_0)^{M-1}} + \ldots ,$$

where $c_{-M}, c_{-M+1}, \ldots \in \mathbb{K}$, $c_{-M} \neq 0$. Substituting to (4.4) yields:

$$\left(\frac{-M c_{-M}}{(x - x_0)^{M+1}} + \ldots\right) + \left(\frac{c_{-M}^2}{(x - x_0)^{2M}} + \ldots\right) = a(x) .$$

Since $a(x)$ has only double poles, $M + 1 \leq 2$ and $2M \leq 2$. Hence $M = 1$. □

Proposition 4.4.4 tells us that if $y(x)$ is a rational solution of the normal Riccati equation (4.4), then $y(x)$ must have the form

$$y(x) = \sum_{i=1}^{n} \frac{r_i}{x - x_i} ,$$

where $n \in \mathbb{N}$ is the number of poles of $y(x)$, $x_1, \ldots, x_n \in \mathbb{C}$ are n distinct poles of $y(x)$, and $r_i \in \mathbb{C}$ is the residue of $y(x)$ at $x = x_i$. Poles of $y(x)$ do not necessarily occur at poles of $a(x)$. This motivates the following definition.

Definition 4.4.5. Assume that $y(x)$ is a rational solution of the normal Riccati equation (4.4). A pole of $y(x)$ is called a *movable pole* if it is not a pole of $a(x)$. Otherwise, it is called a *non-movable pole*.

Theorem 4.4.6. *Assume that $y(x)$ is a rational solution of the normal Riccati equation (4.4) having suitable poles, and $x_0 \in \mathbb{C}$ is a pole of $y(x)$. Let $x_1, \ldots, x_n$ be all poles of $a(x)$ in $\mathbb{C}$. For each $i \in \{1, \ldots, n\}$, denote $s_i := a(x)(x - x_i)^2|_{x=x_i}$, and $s_\infty := a\left(\frac{1}{x}\right) \frac{1}{x^2}|_{x=0}$.*

(i) If x_0 is a non-movable pole, i. e. $x_0 = x_i$ for some $i \in \{1, \ldots, n\}$, then the residue of $y(x)$ at $x = x_0$ is a root of the quadratic equation

$$t^2 + t + s_i = 0 .$$

(ii) If x_0 is a movable pole, then the residue of $y(x)$ at $x = x_0$ is equal to -1.

(iii) The number of movable poles of $y(x)$ does not exceed

$$\frac{1+\sqrt{1-4s_\infty}}{2}+\sum_{i=1}^{n}\frac{-1+\sqrt{1-4s_i}}{2}\,.$$

Proof. Let r be the residue of $y(x)$ at $x = x_0$. The Taylor expansion of $y(x)$ around x_0 is

$$y(x) = \frac{r}{x-x_0}+\sum_{i=0}^{\infty} c_i(x-x_0)^i\,,$$

where $c_i \in \mathbb{C}$. By substituting in (4.4), we have

$$\left(\frac{-r}{(x-x_0)^2}+\dots\right)-\left(\frac{r^2}{(x-x_0)^2}+\dots\right)=a(x)\,.$$

If x_0 is a movable pole, then the term of negative order on the left hand side must be killed. Therefore $r + r^2 = 0$, hence $r = -1$. In case x_0 is also a pole of $a(x)$, we have that $x_0 = x_i$ for some i. In this case, the term of order -2 on the left hand side must be killed by the lowest term on the Taylor expansion of $a(x)$. Thus $-r - r^2 = s_i$. And then (i) and (ii) have been proven.

To prove (iii), we use the Residue Theorem to estimate the possible number of movable poles. The residue theorem says that the total sum of residues of $y(x)$ over the Riemann sphere $\mathbb{C} \cup \{\infty\}$ is equal to zero. Since the residue of $y(x)$ at movable poles is equal to -1, the number of movable poles is exactly the sum of residues of $y(x)$ at non-movable poles and at infinity.

It remains to determine the residue of $y(x)$ at infinity, say r_∞. In order to do so, let us consider the transformation

$$y = -\frac{1}{x^2}z\left(\frac{1}{x}\right)\,.$$

The residue of $y(x)$ at infinity is equal to the residue of $z(x)$ at 0. By this transformation the differential equation (4.4) can be transformed in terms of z to

$$z' - z^2 + \frac{2}{x}z = \frac{1}{x^4}a\left(\frac{1}{x}\right)\,.$$

By using the same technique as before, this implies that the residue of $z(x)$ at 0, which is r_∞, satisfies the quadratic relation $r_\infty^2 - r_\infty + s_\infty = 0$.

Finally, the number of movable poles is equal to the sum of residues of $y(x)$ at infinity and at non-movable poles, which are roots of the quadratic equations $r_\infty^2 - r_\infty + s_\infty = 0$ and $r^2 + r + s_i$ respectively. Hence (iii) is proved. □

Theorem 4.4.6 provides a clear insight into a form of a rational solution of the normal Riccati equation (4.4). First we compute the set of poles of $a(x)$, say $x_1, \dots, x_n$. Then a rational solution $y(x)$ of (4.4) has the form

$$y(x) = \sum_{i=1}^{n}\frac{r_i}{x-x_i}-\sum_{j=1}^{m}\frac{1}{x-c_j}\,, \qquad (4.5)$$

where

- r_i is a root of the quadratic equation $r^2 + r + a(x)(x - x_i)^2|_{x - x_0}=0$.
- m is a positive integer which is not larger than the sum in Theorem 4.4.6 (iii).
- $c_1, \ldots, c_m$ are movable poles.

Note, that if furthermore $y(x)$ is a rational general solution, then the c_j are transcendental over $\mathbb{K}$. Therefore we can always rewrite $y(x)$ in the form

$$y(x) = \sum_{i=1}^{n} \frac{r_i}{x - x_i} - \frac{r_{n+1} + 2r_{n+2}x + \ldots + m r_{n+m} x^{m-1}}{1 + r_{n+1}x + r_{n+2}x^2 + \ldots + r_{n+m}x^m},$$

where $1 + r_{n+1}x + r_{n+2}x^2 + \ldots + r_{n+m}x^m = \frac{(-1)^m}{c_1 c_2 \ldots c_m}(x - c_1) \cdot \ldots \cdot (x - c_m)$.

As a consequence, we propose an algorithm for determining a rational general solution of a classical Riccati equation. Algorithm 4 (page 59) computes for a given Riccati equation a rational general solution or decides that such a solution cannot exist.

Theorem 4.4.7. *Algorithm 4 is correct, i. e. it returns a rational general solution of the given Riccati equation if there is any, and otherwise, it returns "No rational general solution exists".*

Proof. Follows form the discussion above. □

Example 4.4.8. Consider the Riccati equation

$$\omega = \frac{-3x^2 + 2x - 2}{x(x-1)^2} - \frac{6x^2 - x + 3}{x(x-1)}\omega - \frac{3x^2 + 1}{x}\omega^2. \tag{4.6}$$

We normalize the Riccati equation by taking the linear transformation

$$\omega = -\frac{xy}{3x^2 + 1} - \frac{9x^4 - 3x^3 + 9x^2 + 1}{(x-1)(3x^2+1)^2}.$$

The obtained normal Riccati equation is

$$y' - y^2 = -\frac{3(6x^2 - 1)}{(3x^2 + 1)^2}. \tag{4.7}$$

The rational function on the right hand side has double poles at $x_1 = \frac{i\sqrt{3}}{3}$ and $x_2 = -\frac{i\sqrt{3}}{3}$. Assume that $y(x)$ is a rational general solution of (4.6), then its non-movable poles are also x_1 and x_2. The residues of $y(x)$ at non-movable poles are the same and they are the roots of the quadratic equation $t^2 + t - \frac{3}{4} = 0$. The residue of $y(x)$ at infinity is a root of $t^2 - t - 2 = 0$. Therefore, the number of movable poles of $y(x)$ which is equal to the sum of the residues at x_1, x_2 and infinity is at most 3.

Next, we make an ansatz with

$$y(x) = \frac{r_1}{x - \frac{i\sqrt{3}}{3}} + \frac{r_2}{x + \frac{i\sqrt{3}}{3}} - \frac{r_3 + 2r_4 x + 3r_5 x^2}{1 + r_3 x + r_4 x^2 + r_5 x^3},$$

where $r_1^2 + r_1 - \frac{3}{4} = r_2^2 + r^2 - \frac{3}{4} = 0$, and $y(x)$ satisfies the differential equation (4.7). Solving the obtained algebraic system on $r_1, \ldots, r_5$ yields a rational solution

$$y(x) = \frac{1}{2\left(x - \frac{i\sqrt{3}}{3}\right)} + \frac{1}{2\left(x + \frac{i\sqrt{3}}{3}\right)} - \frac{c + 3cx^2}{1 + cx + cx^3},$$

where c is a transcendental constant over $\mathbb{C}$. Hence,

$$\omega(x) = -\frac{1 + 2cx - cx^2 + cx^3}{(x-1)(1 + cx + cx^3)}$$

is a rational general solution of the Riccati equation (4.6).

4.5 Algorithm and Examples

This section is devoted to an algorithm for finding strong rational general solutions of first-order AODEs. As we have seen before, if a first-order AODE has a strong rational general solution, then it is parametrizable, i. e. its corresponding curve is rational. Whenever a first-order AODE is parametrizable, the notions of rational general solution and strong rational general solution coincide. Moreover, in the case of having a strong rational general solution, the associated ODE is either a linear differential equation or a Riccati equation.

In Algorithm 5 we present a full algorithm which computes for a given first-order AODE a strong rational general solution, if it exists. Otherwise it decides that such a solution cannot exist.

Theorem 4.5.1. *Algorithm 5 returns a strong rational general solution of the given first-order AODE, $F(x, y, y') = 0$, if there is any, and it returns "No strong rational general solution exists" if the differential equation has no strong rational general solution.*

Hence, Algorithm 5 decides the existence of strong rational general solutions of the whole class of first-order AODEs. Furthermore, due to Corollary 4.3.5, Algorithm 5 can also be used for determining the existence of rational general solutions of parametrizable first-order AODEs. In the affirmative case it always computes such a solution.

Example 4.5.2 (Example 1.537 in Kamke [23])**.** Consider the differential equation

$$\begin{aligned} F(x, y, y') &= x^3 y'^3 - 3x^2 y y'^2 + (x^6 + 3xy^2) y' - y^3 - 2x^5 y \\ &= (xy' - y)^3 + x^6 y' - 2x^5 y = 0\,. \end{aligned}$$

The associated curve defined by $F(x, y, z) = 0$ can be parametrized by

$$\mathcal{P}(t) = \left(-\frac{t^3x^5 - t^2x^6 + (t-x)^3}{t^3x^5}, -\frac{2t^3x^5 - 2t^2x^6 + (t-x)^3}{t^3x^6} \right).$$

Therefore, the associated differential equation with respect to $\mathcal{P}$ is

$$\omega' = \frac{1}{x^2} \cdot \omega \cdot (2\omega - x)\,,$$

which is a Riccati equation. By applying the algorithm by Kovacic, we can determine a rational general solution of the last differential equation, such as $\omega(x) = \frac{x}{1+cx^2}$. Hence, the differential equation $F(x, y, y') = 0$ has the rational general solution $y(x) = cx(x + c^2)$.

Observe, that this is just an arbitrary example from the collection of Kamke [23]. In total around 64 percent of the listed ODEs there are AODEs and almost all of them are parametrizable and hence suitable for Algorithm 5. For further detail see [16].

Algorithm 4 Rational general solutions of Riccati equations

Require: The Riccati equation $\omega' = a_0 + a_1\omega + a_2\omega^2$ with $a_0, a_1, a_2 \in \mathbb{K}(x)$ and $a_2 \neq 0$.

Ensure: A rational general solution $y(x)$, or "No rational general solution exists".

Compute $a := a_0 a_2 - \frac{a_1^2}{4} + \frac{a_1'}{2} - \frac{3a_2'^2}{4a_2^2} - \frac{a_1 a_2'}{2a_2} + \frac{a_2''}{2a_2}$.

if $a = 0$ **then**

  return $\omega(x) = -\frac{1}{(x-c)a_2(x)} - \frac{a_1(x)}{2a_2(x)} - \frac{a_2'(x)}{2a_2(x)^2}$.

end if

if $a \in \mathbb{K} \setminus \{0\}$ **then**

  return "No rational general solution exists".

end if

Check whether $a(x)$ has suitable poles. If yes, go to the next step. Otherwise, **return** "No rational general solution exists".

Compute the set of poles of $a(x)$, say $\{x_1, \ldots, x_n\}$.

For each pole x_i, compute $s_i := a(x)(x - x_i)^2|_{x=x_i}$.

Compute $s_\infty := \frac{1}{x^2} a\left(\frac{1}{x}\right)|_{x=0}$.

Find the integer part m of

$$\frac{1 + \sqrt{1 - 4s_\infty}}{2} + \sum_{i=1}^{n} \frac{-1 + \sqrt{1 - 4s_i}}{2}.$$

Let

$$y(x) = \frac{r_1}{x - x_1} + \ldots + \frac{r_n}{x - x_n} - \frac{r_{n+1} + 2r_{n+2}x + \ldots + m r_{n+m} x^{m-1}}{1 + r_{n+1}x + \ldots + r_{n+m}x^m}$$

and set up an algebraic system for $r_1, \ldots, r_{n+m}$ by substitution to the normal Riccati equation (4.4) and coefficient comparison. The obtained algebraic system additionally contains the equations of $r_i^2 + r_i + s_i = 0$ for all $i \in \{1, \ldots, n\}$.

Solve the algebraic system from the previous step. Construct for each obtained solution the corresponding rational function $y(x)$. If any of these $y(x)$ contains an arbitrary constant, then go to next step. Otherwise, **return** "No rational general solution exists".

return $\omega(x) = \frac{y(x)}{a_2(x)} - \frac{a_1(x)}{2a_2(x)} - \frac{a_2'(x)}{2a_2(x)^2}$.

Algorithm 5 Strong rational general solutions of first-order AODEs

Require: A first-order AODE, $F(x, y, y') = 0$, where $F \in \mathbb{K}[x, y, z] \setminus \mathbb{K}[x, y]$ is irreducible.

Ensure: A strong rational general solution $y(x)$, or "No strong rational general solution exists".

if genus of the corresponding curve is zero **then**

  Use Algorithm 3 to compute an optimal parametrization of the corresponding curve, say $(p_1(x, t), p_2(x, t)) \in (\mathbb{K}(x)(t))^2$.

  Compute

$$f(x,t) := \frac{p_2(x,t) - \frac{\partial}{\partial x} p_1(x,t)}{\frac{\partial}{\partial t} p_1(x,t)}.$$

  if $f(x, t)$ has the form $a_0(x) + a_1(x)t + a_2(x)t^2$ for some $a_0, a_1, a_2 \in \mathbb{K}(x)$ **then**

    Computing a rational general solution of the linear or Riccati equation $\omega' = f(x, \omega)$.

    if $\omega = \omega(x)$ is a rational general solution **then**

      return $y(x) = p_1(x, \omega(x))$

    end if

  end if

end if

return "No strong rational general solution exists".

Chapter 5

Rational and polynomial solutions of first-order AODEs

The methods we used to study rational general solutions cannot be applied directly to the problem of studying particular solutions, such as rational and polynomial solutions. The reason is that a first-order AODE having a rational general solution usually has a nice geometric property, in the sense that its corresponding curve can be parametrized by rational points. A first-order AODE having no such "nice geometric property" might still have several rational and polynomial solutions.

In this problem, we consider $F(x, y, z)$ as a polynomial in $\overline{\mathbb{K}(x)}[y, z]$ and study a degree bound for rational, polynomial solutions from combinatorial, algebraic and geometric approaches. Whenever a degree bound is found, all rational, polynomial solutions can be computed by the method of using undeterminate coefficient. We first study combinatorial properties of the support of F, i.e. the set of power tuples of monomials having non-zero coefficient in F. In many cases, the set of possible poles, and an upper bound for the order of these poles of a rational solution can be obtained from combinatorial consideration. Secondly, we modified the techniques employed by Eremenko in [12] in order to take an algebraic approach. In particular, some surprising tricks from valuation theory for the algebraic function field $\mathrm{Quot}\left(\overline{\mathbb{K}(x)}[y, z]/\langle F\rangle\right)$ can help us bound the degree of rational solutions for first-order first-degree AODEs. Finally, by combining this with the geometric approach described before, we proposed an algorithm for determining all rational solutions of a large subclass of first-order AODEs which covers all first-order AODEs listed in Kamke's collection [23]. Moreover, for polynomial solutions, we obtained a full decision algorithm. Further details can be found in [17].

5.1 Algebraic function fields

To study global properties of rational solutions, we sometimes pass through algebraic function fields. We recall in this section basic notations and properties in algebraic function field theory for further use.

Let K be an algebraic function field over $\mathbb{K}$ of transcendence degree one. A $\mathbb{K}-$*valuation ring* in K, or briefly, if the ground field is clear, a *valuation ring*, is a ring $\mathcal{O} \subsetneq K$ such that $\mathbb{K} \subset \mathcal{O}$ and for every $x \in K$, either $x \in \mathcal{O}$ or $x^{-1} \in \mathcal{O}$. The valuation ring $\mathcal{O}$ admits a unique maximal ideal, say P, which is the set of all non-invertible elements. Moreover, P is principle, and every ideal of $\mathcal{O}$ forms a power of P. A generator of P, say t, is called a *local uniformizer*. For every $x \in K^*$, there exists a unique pair $(u, n) \in (\mathcal{O}^*, \mathbb{Z})$ such that $x = u \cdot t^n$, where $\mathcal{O}^*$ is the group of units. Notice that the power n does not depend on how the local parameter t is chosen. We call n the valuation of x at P. We define the map $\nu_P : K^* \to \mathbb{Z}$ by $\nu_P(x) = n$.

As a convention, we can extend ν_P to K by setting $\nu_P(0) := \infty$. It is easy to check that ν_P satisfies the following properties for all $x, y \in K$:

(i) $\nu_P(xy) = \nu_P(x) + \nu_P(y)$

(ii) $\min\{\nu_P(x), \nu_P(y)\} \leq \nu_P(x + y) \leq \max\{\nu_P(x), \nu_P(y)\}$

The map ν_P is called a valuation of K (with respect to P).

By $\mathbb{P}_{K/\mathbb{K}}$, or briefly $\mathbb{P}_K$ if the ground field $\mathbb{K}$ is clear, we denote the set of all ideals which are maximal in some valuation ring of K over $\mathbb{K}$. Elements of $\mathbb{P}_K$ are called prime divisors of K. For each prime divisor P, we denote by ν_P the corresponding valuation. Let $x \in K$. If $\nu_P(x) > 0$, we say x has a zero of order $\nu_P(x)$ at P. In case $\nu_P(x) < 0$, we say x has a pole of order $-\nu_P(x)$ at P.

Let $P \in \mathbb{P}_K$ be a prime divisor, $\mathcal{O}_P$ and ν_P the corresponding valuation ring and valuation respectively. Since $\mathbb{K}$ is algebraically closed, the residue field $\mathcal{O}_P/P$ is isomorphic to $\mathbb{K}$. Therefore, there is a natural ring projection $\mathcal{O} \to \mathcal{O}/P \equiv \mathbb{K}$. We usually denote the image of $x \in \mathcal{O}$ in $\mathbb{K}$ by $x(P)$.

We want to collect sufficient information such that an element of K is uniquely determined (up to multiplication by a non-zero constant). The object which gathers all such information about poles and zeroes is called a *divisor*. Formally, a divisor of K is an element of the free abelian group $\bigoplus_{P \in \mathbb{P}_K} \mathbb{Z}P$. For divisors δ, δ', we say $\delta \geq \delta'$ if all coefficients of $\delta - \delta'$ are non-negative. The relation $\geq$ defines a partial order on the set of divisors of K. For a divisor $\delta = n_1 P_1 + n_2 P_2 + \ldots + n_r P_r$, we define $\deg \delta := n_1 + \ldots + n_r$ as the degree of δ. We also denote, by $\operatorname{supp}_{\mathbb{P}_K}(\delta)$ the set of prime divisors having non-zero coefficient in δ.

Among divisors, principle divisors are natural examples. For each $x \in K^*$, the principle divisor of x in K is denoted by $[x] := \sum_{P \in \mathbb{P}_K} \nu_P(x)P$. Notice that, the sum is always finite. For some technical purposes, we sometimes split the negative and positive part of $[x]$, denoted by

$$[x]^- = -\sum_{\substack{P \in \mathbb{P}_K \\ \nu_P(x) < 0}} \nu_P(x)P\,, \qquad [x]^+ = \sum_{\substack{P \in \mathbb{P}_K \\ \nu_P(x) > 0}} \nu_P(x)P\,,$$

respectively. Therefore $[x] = [x]^+ - [x]^-$, and $\deg[x]^+ = \deg[x]^-$.

The simplest algebraic function field over $\mathbb{K}$ is the field $\mathbb{K}(x)$ of rational functions in x. In this case, there is a one-to-one correspondence between prime divisors of $\mathbb{K}(x)$ and the set $\mathbb{K} \cup \{\infty\}$. Therefore, we might identify $\mathbb{P}_{\mathbb{K}(x)}$ with $\mathbb{K} \cup \{\infty\}$. By abuse of notation we denote by ν_{x_0}, for $x_0 \in \mathbb{K} \cup \{\infty\}$ the valuation of the prime divisor corresponding to x_0, and call it the valuation at $x = x_0$. A local uniformizer at $x_0 \in \mathbb{K}$ is $x - x_0$, and at ∞ it is $\frac{1}{x}$. The corresponding valuation rings are the localization $\mathbb{K}[x]_{(x-x_0)}$ and $\mathbb{K}[\frac{1}{x}]_{(\frac{1}{x})}$ respectively.

Let $u(x) = \prod_{i=1}^{r}(x - x_i)^{n_i} \in \mathbb{K}(x)$ be a rational function, where $n_1, \dots, n_r \in \mathbb{Z}$. The valuation of u at $x = x_0 \in \mathbb{K}$ is equal to n_i if $x_0 = x_i$ for some $i = 1, \dots, r$ or is equal to 0 otherwise. The valuation of u at infinity is equal to $-\sum\limits_{i=1}^{r} n_i$, which is also equal to the difference of the degree of the denominator and the degree of the numerator. Therefore the principle divisor of u in $\mathbb{K}(x)$ is $[u] := \sum\limits_{i=1}^{r} n_i q_{x_i} - \left(\sum\limits_{i=1}^{r} n_i\right) q_\infty$, where q_{x_i} denote the prime divisor corresponding with x_i in $\mathbb{K}(x)$. In this particular case we have seen that the degree of a principle divisor is always zero, and we also know that every divisor of degree zero is principle. The first property also holds for a general algebraic function field, while the latter is a particularity of purely transcendental fields of degree one over $\mathbb{K}$.

We also denote by $\operatorname{ord}_{x_0}(u) := -\nu_{x_0}(u)$ and call it the *order of u at $x_0 \in \mathbb{K} \cup \{\infty\}$*. The order satisfies the following properties for all $u, v \in \mathbb{K}(x)$:

(i) $\operatorname{ord}_{x_0}(uv) = \operatorname{ord}_{x_0}(u) + \operatorname{ord}_{x_0}(v)$

(ii) $\min\{\operatorname{ord}_{x_0}(u), \operatorname{ord}_{x_0}(v)\} \leq \operatorname{ord}_{x_0}(u + v) \leq \max\{\operatorname{ord}_{x_0}(u), \operatorname{ord}_{x_0}(v)\}$

(iii) If $\operatorname{ord}_{x_0}(u) \neq 0$, then the order of the derivative is

$$\operatorname{ord}_{x_0}(u') = \begin{cases} \operatorname{ord}_{x_0}(u) + 1\,, & \text{if } x_0 \in \mathbb{K}\,, \\ \operatorname{ord}_{x_0}(u) - 1\,, & \text{if } x_0 = \infty\,. \end{cases}$$

We might extend the domain and the values of a rational function to the affine line $\mathbb{K} \cup \{\infty\}$. Let $u(x) \in \mathbb{K}(x)$ be a rational function. Then the value $u(x_0)$ is defined for all x_0 in $\mathbb{K}$ but roots of the denominator. We may extend the domain of u to the whole affine line $\mathbb{K} \cup \{\infty\}$ as follows.

- If x_0 is a root of the denominator, we define $u(x_0) := \infty$.
- If $x_0 = \infty$ and $u(x) = \frac{a_n x^n + \ldots + a_0}{b_m x^m + \ldots + b_0}$, with $a_n b_m \neq 0$, then

$$u(x_0) := \begin{cases} 0\,, & \text{if } n < m\,, \\ \infty\,, & \text{if } n > m\,, \\ \frac{a_n}{b_m}\,, & \text{if } n = m\,. \end{cases}$$

If $r := \operatorname{ord}_{x_0}(y(x)) > 0$, we say that $y(x)$ has a *pole of order r at $x = x_0$*. In case $r < 0$, x_0 is called a *zero of order r of $y(x)$*. Poles of a rational function are roots of the denominator, and probably at infinity. The degree of a rational function (which is the maximum of the degrees of the numerator and the denominator) is equal to the number of poles in $\mathbb{K} \cup \{\infty\}$ counting multiplicities. We recall partial fraction representation of $y(x)$.

Proposition 5.1.1. *Every rational function $y(x) \in \mathbb{K}(x)$ can be represented in the form*

$$y(x) = \sum_{i=1}^{n} \sum_{j=1}^{r_i} \frac{c_{ij}}{(x - x_i)^j} + \sum_{k=0}^{N} c_i x^i\,. \tag{5.1}$$

In this formula we use the following notation.

n	*number of poles of* $y(x)$ *in* $\mathbb{K}$,
$x_1, \ldots, x_n$	*poles of* $y(x)$ *in* $\mathbb{K}$,
$r_1, \ldots, r_n$	*orders of* $y(x)$ *at* $x_1, \ldots, x_n$ *respectively,*
N	*order of* $y(x)$ *at infinity, and*
c_{ij}, c_k	*coefficients in* $\mathbb{K}$

5.2 Combinatorial approach

In this section we locally study rational solutions of first-order AODEs. We describe how partial fraction decomposition and a bound on the order of a rational solution can be used to define a class of first-order AODEs which can be proven to be without movable poles. For this class the ideas also yield an algorithm for finding all rational solution by coefficient comparison. Moreover we also proposed an algorithm for determining all polynomial solutions for an arbitrary first-order AODEs if there is any.

5.2.1 Bound for the order of the p-adic part

In this subsection, we determine for each $x_0 \in \mathbb{K} \cup \{\infty\}$ an upper bound for the order at $x = x_0$ of a rational solution of a first-order differential equation. An algorithm for finding such a bound is then presented.

Theorem 5.2.1. *Let* $F(x, y, z) = \sum_{i,j} f_{ij}(x) y^i z^j \in \mathbb{K}[x][y, z] \setminus \mathbb{K}[x][y]$. *Assume further that* F *is not homogeneous as a polynomial in* y *and* z *with coefficients in* $\mathbb{K}[x]$. *Then for each* $x_0 \in \mathbb{K} \cup \{\infty\}$, *there is a non-negative integer* $\rho = \rho(x_0, F)$, *depending only on* x_0 *and* F, *such that the order of every rational solution of the differential equation* $F(x, y, y') = 0$ *at* $x = x_0$ *does not exceed* ρ.

Proof. We are going to determine the order bound $\rho(x_0, F)$ in an algorithmic way. Let us fix an x_0 in $\mathbb{K}$. A bound $\rho(\infty, F)$ for the order at infinity is constructed similarly. Assume that $y(x)$ is a rational solution of the differential equation $F(x, y, y') = 0$, and that $y(x)$ has a pole of order $r > 0$ at $x = x_0$. Then $y'(x)$ has a pole of order $r + 1$ at $x = x_0$. For each $(i, j) \in \mathbb{N}^2$, we denote $\alpha_{ij} = \mathrm{ord}_{x_0}(f_{ij})$, i.e. the order at $x = x_0$ for f_{ij}. Note, that α_{ij} is non-positive. Let

$$\begin{aligned}
&E := \{(i, j) \in \mathbb{N}^2 \mid f_{ij} \neq 0\} \\
&n := \max\{i + j \mid (i, j) \in E\} \\
&A := \{(i, j) \in E \mid i + j = n\} \\
&d := \max\{j + \alpha_{ij} \mid (i, j) \in A\} \\
&D := \{(i, j) \in A \mid j + \alpha_{ij} = d\}
\end{aligned}$$

Note, that d is different for the case $x_0 = \infty$. Since F is not homogeneous, $E \setminus A$ is a non-empty set. It is clear that D is also non-empty, and is contained in A. For each $(i, j) \in D$, we rewrite $f_{ij}(x)$ as

$$f_{ij}(x) = a_{ij}(x - x_0)^{-\alpha_{ij}} + h_{ij}(x)\,,$$

where $a_{ij} \in \mathbb{K}$, $h_{ij}(x) \in \mathbb{K}[x]$ such that $\mathrm{ord}_{x_0}(h_{ij}(x)) < \alpha_{ij}$. Since $y = y(x)$ is a solution the differential equation, we have that $F(x, y(x), y'(x)) = 0$. We gather terms of $F(x, y(x), y'(x))$ into different groups as follow:

$$\sum_{(i,j)\in D} f_{ij}(x)y(x)^i y'(x)^j + \sum_{(i,j)\in A\setminus D} f_{ij}(x)y(x)^i y'(x)^j + \sum_{(i,j)\in E\setminus A} f_{ij}(x)y(x)^i y'(x)^j = 0\,.$$

Therefore,

$$\sum_{(i,j)\in D} a_{ij}(x-x_0)^{-\alpha_{ij}} y(x)^i y'(x)^j + \sum_{(i,j)\in D} h_{ij}(x)y(x)^i y'(x)^j + \\ + \sum_{(i,j)\in A\setminus D} f_{ij}(x)y(x)^i y'(x)^j = -\sum_{(i,j)\in E\setminus A} f_{ij}(x)y(x)^i y'(x)^j\,. \quad (5.2)$$

The orders at $x = x_0$ of terms in the first sum are equal to $nr + d$, which is always larger than the order of terms in the second and the third sum. It does not mean that the order of the left hand side is equal to $nr + d$. Two cases arise.

Case 1: The order at $x = x_0$ of the first sum in (5.2) is equal to $nr + d$: Then the order of the left hand side of (5.2) is exactly $nr + d$. By comparing with the order of terms on the right hand side, we obtain

$$nr + d \le \max\left\{(i + j)r + j + \alpha_{ij} \mid (i, j) \in E \setminus A\right\}\,.$$

Therefore,

$$r \le \max\left\{\frac{j + \alpha_{ij} - d}{n - i - j} \mid (i, j) \in E \setminus A\right\}\,.$$

Case 2: The order at $x = x_0$ of the first sum in (5.2) is smaller than $nr + d$: Let

$$g(x) := (x - x_0)^{nr+d} \sum_{(i,j)\in D} a_{ij}(x - x_0)^{-\alpha_{ij}} y(x)^i y'(x)^j\,.$$

Then $g(x_0) = 0$. This property of g leads to an upper bound for r. In order to do that, let $z(x) := (x - x_0)^r y(x)$. Then $z(x_0)$ is neither 0 nor ∞, and

$$y'(x) = \frac{z'(x)(x - x_0) - rz(x)}{(x - x_0)^{r+1}}\,.$$

Rewriting $g(x)$ in terms of $z(x)$ and then simplifying the result yields

$$g(x) = \sum_{(i,j)\in D} a_{ij} z(x)^i (z'(x)(x - x_0) - rz(x))^j\,.$$

By substituting $x = x_0$ and dividing by $z(x_0)^n$, we see that r must be a positive integer root of the algebraic equation

$$\sum_{(i,j)\in D} a_{ij}(-1)^j t^j = 0\,. \quad (5.3)$$

Hence, in any case, r must be smaller than either

$$\max\left\{\frac{j + \alpha_{ij} - d}{n - i - j} \mid (i, j) \in E \setminus A\right\}\,,$$

or the largest positive integer root of the algebraic equation (5.3). □

Theorem 5.2.1 itself is not interesting. What is useful for us is the proof. There, the bound for the order at $x = x_0$ of rational solutions of the differential equation is constructed in an algorithmic way. We summarize this result in Algorithm 6.

Algorithm 6 Order bound for the p-adic part

Require: $x_0 \in \mathbb{K} \cup \{\infty\}$; $F = \sum f_{ij}(x) y^i z^j \in \mathbb{K}[x,y,z] \setminus \mathbb{K}[x,y]$ non-homogeneous in y and z.

Ensure: An upper bound $\rho(x_0, F)$ for the order at $x = x_0$ of all rational solutions of the ODE, $F(x, y, y') = 0$.

```
E = {(i,j) ∈ ℕ² | f_ij ≠ 0}
n = max {i + j | (i,j) ∈ E}
A = {(i,j) ∈ E | i + j = n}
α_ij = ord_{x0}(f_ij(x)), for (i,j) ∈ E
if x0 ∈ 𝕂 then
    d = max {j + α_ij | (i,j) ∈ A}
    D = {(i,j) ∈ A | j + a_ij = d}
    a_ij = f_ij(x) / (x − x0)^{−α_ij} |_{x=x0}  for (i,j) ∈ D
    R = the set of positive integer roots of the equation Σ_{(i,j)∈D} a_ij (−t)^j = 0
    ρ̄ = ⌊max { (j + α_ij − d)/(n − i − j) | (i,j) ∈ E \ A }⌋
else if x0 = ∞ then
    d = max {α_ij − j | (i,j) ∈ A}
    D = {(i,j) ∈ A | α_ij − j = d}
    a_ij = the leading coefficient of f_ij(x), for (i,j) ∈ D
    R = the set of positive integer roots of the algebraic equation Σ_{(i,j)∈D} a_ij t^j = 0
    ρ̄ = ⌊max { (α_ij − j − d)/(n − i − j) | (i,j) ∈ E \ A }⌋
end if
return ρ = max (R ∪ {ρ̄, 0}).
```

5.2.2 First-order AODEs without movable poles

As we have seen in the previous subsection, once a pole is given, one can compute an upper bound for the order of rational solutions of a given first-order AODE. Unfortunately it is not always easy to find possible candidates for the positions of poles. For a linear ODE, poles of a (rational) solution can be easily determined from the coefficients of the differential equation itself. In general, it is no longer true when we pass to the class of non-linear first-order AODEs. In fact, poles of a rational solution of a first-order AODE may occur at an arbitrary point. For example, for every $c \in \mathbb{K}$, the function $y(x) = \frac{1}{x-c}$ is a rational solution of the Riccati equation $y' + y^2 = 0$.

The task of this section is to collect first-order AODEs for which no unexpected poles occur in their rational solutions. The core of the idea is that we equip the support of F with a certain partial order. The existence of the greatest element decides whether the differential equation is in the class of AODEs we are interested in.

Definition 5.2.2. On $\mathbb{N}^2$ we define a relation $\gg$ as follows. For $(i_1, j_1), (i_2, j_2) \in \mathbb{N}^2$, we say $(i_1, j_1) \gg (i_2, j_2)$ iff either $i_1 + j_1 = i_2 + j_2$ and $j_1 > j_2$, or $(i_1 + j_1) - (i_2 + j_2) > \max\{0, j_2 - j_1\}$.

It is easy to check that $\gg$ is a strict partial ordering on $\mathbb{N}^2$, i.e. the following properties hold for all $u, v, w \in \mathbb{N}^2$:

(i) irreflexivity: $u \not\gg u$

(ii) transitivity: if $u \gg v$ and $v \gg w$ then $u \gg w$

(iii) asymmetry: if $u \gg v$ then $v \not\gg u$.

For $u, v \in \mathbb{N}^2$, we say u and v are *comparable* if either $u \gg v$ or $v \gg u$. Otherwise, they are called *incomparable*. Not every pair of elements from $\mathbb{N}^2$ is comparable. In other words, the order $\gg$ is not a total order on $\mathbb{N}^2$. For example, $(2, 0)$ and $(0, 1)$ are incomparable.

In Figure 5.1 we consider a given point ✖ (in black) and show all the comparable and incomparable points in some surrounding. All smaller points in that eary are highlighted in green (with green background), the greater points are in blue. The symbol at the respective points ilustrates the class of points according to the inequalities in the definiton. The points with symbol ✖ are incomparable to the given one. All such incomparable points except the given one are drawn in red. Let (i_1, j_1) be our given point, and (i_2, j_2) a smaller point. Then one of the following cases has to be fulfilled.

- ◆ $i_1 + j_1 = i_2 + j_2$ and $j_1 > j_2$,

or $(i_1 + j_1) - (i_2 + j_2) > \max\{0, j_2 - j_1\}$ and furthermore

- ● $j_2 < j_1$, or
- ▲ $j_2 = j_1$, or
- ■ $j_2 > j_1$.

Note, that this shows, that for a given point, the number of points which are incomparable to this one, is finite.

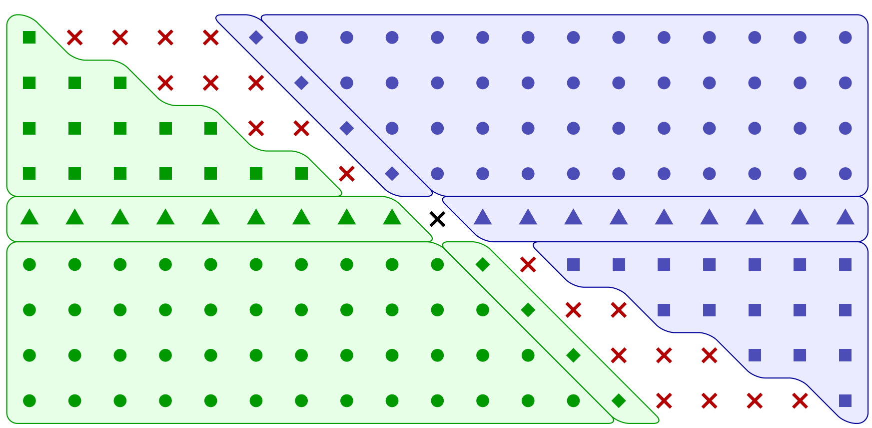

Figure 5.1: Comparable and incomparable points for a given point

Let S be a subset of $\mathbb{N}^2$. An element $u \in S$ is called a greatest element of S if $u \gg v$ for every $v \in S \setminus \{u\}$. The set S has at most one greatest element. Since $\gg$ is not a total order, a subset of $\mathbb{N}^2$ may have no greatest element. This motivates the following definition.

Definition 5.2.3. Let $F = \sum_{i,j} f_{ij}(x) y^i z^j$ be a polynomial in $\mathbb{K}[x][y,z]$. Let $\mathcal{E}(F) \subseteq N^2$ be the support of F, i. e.

$$\mathcal{E}(F) := \left\{ (i,j) \in \mathbb{N}^2 \mid f_{ij} \neq 0 \right\} .$$

A polynomial F for which the set $\mathcal{E}$ admits a greatest element is called *maximally comparable*. We also call the corresponding differential equation maximally comparable.

Theorem 5.2.4. *Let $F = \sum_{i,j} f_{ij}(x) y^i z^j \in \mathbb{K}[x][y,z] \setminus \mathbb{K}[x][y]$ be a polynomial, which is non-homogeneous in y and z. We assume that F is maximally comparable and the greatest element of $\mathcal{E} = \mathcal{E}(F)$ is (i_0, j_0). Then poles in $\mathbb{K}$ of a rational solution of the differential equation $F(x, y, y') = 0$ only occur at the zeros of $f_{i_0 j_0}(x)$.*

Proof. We prove the theorem by contradiction. Assume that $y(x) \in \mathbb{K}(x)$ is a rational solution of the differential equation $F(x, y, y') = 0$, that $x_0 \in \mathbb{K}$ is a pole of order $r \geq 1$ of $y(x)$, and that $f_{i_0 j_0}(x_0) \neq 0$. Let $A := \{(i,j) \in \mathcal{E} \mid i + j = i_0 + j_0\}$. It is clear that $(i_0, j_0) \in A$ and $A \subsetneq \mathcal{E}$. Let us substitute $y(x)$ to the differential equation $F(x, y, y') = 0$ and group terms on the left hand side as follows.

$$\begin{aligned} f_{i_0 j_0}(x) y(x)^{i_0} y'(x)^{j_0} + \sum_{(i,j) \in A \setminus \{(i_0, j_0)\}} f_{i,j}(x) y(x)^i y'(x)^j = \\ = - \sum_{(i,j) \in E \setminus A} f_{i,j}(x) y(x)^i y'(x)^j \end{aligned} \tag{5.4}$$

where the sum on the left hand side is just zero if $A \setminus \{(i_0, j_0)\}$ is the empty set. The order at $x = x_0$ of the first term in (5.4) is equal to $(i_0 + j_0)r + j_0$, while the orders of terms in the sum on the left hand side are $(i + j)r + j - \nu_{x_0}(f_{ij}) = (i_0 + j_0)r + j - \nu_{x_0}(f_{ij})$. Since (i_0, j_0) is the greatest element in A, the order of the left hand side is always equal to the order of the first term, $(i_0 + j_0)r + j_0$. By comparing with the order of terms on the right hand side, we have

$$(i_0 + j_0)r + j_0 \leq \max \left\{ (i + j)r + j - \nu_{x_0}(f_{ij}) \mid (i,j) \in \mathcal{E} \setminus A \right\} .$$

Therefore,

$$r \leq \max \left\{ \frac{j - j_0 - \nu_{x_0}(f_{ij})}{(i_0 + j_0) - (i + j)} \mid (i,j) \in \mathcal{E} \setminus A \right\} . \tag{5.5}$$

For each $(i,j) \in E \setminus A$, we have $(i_0, j_0) \gg (i,j)$ and $i_0 + j_0 \neq i + j$. Thus $(i_0 + j_0) - (i + j) > \max\{0, j - j_0\} \geq j - j_0 - \nu_{x_0}(f_{ij})$. Combination with (5.5) yields $r < 1$. This contradicts with the assumption. □

The theorem gives us a necessary condition for a first-order AODE having no ”unexpected” poles. Once the condition is fulfilled, candidates for poles will be easily determined. Note, that by Theorem 5.2.4 maximally comparable AODEs cannot have movable poles. It is not clear whether the inverse direction also holds, but it is not important to us. The previous subsection provides a bound for the order of these pole candidates. Thus we have enough ingredients to determine the form of a rational solution in a certain finite number of indeterminate coefficients. By an ansatz we find all possible rational solutions.

Before we give an algorithm for finding all rational solutions of first-order AODEs without movable poles, we analyze different features of Theorem 5.2.4 by discussing the following questions.

1. What if F is homogeneous as a polynomial in y and z with coefficients in $\mathbb{K}[x]$?
2. How can we check the existence of the greatest element of $\mathcal{E}$ effectively, and find it in the affirmative case?
3. How likely is an $\mathcal{E}$ which admits a greatest element?

To answer Question 1, let us consider a homogeneous polynomial of degree $n \geq 1$ as a polynomial in y and z,

$$F(x, y, z) := f_{n,0}(x)y^n + f_{n-1,1}(x)y^{n-1}z + \ldots + f_{0,n}(x)z^n\,,$$

where $f_{i,j}(x) \in \mathbb{K}[x]$. We assume F to be irreducible as a polynomial in $\mathbb{K}[x, y, z]$, and consider the differential equation $F(x, y, y') = 0$. The differential equation always has the solution 0. Assume that the differential equation admits a non-zero rational solution $y(x)$, then $\frac{y'(x)}{y(x)}$ is a rational solution of the algebraic equation

$$f_{n,0}(x) + f_{n-1,1}(x)t + \ldots + f_{0,n}(x)t^n = 0\,.$$

Since F is irreducible, this is only possible if $n = 1$. In case $n = 1$, the differential equation is linear, therefore it can be solved easily by known methods.

For the Question 2, the naive way would be to check the existence of the greatest element of $\mathcal{E}$ by comparing all pairs of its elements.

However, there is a much simpler and intuitive way. Figure 5.2 shows how to proceed. We first take the set of points which have the greatest total degree. Within these we take the element, say $p = (p_1, p_2)$ which has the smallest first component. Now we check for each remaining point (x, y) whether or not it satisfies $y \geq \frac{2p_2+p_1-x}{2}$. If one point does, it is incomparable to the point p and hence, there is no greatest element. Otherwise p is the greatest element.

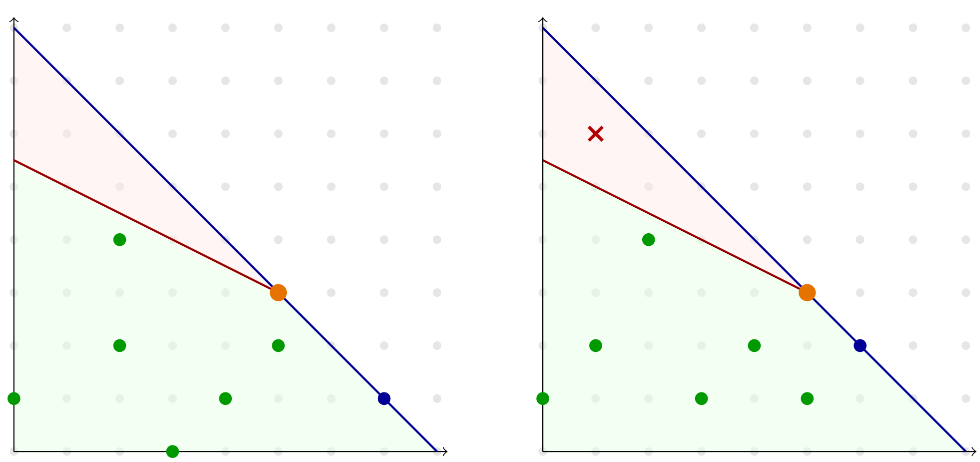

Figure 5.2: Check the existence of a greatest element

We are going to answer Question 3. We rewrite the differential equation in the form $F_0(x, y) + F_1(x, y)y' + \ldots + F_n(x, y)y'^n = 0$, where n is the degree of F in z and $F_0, \ldots, F_n \in \mathbb{K}[x, y]$. Then $\mathcal{E}$ admits a greatest element if and only if the smaller set

$$\{(\deg_y F_0, 0), (\deg_y F_1, 1), \ldots, (\deg_y F_n, n)\}$$

does. Note, that if $F_i = 0$ for some $i < n$, the point $(\deg_y F_i, i)$ is smaller than $(\deg_y F_n, n)$. Therefore, we can state Question 3 in a different way: Given $n \in \mathbb{N}$, and let $m_0, m_1, \ldots, m_n$ be arbitrary natural numbers, how likely has the set $\{(m_0, 0), (m_1, 1), \ldots, (m_n, n)\}$ a greatest element? In order to answer the last question, we denote

$$A_{ijk} := \Big\{(m_0, m_1, ..., m_n) \in \mathbb{N}^{n+1} \,|\, m_i - m_j = j - i + k\Big\}$$
$$A_{ij} := \bigcup_{k=1}^{j-i} A_{ijk}$$
$$A := \bigcup_{0 \le i < j \le n} A_{ij}$$

It follows immediately from the definition of $\gg$ that for $0 \le i < j \le n$, the pairs (m_i, i) and (m_j, j) are incomparable if and only if $j - i < m_i - m_j \le 2(j - i)$. Therefore, a point $(m_0, m_1, \ldots, m_n) \in \mathbb{N}^{n+1}$ is in A_{ij} if and only if (m_i, i) and (m_j, j) are incomparable. Hence, for every point $(m_0, m_1, \ldots, m_n) \in \mathbb{N}^{n+1} \setminus A$, the set $S := \{(m_0, 0), (m_1, 1), \ldots, (m_n, n)\}$ has no incomparable points, or consequently, S admits a greatest element. We might consider each A_{ijk} as a hyperplane in $\mathbb{N}^{n+1}$. The set A is a union of $\sum_{0 \le i < j \le n} j - i = \frac{1}{6} n(n+1)(n+2)$ such hyperplanes. Therefore, almost all first-order AODEs for a given degree n are maximally comparable.

For instance, let us consider $m, n \in \mathbb{N}$, the pairs $(n, 0)$ and $(m, 1)$ are comparable iff $n - m \neq 2$. In other words, the quasi-linear ODE, $F(x, y, y') = F_0(x, y) + F_1(x, y)y' = 0$, with $F_0, F_1 \in \mathbb{K}[x, y]$, is out of the scope of Theorem 5.2.4 iff $\deg_y F_0 - \deg_y F_1 = 2$. In the next section, we introduce another approach which covers such a differential equation.

Algorithm 7 results from the above discussion. It finds all rational solutions of first-order maximally comparable AODEs. Example 5.2.5 illustrates Algorithm 7.

Example 5.2.5. We consider the differential equation

$$F(x, y, y') = 3x^8 y y'^2 - 2x^9 y' - x^6 y^3 + 4x^4(x^3 + 2)y + 52x^3 + 152 = 0\,. \tag{5.6}$$

Among the terms of F as a polynomial in y and y', the term $3x^8 y y'^2$ has the largest power (which is $(1, 2)$) with respect to $\gg$. Therefore, poles of a rational solution of the differential equation (5.6) occur only at $x = 0$ and probably at infinity. By applying Algorithm 6, the orders of a rational solution at $x = 0$ and infinity are at most 2 and 1 respectively. Making an ansatz with $y(x) = \frac{c_1}{x^2} + \frac{c_1}{x} + c_3 + c_4 x$, and then solving the obtained algebraic system in c_1, c_2, c_3, c_4, we see that the differential equation (5.6) has only a rational solution $y(x) = \frac{-2}{x^2} - x$.

5.2.3 Polynomial solutions of first-order AODEs

Another interesting result of the combinatorial aspect is that it provides an algorithm for determining all polynomial solutions of an arbitrary first-order AODE. It is based on the fact that a non-constant polynomial has only a pole at infinity, and the order of this pole is exactly the degree of the polynomial. Assume that we are looking for all polynomial solutions of the differential equation $F(x, y, y') = 0$. If F is homogeneous as a polynomial in y and y' then either the differential equation

Algorithm 7 Rational solutions of maximally comparable first-order AODEs

Require: $F = \sum f_{ij}(x) y^i z^j \in \mathbb{K}[x,y,z] \setminus \mathbb{K}[x,y]$ maximally comparable, and (i_0, j_0) the greatest element of the set $\mathcal{E}(F) = \{(i,j) \mid f_{ij} \neq 0\}$

Ensure: All rational solutions of the ODE, $F(x, y, y') = 0$.

$Sol = \emptyset$
if F is homogeneous of order n as polynomial in y and z **then**
  if $n > 1$ **then**
    $Sol = \{0\}$
  else if $n = 1$ **then**
    Solve the linear ODE, $F(x, y, y') = 0$, and append all rational solutions to Sol.
  end if
else
  $Pol = \{x_0 \in \mathbb{K} \mid f_{i_0 j_0}(x_0) = 0\}$
  Use Algorithm 6 to compute $\rho(x_0, F)$ for all $x_0 \in Pol$
  Use Algorithm 6 to compute $\rho(\infty, F)$, a bound for the degree of the polynomial part.
  Make an ansatz for

$$y(x) = \sum_{x_0 \in Pol} \sum_{j=1}^{\rho(x_0,F)} \frac{c_{x_0,j}}{(x - x_0)^j} + \sum_{k=0}^{\rho(\infty,F)} c_k x^k ,$$

  with indeterminate coefficients $c_{x_0,j}, c_k$
  Solve the obtained algebraic system in $c_{x_0,j}$ and c_k, and append the results to Sol
end if
return Sol

has only a zero polynomial solution or it is a linear first-order ODE which can be easily solved by known methods. Otherwise Algorithm 6 provides a bound for the degree of a polynomial solution. All polynomial solutions can be computed by the undeterminate coefficient method.

The question of finding all polynomial solutions of first-order AODEs was already addressed and solved in [26]. In fact this paper considers also higher order AODEs, but for higher order there is no full decision. The methods and results are rather similar to those presented here both restricted to polynomial solutions of first-order AODEs. We leave here the algorithm for further use.

5.3 Algebraic approach

A quasi-linear first-order ODE is a differential equation of the form $y' = f(x, y)$ for some $f \in \mathbb{K}(x, y)$. By multiplying both sides of the differential equation with the denominator of f, we also view it as a first-order AODE of degree 1 in y'. Although Algorithm 7 works on a generic class of first-order AODEs, its scope does not cover the class of quasi-linear first-order ODEs. In particular if the subtraction of the degree of the numerator of f by the degree of the denominator is equal to 2, then

Algorithm 8 Polynomial solutions of first-order AODEs

Require: $F = \sum f_{ij}(x) y^i z^j \in \mathbb{K}[x,y,z] \setminus \mathbb{K}[x,y]$.
Ensure: All polynomial solutions of the ODE, $F(x, y, y') = 0$.

```
Sol = ∅
if F is homogeneous of order n as polynomial in y and z then
    if n > 1 then
        Sol = {0}
    else if n = 1 then
        Solve the linear ODE, F(x, y, y') = 0, and
        append all rational solutions to Sol.
    end if
else
    Use Algorithm 6 to compute ρ(∞, F), a bound for the degree of a polynomial
solution.
    Make an ansatz for
```

$$y(x) = \sum_{k=0}^{\rho(\infty,F)} c_k x^k ,$$

```
    with indeterminate coefficients c_k
    Solve the obtained algebraic system in c_k, and
    append the results to Sol
end if
return Sol
```

Algorithm 7 is invalid. In this section we approach rational solutions of first-order AODEs in the global meaning by using tools from algebraic function field theory. Together with Algorithm 7, the new approach provides another puzzle piece for an algorithm covering the whole class of quasi-linear first-order ODEs.

The following idea is derived from [12]. In order to find all rational solutions of a quasi-linear first-order ODE, we first study a degree bound for all rational solutions. Once a degree bound is determined, we can make an ansatz for rational solutions with undeterminate coefficients. In [12], Eremenko studies a degree bound for rational solutions of first-order AODEs. He corresponds each first-order AODE with an algebraic function field over $\overline{\mathbb{K}(x)}$. The function field is moreover a differential field with the derivation extended from the usual derivation of $\overline{\mathbb{K}(x)}$. In case the differential function field satisfies the Fuchs condition (without movable critical points), it can be classified up to an isomorphism of differential fields by using the theory of Matsuda [28]. Hence, Eremenko reduces the differential equation to several simpler ones according to the classification, and then estimates a degree bound for rational solutions of such particular cases.

Eremenko [12] theoretically investigates the determination of a degree bound of rational solutions of first-order AODEs. Based on some of his ideas we give a different and more explicit algorithm for actually computing this bound. In the scope of this paper, we study such an algorithm for the class of quasi-linear first-order ODEs. Although our idea is based on Eremenko's results, we study the problem without the theory of Matsuda on classification of differential function fields. Our algorithm is therefore much simpler.

5.3.1 Preparation

We start the discussion by giving an important class of algebraic function fields over the field $\overline{\mathbb{K}(x)}$ of algebraic functions. Let $H \in \mathbb{K}[x, Y, Z]$ be a trivariate polynomial such that H is irreducible as an element in $\overline{\mathbb{K}(x)}[Y, Z]$. The algebraic equation $H(x, Y, Z) = 0$ defines an irreducible algebraic curve in the affine plane $\mathbb{A}^2(\overline{\mathbb{K}(x)})$. The set of all rational functions over the curve is a field and it is isomorphic to the fraction field of the coordinate ring, i. e. $K := \mathrm{Frac}\left(\overline{\mathbb{K}(x)}[Y, Z]/(H)\right)$. The field K is an algebraic function field of degree one over $\overline{\mathbb{K}(x)}$. The set $\mathbb{P}_{K/\overline{\mathbb{K}(x)}}$ of all prime divisors of K over $\overline{\mathbb{K}(x)}$ is in general not easy to determine.

Lemma 5.3.1 (Eremenko [12])**.** *Let K be an algebraic function field over $\overline{\mathbb{K}(x)}$ of transcendence degree one, and $y, z \in K$ such that $[y]^- \leq [z]^-$. Let $m := \left|\overline{\mathbb{K}(x)}(y, z) : \overline{\mathbb{K}(x)}(z)\right|$ be the degree of the field extension. Then there exists a unique irreducible polynomial $G \in \overline{\mathbb{K}(x)}[Y, Z]$ of the form*

$$G(Y, Z) = Y^m - \sum_{\substack{i+j \leq m \\ 0 \leq i \leq m-1 \\ 0 \leq j \leq m}} g_{ij}(x) Y^i Z^j\,,$$

with $g_{ij} \in \overline{\mathbb{K}(x)}$, such that $G(y, z) = 0$ in K.

The result of the following lemma is used in the proof of Lemma 1 in [12] without giving a detailed proof.

Lemma 5.3.2. *Let K be an algebraic function field over $\overline{\mathbb{K}(x)}$ of transcendence degree one, and $y, z \in K$ such that $[y]^- \leq [z]^-$. Since Lemma 5.3.1 is applicable we let $G \in \overline{\mathbb{K}(x)}[Y, Z]$ be a polynomial with this property. Denote by $L \subset \overline{\mathbb{K}(x)}$ an algebraic function field over $\mathbb{K}$ containing x and all coefficients g_{ij} of G. Let $P \in \mathbb{P}_{K/\overline{\mathbb{K}(x)}}$ be a prime divisor. If $y(P), z(P)$ belong to L, then*

$$\mathrm{supp}_{\mathbb{P}_{L/\mathbb{K}}}\left([y(P)]^-\right) \subseteq \mathrm{supp}_{\mathbb{P}_{L/\mathbb{K}}}\left([z(P)]^-\right) \bigcup \left(\bigcup_{i,j} \mathrm{supp}_{\mathbb{P}_{L/\mathbb{K}}}\left([g_{ij}]^-\right) \right)\,.$$

Proof. We prove the lemma by contradiction. Assume that the conclusion does not hold. Then there is a prime divisor $q \in \mathbb{P}_{L/\mathbb{K}}$ not in

$$\mathrm{supp}_{\mathbb{P}_{L/\mathbb{K}}}\left([z(P)]^-\right) \bigcup \left(\bigcup_{i,j} \mathrm{supp}_{\mathbb{P}_{L/\mathbb{K}}}\left([g_{ij}]^-\right) \right)$$

but in $\mathrm{supp}_{\mathbb{P}_{L/\mathbb{K}}}\left([y(P)]^-\right)$. This implies that $\nu_q(y(P)) < 0$, and furthermore, we have $\nu_q(z(P)) \geq 0$ and $\nu_q(g_{ij}) \geq 0$ for all i, j. From Lemma 5.3.1 we have

$$y^m = \sum_{\substack{i+j \leq m \\ 0 \leq i \leq m-1 \\ 0 \leq j \leq m}} g_{ij}(x) y^i z^j\,.$$

Therefore,

$$\begin{aligned} m \cdot \nu_q(y(P)) &\geq \min_{i,j} \{i \cdot \nu_q(y(P)) + j \cdot \nu_q(z(P)) + \nu_q(g_{ij})\} \\ &\geq (m-1) \cdot \nu_q(y(P)) + \min_{i,j} \{j \cdot \nu_q(z(P)) + \nu_q(g_{ij})\}. \end{aligned}$$

It implies that

$$\nu_q(y(P)) \geq \min_{i,j} \{j \cdot \nu_q(z(P)) + \nu_q(g_{ij})\}.$$

Because the right hand side is a non-negative integer, the last inequality can not happen. □

The next theorem reads similar to Lemma 1 in [12]. However, it yields more information on the poles and their order and it explicitly describes the constant. In fact Theorem 5.3.3 implies Lemma 1 in [12].

Theorem 5.3.3. *With notations as above. Let* $A := \bigcup_{i,j} \operatorname{supp}_{\mathbb{P}_{L/\mathbb{K}}}([g_{ij}]^-) \subseteq \mathbb{P}_{L/\mathbb{K}}$*, and for each* $q \in A$*, let* $\sigma_q = \left\lfloor \max_{i,j} \left\{ \frac{-\nu_q(g_{ij})}{m-i} \right\} \right\rfloor$*. Then,* $\sigma := \sum_{q \in A} \sigma_q \cdot q$ *is an effective divisor in* L*, i. e.* $\sigma \geq 0$*. Let* $P \in \mathbb{P}_{K/\overline{\mathbb{K}(x)}}$ *be a prime divisor. If* $y(P), z(P)$ *belong to* L*, then*

$$[y(P)]^- \leq [z(P)]^- + \sigma$$

as divisors in $\mathbb{P}_{L/\mathbb{K}}$*.*

Proof. Let q be an arbitrary prime divisor of L such that $q \in \operatorname{supp}_{\mathbb{P}_{L/\mathbb{K}}}([y(P)]^-)$. Then $\nu_q(y(P)) < 0$. We need to prove that:

$$-\nu_q(y(P)) \leq \max_{i,j}\{0, -\nu_q(z(P))\} + \max_{i,j} \left\{ \frac{-\nu_q(g_{ij})}{m-i} \right\}.$$

Due to Lemma 5.3.1, we have

$$m \cdot \nu_q(y(P)) \geq \min_{i,j} \{i \cdot \nu_q(y(P)) + j \cdot \nu_q(z(P)) + \nu_q(g_{ij})\}.$$

Hence,

$$\begin{aligned} \nu_q(y(P)) &\geq \min_{i,j} \left\{ \frac{j}{m-i} \nu_q(z(P)) + \frac{\nu_q(g_{ij})}{m-i} \right\} \\ &\geq \min_{i,j}\{0, \nu_q(z(P))\} + \min_{i,j} \left\{ \frac{\nu_q(g_{ij})}{m-i} \right\}. \end{aligned}$$

The last inequality concludes the theorem. □

Theorem 5.3.3 works for algebraic functions. In fact, we need the result only for rational functions. Application of Theorem 5.3.3 to rational functions can be seen in Corollary 5.3.4. We get a result comparable to Lemma 1 in [12] for the case of rational functions but we are able to explicitly determine the constant C.

Corollary 5.3.4. *Let* $y = \frac{y_1}{y_2}, z = \frac{z_1}{z_2} \in \mathbb{K}(x,t)$*, where* $y_1, y_2, z_1, z_2 \in \mathbb{K}[x,t]$ *such that* $\gcd(y_1, y_2) = \gcd(z_1, z_2) = 1$*. Assume that* y_2 *divides* z_2 *and* $\deg_t y_1 - \deg_t y_2 \leq \max\{0, \deg_t z_1 - \deg_t z_2\}$*. Then there exists a constant* $C = C(y,z)$ *depending only on* y *and* z *such that: for every* $t(x) \in \mathbb{K}(x)$*, with* $z_2(x, t(x)) \neq 0$*, we have* $\deg y(x, t(x)) \leq \deg z(x, t(x)) + C$*.*

Proof. We consider y, z as elements of the algebraic function field $K := \overline{\mathbb{K}(x)}(t)$ over $\overline{\mathbb{K}(x)}$. From this point of view, the condition $[y]^- \leq [z]^-$ as divisors in K is equivalent to the assumption that y_2 divides z_2 and $\deg_t y_1 - \deg_t y_2 \leq \max\{0, \deg_t z_1 - \deg_t z_2\}$.

We apply Theorem 5.3.3 with K as above and $L := \mathbb{K}(x)$. The sets $\mathbb{P}_{K/\overline{\mathbb{K}(x)}}$ and $\mathbb{P}_{L/\mathbb{K}}$ of prime divisors are exactly $\overline{\mathbb{K}(x)} \cup \{\infty\}$ and $\mathbb{K} \cup \{\infty\}$ respectively. For each $t(x) \in \mathbb{K}(x)$, the image of y in L is $y(x, t(x))$. Since the assumptions of Lemma 5.3.1 are fulfilled, we know that there exists $G \in \overline{\mathbb{K}(x)}[Y, Z]$, determined by coefficients g_{ij}, such that $G(y, z) = 0$. By Remark 5.3.5 we even know that $G \in \mathbb{K}(x)[Y, Z]$. Note, that A, as defined in Theorem 5.3.3, is the set of poles of g_{ij} in $\mathbb{K} \cup \{\infty\}$. Theorem 5.3.3 yields

$$[y(x, t(x))]^- \leq [z(x, t(x))]^- + \sum_{x_0 \in A} \sigma_{x_0} q_{x_0} \,,$$

for all $t(x) \in \mathbb{K}(x)$ such that $z_2(x, t(x)) \neq 0$, where σ_{x_0} is the largest integer which does not exceed $\max_{i,j} \left\{ \frac{-\nu_{x_0}(g_{ij})}{m-i} \right\}$, and q_{x_0} the prime divisor corresponding with x_0 in L. By taking the degree of divisors on both sides, we obtain

$$\deg y(x, t(x)) \leq \deg z(x, t(x)) + \sum_{x_0 \in A} \sigma_{x_0} \,.$$

The constant $C := \sum_{x_0 \in A} \sigma_{x_0}$ depends only on y and z and it is independent of $t(x)$. □

Remark 5.3.5. The polynomial G defined in the last proof can be constructed by using Gröbner bases. We first compute a reduced Gröbner basis of the ideal $\langle y_2 Y - y_1, z_2 Z - z_1 \rangle$ in $\mathbb{K}[x, t, Y, Z]$ with respect to lexicographic order such that $t > Y > Z > x$. Let H be an element in the basis with the smallest leading term. Then H must be in $\mathbb{K}[x, Y, Z]$. Finally we consider H as a polynomial in $\mathbb{K}[x][Y, Z]$ and divide H by the leading coefficient (with respect to lexicographic order such that $Y > Z$). The result, which is an irreducible polynomial in $\mathbb{K}(x)[Y, Z]$, is the polynomial G we are looking for.

We summarize the discussion on Corollary 5.3.4 and Remark 5.3.5 as the following algorithm for further use.

5.3.2 Rational solutions of first-order first-degree AODEs

As a nice application of Corollary 5.3.4, we present here an algorithm for determining all rational solutions of a first-order first-degree AODEs.

In what follows we will need the following technical lemma.

Lemma 5.3.6. *Let $f(x, y) \in \mathbb{K}(x, y) \setminus \mathbb{K}(x)$ be a rational function of degree d in y. Then there exists $C = C(f) > 0$, depending only on f, such that for every $y(x) \in \mathbb{K}(x)$ with $f(x, y(x)) \neq 0, \infty$, we have*

$$d \cdot \deg y(x) \leq \deg f(x, y(x)) + C$$

Proof. We will construct such an C by using Corollary 5.3.4. We rewrite $f = \frac{P}{Q}$ in the reduced form, i.e. $P, Q \in \mathbb{K}[x, y]$ and $\gcd(P, Q) = 1$. Since $\deg_y f = \deg_y \frac{1}{f}$,

Algorithm 9 Degree of rational functions

Require: $y = \frac{y_1}{y_2}, z = \frac{z_1}{z_2} \in \mathbb{K}(x,t)$ such that y_2 divides z_2 and $\deg_t y_1 - \deg_t y_2 \leq \max\{0, \deg_t z_1 - \deg_t z_2\}$

Ensure: $C = C(y,z) > 0$ such that $\deg y(x,t(x)) \leq \deg z(x,t(x)) + C$ for every $t(x) \in \mathbb{K}(x)$ with $z_2(x,t(x)) \neq 0$.

1: Compute a reduced Gröner basis, say $\mathcal{G}$, for the ideal

$$\langle y_2(x,t)Y - y_1(x,t), z_2(x,t)Z - z_1(x,t)\rangle \subset \mathbb{K}[t,Y,Z,x]$$

with respect to the lexicographic order such that $t > Y > Z > x$.

2: Let $H(x,Y,Z)$ be an element in $\mathcal{G} \cap K[x,Y,Z]$ with the smallest leading term.

3: $h(x) :=$ leading term of $H(x,Y,Z)$ considered as an element in $\mathbb{K}(x)[Y,Z]$ with $Y > Z$.

4: Let $G(x,Y,Z) := \frac{H(x,Y,Z)}{h(x)}$, and rewrite G as the form $G = Y^m - \sum_{i,j \leq m} g_{i,j} Y^i Z^j$, where Y^m is the leading term of G and $g_{i,j} \in \mathbb{K}(x)$.

5: $A :=$ the set of poles of $g_{i,j}$ in $\mathbb{K} \cup \{\infty\}$.

6: For each $x_0 \in A$, compute $\sigma_{x_0} := \max_{i,j} \frac{-\nu_{x_0}(g_{i,j})}{m-i}$.

7: Return $C := \sum_{x_0 \in A} \sigma_{x_0}$.

---

we might assume that $d = \deg_y P \geq \deg_y Q$. Consider y^d and $P(x,y)$ as elements in $\mathbb{K}(x,y)$, they meets the requirement of Corollary 5.3.4. Therefore, there exists a constant $C_1 = C(y^d, P)$ such that for every $y(x) \in \mathbb{K}(x)$, we have

$$d \cdot \deg f(x,y(x)) \leq \deg P(x,y(x)) + C_1 \tag{5.7}$$

Similarly, the rational functions $\frac{1}{P}$ and $\frac{Q}{P}$ satisfy the requirement of Corollary 5.3.4. There exists a constant $C_2 = C(\frac{1}{P}, \frac{Q}{P})$ such that

$$\deg \frac{1}{P(x,y(x))} \leq \deg \frac{Q(x,y(x))}{P(x,y(x))} + C_2 \tag{5.8}$$

for every $y(x) \in \mathbb{K}(x)$ with $P(x,y(x)) \neq 0$. The constants C_1 and C_2 can be computed by using Algorithm 9. By combining (5.7) and (5.8), we obtain

$$d \cdot \deg y(x) \leq \deg f(x,y(x)) + C_1 + C_2$$

for every $y(x) \in \mathbb{K}(x)$ such that $f(x,y(x)) \neq 0, \infty$. □

Let us return to the problem of determining all rational solutions of a first-order first-degree AODE. Assume that $y(x) \in \mathbb{K}(x)$ is a non-constant rational solution of the differential equation $y' = f(x,y)$. By applying the above lemma, one can determine a constant C depending only on f such that

$$\deg_y f \cdot \deg y(x) \leq \deg f(x,y(x)) + C$$

Since $\deg f(x,y(x)) = \deg y'(x) \leq 2 \deg y(x)$, we imply that

$$(\deg_y f - 2) \deg y(x) \leq C$$

Hence if $\deg_y f \geq 3$, the degree of every rational solution of the differential equation $y' = f(x, y)$ does not exceed $\frac{C}{\deg_y f - 2}$. This bound can be used for making an ansatz and then computing all rational solutions.

Most of such quasi-linear ODEs are also in the scope of Algorithm 7. Note, that in these cases the constants C_1 and C_2 from Corollary 5.3.4 might be rather high and hence Algorithm 7 is recommended.

Now let us gather the two approaches to propose an algorithm for computing rational solutions of the first-order first-degree AODE $y' = f(x, y)$. Let n and m be the degree of the numerator and the denominator of f, respectively. As we have seen, Algorithm 7 can rationally solve such an ODE in all cases but $n - m = 2$. If $n - m = 2$ and $n \geq 3$, the above discussion gives an upper bound for the degree of a rational solution, and then gives a chance to find all of them. The only remaining case is $(n, m) = (2, 0)$. In this case, the differential equation is a rational Riccati equation. Fortunately algorithms for finding rational solutions of a rational Riccati equation are well established in literature. In the next section, we recall such an algorithm.

We conclude this section by summarizing an algorithm for finding all rational solutions of a given quasi-linear first-order ODE (see Algorithm 10) and giving an example thereof.

Algorithm 10 Rational solutions of first-order first-degree AODEs

Require: A quasi-linear ODE, $y' = f(x, y)$, where $f = \frac{P}{Q} \in \mathbb{K}(x, y)$.
Ensure: All rational solutions.

$n = \deg_y P$ and $m = \deg_y Q$
if $n - m \neq 2$ **then**
  return the result of Algorithm 7
else if $(n, m) = (2, 0)$ **then**
  return the result of Algorithm 11
else
  Using Algorithm 9 to determine $C_1 := C(y^n, P)$ and $C_2 := C(\frac{1}{P}, \frac{Q}{P})$.
  $r = \left\lfloor \frac{C_1 + C_2}{n - 2} \right\rfloor$
  Make an ansatz

$$y(x) = \frac{a_0 + a_1 x + \ldots + a_r x^r}{b_0 + b_1 x + \ldots + b_r x^r}$$

  with indeterminate coefficients a_i, b_j, and solve the obtained algebraic system.
  return all solutions $y(x)$
end if

Example 5.3.7. We consider the differential equation

$$y' = f(x, y) = \frac{x^3 y^4 - 5xy - x^3 + 5x^2 - 3}{x^3(y^2 + x)}. \tag{5.9}$$

Let P and Q be the numerator and the denominator of f, respectively. Since $\deg_y P - \deg_y Q = 2$, the differential equation (5.9) is out of the scope of Algorithm 7. We use the algebraic method described in this section to compute a degree bound for a rational solution. Keeping the notations as in Algorithm 10, we first using Algorithm 9 to compute $C_1 = C(y^4, P)$. In order to do that, let us determine a polynomial $G_1 \in \mathbb{K}(x)[Y, Z]$ such that $G_1\left(y^4, P(x, y)\right) = 0$. It can be done by Gröbner bases. In fact, we compute a reduced Gröbner basis of the ideal $\langle Y - y^4, Z - P\rangle$ in $\mathbb{K}[y, Y, Z, x]$ with the lexicographic order $y > Y > Z > x$. The polynomial in the basis with the smallest leading term, say G, is the unique one containing only x, Y, Z, and it has the form

$$G = x^{12}Y^4 + \text{ terms of smaller lex order}$$

Therefore, $G_1 = \frac{G}{x^{12}}$. Poles of coefficients of G_1 occur only at $x = 0$. The constant C_1, as defined above, is equal to 1. Similarly, G_2 is a polynomial in $\mathbb{K}(x)[Y, Z]$ such that $G_2\left(\frac{1}{P(x,y)}, \frac{Q(x,y)}{P(x,y)}\right) = 0$ and thus $C_2 := C(\frac{1}{P}, \frac{Q}{P}) = 11$. Then the degree of a rational solution of the differential equation (5.9) does not exceed $\frac{C_1+C_2}{\deg_y f - 2} = 6$. By using the indeterminate coefficient method, we see that $y(x) = \frac{-1+x}{x}$ is the only rational solution of the differential equation (5.9).

5.3.3 Riccati equations

In this subsection, we restrict our work to the class of Riccati equations. A Riccati equation is a differential equation of the form (4.3)

$$\omega' = b_0(x) + b_1(x)\omega + b_2(x)\omega^2\,, \tag{5.10}$$

where $b_0, b_1, b_2 \in \mathbb{K}(x)$, $b_2 \neq 0$. We normalize (4.3) by transforming with $y = -b_2(x)\omega - \frac{b_2'(x)}{2b_2(x)} - \frac{b_1(x)}{2}$. The obtained differential equation is

$$y' + y^2 = a(x)\,, \tag{5.11}$$

where $a = \frac{1}{4}\left(\frac{b_2'}{b_2} + b_1\right)^2 - \frac{1}{2}\left(\frac{b_2'}{b_2} + b_1\right)' - b_0 b_2$. A differential equation of the form (5.11) is called a rational normal Riccati equation. Since this is always possible we only consider Riccati equations in normal form and study their rational solutions.

The problem of finding rational solutions of Riccati equations has been intensively studied. An algorithm for finding rational solutions of a Riccati equation can be found for instance in [37, Alg. 2.2,p. 21]. In [25, Case 1] Kovacic also considers rational solutions of Riccati equations, as a step in the computation of Liouvillian solutions of second-order ODEs. Here we summarize the most important aspects of Kovacic's algorithm for Riccati equations.

First we collect necessary conditions for a rational normal Riccati equation having a rational solution. To avoid triviality we always assume that a is not a constant, or equivalently, $a(x)$ has at least one pole in $\mathbb{K} \cup \{\infty\}$.

Proposition 5.3.8 (Kovacic [25])**.** *If the rational normal Riccati equation* (5.11) *has a rational solution, then*

(i) every pole of $a(x)$ on $\mathbb{K}$ must be either a simple pole or a multiple pole of even order,

(ii) the valuation of $a(x)$ at infinity $\nu_\infty(a(x))$ must be even or be greater than or equal to 2.

Assume that $y(x) \in \mathbb{K}(x)$ is a rational solution of the differential equation (5.11). A pole of $y(x)$ which is also a pole of $a(x)$ is called a non-movable pole. Otherwise, it is called a movable pole. According to Kovacic's algorithm (see [25]), $y(x)$ must have the form

$$y(x) = \sum_{i=1}^{n}\sum_{j=1}^{r_i} \frac{c_{ij}}{(x-x_i)^j} + \sum_{i=1}^{m} \frac{1}{x-\chi_i} + \sum_{i=0}^{N} d_i x^i \,, \tag{5.12}$$

where $x_1, \ldots, x_n \in \mathbb{K}$ are poles of $a(x)$, $\chi_1, \ldots, \chi_m \in \mathbb{K}$ are movable poles of $y(x)$, the indices $N, n, r_1, \ldots, r_n$ can be determinded from $a(x)$, and each of the vectors $(d_0, d_1, \ldots, d_N)$, $(c_{i1}, \ldots, c_{ir_i})$ over $\mathbb{K}$ is determined up to at most 2 choices. Every such choice determines the potential number m of movable poles.

For details on the possible choices, we separate into several cases depending on the property of $a(x)$ at the pole $x_0 \in \mathbb{K} \cup \{\infty\}$. The first sum in (5.12) corresponds to the property of $a(x)$ at finite poles $x_1, \ldots, x_n$, while the last sum is defined by the property of $a(x)$ at infinity. In all cases the coefficents can be obtained from Laurent series expansion of a and y. Then we substitute into the ODE and compare coefficients.

Case 1: $x_0 = x_i \in \mathbb{K}$ is a double pole of $a(x)$.
In this case, $y(x)$ has a simple pole at $x = x_i$. The Laurent series expansion of $a(x)$ and $y(x)$ at $x = x_0$ are

$$a(x) = \frac{a_{i2}}{(x-x_i)^2} + \frac{a_{i1}}{(x-x_i)} + \sum_{k=0}^{\infty} a_{i,-k}(x-x_i)^k \,,$$
$$y(x) = \frac{c_{i1}}{(x-x_i)} + \sum_{k=0}^{\infty} c_{i,-k}(x-x_i)^k \,.$$

By substituting these Laurent series to (5.11) and comparing coefficients of $(x-x_i)^{-2}$ both sides, we see that the c_{i1} has two options

$$c_{i1} = \frac{1 \pm \sqrt{1+4a_{i2}}}{2} \,. \tag{5.13}$$

Notice that c_{i1} is also the residue of $y(x)$ at $x = x_i$.

Case 2: $x_0 = x_i \in \mathbb{K}$ is a multiple pole of degree $2r_i$ of $a(x)$, $(r_i \geq 2)$.
In this case, $y(x)$ has a pole of degree r_i at $x = x_i$. The Laurent series expansion of $a(x)$ and $y(x)$ at $x = x_0$ are

$$a(x) = \frac{a_{i,2r_i}}{(x-x_i)^{2r_i}} + \frac{a_{i,2r_i-1}}{(x-x_i)^{2r_i-1}} + \sum_{k=2}^{\infty} \frac{a_{i,2r_i-k}}{(x-x_i)^{2r_i-k}} \,,$$
$$y(x) = \frac{c_{i,r_i}}{(x-x_i)^{r_i}} + \ldots + \frac{c_{i,1}}{(x-x_i)} + \sum_{k=0}^{\infty} c_{i,-k}(x-x_i)^k \,.$$

Again, we substitute these Laurent series to the differential equation (5.11) and then identify the coefficients of $(x-x_i)^j$ on both sides with $j = 2r_i, 2r_i - 1, \ldots, r_i + 1$. We see that c_{i,r_i} has two possibilities, and once a choices is fixed,

$c_{i,r_i-1},\ldots,c_{i1}$ are determined uniquely. In particular, the vector $(c_{i,r_i},\ldots,c_{i1})$ is determined by

$$\begin{cases} c_{i,r_i} = \pm\sqrt{a_{i,2r_i}}\,, \\ c_{i,s} = \dfrac{1}{2c_{i,r_i}}\left(a_{i,r_i+s} - \displaystyle\sum_{j=s+1}^{r_i-1} c_{i,j}c_{i,r_i+s-j}\right), & s\in\{2,\ldots,r_i-1\}\,, \\ c_{i,1} = \dfrac{1}{2c_{i,r_i}}\left(a_{i,r_i+1} - \displaystyle\sum_{j=2}^{r_i-1} c_{i,j}c_{i,r_i+1-j} - r_i c_{i,r_i}\right). \end{cases} \tag{5.14}$$

The residue of $y(x)$ at $x = x_i$ in this case is c_{i1}.

Case 3: $x_0 = x_i$ is either a simple pole of $a(x)$ or a movable pole.
Then $y(x)$ has a simple pole at $x = x_i$ with the residue 1. The Laurent series expansion of $y(x)$ at x_i is $y(x) = \frac{1}{x-x_i} + \sum_{k=0}^{\infty} c_{i,-k}(x-x_i)^k$.

Case 4: $x_0 = \infty$ satisfying $\nu_\infty(a(x)) = -2N < 0$ for some $N \in \mathbb{N}$.
Then the valuation of $y(x)$ at infinity is $-N$. The Laurent series expansion of $a(x)$ and $y(x)$ at infinity, respectively, are

$$a(x) = a_{2N}x^{2N} + a_{2N-1}x^{2N-1} + \sum_{k=2}^{\infty} a_{2N-k}x^{2N-k}\,,$$
$$y(x) = d_N x^N + d_{N-1}x^{N-1} + \sum_{k=2}^{\infty} d_{N-k}x^{N-k}\,.$$

With the same technique as above, by substituting these Laurent series to the differential equation (5.11) and comparing corresponding coefficients on both sides, we obtain the following two possibilities for the vector $(d_0, d_1, \ldots, d_N)$:

$$\begin{cases} d_N = \pm\sqrt{a_{2N}}\,, \\ d_s = \dfrac{1}{2d_N}\left(a_{N+s} - \displaystyle\sum_{j=s+1}^{N-1} d_j d_{N+s-j}\right), \quad s\in\{0,\ldots,N-1\}\,. \end{cases} \tag{5.15}$$

Notice that the residue of $y(x)$ at infinity in this case is

$$-d_{-1} = \frac{-1}{2d_N}\left(a_{N-1} - d_N - \sum_{j=0}^{N-1} d_j d_{N-1-j}\right).$$

Case 5: $x_0 = \infty$ satisfying $\nu_\infty(a(x)) = 0$.
We take $N = 0$. Then the valuation of $y(x)$ at infinity is 0. The Laurent series expansion of $a(x)$ and $y(x)$ at infinity, respectively, are

$$a(x) = a_0 + \frac{a_{-1}}{x} + \sum_{k=2}^{\infty} \frac{a_{-k}}{x^k}\,,$$
$$y(x) = d_0 + \frac{d_{-1}}{x} + \sum_{k=2}^{\infty} \frac{d_{-k}}{x^k}\,.$$

These Laurent series must satisfy the differential equation (5.11). Therefore $d_0 = \pm\sqrt{a_0}$ and $d_{-1} = \frac{a_{-1}}{2d_0}$. The residue of $y(x)$ at infinity in this case is $-d_{-1} = -\frac{a_{-1}}{2d_0}$

Case 6: $x_0 = \infty$ satisfying $\nu_\infty(a(x)) \geq 2$.
In this case, the valuation of $y(x)$ at infinity is positive. Therefore, we just simply replace the last sum in (5.12) by zero, i. e. $N = -1$. The Laurent series of $a(x)$ and $y(x)$ at infinity are of the form

$$\begin{aligned} a(x) &= \frac{a_{-\nu_\infty(a(x))}}{x^{\nu_\infty(a(x))}} + \sum_{k=1}^{\infty} \frac{a_{-\nu_\infty(a(x))-k}}{x^{\nu_\infty(a(x))+k}}\,, \\ y(x) &= \frac{d_{-1}}{x} + \sum_{k=2}^{\infty} \frac{d_{-k}}{x^k}\,, \end{aligned}$$

respectively. The possible residue at infinity of $y(x)$ in this case is $-d_{-1} = -\frac{1\pm\sqrt{1+4s_\infty}}{2}$, where $s_\infty = \lim\limits_{x\to 0} \frac{1}{x^2} a\left(\frac{1}{x}\right)$.

Once vectors $(d_{-1}, d_0, \ldots, d_N)$ and $(c_{i1}, \ldots, c_{i,r_i})$ for $i = 1, \ldots, n$ are chosen, the number of non-movable poles can be estimated. By the residue theorem, the sum of all residues of $y(x)$ over $\mathbb{K} \cup \{\infty\}$ is equal to zero. Since the residue of $y(x)$ at movable poles is always equal to 1, the number of movable poles is $m = d_{-1} - \sum_{i=1}^{n} c_{i1}$.

After determining the number m of movable poles, we can make an ansatz. Let $\overline{y}(x) := y(x) - \sum\limits_{i=1}^{m} \frac{1}{x-\chi_i}$, and let $P(x) := (x-\chi_1) \cdot \ldots \cdot (x-\chi_m)$. Then $y(x)$ can be written in the form $\overline{y}(x) + \frac{P'(x)}{P(x)}$. By substituting to the differential equation (5.11), P must be a polynomial solution of degree m of the following linear second-order ODE:

$$P''(x) + 2\overline{y}(x)P'(x) + (\overline{y}'(x) + \overline{y}(x)^2 - a(x))P(x) = 0\,. \tag{5.16}$$

Finding all polynomial P of degree m of the differential equation (5.16) can be done by linear algebra.

Summarize this discussion we can recall Kovacic's algorithm for rationally solving Riccati equations.

Algorithm 11 Rational Solutions of Riccati equations

Require: The Riccati equation $\omega' = b_0(x) + b_1(x)\omega + b_2(x)\omega^2$, with $b_i \in \mathbb{K}(x)$ and $b_2 \neq 0$

Ensure: The set Sol of all rational solutions.

Set $y = -b_2\omega - \frac{b_2'}{2b_2} - \frac{b_1}{2}$ and transform to the rational normal Riccati equation: $y' + y^2 = a(x)$, where $a = \frac{1}{4}\left(\frac{b_2'}{b_2} + b_1\right)^2 - \frac{1}{2}\left(\frac{b_2'}{b_2} + b_1\right)' - b_0 b_2$

$Sol = \emptyset$, $preSol = \emptyset$

If $a = 0$, then $preSol = preSol \cup \left\{\frac{1}{x-c}\right\}$.

If $a \in \mathbb{K} \setminus \{0\}$, then $preSol := preSol \cup \{\pm\sqrt{a}\}$.

Determine poles of $a(x)$ in $\mathbb{K}$, say $x_1, \ldots, x_n$, and their orders. Compute $\nu_\infty(a(x))$.

Based on the results of the previous step, determine all possible vectors $(d_{-1}, d_0, \ldots, d_N)$, and $(c_{i1}, \ldots, c_{ir_i})$ for $i \in \{1, \ldots, n\}$ as discussed above.

for all possible combinations of these vectors **do**

  Compute $m = d_{-1} - \sum_{i=1}^{n} c_{i1}$.

  Denote $\overline{y}(x) = \sum_{i=1}^{n} \sum_{j=1}^{r_i} \frac{c_{ij}}{(x-x_i)^j} + \sum_{i=0}^{N} d_i x^i$.

  if m is a non-negative integer **then**

    Find polynomial solutions of degree m, say $P(x)$, of the differential equation

$$P'' + 2\overline{y}(x)P' + (\overline{y}'(x) + \overline{y}(x)^2 - a(x))P = 0\,.$$

    For each P, append $\overline{y}(x) + \frac{P'(x)}{P(x)}$ to $preSol$

  end if

end for

For each $y(x) \in preSol$, append $\omega := \frac{-1}{b_2}\left(y + \frac{b_2'}{2b_2} + \frac{b_1}{2}\right)$ to Sol

Return Sol.

5.4 Geometric approach

This section is devoted to studying an algorithm for determining all rational solutions for a parametrizable first-order AODE. In Section 4.3, we demonstrated a transformation of a parametrizble first-order AODE to a first-order first-degree AODE by using optimal parametrization of rational curves over the field of rational functions. General solutions are invariant under these transformations. But for specific rational solutions, information might be lost. However, the following lemma shows that lost rational solutions can be recovered.

Lemma 5.4.1 (see Lemma 4.3.2)**.** *Let $F(x, y, y') = 0$ be a parametrizable first-order AODE, and $\mathcal{P} := (p_1(x,t), p_2(x,t)) \in \mathbb{K}(x)(t)^2$ an optimal parametrization of the corresponding algebraic curve $\mathcal{C}$. Then $y = y(x) \in \mathbb{K}(x)$ is a rational solution of the differential equation $F(x, y, y') = 0$ if and only if one of the following holds:*

i. The rational point $(y(x), y'(x))$ lies in the finite set $\mathcal{C} \setminus \mathrm{im}(\mathcal{P})$.

ii. $y(x) = p_1(x, \omega(x))$ *for some algebraic solution* $\omega(x)$ *of the algebraic system:*

$$\begin{cases} \dfrac{\partial p_1}{\partial t}(x,\omega) = 0\,, \\ \dfrac{\partial p_1}{\partial x}(x,\omega) = p_2(x,\omega)\,. \end{cases}$$

iii. $y(x) = p_1(x, \omega(x))$ *for some algebraic solution* $\omega(x)$ *of the associated differential equation.*

For the class of autonomous first-order AODEs, the following proposition shows that being parametrizable is a necessary condition for such a differential equation having a rational solution.

Proposition 5.4.2. *If an autonomous AODE,* $F(y, y') = 0$*, has a rational solution then it is parametrizable.*

Proof. Let $y(x)$ be a rational solution of the AODE, then $y(x+c)$ is also a solution of the AODE (see [13]) and hence, $(y(x+c), y'(x+c))$ is a parametrization. □

As a consequence of Lemma 4.3.2, the problem of determining all rational solutions of a parametrizable first-order AODE reduced to three smaller ones: determining the complement of $\operatorname{im}\mathcal{P}$ in $\mathcal{C}$, finding rational solutions of a system of two algebraic equations, and finding rational solutions of a quasilinear first-order ODE. The first two problems are well-known, and therefore, quite easy. The last one has been investigated in the previous section. Thus Lemma 4.3.2 yields the following algorithm for determining all rational solutions of a parametrizable first-order AODE (see Algorithm 12).

Example 5.4.3. Consider the differential equation:

$$F(x, y, y') = -y^5 - xy^4y' + y'^3 = 0\,.$$

The corresponding curve, say $\mathcal{C}$, has an optimal parametrization

$$\mathcal{P}(t) := \left(\frac{t^3}{x^3(t^2 - x^3)}, \frac{-t^5}{x^4(t^2 - x^3)^2} \right).$$

The rational map $\mathcal{P}^{-1} : \mathcal{C} \to \mathbb{A}^1(\overline{\mathbb{K}(x)})$ given by $\mathcal{P}^{-1}(y, z) = -\frac{x^2y^2}{z}$ is an inverse of $\mathcal{P}$. Therefore, $\mathcal{C} \setminus \operatorname{im}(\mathcal{P}) = \emptyset$. The associated algebraic system and the associated differential equation with respect to $\mathcal{P}$ are

$$\begin{cases} -\frac{\omega^4(\omega^2 - 5x^3)}{x^4(\omega^2 - x^3)^3} = 0\,, \\ -\frac{3\omega^3(\omega^2 - 2x^3)}{x^4(\omega^2 - x^3)^2} = -\frac{\omega^5}{x^4(\omega^2 - x^3)^2}\,, \end{cases}$$

and

$$\omega' = \frac{2\omega}{x}\,,$$

respectively. This algebraic system and the differential equation are quite easy to solve. The algebraic system has only the zero solution, while the differential equation has a one-parameter class of rational solution, $\omega(x) = cx^2$. Hence, the set of all rational solutions of the given differential equation is the one-parameter class of functions $y(x) = \frac{c^3}{c^2x - 1}$.

Algorithm 12 Rational Solutions of parametrizable first-order AODEs

Require: A parametrizable first-order AODE, $F(x, y, y') = 0$, with an optimal parametrization $\mathcal{P} = (p_1, p_2)$ of the associated curve.

Ensure: All rational solutions.

1: $Sol = \emptyset$

2: Determine the finite set $\mathcal{C} \setminus \mathcal{P}$

3: **for all** $y(x) \in \mathcal{C} \setminus \mathcal{P}$ with $y(x) \in \mathbb{K}(x)$ and $F(x, y(x), y'(x)) = 0$ **do**

4:     append $y(x)$ to Sol

5: **end for**

6: Find all rational solutions $\omega = \omega(x)$ of the associated algebraic system

$$\begin{cases} \frac{\partial p_1}{\partial x}(x, \omega) = 0\,, \\ \frac{\partial p_1}{\partial t}(x, \omega) = p_2(x, \omega)\,, \end{cases}$$

append $p_1(x, \omega(x))$ to Sol

7: Use Algorithm 10 to rationally solve the associated differential equation

$$\omega' = \frac{p_2(x, \omega) - \frac{\partial p_1}{\partial x}(x, \omega)}{\frac{\partial p_1}{\partial t}(x, \omega)}\,.$$

For each of the rational solution $\omega = \omega(x)$, append $p_1(x, \omega(x))$ to Sol

8: **return** Sol.

Chapter 6

Conclusion and Future work

We have considered the class of first-order AODEs and studied their specific solutions, such as: algebraic general solutions, rational general solutions, particular rational and polynomial solutions. Several method have been proposed to attack the problem of determining these kinds of solutions for a first-order AODE.

In order to determine an algebraic/rational general solution for a first-oder AODE, we view the differential equation as an algebraic surface/curve over a suitable ground field. By using birational transformation of algebraic surfaces/curves, we transform the differential equation to a new one which we hope that it is easier to solve. Following this geometric approach, we proposed a full algorithm for deciding the existence of a strong rational general solution of a first-order AODE, and actually compute it in affirmative case (see Algorithm 5). The problem of determining an algebraic general solution is much harder. We proved that this problem is equivalent to the Poincaré problem. Beside, we presented a procedure for determining an algebraic general solution.

For the problem of determining all particular rational and polynomial solutions, we combine combinatorial, algebraic and geometric methods. A combinatorial consideration leads us to an algorithm for determining all rational solutions for a maximally comparable first-order AODE. An approach based on algebraic function field theory has been presented to study rational solutions of a first-order first-degree AODE. By combining all above approaches, we provided an algorithm for determining all rational solutions for a first-order AODE which is maximally comparable or parametrizable. This class covers all first-order AODEs from the collection by Kamke [23]. For computing polynomial solutions, we proposed a full algorithm.

The following is a short description of our ongoing and future research.

1. **Computing a rational general solution or all particular rational solutions of a first-order AODE**. Our previous algorithms already solve the problem for the class of first-order AODEs whose corresponding curves are of genus zero. To attack the positive genus cases, we are working on a computational modification of results by Eremenko in [12] and in related papers.

2. **An application: Effective Zolotarev Polynomial**. By using our algorithm, we expect to obtain an efficient algorithm to determine effective Zolotarev polynomials.

3. **Studying algebraic general solutions of first-order first-degree AODEs**. The problem of determining an algebraic general solution of a first-order AODE and the Poincaré problem are well known difficult problems (see [41]). We are studying different aspects of a generalized version of the Poincaré problem.

4. **Higher order AODEs and other kinds of symbolic solutions**. Developing algorithms for first-order AODEs is a natural first step before crafting algorithms for solving higher order AODEs. Key concepts needed to pass from first-order to higher order are those of differential divisors and Differential Nullstallensatz. Since the definition of differential divisors is based on the inclusion of general solution sets, Differential Nullstellensatz is naturaly relevant in investigating them. On the other hand, we also intend to investigate other types of symbolic solutions in the future, such as radical solutions, closed from solutions.

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Publications

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Schools, Conferences, and Talks

- WWCA 2016, Waterloo Workshop in Computer Algebra, July 23-24
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