Generalizing Some Results in Field Theory for Rings

Capco, Jose*
Email: jcapco@risc.jku.at

Abstract

In this paper we introduce some Galois-like theory for commutative reduced Baer rings. We show that the splitting ring of a polynomial over a Bear reduced ring is a finitely generated module. These rings will not always induce a finitely generated group of automorphisms, but the group will be a torsion group with finite exponent. Finally we show a generalized Artin-Schreier theorem: if the algebraic closure of a von Neumann, Baer reduced normal real ring is a finitely generated module then the ring is real closed and adjoining the base ring with $\sqrt{-1}$ will even give us its algebraic closure.

Keywords: Artin-Schreier Theorem, splitting rings, total integral closure, Galois theory

1 Introduction

There are two Artin-Schreier theorems known in literature [17, 18, 19]. One, more commonly known to number theorists, deals with finite Galois extensions of fields of non-zero characteristics for which the Galois group has the same order as the characteristic. This leads to a statement on the minimal polynomial of a primitive element for the field extension. A second, more commonly known to real algebraists, deals with fields which are not algebraically closed but whose algebraic closure are finite extension of them. This second result, leads to a characterization of real closed fields. Both were published in a 1927 paper written by Emil Artin and Otto Schreier (see [2]). The second version is what we are concerned with. However, one should not forget the fact that the second theorem is a consequence of the first one (proof by contradiction). One application of the second theorem is given in a work of Artin in 1927 leading to the solution of the celebrated Hilbert's 17-th problem (see [1]).

Our goal in this paper is to give a generalized Artin-Schreier theorem for von Neumann regular rings, by using a generalization of algebraic closure for reduced commutative rings first proposed by Enochs [12] in the 60's:

> The maximum integral and essential extension of a reduced commutative ring exists and is unique up to isomorphism and is called the *algebraic closure* of the ring.

Specifically we will prove the following theorem:

Theorem 1. Let A be a Baer real ring integrally closed in its total quotient and let $\bar{A}$ be its algebraic closure. Suppose $\bar{A}$ is a finitely generated A-module, then

i) A is a real closed ring

ii) If A is von Neumann regular, then $A[\sqrt{-1}] = \bar{A}$

*Supported and funded by the Austrian Science Fund (FWF): Project P28349-N32 and W1214-N14 Project DK9

To prove the above theorem we would need some Galois-like theory for reduced rings. There has been several attempts extending some Galois theory of fields to commutative rings. The most recent one can be found in [3, 14]. However, one school of thought on the Galois theory for commutative rings, remains most popular and was introduced by Chase, Harrison and Rosenberg [9]. In fact, we rely heavily on the work of Raphael who contributed a great deal in the study of algebraic closure of commutative von Neumann regular ring and with Desrochers (in [11]) he investigates this idea from [9] to develop theory on generalized *Galois extensions* (and its variations) of von Neumann regular commutative rings. For the purpose of proving the above Theorem, we need not go too deep. We would only need a generalization of normal extension of fields to the category of reduced commutative rings for proving Theorem 1. So we will dedicate a chapter on *splitting rings*. For a reduced commutative ring A, the *splitting ring* of a univariate monic non-constant $f \in A[x]$ is defined as the extension ring of A consisting of A adjoined with all the roots of f in the algebraic closure of A. We will see in the chapter that the splitting ring of A is in fact a finitely generated A-module if A is a reduced Baer von Neumann regular ring.

Finally we also stress that Theorem 1 would not hold if we remove the condition for A being Baer. This is the case even if all the other conditions of the Theorem applies. This will be shown by a constructive example at the end of this paper.

2 Preliminaries

Note that some of the results or definitions that we will give in this section can be generalized to other categories (e.g. rings that are not reduced, modules etc.). If σ is any endomorphism (between the same object in a given category) and if $k \in \mathbb{N}$, then by σ^k we mean the endomorphism created by composing σ with itself k-times.

For a commutative ring A, $T(A)$ denotes the total quotient ring of A. Unless otherwise stated, all rings in this papers are commutative unitary and reduced. All ring homomorphisms are such that 1 is mapped to 1. Let A be a ring, then $\operatorname{Spec} A$ is the set of prime ideals of A endowed with the Zariski topology and $\operatorname{Min} A$ is the set of minimal prime ideals of A considered as a subspace of $\operatorname{Spec} A$.

Here are some facts and more definitions

- A commutative ring A with multiplicative identity 1 is von Neumann regular iff for all $a \in A$ there is a $b \in B$ (called a *quasi-inverse* of a) such that $a^2 b = a$ (this is equivalent to the ring being reduced and its prime spectrum having a zero Krull dimension). Products of fields are easy examples of such rings. A less trivial example would be the ring of *locally constant* real-valued function with domain $\alpha\mathbb{N}$, i.e.

$$\{f : \alpha\mathbb{N} \to \mathbb{R} : f^{-1}(a) \text{ is open for all } a \in \mathbb{R}\}$$

  where $\alpha\mathbb{N}$ is the Alexandroff one-point compactification of $\mathbb{N}$ with discrete topology. The ring in the last example has a prime spectrum which is canonically isomorphic to $\alpha\mathbb{N}$.

- A commutative ring A is said to be Baer if for any set $S \subset A$ there is an idempotent $e \in A$ (i.e. $e = e^2$) such that the annihilator of S is generated by it, i.e. $\operatorname{Ann}_A(S) = eA$. Trivial example of Baer rings are integral domains (in particular also fields) and products of them. A less trivial example would be the ring of *locally constant* real-valued function with domain $\beta\mathbb{N}$, i.e.

$$\{f : \beta\mathbb{N} \to \mathbb{R} : f^{-1}(a) \text{ is open for all } a \in \mathbb{R}\}$$

  where $\beta\mathbb{N}$ is the Stone-ech compactification of $\mathbb{N}$ with discrete topology. The last example is because $\beta\mathbb{N}$ is an *extremally disconnected space* (i.e. a space for which the closure of any open set is again open) and is canonically isomorphic to the prime spectrum of the ring and extremally disconnected prime spectra characterizes Baer von Neumann regular rings (see [20] Propositon 2.1).

- Let A be a subring of a ring B, then B is an *essential extension of A* (in the category of commutative rings) if for all $b \in B\backslash\{0\}$ there exists a $c \in B$ such that $bc \in A\backslash\{0\}$. There is a generalized definition for essential extensions in any category $\mathcal{C}$. Let $\mathcal{C}$ be a category, then a monomorphism $f : a \to b$ is said to be an *essential monomorphism* (or extension) provided that for all morphism/arrow $g : b \to c$ such that the composition $g \circ f : a \to c$ is a monomorphism, it follows that g is a monomorphism. Some authors (e.g. Hochster in [15]) also use *tight extension* to mean essential extension.

- Let A be a subring of a reduced ring B, then B is said to be a *rational extension* of A if for all $b \in B\backslash\{0\}$ there exists an $a \in A$ such that $ab \in A\backslash\{0\}$. This definition can be generalized for non-reduced rings but we will confine ourselves to reduced commutative unitary rings. The study of such ring extensions became quite popular in the 50's (notably by Utumi, Lambek, Findlay and Johnson).

- It is known (see [13]) that if A is reduced, commutative and unitary then there is a maximum rational extension $Q(A)$, i.e. a rational extension $Q(A)$ of A such that any other rational extension of A is A-isomorphic (i.e. isomorphism fixing A) to a subring of $Q(A)$. $Q(A)$ is called the *rational completion* or the *complete ring of quotients* of A. If $Q(A)$ is ring-isomorphic to A, then A is said to be *rationally complete.*

- As used by Raphael (see [11], [22] and [23]), we shall call an essential and integral extension of a commutative ring the *algebraic extension* of this ring. This definition coincides with the classical definition of algebraic extension when working with the category fields. It has been shown (see [5], [12] and [15]) that a commutative reduced unitary ring A have a maximum algebraic extension $\bar{A}$, in the sense that any algebraic extension of $\bar{A}$ is A-isomorphic to $\bar{A}$, which we shall call the *algebraic closure* of the ring A. When dealing with fields, this definition of algebraic closure also coincides with the classical definition of algebraic closure. Most authors also call the *algebraic closure* the *total integral closure* of the ring and in this paper we may sometimes use this name.

- A commutative ring A is said to be *real* iff for all $n \in \mathbb{N}$ the following holds

  $$a_1^2 + \cdots + a_n^2 = 0 \Leftrightarrow a_1, \dots, a_n = 0 \qquad \forall a_1, \dots, a_n \in A$$

  Clearly, every real ring is reduced and the total quotient of a real ring is real as well (specifically, the quotient field of a real domain is real). A real ring A is said to be *real closed* iff there is no strict algebraic extension of A that is also real. By Zorn's lemma every real ring has an algebraic extension that is real closed. Real closed rings (using this definition) were first introduced by Sankaran and Varadarajan in [24]. It was then more extensively studied in the PhD thesis of Capco (who defined this originally as *real closed* $*$ to distinguish with other definitions with similar name). Real closed rings are also Baer rings (see [7] Remark 28). Integral domains are real closed iff they are integrally closed in their quotient fields and their quotient fields are real closed fields (see [24] Proposition 2). Commutative von Neumann regular rings are real closed iff it is Baer and all the residue fields are real closed (see [7] Theorem 34).

The proposition below illustrates that one is able to arbitrarily strictly extend any reduced ring integrally (the proposition uses a field but this can be generalized). Thus, as discussed by Raphael [22], Borho [5], Enochs [12] and Hochster [15], it is necessary to involve *essential extensions* when defining algebraic extensions and algebraic closures of commutative reduced rings.

Proposition 2. Let K be a field and L be an algebraic extension of K then for any $n \in \mathbb{N}$ the ring L^n (componentwise addition and multiplication) is an integral extension of K.

Proof. There is a natural monomorphism from K to L^n (diagonal homomorphism) that brings each $k \in K$ to $(k, k, \dots, k) \in L^n$. In this way we identify K as a subring of L^n, but to avoid confusion we write $\bar{k}$ (instead of $k \in K$) to denote $(k, k, \dots, k)$. Furthermore, there is a canonical ring homomorphism $K[x] \to L^n[x]$, between polynomial rings, that maps each $\sum_{i=1}^m a_i x^i \in K[x]$ to $\sum_{i=1}^m \bar{a}_i x^i \in L^n[x]$. We abuse notation and denote the image of $f \in K[x]$ (by this homomorphism) by $\bar{f} \in L^n[x]$. Consider now $b_i \in L$ for $i = 1, \ldots, n$ (i.e. an arbitrary $(b_1, b_2, \ldots, b_n) \in L^n$). There exists non-constant monic polynomials $f_1, f_2, \ldots, f_n \in L$ such that b_i is a zero of f_i for $i = 1, \ldots, n$. So if we consider the polynomial $f := \prod_{i=1}^n f_i \in K[x]$, then the canonical image $\bar{f}$ in $L^n[x]$ has a zero $(b_1, b_2, \ldots, b_n)$. Thus, L^n is an integral extension of K. $\square$

Observe that we have made several use of the *overline* symbol. We will continue to make occasional use of this symbol (with its various meaning) where they are needed and whenever this will not cause confusion. To summarize: For a reduced commutative unitary ring A, by $\bar{A}$ we mean the algebraic closure (or sometimes called total integral closure) of the ring. For an element $a \in A$ if there is a canonical diagonal monomorphism (as in the above proof) of A into product of rings (or fields) then the canonical image of a is denoted by $\bar{a}$ and in this way we identify A as a subring of this product of rings (this should be clear as long as the domain and codomain are clear to the reader). Similarly, there is a canonical map (as in the above proof) that brings a polynomial $f \in A[x]$ into a polynomial $\bar{f}$ over product of rings (again, we use this only if the domain and codomain of this monomorphism is clear to the reader). With this, we hope that we have avoided unnecessary symbolic clutter in this paper without sacrificing clarity.

Notation. Let A be a subring of a ring B then the group $\mathrm{Aut}(B/A)$ is the set of ring automorphisms of B fixing A, i.e. the group of A-automorphisms of B.

Notation. Let A be a commutative ring and $f \in A[x]$ be a polynomial. Suppose that f is written as follows

$$f(x) := \sum_{i=0}^n a_i x^i$$

then for any ideal $I \trianglelefteq A$, we abuse notation and denote $f \bmod I$ (or $f(x) \bmod I$) as the polynomial

$$f \bmod I := \sum_{i=0}^n (a_i \bmod I) x^i \in (A/I)[x]$$

3 Splitting Rings

Let A be a reduced commutative unitary ring with algebraic closure B and suppose that $f \in A[x]$ is monic. Then for $S := \{b \in B : f(b) = 0\}$, we want to study the ring extension $A \hookrightarrow A[S]$. We start with a Theorem that claims that if A is Baer than we can at least conclude that B is a finitely generated A-module ...

Theorem 3. (Splitting Ring) Let A be a commutative reduced Baer ring. Suppose $B = \bar{A}$ and that $f \in A[x]$ is a non-constant monic polynomial with coefficients in A. Consider the set

$$S := \{b \in B \,:\, f(b) = 0\}$$

then $A[S]$ is a finitely generated module. Specifically, there are $b_1, \ldots, b_n \in B$ integral over A (where $n \leq \deg(f)$) such that

$$A[S] = A[b_1, \ldots, b_n]$$

Proof. Define $K := \prod_{\mathfrak{p} \in \mathrm{Min}\, B} \mathrm{Quot}(B/\mathfrak{p})$, we have the following diagram (with all canonical maps)

where I is an ideal maximal (Zorn's Lemma) with the property that $I \cap A = \langle 0 \rangle$. This allows the lower right ring monomorphism in the diagram to be an essential extension. Let $n := \deg(f) \in \mathbb{N}$ and for each $\mathfrak{p} \in \mathrm{Min}\, B$ arbitrarily arrange all n (not necessarily distinct) zeros of $f \bmod \mathfrak{p}$ in

$\mathrm{Quot}(B/\mathfrak{p})$ (which is an algebraically closed field, see [15] Corollary 1) as $k_{i,\mathfrak{p}}$ for $i = 1, \ldots, n$. Suppose that $\pi_\mathfrak{p} : K \twoheadrightarrow \mathrm{Quot}(B/\mathfrak{p})$ is the canonical projection for every $\mathfrak{p} \in \mathrm{Min}\, B$. We define $k_i \in K$, for $i = 1, \ldots, n$, by setting the projection of k_i for each $\mathfrak{p}$ to be $\pi_\mathfrak{p}(k_i) := k_{i,\mathfrak{p}}$. By the construction we get $f(\pi(k_i)) = 0$ for all $i = 1, \ldots, n$. Thus, since B is totally integrally closed, $\pi(k_i) \in B$ for all $i = 1, \ldots, n$. We define $b_i := \pi(k_i)$ for $i = 1, \ldots, n$ and claim that $A[S] = A[b_1, \ldots, b_n]$ (and because $b_1, \ldots, b_n$ are integral elements of A, $A[S]$ is a finitely generated A-module). Let $b \in S$, then define $e_i \in K$ for $i = 1, \ldots, n$ by

$$\pi_\mathfrak{p}(e_i) := \begin{cases} 1 & b \equiv k_{i,\mathfrak{p}} \bmod \mathfrak{p} \text{ and } b \not\equiv k_{j,\mathfrak{p}} \bmod \mathfrak{p} \text{ for } j > i \\ 0 & \text{otherwise} \end{cases}$$

Note that the e_i's are well-defined because $b \bmod \mathfrak{p}$ is a zero of $f \bmod \mathfrak{p}$ and the $k_{i,\mathfrak{p}}$'s are all the zeros of $f \bmod \mathfrak{p}$.

The condition used in defining $\pi_\mathfrak{p}(e_i) = 1$ is necessary because we may have roots of $f \bmod \mathfrak{p}$ that are not simple and because, in the ring K, we want to have the identity $b = \sum_{i=1}^n e_i k_i$. Taking the image of this with respect to π we get $b = \sum_{i=1}^n \pi(e_i) b_i$ in the ring K/I. Now e_i is an idempotent in K so the projection $\pi(e_i)$ is an idempotent in K/I. Since B is integrally closed in K/I (this is by the definition of total integral closure, see also [7] Theorem 29.) we also have $b_i, \pi(e_i) \in B$ (all idempotents satisfy the equation $x^2 - x = 0$) for all $i = 1, \ldots, n$. Since A is Baer, it contains all the idempotents of its total integral closure and hence $\pi(e_i) \in A$ (see [22] Lemma 1.6). We thus have shown that $b \in A[b_1, \ldots, b_n]$ for all $b \in S$, hence $A[S] = A[b_1, \ldots, b_n]$. □

We shall call the ring $A[S]$, in the theorem above, the *splitting ring*[1] *of* $f \in A[x]$ *(over* A*)*. The above result, however, does not gaurantee us an A-automorphism group that is finitely generated. We give an example of such a group that is not finitely generated and for that we use this very easy Lemma (whose proof we leave to the reader) ...

Lemma 4. Suppose A is a Baer reduced ring, B is a splitting ring of a monic non-constant polynomial $f \in A[x]$ and $b \in B$ is a zero of f then for any $\sigma \in \mathrm{Aut}(B/A)$, $\sigma(b)$ is a zero of f.

Example. In this example, we show a splitting ring B of some $f \in A[x]$ (where A is a Baer reduced ring) that provides a group of A-automorphisms of B, $\mathrm{Aut}(B/A)$, that is not finitely generated. Let $A = \mathbb{Q}^\mathbb{N}$ (this is a Baer reduced ring!) and consider $f(x) := x^2 - \bar{2} \in A[x]$. Note that the algebraic closure of A is A-isomorphic to the integral closure of A in $\bar{\mathbb{Q}}^\mathbb{N}$ (hints for this can be found in the proof of Theorem 38 [7]). So the ring $B := \mathbb{Q}(\sqrt{2})^\mathbb{N}$ is, in fact, the splitting ring of f (observe that $B = A + \sqrt{2}A$). We first claim that the group $G := \mathrm{Aut}(B/A)$ has exponent 2 (thus a torsion group).

Let, for each $j \in \mathbb{N}$, $\pi_j : B \to \mathbb{Q}(\sqrt{2})$ be the canonical projection on the j-th coordinate. Let $\sigma \in G$, then, one checks that, this induces a well-defined field automorphism $\sigma_j \in \mathrm{Aut}(\mathbb{Q}(\sqrt{2})/\mathbb{Q})$ for each $j \in \mathbb{N}$ that maps each $a + b\sqrt{2}$ ($a, b \in \mathbb{Q}$) to $a + b\pi_j(\sigma(\sqrt{2}))$ (by Lemma 4, $\pi_j(\sigma(\sqrt{2})) \in \{\sqrt{2}, -\sqrt{2}\}$). But $\sigma \circ \sigma \in G$ induces (in the same manner) the automorphism $\sigma_j \circ \sigma_j \in \mathrm{Aut}(\mathbb{Q}(\sqrt{2})/\mathbb{Q})$, for all $j \in \mathbb{N}$, and this can only be the identity map. Thus, the group G has a finite exponent 2 and so it is a torsion group.

Now suppose that G is finitely generated. Because it is a torsion group with exponent 2, this becomes a trivial Burnside problem and for this case G must necessarily be finite and even commutative. This gives us a contradiction because we know that G is an infinite group: Consider, for every $i \in \mathbb{N}$, the element $\sigma \in G$ that induces

$$\sigma_j := \begin{cases} \mathrm{id} & j \neq i \\ \tau & \text{otherwise} \end{cases} \qquad j \in \mathbb{N}$$

where $\tau \in \mathrm{Aut}(\mathbb{Q}(\sqrt{2})/\mathbb{Q})$ is the $\mathbb{Q}$-automorphism of $\mathbb{Q}(\sqrt{2})$ such that $\tau(\sqrt{2}) = -\sqrt{2}$ and $\mathrm{id} : \mathbb{Q}(\sqrt{2}) \to \mathbb{Q}(\sqrt{2})$ is the identity map. This gives us countably but infinitely many number of automorphisms in G.

[1] this definition is also in accordance to [12], where it probably first appeared

The above example gave us an automorphism group $\mathrm{Aut}(B/A)$ for which the exponent is finite. In general, this is true for any Baer commutative reduced ring A and splitting ring B. To prove this we first show a few nice results about the total quotients $T(A)$ and $T(B)$.

Theorem 5. Let A be a reduced commutative unitary ring. Suppose that A has the property that for any $a \in A$, there exists an idempotent $e \in A$ such that

$$\mathrm{Ann}(aA) = eA$$

then $T(A)$ is von Neumann regular.

Proof. Let $a \in A$ and suppose that $e \in A$ be an idempotent such that $\mathrm{Ann}(aA) = eA$. We claim tha $a + e$ is a regular element of A. In fact, if we prove this claim then the result follows from Theorem 9.3 in [26] (they proved a more general result). For completenses, we provide the proof for our specific case. Pick a $b \in A$ such that $(a + e)b = 0$. Then after multiplying by a we get $a^2b = 0$. Since A is reduced, we know that $b \in \mathrm{Ann}(aA)$. Thus, there is a $c \in A$ with $ce = b$. This gives us

$$0 = ce(a + e) = ce^2$$

So $ce = 0$ and this leads us to conclude that $ce = b = 0$. So the annihilator of $(a + e)A$ is 0. In other words, $a + e$ is a regular element.

We now know that $a + e$ is invertible in $T(A)$. So we clearly have

$$a + e = (a + e)^2(a + e)^{-1} = (a^2 + e)(a + e)^{-1} \Rightarrow a^2(a + e)^{-1} = a + e - e(a + e)^{-1}$$

But

$$e = e((a + e)(a + e)^{-1}) = (e(a + e))(a + e)^{-1} = e^2(a + e)^{-1} = e(a + e)^{-1}$$

So

$$a^2(a + e)^{-1} = a + e - e = a$$

We can therefore conclude that every element of A has a *quasi-inverse* in $T(A)$.

Now let $\frac{a}{b} \in T(A)$, with $a \in A$ and b a regular element of A. Also set a' to be the quasi-inverse of a in $T(A)$. Then

$$\left(\frac{a}{b}\right)^2 ba' = \frac{1}{b}a^2a' = \frac{1}{b}a = \frac{a}{b}$$

Thus, any element of $T(A)$ has also a quasi-inverse in $T(A)$. In other words, $T(A)$ is a von Neumann regular ring. □

Corollary 6. Let A be a reduced commutative unitary Baer ring then $T(A)$ is Baer, von Neumann regular and $\mathrm{Spec}\, T(A)$ is canonically homeomorphic to the $\mathrm{Min}\, A$.

Proof. The ring A satisfies (because it is Baer) the condition in Theorem 5, so $T(A)$ is von Neumann regular ring. Now, the smallest von Neumann regular intermediate ring of A and $Q(A)$ has a prime spectrum that is canonically isomorphic to $\mathrm{Min}\, A$ (see [21] Theorem 4.4). And the result follows since any two essential extension of a Baer von Neumann regular ring that are von Neumann regular will have homeomorphic prime spectra (see [22] Remark 1.17). □

Remark 7. The Corollary above will give us even more information. If A is a reduced commutative Baer ring and if B is an essential extension of A (then B must necessarily be reduced by Lemma 1.3 in [22]) then, by Storrer's Satz (see [25] 10.1), there is a canonical essential extension $Q(A) \hookrightarrow Q(B)$ and, by [22] Lemma 1.7, $Q(A)$ contains all of the idempotents of $Q(B)$ and thus, by [20] Proposition 2.5 and Storrer's Satz, both A and B are Baer. Thus, by the above Corollary, both $T(B)$ and $T(A)$ are von Neumann regular. Now, there is a canonical essential extension $T(A) \hookrightarrow T(B)$ and if we use [22] Remark 1.17 we conclude that $T(A)$ and $T(B)$ have homeomorphic prime spectra which are homeomorphic to both $\mathrm{Min}\, A$ and $\mathrm{Min}\, B$.

In short: If A is a reduced commutative Baer ring and B is an essential extension of A then

- B is Baer
- $T(A)$ and $T(B)$ are von Neumann regular
- We have the canonical homeomorphisms

$$\mathrm{Min}\, A \cong \mathrm{Spec}\, T(A) \cong \mathrm{Min}\, B \cong \mathrm{Spec}\, T(B)$$

We first only state the theorem that we want to prove:

Theorem 8. Let A be a Baer ring and B be the splitting ring of a non-constant monic polynomial $f \in A[x]$, then the automorphism group $\mathrm{Aut}(B/A)$ is a torsion group with finite exponent.

Now, before proving the theorem, we give and prove two lemmas on splitting rings that will be used in the main proof of the theorem:

Lemma 9. Let A be a domain and B be the splitting ring of a non-constant monic $f \in A[x]$ then the following holds

- B is a domain
- $\mathrm{Quot}(B)$ is the splitting field of f over $\mathrm{Quot}(A)$

Proof. Any essential extension of A is a domain and a field containing A is an essential extension of A. It easily follows that $\bar{A}$ is the integral closure of A in the algebraic closure of $\mathrm{Quot}(A)$ (see also [15] Corollary 1, p.774). Since the splitting ring lies between $\bar{A}$ and A, it must be a domain.

For the second part, let K be the splitting field of f over $\mathrm{Quot}(A)$ then one easily sees that

$$K = \mathrm{Quot}(A)(b_1, \ldots, b_n) = \mathrm{Quot}(A)[b_1, \ldots, b_n] = \mathrm{Quot}(A[b_1, \ldots, b_n])$$

where b_i, for $i = 1, \ldots, n$, are all the zeros of f in the algebraic closure of $\mathrm{Quot}(A)$. This proves the Lemma since $B = A[b_1, \ldots, b_n]$. □

Lemma 10. Suppose that B is the splitting ring of a non-constant monic $f \in A[x]$ over a Baer ring A. It follows that for any $\mathfrak{p} \in \mathrm{Min}\, B$, $B/\mathfrak{p}$ is the splitting ring of $A/(\mathfrak{p} \cap A)$.

Proof. In this proof, for simplicity, (because A is Baer) we identify all the minimal prime spectra of rings between A and $\bar{A}$ with $\mathrm{Spec}\, T(A)$. Let $\mathfrak{p} \in \mathrm{Min}\, B$, then we have $B/\mathfrak{p} = A[S]/\mathfrak{p} = (A/\mathfrak{p})[S/\mathfrak{p}]$ where

$$S := \{b \in B \,:\, f(b) = 0\}$$

Clearly any element in $S/\mathfrak{p}$ is a zero of $f \bmod \mathfrak{p}$, so it suffices to show that $S/\mathfrak{p}$ contains all the zeros of $f \bmod \mathfrak{p}$. Let $k \in \bar{A}/\mathfrak{p}$ be a zero of $f \bmod \mathfrak{p}$ (observe that $\bar{A}/\mathfrak{p}$ is integrally closed and has an algebraically closed quotient field, see e.g. [15] Corollary 1). There is a $b \in \bar{A}$ such that b is canonically mapped to k. Define the disjoint clopen sets (recall that $T(A)$ is von Neumann regular)

$$\begin{aligned} U_1 &:= \{\mathfrak{q} \in \mathrm{Spec}\, T(A) \,:\, f(b) \bmod \mathfrak{q} \equiv 0\} \\ U_2 &:= \mathrm{Spec}\, T(A) \backslash U_1 \end{aligned}$$

that cover $\mathrm{Spec}\, T(A)$ and from this we can define idempotents in e_1 and e_2 in $T(A)$ by

$$e_i \bmod \mathfrak{q} := \begin{cases} 1 & \mathfrak{q} \in U_i \\ 0 & \mathfrak{q} \notin U_i \end{cases} \qquad i = 1, 2$$

This implies that $c := be_1 + se_2 \in S$ for any $s \in S$ (since $f(c) \equiv 0 \bmod \mathfrak{q}$ for all $\mathfrak{q} \in \mathrm{Spec}\, T(A)$). Furthermore, $c \bmod \mathfrak{p} = k$ and we so are done. □

Finally the proof of Theorem 8...

Proof of Theorem 8. Denote $G := \operatorname{Aut}(B/A)$ and suppose $\sigma \in G$. Since σ is an automorphism, for any minimal prime ideal $\mathfrak{p} \in \operatorname{Min} B$ the set $\sigma(\mathfrak{p})$ is also a minimal prime ideal of B. Because A is Baer and B is an essential extension of A, we know that there is a canonical homeomorphism from $\operatorname{Min} B$ to $\operatorname{Min} A$ (see Remark 7) given by

$$\operatorname{Min} B \to \operatorname{Min} A \qquad \mathfrak{q} \mapsto \mathfrak{q} \cap A$$

Since $\mathfrak{p} \cap A$ is a minimal prime ideal of A and σ is an A-automorphism we have

$$\sigma(\mathfrak{p} \cap A) = \mathfrak{p} \cap A \subset \sigma(\mathfrak{p}) \cap A \subset \sigma(\mathfrak{p})$$

which would mean, by the above homeomorphism, that $\sigma(\mathfrak{p}) = \mathfrak{p}$.

Set $\mathfrak{p}_A := \mathfrak{p} \cap A$ then, since $\sigma(\mathfrak{p}) = \mathfrak{p}$, σ induces a well-defined $A/\mathfrak{p}_A$-automorphism

$$B/\mathfrak{p} \to B/\mathfrak{p} \qquad b \bmod \mathfrak{p} \mapsto \sigma(b) \bmod \mathfrak{p}$$

This in turn canonically induces a $\operatorname{Quot}(A/\mathfrak{p}_A)$-automorphism

$$\operatorname{Quot}(B/\mathfrak{p}) \xrightarrow{\sim} \operatorname{Quot}(B/\mathfrak{p})$$

The minimal prime spectra $\operatorname{Min} \bar{A}, \operatorname{Min} B$ and $\operatorname{Min} A$ are all homeomorphic (see Remark 7) and they can all be canonically identified. Now suppose $\mathfrak{p} \in \operatorname{Min} \bar{A}$ and, for simplicity, write $\mathfrak{p}_A := \mathfrak{p} \cap A$ and $\mathfrak{p}_B := \mathfrak{p} \cap B$. By the Lemmas 9 and 10, $B/\mathfrak{p}_B \cong A[b_1, \ldots, b_n]/\mathfrak{p}$ for some $b_i \in B$ such that $b_i \bmod \mathfrak{p}$ (i=1,...,n) are all the roots of $f \bmod \mathfrak{p}$ in the algebraic closed field $\operatorname{Quot}(\bar{A}/\mathfrak{p})$ and $\operatorname{Quot}(B/\mathfrak{p}_B)$ is the splitting field of $f \bmod \mathfrak{p}$ over $\operatorname{Quot}(A/\mathfrak{p}_A)$.

Now, let us write for short $K_\mathfrak{p} := \operatorname{Quot}(A/\mathfrak{p}_A)$ and $L_\mathfrak{p} := \operatorname{Quot}(B/\mathfrak{p}_B)$. We have already seen that $\sigma \in \operatorname{Aut}(A/B)$ induces a $K_\mathfrak{p}$-automorphism $\sigma_\mathfrak{p} \in \operatorname{Aut}(L_\mathfrak{p}/K_\mathfrak{p})$. Similarly, for any $k \in \mathbb{N}$, $\sigma^k \in \operatorname{Aut}(A/B)$ will induce $\sigma_\mathfrak{p}^k \in \operatorname{Aut}(L_\mathfrak{p}/K_\mathfrak{p})$. But $L_\mathfrak{p}$ is the splitting field for $f \bmod \mathfrak{p}$, so if n is the degree of f, the order of $\sigma_\mathfrak{p}$ will divide $n!$ (e.g. [10] Theorem 3 p.176 discusses this). Thus $\sigma^{n!}$ induces the identity $L_\mathfrak{p} \to L_\mathfrak{p}$. This is true for any $\mathfrak{p} \in \operatorname{Min} \bar{A}$.

In other words, for any $\mathfrak{p} \in \operatorname{Min} \bar{A}$ (or $\mathfrak{p}_A \in \operatorname{Min} A$) and for any $\sigma \in \operatorname{Aut}(B/A)$ we have the equation

$$\sigma^{n!}(b) \equiv b \bmod \mathfrak{p} \qquad \forall b \in B$$

Since B is reduced, $\sigma^{n!}(b) = b$ for all $b \in B$ and this means that $n!$ is an exponent of the group $\operatorname{Aut}(A/B)$.

□

4 Generalized Artin Schreier Theorem

We recall and reformulate Remark 109 in [7]

Remark 11. Let A be a reduced ring and B be an overring of A, then there is a canonical map $\operatorname{Spec} B \to \operatorname{Spec} A$. This map has the property that its image contains $\operatorname{Min} A$.

Recall now Theorem 101 in [7]

Theorem 12. Let A be a Baer reduced ring, then A is integrally closed in $T(A)$ iff for any $\mathfrak{p} \in \operatorname{Spec} T(A)$ we have $A/(\mathfrak{p} \cap A)$ is integrally closed in $T(A)/\mathfrak{p}$.

Proof. See [7] Theorem 101. □

We can combine a few results in [7] to arrive to the following Theorem

Theorem 13. Let A be a Baer reduced ring and suppose that $T(A)$ is a subring of a von Neumann regular real closed B. If, furthermore, A is integrally closed in B then $T(A)$ is a real closed von Neumann regular ring.

Proof. This follows immediately from Theorem 99(iii) and Theorem 34 in [7]. □

Corollary 14. A real Baer ring B is real closed iff for every minimal prime ideal $\mathfrak{p} \in \operatorname{Min} B$ one has $B/\mathfrak{p}$ is a real closed integral domain.

Proof. "⇒" If B is real closed then it is integrally closed in $Q(B)$ and furthermore $Q(B)$ is real closed (see [7] Remark 28). Now, by Theorem 13, $T(B)$ is also a real closed von Neumann regular ring. It follows from [24] Proposition 2 and Theorem 12 that $B/\mathfrak{p}$ is a real closed integral domain for any minimal prime ideal $\mathfrak{p}$ in $\operatorname{Spec} B$ ($\mathfrak{p} \in \operatorname{Min} B$ is a restriction of a prime ideal in $T(B)$, see Remark 7).

"⇐" Since B is Baer, $T(B)$ is von Neumann regular. So, by the hypothesis, the residue domains of B with respect to prime ideals in $\operatorname{Min} B$ is integrally closed in their real closed quotient fields (see [24] Proposition 2), and these quotient fields are the residue fields of $T(B)$. Thus the residue fields of the Baer von Neumann regular ring $T(B)$ are real closed and so (see [7] Theorem 34) $T(B)$ is real closed. Finally, Theorem 12 implies that B is integrally closed in $T(B)$ and so all conditions for [27] Theorem 3 are satisfied (B is real closed iff $Q(B)$ is real closed and B is integrally closed in $Q(B)$. Note that $Q(B)$ is also the complete ring of quotient of $T(B)$.) and so B is real closed. □

The minimal prime ideals of real rings are actually quite important in real algebra. We recall a well-known result that gives us an insight to this

Proposition 15. Let A be a real ring and $\mathfrak{p} \in \operatorname{Min} A$ then $A/\mathfrak{p}$ is a real ring (i.e. the prime ideal $\mathfrak{p}$ is a *real ideal*).

Proof. A proof can be found in [17] Kapitel III. §Satz 1. p.104. □

In the following easy lemma we see the relationship between integral extension of (containing) domains and their quotient fields and how essential extension plays a role in this relationship...

Lemma 16. Let A and B be integral domains and $A \subset B$ as rings, then

i. If B is integral over A, then $\operatorname{Quot}(B)$ is algebraic over $\operatorname{Quot}(A)$.

ii. If B is an essential extension of A and A is integrally closed in B, then $\operatorname{Quot}(A)$ is algebraically closed in $\operatorname{Quot}(B)$.

Proof. i. This is left as an easy exercise for the reader.

ii. Let $\frac{b}{c}$ be an element of $\operatorname{Quot}(B)$, with $b, c \in B$ and $c \neq 0$. Because B is essential over A, without loss of generality we may assume that $c \in A$. Suppose furthermore that $\frac{b}{c}$ is an algebraic element of $\operatorname{Quot}(A)$. Then there is a polynomial

$$f(T) = T^n + \sum_{i=0}^{n-1} T^i \frac{a_i}{x} \in \operatorname{Quot}(A)[T]$$

with $a_i \in A$ and $x \in A \backslash \{0\}$ and such that $\frac{b}{c}$ a zero of f. So

$$c^n x^n f(b/c) = (bx)^n + \sum_{i=0}^{n-1} (bx)^i c^{n-i} x^{n-i-1} a_i = 0$$

and therefore $bx \in B$ is a zero of

$$T^n + \sum_{i=0}^{n-1} b_i T^i \in A[T]$$

where $b_i := c^{n-i} x^{n-i-1} a_i \in A$ for $i = 0, \ldots, n-1$. But A is integrally closed in B, so $bx \in A$ and thus $b/c \in \operatorname{Quot}(A)$ □

Now we have sufficient tools to prove Theorem 1 ...

Proof of Theorem 1. i) Let $\mathfrak{p}$ be a minimal prime ideal of A, then both $A/\mathfrak{p}$ and $\mathrm{Quot}(A/\mathfrak{p})$ are real (see Proposition 15). There is a $\tilde{\mathfrak{p}} \in \mathrm{Spec}\,\bar{A}$ such that $\tilde{\mathfrak{p}} \cap A = \mathfrak{p}$ (see Remark 11). Since $\bar{A}/\tilde{\mathfrak{p}}$ is a finite integral extension of $A/\mathfrak{p}$, we see that $\mathrm{Quot}(\bar{A}/\tilde{\mathfrak{p}})$ is a finite field extension of $\mathrm{Quot}(A/\mathfrak{p})$ (see Lemma 16). We know by [15] Theorem 1 that $\mathrm{Quot}(\bar{A}/\tilde{\mathfrak{p}})$ is an algebraically closed field and so by the classical Artin-Schreier Theorem (see [17] §1 Theorem in p.18) $\mathrm{Quot}(A/\mathfrak{p})$ is a real closed field.

By Theorem 12 and Remark 7, for any $\mathfrak{p} \in \mathrm{Min}\,A$, $A/\mathfrak{p}$ is integrally closed in its quotient field. This quotient field, we have shown, is real closed. Thus, for any $\mathfrak{p} \in \mathrm{Min}\,A$ the integral domain $A/\mathfrak{p}$ is real closed ([24] Proposition 2). Employ Corollary 14 to conclude that the ring A is also real closed.

ii) Let i (or $\sqrt{-1}$) be a zero of $T^2 + 1 \in A[T]$ in the splitting ring of A. Note that $A[i]$ and $\bar{A}$ are von Neumann regular because an integral extension of a von Neumann regular ring that is reduced is also von Neumann regular (see [22] Lemma 1.9). A being Baer implies that $\mathrm{Spec}\,A$,$\mathrm{Spec}\,A[i]$ and $\mathrm{Spec}\,\bar{A}$ are canonically homeomorphic. We have also previously seen that, for all $\mathfrak{p} \in \mathrm{Min}\,A = \mathrm{Spec}\,A$ and (unique) $\tilde{p} \in \mathrm{Min}\,A[i] = \mathrm{Spec}\,\bar{A}[i]$ lying over $\mathfrak{p}$, $A/\mathfrak{p} = \mathrm{Quot}(A/\mathfrak{p})$ is real closed and have algebraic closure

$$(A/\mathfrak{p})[i \bmod \tilde{p}] = (A/\mathfrak{p})[\sqrt{-1}] = A[i]/\tilde{\mathfrak{p}}$$

Thus, for all $\tilde{\mathfrak{p}} \in \mathrm{Spec}\,A[i]$, $A[i]/\tilde{p}$ is an algebraically closed field and this is a characterization of algebraically closed von Neumann regular reduced rings (see [15] Proposition 5) and thus $A[i] \cong \tilde{A}$ ☐

Unfortunately, we do not yet know whether the second part of the theorem above is true for reduced rings in general. The complication lies on the fact that we used the characterization for algebraically closed rings (i.e. for each prime ideal, residue domains are algebraically closed). We do not know this in general, because normality does not in general hold for residue domain of non-minimal prime ideals (Proposition 5 in [15] requires algebraic closedness for *all* residue domain). For von Neumann regular rings, we had the convenience that the residue domains where themselves fields.

In the following, we show why we cannot remove the precondition *Baer* from the theorem above...

Example. Define the $X := \beta\mathbb{N} \times \{0, 1\}$ with $\{0, 1\}$ having the discrete topology. Define also $Y := X/\sim$ where

$$(x, 1) \sim (y, 0) \Leftrightarrow x, y \in \beta\mathbb{N}\backslash\mathbb{N} \text{ and } x = y$$

with the usual quotient topology. Then clearly both X and Y are Stone spaces with X being extremally disconnected. For brevity, we write the image of any $(x, i) \in X$ in Y also as (x, i). And we define $\psi : X \to Y$ to be the canonical surjection from X to Y.

Now we state a few facts:

- Y is not extremally disconnected because the closure of the open set

$$\{(x, 0) : x \in \mathbb{N}\} \subset Y$$

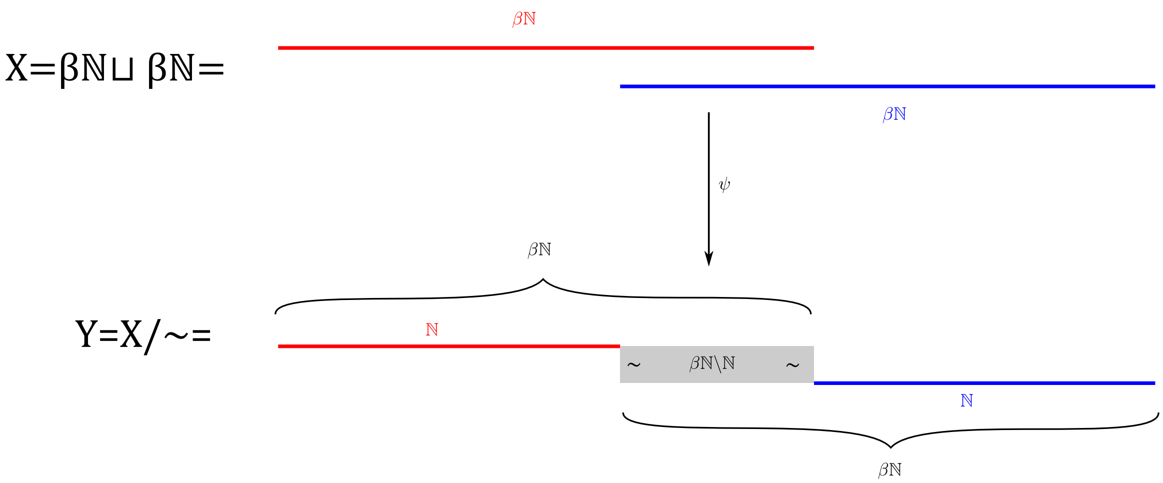
  is not open in Y.

- Let K be a real closed field and consider the von Neumann regular rings

$$A_Y := \{f : Y \to K : f^{-1}(k) \text{ is open for all } k \in K\}$$

  and

$$A_X := \{f : X \to K : f^{-1}(k) \text{ is open for all } k \in K\}$$



Then both A_Y and A_X are von Neumann regular rings (see [15] p.779) with prime spectra Y and X respectively. Because of the surjection ψ, we know that the (canonical) map defined by

$$\phi : A_Y \to A_X \qquad f \mapsto f \circ \psi$$

is injective. It is clear that ϕ is a ring monomorphism and so we may identify A_Y as a subring of A_X.

- Define now $e : X \to K$ the following way

$$e(x,i) := \begin{cases} 1 & i = 0 \\ 0 & i = 1 \end{cases}$$

Let $k \in K$, then if $k \notin \{0,1\}$ we have $e^{-1}(k) = \emptyset$ which is clearly open in X. For $k \in \{0,1\}$ we have

$$e^{-1}(0) = \{(x,i) \,:\, x \in \beta\mathbb{N},\ i = 1\}$$
$$e^{-1}(1) = \{(x,i) \,:\, x \in \beta\mathbb{N},\ i = 0\}$$

which are also open in X. Thus $e \in A_X$.

- We claim that $A_Y[e] = A_X$
Let $f \in A_X$, then define $g_1 : X \to K$ by

$$g_1(x,i) := f(x,0) \qquad x \in \beta\mathbb{N}, i = 0,1$$

and $g_2 : X \to K$ by

$$g_2(x,i) := f(x,1) \qquad x \in \beta\mathbb{N}, i = 0,1$$

First we claim that g_1 and g_2 are in A_Y, but this is clear since

$$g_j(x,0) = g_j(x,1) \qquad \forall x \in \beta\mathbb{N}, j = 1,2$$

We can then easily check that $f = g_1 e + g_2(1-e)$ and thus conclude that $f \in A_Y[e]$.

Because X is extremally disconnected and Y is not, we know that A_X is Baer and A_Y is not Baer. Now define $e' : X \to K$ the following way (we may choose $x = 7$ or any value in $\mathbb{N} \subset \beta\mathbb{N}$)

$$e'(x,i) := \begin{cases} 1 & x = 7 \text{ and } i = 0 \\ 0 & \text{else} \end{cases}$$

Then $e' \in A_Y$ because $\psi(7,0) \in Y$ is an isolated point. Finally, we also see that $A_X = A_Y[e]$ is a rational extension of A_Y, since $e'e \in A_Y \backslash \{0\}$. Since we are dealing with von Neumann regular rings A_X and A_Y, we know that these rings are integrally closed in their respective total quotients. Because of the preceding Theorem we also know that $A_X[\sqrt{-1}]$ is a total integral closure of A_Y (since A_X is Baer). We also note that e is not in $A_Y[\sqrt{-1}]$ and so the result of the Theorem above does not hold if we remove the condition that the ring should be Baer.

References

[1] **E. Artin**, *Über die Zerlegung definiter Funktionen in Quadrate*, Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg, Vol. 5, No. 1, p. 100-115.

[2] **E. Artin, O. Schreier**, *Eine Kennzeichnung der reell abgeschlossenen Körper*, Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg, Vol. 5, No. 1, p. 225-231.

[3] **M. Bhargava, M. Satriano**, *On a notion of "Galois closure" for extensions of rings*, Journal of European Mathematical Society 2014, Vol. 16, No. 9, p. 1881-1913

[4] **J. Bochnak, M. Coste, M-F. Roy**, *Real Algebraic Geometry*, Springer-Verlag 1998

[5] **W. Borho, V. Weber**, *Zur Existenz total ganz abgeschlossener Ringerweiterungen*, Math. Zeitschrift 1971, Vol. 121, p. 55-57.

[6] **G.W. Brumfiel**, *Partially Ordered Rings and Semi-Algebraic Geometry*, London Math. Soc. Lecture Note Series **37**, Cambridge University Press 1979

[7] **J. Capco**, *Real Closed $*$ Rings*, PhD Dissertation, October 2010, Universität Passau, Germany. Online: http://www.opus-bayern.de/uni-passau/volltexte/2012/2591. Accessed: 05.2012.

[8] **J. Capco**, *Real closed rings and real closed $*$ rings*, Journal of Pure and Applied Algebra 2014, Vol. 218, No. 1, p.101-111

[9] **S.U. Chase, D.K. Harrison and A. Rosenberg**, *Galois theory and Galois cohomology of commutative rings*, Memoirs of the American Mathematical Society 1964, No. 52.

[10] **P.M. Cohn**, *Algebra - Volume 2*, John Wiley & Sons Ltd. 1977.

[11] **M.C. Desrochers, R.M. Raphael**, *Galois Theory and Algebraic Extensions of Commutative Semiprime Rings*, Communications in Algebra 1990, Vol. 18, No. 3, p. 927-946

[12] **E. Enochs**, *Totally Integrally Closed Rings*, Proceedings of the American Mathematical Society 1968, Vol. 19, No. 3, p.701-706.

[13] **N.J. Fine, L. Gillman, J. Lambek**, *Rings of Quotients of Rings of Functions*, Editors: M. Barr, R. Raphael. Transcribed and edited into PDF from the original 1966 McGill University Press book. Online: http://at.yorku.ca/i/a/a/l/94.htm. Accessed: 10.2007

[14] **A. Gioia**, *On the Galois closure of commutative algebras*, PhD Thesis, Sept. 2013, Mathematical Institute Leiden, Faculty of Science, Leiden University. Online: http://hdl.handle.net/1887/21633. Accessed: 12.2017.

[15] **M. Hochster**, *Totally Integrally Closed Rings and Extremal Spaces*, Pacific Journal of Mathematics 1970, Vol. 32, No. 3, p.767-779.

[16] **J.A. Huckaba**, *Commutative rings with zero divisors*, Marcel Dekker 1988

[17] **M. Knebusch, C. Scheiderer**, *Einführung in die reelle Algebra*, Vieweg 1989. Also available Online: epub.uni-regensburg.de/id/eprint/12857/contents. Accessed: 01.2013

[18] **T.Y. Lam**, *Introduction to Quadratic Forms over Fields*, Graduate Studies in Mathematics, Vol. 67, American Mathematical Society 2005.

[19] **S. Lang**, Algebra, Graduate Texts in Mathematics, Vol. 211 (Revised 3rd. ed.), Springer-Verlag 2002

[20] **A.C. Mewborn**, *Regular Rings and Baer Rings*, Math. Z. 1971, Vol. 121, p. 211-219

[21] **A.C. Mewborn**, *Some Conditions on Commutative Semiprime Rings*, Journal of Algebra 1969, Vol. 13, p. 422-431

[22] **R.M. Raphael**, *Algebraic Extensions of Commutative Regular Rings*, Canad. J. Math. 1970, Vol. 22, p. 1133-1155

[23] **R. Raphael**, *On Algebraic Closures*, Publicacions Matemàtiques 1992, Vol. 36, p. 913-924

[24] **N. Sankaran, K. Varadarajan**, *Formally real rings and their real closures*, Acta Math. Hungarica 1996, Vol. 19, p. 101-120

[25] **H.H. Storrer**, *Epimorphismen von kommutativen Ringen*, Comm. Math. Helvetici 1968, Vol. 43, p. 378-401

[26] **N. Schwartz, M. Tressl**, *Elementary properties of minimal and maximal points in Zariski spectra*, Journal of Algebra 2010, Vol. 323, p. 698-728

[27] **Zhizhong Dai**, *On Real Closed Rings*, Algebra Colloquium 2001, Vol. 8:2, p. 237-240