

Leipzig, April 25, 2016

Loops and Legs in Quantum Field Theory (LL2016)

Algorithms to solve coupled differential systems in terms of power series

Carsten Schneider

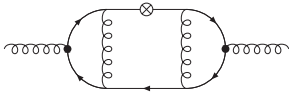
Research Institute for Symbolic Computation (RISC)
Johannes Kepler University Linz

joint with A. Behring, J. Blümlein, A. De Freitas (DESY)
and J. Ablinger (RISC)

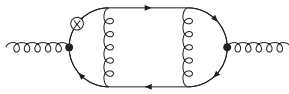


Completed project: 3-loop massive ladder and V-diagrams

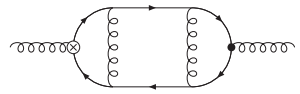
Ablinger, Behring, Blümlein, De Freitas, von Manteuffel, CS; Comput. Phys. Comm. 202 [arXiv:1509.08324 [hep-ph]]



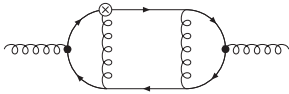
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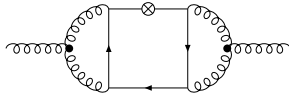
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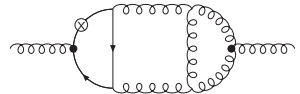
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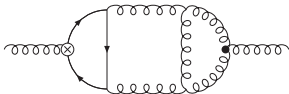
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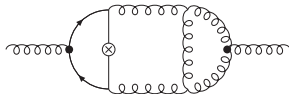
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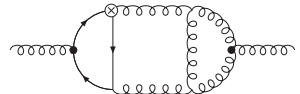
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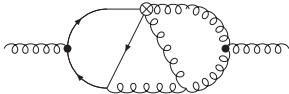
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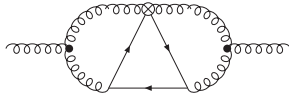
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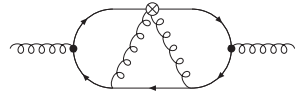
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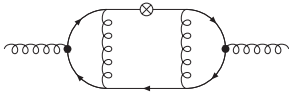


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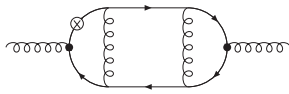
All diagrams are produced with axodraw (J. Vermaseren)

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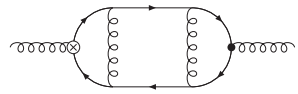
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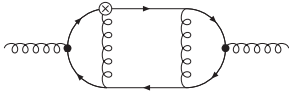
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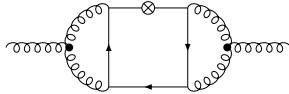
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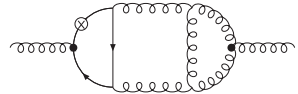
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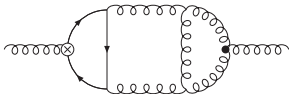
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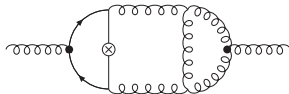
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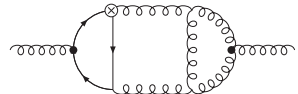
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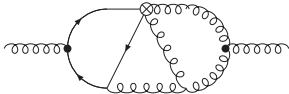
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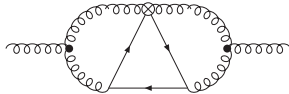
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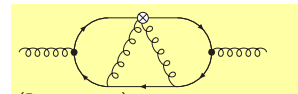
9



10



11



(5 propagators)

12

All diagrams are produced with axodraw (J. Vermaseren)

Used technologies (and mathematical software)

1. simplification of multi-sums (Sigma.m, EvaluateMultiSums.m)
2. computing recurrences for multi-integrals (Ablinger's MutliIntegrate.m package)
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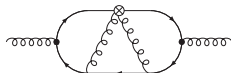
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Example:



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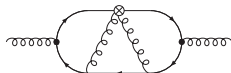
expression in terms of

harmonic sums (Vermaseren, 1998; Blümlein, Kurth, 1998)

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nested binomial sums (Ablinger, Blümlein, Raab, CS 2014)

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Goal of this talk: present the underlying algorithm and elaborate new ideas

Problem 1: Solving recurrences

Solving recurrences

$$\begin{aligned} \ln[1]:= \text{rec} &= -2(N+1)(N+2)^2 I[N] - (N+2)(-6N^2 - 28N - 32) I[N+1] \\ &+ (-6N^3 - 50N^2 - 136N - 120) I[N+2] \\ &- (-N-2)(N+4)(2N+8) I[N+3] == -\frac{4(N+2)}{3(N+3)}; \end{aligned}$$

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$$\begin{aligned} \text{In}[1]:= \text{rec} = & -2(N+1)(N+2)^2 I[N] - (N+2)(-6N^2 - 28N - 32) I[N+1] \\ & + (-6N^3 - 50N^2 - 136N - 120) I[N+2] \\ & - (-N-2)(N+4)(2N+8) I[N+3] == -\frac{4(N+2)}{3(N+3)}; \end{aligned}$$

In[2]:= << Sigma.m

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[3]:= recSol = SolveRecurrence[rec, I[N]]

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Solving recurrences

$$\begin{aligned} \text{In}[1]:= \text{rec} = & -2(N+1)(N+2)^2 I[N] - (N+2)(-6N^2 - 28N - 32) I[N+1] \\ & + (-6N^3 - 50N^2 - 136N - 120) I[N+2] \\ & - (-N-2)(N+4)(2N+8) I[N+3] == -\frac{4(N+2)}{3(N+3)}; \end{aligned}$$

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$$\begin{aligned} \text{General solution: } & \left\{ c_1 \frac{1}{-N-1} + c_2 \frac{-N}{N+1} + c_3 \left(\frac{1}{(N+1)^2} + \frac{1}{N+1} \right) \sum_{i=1}^N \frac{1}{i} \right. \\ & \left. + \frac{2(N^2+N-1)}{3(N+1)^2} - \frac{2(N+2)}{3(N+1)} \sum_{i=1}^N \frac{1}{i} \mid c_1, c_2, c_3 \in \mathbb{R} \right\} \end{aligned}$$

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$$\begin{aligned} \text{In}[1]:= \text{rec} = & -2(N+1)(N+2)^2 I[N] - (N+2)(-6N^2 - 28N - 32) I[N+1] \\ & + (-6N^3 - 50N^2 - 136N - 120) I[N+2] \\ & - (-N-2)(N+4)(2N+8) I[N+3] == -\frac{4(N+2)}{3(N+3)}; \end{aligned}$$

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Solving recurrences

$$\begin{aligned} \text{In}[1]:= \text{rec} = & -2(N+1)(N+2)^2 I[N] - (N+2)(-6N^2 - 28N - 32) I[N+1] \\ & + (-6N^3 - 50N^2 - 136N - 120) I[N+2] \\ & - (-N-2)(N+4)(2N+8) I[N+3] == -\frac{4(N+2)}{3(N+3)}; \end{aligned}$$

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$$I(1) = 5, I(2) = \frac{130}{27}, I(3) = \frac{169}{36}$$

$$\Downarrow c_1 = -\frac{49}{9}, c_2 = -\frac{41}{9}, c_3 = -\frac{2}{3}$$

$$I(N) = \frac{59N^2+120N+49}{9(N+1)^2} - \frac{2(N+3)}{3(N+1)} \sum_{i=1}^N \frac{1}{i}$$

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$$\begin{aligned} \text{In}[1]:= \text{rec} = & -2(N+1)(N+2)^2 I[N] - (N+2)(-6N^2 - 28N - 32) I[N+1] \\ & + (-6N^3 - 50N^2 - 136N - 120) I[N+2] \\ & - (-N-2)(N+4)(2N+8) I[N+3] == -\frac{4(N+2)}{3(N+3)}; \end{aligned}$$

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In[4]:= sol = FindLinearCombination[recSol, {1, {5, $\frac{130}{27}$, $\frac{169}{36}$ }}}, N, 3]

$$\text{Out}[4]= \frac{59N^2 + 120N + 49}{9(N+1)^2} - \frac{2(N+3)}{3(N+1)} \sum_{i=1}^N \frac{1}{i}$$

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In[5]:= << **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[6]:= **sol = ReduceToBasis[TransformToSSums[sol], Dynamic → Automatic]**

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Sigma - A summation package by Carsten Schneider © RISC-Linz

In[3]:= **recSol** = **SolveRecurrence**[**rec**, **I**[**N**]]

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$$\text{Out}[6]= \frac{59N^2 + 120N + 49}{9(N+1)^2} - \frac{2(N+3)}{3(N+1)} S_1[N]$$

Theorem 1. Given $\langle I(N) \rangle_{N \geq 0}$ specified by the initial values $I(0), \dots, I(d-1)$ and the recurrence

$$a_0(N)I(N) + a_1(N)I(N+1) + \dots + a_d(N)I(N+d) = r(N)$$

$a_0(N), \dots, a_d(N)$: polynomials in N ;

$r(N)$: expression in terms of nested sums and products

Sigma computes, if possible, a representation of $I(N)$ in terms of nested sums and products.

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Selected articles that make this machinery possible:

1. M. Karr. *J. ACM*, 28:305–350, 1981.
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17. S.A. Abramov, M. Bronstein, M. Petkovšek, CS, in preparation

Problem 2: Computing the ε -expansion from a recurrence relation

Computing the ε -expansion from a recurrence relation

$$I(1) = \frac{5}{\varepsilon^3} - \frac{163}{12\varepsilon^2} + O(\varepsilon^{-1}), \quad I(2) = \frac{130}{27\varepsilon^3} - \frac{695}{54\varepsilon^2} + O(\varepsilon^{-1}), \quad I(3) = \frac{169}{36\varepsilon^3} - \frac{395}{32\varepsilon^2} + O(\varepsilon^{-1})$$

+

$$\begin{aligned} & - 2(N+1)(N+2)(2+\varepsilon+N)I(N) \\ & \quad - (N+2)(-32-7\varepsilon+2\varepsilon^2-28N-5\varepsilon N-6N^2)I(N+1) \\ & - (120+3\varepsilon-14\varepsilon^2-\varepsilon^3+136N+13\varepsilon N-4\varepsilon^2 N+50N^2+4\varepsilon N^2+6N^3)I(N+2) \\ & \quad + (2-\varepsilon+N)(4+\varepsilon+N)(8+\varepsilon+2N)I(N+3) \\ & = \frac{1}{\varepsilon^3} \frac{-4(N+2)}{3(N+3)} + \frac{1}{\varepsilon^2} \left[-\frac{2(2N+7)S_1}{3(N+3)} - \frac{2(4N^4+35N^3+101N^2+105N+25)}{3(N+1)(N+2)(N+3)^2} \right] + O(\varepsilon^{-1}) \end{aligned}$$

Computing the ε -expansion from a recurrence relation

$$I(1) = \frac{5}{\varepsilon^3} - \frac{163}{12\varepsilon^2} + O(\varepsilon^{-1}), \quad I(2) = \frac{130}{27\varepsilon^3} - \frac{695}{54\varepsilon^2} + O(\varepsilon^{-1}), \quad I(3) = \frac{169}{36\varepsilon^3} - \frac{395}{32\varepsilon^2} + O(\varepsilon^{-1})$$

+

$$\begin{aligned} & - 2(N+1)(N+2)(2+\varepsilon+N)I(N) \\ & \quad - (N+2)(-32-7\varepsilon+2\varepsilon^2-28N-5\varepsilon N-6N^2)I(N+1) \\ & - (120+3\varepsilon-14\varepsilon^2-\varepsilon^3+136N+13\varepsilon N-4\varepsilon^2 N+50N^2+4\varepsilon N^2+6N^3)I(N+2) \\ & \quad + (2-\varepsilon+N)(4+\varepsilon+N)(8+\varepsilon+2N)I(N+3) \\ & = \frac{1}{\varepsilon^3} \frac{-4(N+2)}{3(N+3)} + \frac{1}{\varepsilon^2} \left[-\frac{2(2N+7)S_1}{3(N+3)} - \frac{2(4N^4+35N^3+101N^2+105N+25)}{3(N+1)(N+2)(N+3)^2} \right] + O(\varepsilon^{-1}) \end{aligned}$$

↓

$$\begin{aligned} I(N) &= \varepsilon^{-3} \left[\frac{59N^2+120N+49}{9(N+1)^2} - \frac{2(N+3)}{3(N+1)} S_1(N) \right] \\ &+ \varepsilon^{-2} \left[\frac{-2(20N^3+58N^2+57N+22)}{3(N+1)^3} + \frac{2(N+2)(2N-1)S_1(N)}{3(N+1)^2} - \frac{S_1(N)^2}{N+1} - \frac{S_2(N)}{N+1} \right] + O(\varepsilon^{-1}) \end{aligned}$$

Computing the ε -expansion from a recurrence relation

$$\begin{aligned} & a_0(\varepsilon, N) [I(N)] \\ & + a_1(\varepsilon, N) [I(N + 1)] \\ & + \\ & \vdots \\ & + a_d(\varepsilon, N) [I(N + d)] \end{aligned}$$

$= h_{-3}(N)\varepsilon^{-3} + h_{-2}(N)\varepsilon^{-2} + h_{-1}(N)\varepsilon^{-1} + \dots$

given (in terms of indefinite nested sums and products)

Computing the ε -expansion from a recurrence relation

$$\begin{aligned} & a_0(\varepsilon, N) \left[I_{-3}(N)\varepsilon^{-3} + I_{-2}(N)\varepsilon^{-2} + I_{-1}(N)\varepsilon^{-1} + \dots \right] \\ & + a_1(\varepsilon, N) \left[I(N+1) \right] \\ & + \\ & \vdots \\ & + a_d(\varepsilon, N) \left[I(N+d) \right] \\ & \qquad = h_{-3}(N)\varepsilon^{-3} + h_{-2}(N)\varepsilon^{-2} + h_{-1}(N)\varepsilon^{-1} + \dots \end{aligned}$$

given (in terms of indefinite nested sums and products)

Computing the ε -expansion from a recurrence relation

$$\begin{aligned} & a_0(\varepsilon, N) \left[I_{-3}(N)\varepsilon^{-3} + I_{-2}(N)\varepsilon^{-2} + I_{-1}(N)\varepsilon^{-1} + \dots \right] \\ & + a_1(\varepsilon, N) \left[I_{-3}(N+1)\varepsilon^{-3} + I_{-2}(N+1)\varepsilon^{-2} + I_{-1}(N+1)\varepsilon^{-1} + \dots \right] \\ & + \\ & \vdots \\ & + a_d(\varepsilon, N) \left[I(N+d) \right] \\ & \qquad \qquad \qquad = h_{-3}(N)\varepsilon^{-3} + h_{-2}(N)\varepsilon^{-2} + h_{-1}(N)\varepsilon^{-1} + \dots \end{aligned}$$

given (in terms of indefinite nested sums and products)

Computing the ε -expansion from a recurrence relation

$$\begin{aligned} & a_0(\varepsilon, N) \left[I_{-3}(N)\varepsilon^{-3} + I_{-2}(N)\varepsilon^{-2} + I_{-1}(N)\varepsilon^{-1} + \dots \right] \\ & + a_1(\varepsilon, N) \left[I_{-3}(N+1)\varepsilon^{-3} + I_{-2}(N+1)\varepsilon^{-2} + I_{-1}(N+1)\varepsilon^{-1} + \dots \right] \\ & + \\ & \vdots \\ & + a_d(\varepsilon, N) \left[I_{-3}(N+d)\varepsilon^{-3} + I_{-2}(N+d)\varepsilon^{-2} + I_{-1}(N+d)\varepsilon^{-1} + \dots \right] \\ & \qquad \qquad \qquad = h_{-3}(N)\varepsilon^{-3} + h_{-2}(N)\varepsilon^{-2} + h_{-1}(N)\varepsilon^{-1} + \dots \end{aligned}$$

given (in terms of indefinite nested sums and products)

Computing the ε -expansion from a recurrence relation

$$\begin{aligned} & a_0(\varepsilon, N) \left[I_{-3}(N)\varepsilon^{-3} + I_{-2}(N)\varepsilon^{-2} + I_{-1}(N)\varepsilon^{-1} + \dots \right] \\ & + a_1(\varepsilon, N) \left[I_{-3}(N+1)\varepsilon^{-3} + I_{-2}(N+1)\varepsilon^{-2} + I_{-1}(N+1)\varepsilon^{-1} + \dots \right] \\ & + \\ & \vdots \\ & + a_d(\varepsilon, N) \left[I_{-3}(N+d)\varepsilon^{-3} + I_{-2}(N+d)\varepsilon^{-2} + I_{-1}(N+d)\varepsilon^{-1} + \dots \right] \\ & \qquad \qquad \qquad = h_{-3}(N)\varepsilon^{-3} + h_{-2}(N)\varepsilon^{-2} + h_{-1}(N)\varepsilon^{-1} + \dots \end{aligned}$$

↓ lowest terms must agree

$$a_0(0, N)I_{-3}(N) + a_1(0, N)I_{-3}(N+1) + \dots + a_d(0, N)I_{-3}(N+d) = h_{-3}(N)$$

Computing the ε -expansion from a recurrence relation

$$\begin{aligned} & a_0(\varepsilon, N) \left[I_{-3}(N)\varepsilon^{-3} + I_{-2}(N)\varepsilon^{-2} + I_{-1}(N)\varepsilon^{-1} + \dots \right] \\ & + a_1(\varepsilon, N) \left[I_{-3}(N+1)\varepsilon^{-3} + I_{-2}(N+1)\varepsilon^{-2} + I_{-1}(N+1)\varepsilon^{-1} + \dots \right] \\ & + \\ & \vdots \\ & + a_d(\varepsilon, N) \left[I_{-3}(N+d)\varepsilon^{-3} + I_{-2}(N+d)\varepsilon^{-2} + I_{-1}(N+d)\varepsilon^{-1} + \dots \right] \\ & \qquad \qquad \qquad = h_{-3}(N)\varepsilon^{-3} + h_{-2}(N)\varepsilon^{-2} + h_{-1}(N)\varepsilon^{-1} + \dots \end{aligned}$$

↓ lowest terms must agree

$$a_0(0, N)I_{-3}(N) + a_1(0, N)I_{-3}(N+1) + \dots + a_d(0, N)I_{-3}(N+d) = h_{-3}(N)$$

REC solver: Given the initial values $I_{-3}(1), I_{-3}(2), \dots, I_{-3}(d)$,
decide if $I_{-3}(N)$ can be written in terms of indefinite
nested sums and products.

Computing the ε -expansion from a recurrence relation

$$I(1) = \frac{5}{\varepsilon^3} - \frac{163}{12\varepsilon^2} + O(\varepsilon^{-1}), \quad I(2) = \frac{130}{27\varepsilon^3} - \frac{695}{54\varepsilon^2} + O(\varepsilon^{-1}), \quad I(3) = \frac{169}{36\varepsilon^3} - \frac{395}{32\varepsilon^2} + O(\varepsilon^{-1})$$

+

$$\begin{aligned} & - 2(N+1)(N+2)(2+\varepsilon+N)I(N) \\ & \quad - (N+2)(-32-7\varepsilon+2\varepsilon^2-28N-5\varepsilon N-6N^2)I(N+1) \\ & - (120+3\varepsilon-14\varepsilon^2-\varepsilon^3+136N+13\varepsilon N-4\varepsilon^2 N+50N^2+4\varepsilon N^2+6N^3)I(N+2) \\ & \quad + (2-\varepsilon+N)(4+\varepsilon+N)(8+\varepsilon+2N)I(N+3) \\ & = \frac{1}{\varepsilon^3} \frac{-4(N+2)}{3(N+3)} + \frac{1}{\varepsilon^2} \left[-\frac{2(2N+7)S_1}{3(N+3)} - \frac{2(4N^4+35N^3+101N^2+105N+25)}{3(N+1)(N+2)(N+3)^2} \right] + O(\varepsilon^{-1}) \end{aligned}$$

Solving recurrences

$$\begin{aligned} \text{In}[1]:= \text{rec} = & -2(N+1)(N+2)^2 I[N] - (N+2)(-6N^2 - 28N - 32) I[N+1] \\ & + (-6N^3 - 50N^2 - 136N - 120) I[N+2] \\ & - (-N-2)(N+4)(2N+8) I[N+3] == -\frac{4(N+2)}{3(N+3)}; \end{aligned}$$

In[2]:= << Sigma.m

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[3]:= recSol = SolveRecurrence[rec, I[N]]

$$\text{Out}[3]=\left\{ \left\{ 0, \frac{-1}{N+1} \right\}, \left\{ 0, \frac{-N}{N+1} \right\}, \left\{ 0, \frac{1}{(N+1)^2} + \frac{1}{N+1} \sum_{i=1}^N \frac{1}{i} \right\}, \left\{ 1, \frac{2(N^2+N-1)}{3(N+1)^2} - \frac{2(N+2)}{3(N+1)} \sum_{i=1}^N \frac{1}{i} \right\} \right\}$$

$$\begin{aligned} \text{General solution: } & \left\{ c_1 \frac{1}{-N-1} + c_2 \frac{-N}{N+1} + c_3 \left(\frac{1}{(N+1)^2} + \frac{1}{N+1} \right) \sum_{i=1}^N \frac{1}{i} \right. \\ & \left. + \frac{2(N^2+N-1)}{3(N+1)^2} - \frac{2(N+2)}{3(N+1)} \sum_{i=1}^N \frac{1}{i} \mid c_1, c_2, c_3 \in \mathbb{R} \right\} \end{aligned}$$

$$I(1) = 5, I(2) = \frac{130}{27}, I(3) = \frac{169}{36}$$

$$\Downarrow c_1 = -\frac{49}{9}, c_2 = -\frac{41}{9}, c_3 = -\frac{2}{3}$$

$$I(N) = \frac{59N^2+120N+49}{9(N+1)^2} - \frac{2(N+3)}{3(N+1)} \sum_{i=1}^N \frac{1}{i}$$

Computing the ε -expansion from a recurrence relation

$$I(1) = \frac{5}{\varepsilon^3} - \frac{163}{12\varepsilon^2} + O(\varepsilon^{-1}), \quad I(2) = \frac{130}{27\varepsilon^3} - \frac{695}{54\varepsilon^2} + O(\varepsilon^{-1}), \quad I(3) = \frac{169}{36\varepsilon^3} - \frac{395}{32\varepsilon^2} + O(\varepsilon^{-1})$$

+

$$\begin{aligned} & - 2(N+1)(N+2)(2+\varepsilon+N)I(N) \\ & \quad - (N+2)(-32-7\varepsilon+2\varepsilon^2-28N-5\varepsilon N-6N^2)I(N+1) \\ & - (120+3\varepsilon-14\varepsilon^2-\varepsilon^3+136N+13\varepsilon N-4\varepsilon^2 N+50N^2+4\varepsilon N^2+6N^3)I(N+2) \\ & \quad + (2-\varepsilon+N)(4+\varepsilon+N)(8+\varepsilon+2N)I(N+3) \\ & = \frac{1}{\varepsilon^3} \frac{-4(N+2)}{3(N+3)} + \frac{1}{\varepsilon^2} \left[-\frac{2(2N+7)S_1}{3(N+3)} - \frac{2(4N^4+35N^3+101N^2+105N+25)}{3(N+1)(N+2)(N+3)^2} \right] + O(\varepsilon^{-1}) \end{aligned}$$

↓

$$I(N) = \varepsilon^{-3} \left[\frac{59N^2+120N+49}{9(N+1)^2} - \frac{2(N+3)}{3(N+1)} S_1(N) \right]$$

Computing the ε -expansion from a recurrence relation

$$\begin{aligned} & a_0(\varepsilon, N) \left[I_{-3}(N)\varepsilon^{-3} + I_{-2}(N)\varepsilon^{-2} + I_{-1}(N)\varepsilon^{-1} + \dots \right] \\ & + a_1(\varepsilon, N) \left[I_{-3}(N+1)\varepsilon^{-3} + I_{-2}(N+1)\varepsilon^{-2} + I_{-1}(N+1)\varepsilon^{-1} + \dots \right] \\ & + \\ & \vdots \\ & + a_d(\varepsilon, N) \left[I_{-3}(N+d)\varepsilon^{-3} + I_{-2}(N+d)\varepsilon^{-2} + I_{-1}(N+d)\varepsilon^{-1} + \dots \right] \\ & \qquad \qquad \qquad = h_{-3}(N)\varepsilon^{-3} + h_{-2}(N)\varepsilon^{-2} + h_{-1}(N)\varepsilon^{-1} + \dots \end{aligned}$$

↓ lowest terms must agree

$$a_0(0, N)I_{-3}(N) + a_1(0, N)I_{-3}(N+1) + \dots + a_d(0, N)I_{-3}(N+d) = h_{-3}(N)$$

REC solver: Given the initial values $I_{-3}(1), I_{-3}(2), \dots, I_{-3}(d)$,
decide if $I_{-3}(N)$ can be written in terms of indefinite nested sums and products.

Computing the ε -expansion from a recurrence relation

$$\begin{aligned} & a_0(\varepsilon, N) \left[I_{-3}(N)\varepsilon^{-3} + I_{-2}(N)\varepsilon^{-2} + I_{-1}(N)\varepsilon^{-1} + \dots \right] \\ & + a_1(\varepsilon, N) \left[I_{-3}(N+1)\varepsilon^{-3} + I_{-2}(N+1)\varepsilon^{-2} + I_{-1}(N+1)\varepsilon^{-1} + \dots \right] \\ & + \\ & \vdots \\ & + a_d(\varepsilon, N) \left[I_{-3}(N+d)\varepsilon^{-3} + I_{-2}(N+d)\varepsilon^{-2} + I_{-1}(N+d)\varepsilon^{-1} + \dots \right] \\ & \qquad \qquad \qquad = h_{-3}(N)\varepsilon^{-3} + h_{-2}(N)\varepsilon^{-2} + h_{-1}(N)\varepsilon^{-1} + \dots \end{aligned}$$

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$$a_0(0, N)I_{-3}(N) + a_1(0, N)I_{-3}(N+1) + \dots + a_d(0, N)I_{-3}(N+d) = h_{-3}(N)$$

Computing the ε -expansion from a recurrence relation

$$\begin{aligned} & a_0(\varepsilon, N) \left[I_{-3}(N)\varepsilon^{-3} + I_{-2}(N)\varepsilon^{-2} + I_{-1}(N)\varepsilon^{-1} + \dots \right] \\ & + a_1(\varepsilon, N) \left[I_{-3}(N+1)\varepsilon^{-3} + I_{-2}(N+1)\varepsilon^{-2} + I_{-1}(N+1)\varepsilon^{-1} + \dots \right] \\ & + \\ & \vdots \\ & + a_d(\varepsilon, N) \left[I_{-3}(N+d)\varepsilon^{-3} + I_{-2}(N+d)\varepsilon^{-2} + I_{-1}(N+d)\varepsilon^{-1} + \dots \right] \\ & \qquad \qquad \qquad = h_{-3}(N)\varepsilon^{-3} + h_{-2}(N)\varepsilon^{-2} + h_{-1}(N)\varepsilon^{-1} + \dots \end{aligned}$$

↓ lowest terms must agree

$$a_0(0, N)I_{-3}(N) + a_1(0, N)I_{-3}(N+1) + \dots + a_d(0, N)I_{-3}(N+d) = h_{-3}(N)$$

Computing the ε -expansion from a recurrence relation

$$\begin{aligned} & a_0(\varepsilon, N) \left[I_{-2}(N)\varepsilon^{-2} + I_{-1}(N)\varepsilon^{-1} + \dots \right] \\ & + a_1(\varepsilon, N) \left[I_{-2}(N+1)\varepsilon^{-2} + I_{-1}(N+1)\varepsilon^{-1} + \dots \right] \\ & + \\ & \vdots \\ & + a_d(\varepsilon, N) \left[I_{-2}(N+d)\varepsilon^{-2} + I_{-1}(N+d)\varepsilon^{-1} + \dots \right] \\ & \qquad \qquad \qquad = h'_{-3}(N)\varepsilon^{-3} + h'_{-2}(N)\varepsilon^{-2} + h'_{-1}(N)\varepsilon^{-1} + \dots \end{aligned}$$

Computing the ε -expansion from a recurrence relation

$$\begin{aligned} & a_0(\varepsilon, N) \left[I_{-2}(N)\varepsilon^{-2} + I_{-1}(N)\varepsilon^{-1} + \dots \right] \\ & + a_1(\varepsilon, N) \left[I_{-2}(N+1)\varepsilon^{-2} + I_{-1}(N+1)\varepsilon^{-1} + \dots \right] \\ & + \\ & \vdots \\ & + a_d(\varepsilon, N) \left[I_{-2}(N+d)\varepsilon^{-2} + I_{-1}(N+d)\varepsilon^{-1} + \dots \right] \\ & \qquad = \underbrace{h'_{-3}(N)}_{=0} \varepsilon^{-3} + h'_{-2}(N)\varepsilon^{-2} + h'_{-1}(N)\varepsilon^{-1} + \dots \end{aligned}$$

Computing the ε -expansion from a recurrence relation

$$\begin{aligned} & a_0(\varepsilon, N) \left[I_{-2}(N)\varepsilon^{-2} + I_{-1}(N)\varepsilon^{-1} + \dots \right] \\ & + a_1(\varepsilon, N) \left[I_{-2}(N+1)\varepsilon^{-2} + I_{-1}(N+1)\varepsilon^{-1} + \dots \right] \\ & + \\ & \vdots \\ & + a_d(\varepsilon, N) \left[I_{-2}(N+d)\varepsilon^{-2} + I_{-1}(N+d)\varepsilon^{-1} + \dots \right] \\ & \qquad \qquad \qquad = h'_{-2}(N)\varepsilon^{-2} + h'_{-1}(N)\varepsilon^{-1} + \dots \end{aligned}$$

Now repeat for $I_{-2}(N), I_{-1}(N), \dots$

Blümlein, Klein, CS, Stan, J. Symbol. Comput. 2012; arXiv:1011.2656[cs.SC]

Ablinger, Blümlein, Round, CS, LL2012, arXiv:1210.1685 [cs.SC]

Computing the ε -expansion from a recurrence relation

$$I(1) = \frac{5}{\varepsilon^3} - \frac{163}{12\varepsilon^2} + O(\varepsilon^{-1}), \quad I(2) = \frac{130}{27\varepsilon^3} - \frac{695}{54\varepsilon^2} + O(\varepsilon^{-1}), \quad I(3) = \frac{169}{36\varepsilon^3} - \frac{395}{32\varepsilon^2} + O(\varepsilon^{-1})$$

$$\begin{aligned} \ln[7]:= \text{recEp} &= -2(N+1)(N+2)(2+\varepsilon+N)I[N] \\ &\quad - (N+2)(-32-7\varepsilon+2\varepsilon^2-28N-5\varepsilon N-6N^2)I[N+1] \\ &\quad - (120+3\varepsilon-14\varepsilon^2-\varepsilon^3+136N+13\varepsilon N-4\varepsilon^2 N+50N^2+4\varepsilon N^2+6N^3)I[N+2] \\ &\quad + (2-\varepsilon+N)(4+\varepsilon+N)(8+\varepsilon+2N)I[N+3] \\ &== \frac{1}{\varepsilon^3} \frac{-4(N+2)}{3(N+3)} + \frac{1}{\varepsilon^2} \left[-\frac{2(2N+7)S_1}{3(N+3)} - \right. \\ &\quad \left. \frac{2(4N^4+35N^3+101N^2+105N+25)}{3(N+1)(N+2)(N+3)^2} \right] + O(\varepsilon^{-1}). \end{aligned}$$

Computing the ε -expansion from a recurrence relation

$$I(1) = \frac{5}{\varepsilon^3} - \frac{163}{12\varepsilon^2} + O(\varepsilon^{-1}), \quad I(2) = \frac{130}{27\varepsilon^3} - \frac{695}{54\varepsilon^2} + O(\varepsilon^{-1}), \quad I(3) = \frac{169}{36\varepsilon^3} - \frac{395}{32\varepsilon^2} + O(\varepsilon^{-1})$$

$$\begin{aligned} \text{In[7]:= recEp} &= -2(N+1)(N+2)(2+\varepsilon+N)I[N] \\ &\quad - (N+2)(-32-7\varepsilon+2\varepsilon^2-28N-5\varepsilon N-6N^2)I[N+1] \\ &\quad - (120+3\varepsilon-14\varepsilon^2-\varepsilon^3+136N+13\varepsilon N-4\varepsilon^2 N+50N^2+4\varepsilon N^2+6N^3)I[N+2] \\ &\quad + (2-\varepsilon+N)(4+\varepsilon+N)(8+\varepsilon+2N)I[N+3] \\ &== \frac{1}{\varepsilon^3} \frac{-4(N+2)}{3(N+3)} + \frac{1}{\varepsilon^2} \left[-\frac{2(2N+7)S_1}{3(N+3)} - \right. \\ &\quad \left. \frac{2(4N^4+35N^3+101N^2+105N+25)}{3(N+1)(N+2)(N+3)^2} \right] + O(\varepsilon^{-1}). \end{aligned}$$

$$\begin{aligned} \text{In[8]:= GenerateExpansion[recEp[[1]],} \\ &\quad \{\text{Coefficient[recEp[[2]], } \varepsilon^{-3}\}, \text{Coefficient[recEp[[2]], } \varepsilon^{-2}\}\}, I[N], \\ &\quad \{\varepsilon, -3, -2\}, \{\{5, \frac{130}{27}, \frac{169}{36}\}, \{-\frac{163}{12}, -\frac{695}{54}, -\frac{395}{32}\}\}, \text{MinInitialValue} \rightarrow 1] \end{aligned}$$

Computing the ε -expansion from a recurrence relation

$$I(1) = \frac{5}{\varepsilon^3} - \frac{163}{12\varepsilon^2} + O(\varepsilon^{-1}), \quad I(2) = \frac{130}{27\varepsilon^3} - \frac{695}{54\varepsilon^2} + O(\varepsilon^{-1}), \quad I(3) = \frac{169}{36\varepsilon^3} - \frac{395}{32\varepsilon^2} + O(\varepsilon^{-1})$$

$$\begin{aligned} \text{In}[7]:= \text{recEp} &= -2(N+1)(N+2)(2+\varepsilon+N)I[N] \\ &\quad - (N+2)(-32-7\varepsilon+2\varepsilon^2-28N-5\varepsilon N-6N^2)I[N+1] \\ &\quad - (120+3\varepsilon-14\varepsilon^2-\varepsilon^3+136N+13\varepsilon N-4\varepsilon^2 N+50N^2+4\varepsilon N^2+6N^3)I[N+2] \\ &\quad + (2-\varepsilon+N)(4+\varepsilon+N)(8+\varepsilon+2N)I[N+3] \\ &== \frac{1}{\varepsilon^3} \frac{-4(N+2)}{3(N+3)} + \frac{1}{\varepsilon^2} \left[-\frac{2(2N+7)S_1}{3(N+3)} - \right. \\ &\quad \left. \frac{2(4N^4+35N^3+101N^2+105N+25)}{3(N+1)(N+2)(N+3)^2} \right] + O(\varepsilon^{-1}). \end{aligned}$$

$$\begin{aligned} \text{In}[8]:= \text{GenerateExpansion}[\text{recEp}[[1]], \\ &\quad \{\text{Coefficient}[\text{recEp}[[2]], \varepsilon^{-3}], \text{Coefficient}[\text{recEp}[[2]], \varepsilon^{-2}]\}, I[N], \\ &\quad \{\varepsilon, -3, -2\}, \{\{5, \frac{130}{27}, \frac{169}{36}\}, \{-\frac{163}{12}, -\frac{695}{54}, -\frac{395}{32}\}\}, \text{MinInitialValue} \rightarrow 1] \end{aligned}$$

$$\begin{aligned} \text{Out}[8]= &\left\{ \frac{59N^2+120N+49}{9(N+1)^2} - \frac{2(N+3)S_1[N]}{3(N+1)}, \right. \\ &\left. -\frac{2(20N^3+58N^2+57N+22)}{3(N+1)^3} + \frac{2(N+2)(2N-1)S_1[N]}{3(N+1)^2} - \frac{S_1[N]^2}{N+1} - \frac{S_2[N]}{N+1} \right\} \end{aligned}$$

Theorem 2. Given $\langle I(N) \rangle_{N \geq 0}$ with

$$I(N) = I_{-3}(N)\varepsilon^{-3} + I_{-2}(N)\varepsilon^{-2} + \cdots + I_l(N)\varepsilon^l + O(\varepsilon^{l+1})$$

specified by the initial values $I(0), \dots, I(d-1)$ and the recurrence

$$\begin{aligned} a_0(\varepsilon, N)I(N) + a_1(\varepsilon, N)I(N+1) + \cdots + a_d(\varepsilon, N)I(N+d) \\ = \varepsilon^{-3}r_{-3}(N) + \varepsilon^{-2}r_{-2}(N) + \cdots + \varepsilon^l r_l(N) + O(\varepsilon^{l+1}) \end{aligned}$$

$a_0(\varepsilon, N), \dots, a_d(\varepsilon, N)$: polynomials in N and ε ;
 $r_i(N)$: nested sums and products.

Sigma computes, if possible, a representation of $I_{-3}(N), \dots, I_l(N)$ in terms of nested sums and products.

Theorem 2. Given $\langle I(N) \rangle_{N \geq 0}$ with

$$I(N) = I_{-3}(N)\varepsilon^{-3} + I_{-2}(N)\varepsilon^{-2} + \cdots + I_l(N)\varepsilon^l + O(\varepsilon^{l+1})$$

specified by the initial values $I(0), \dots, I(d-1)$ and the recurrence

$$\begin{aligned} a_0(\varepsilon, N)I(N) + a_1(\varepsilon, N)I(N+1) + \cdots + a_d(\varepsilon, N)I(N+d) \\ = \varepsilon^{-3}r_{-3}(N) + \varepsilon^{-2}r_{-2}(N) + \cdots + \varepsilon^l r_l(N) + O(\varepsilon^{l+1}) \end{aligned}$$

$a_0(\varepsilon, N), \dots, a_d(\varepsilon, N)$: polynomials in N and ε ;
 $r_i(N)$: nested sums and products.

Sigma computes, if possible, a representation of $I_{-3}(N), \dots, I_l(N)$ in terms of nested sums and products.

Problem 3: Solving a coupled recurrence system

Solving a coupled recurrence system

$$A_0 \begin{pmatrix} I_1(N) \\ I_2(N) \\ I_3(N) \end{pmatrix} + A_1 \begin{pmatrix} I_1(N+1) \\ I_2(N+1) \\ I_3(N+1) \end{pmatrix} = \begin{pmatrix} r_1(N) \\ r_2(N) \\ r_3(N) \end{pmatrix}$$

with

$$A_0 = \begin{pmatrix} N+1 & 0 & 0 \\ \varepsilon(3\varepsilon+2) & -2(3\varepsilon+1) & -2(-1+\varepsilon-2N) \\ -\varepsilon(3\varepsilon+2) & 2(3+3\varepsilon+2N) & 2(\varepsilon+1) \end{pmatrix},$$

$$A_1 = \begin{pmatrix} -2-\varepsilon-N & 2 & 0 \\ -2\varepsilon(3\varepsilon+2) & 2(5\varepsilon+2) & 4(-1+\varepsilon-N) \\ 0 & -2(4+\varepsilon+2N) & 0 \end{pmatrix}$$

and

$$r_1(N) = \left(-\frac{4(N+3)}{3(N+2)}\right)\varepsilon^{-3} + \left(\frac{2}{3} \frac{6N^3+29N^2+45N+21}{(N+1)(N+2)^2} - \frac{2(2N+3)S_1(N)}{3(N+2)}\right)\varepsilon^{-2} + O(\varepsilon^{-1})$$

$$r_2(N) = -\frac{8}{3}\varepsilon^{-3} + \left(\frac{4(3N+1)}{3(N+1)} - \frac{8S_1(N)}{3}\right)\varepsilon^{-2} + O(\varepsilon^{-1}),$$

$$r_3(N) = \frac{8}{3}\varepsilon^{-3} + \left(\frac{-4(3N+1)}{3(N+1)} + \frac{8S_1(N)}{3}\right)\varepsilon^{-2} + O(\varepsilon^{-1})$$

Solving a coupled recurrence system

$$A_0 \begin{pmatrix} I_1(N) \\ I_2(N) \\ I_3(N) \end{pmatrix} + A_1 \begin{pmatrix} I_1(N+1) \\ I_2(N+1) \\ I_3(N+1) \end{pmatrix} = \begin{pmatrix} r_1(N) \\ r_2(N) \\ r_3(N) \end{pmatrix}$$

uncoupling algorithms \downarrow Gerhold's OreSys.m (using Zürcher's algorithm)

$$\begin{aligned} a_0(\varepsilon, N)I_1(N) + a_1(\varepsilon, N)I_1(N+1) + a_2(\varepsilon, N)I_1(N+2) + a_3(\varepsilon, N)I_1(N+3) \\ = h_{-3}\varepsilon^{-3} + h_{-2}\varepsilon^{-2} + h_{-1}\varepsilon^{-1} + O(\varepsilon^0) \end{aligned}$$

$$I_2(N) = \text{LinearCombination}(I_1(N), I_1(N+1), I_1(N+2))$$

$$I_3(N) = \text{LinearCombination}(I_1(N), I_1(N+1), I_1(N+2))$$

Computing the ε -expansion from a recurrence relation

$$I(1) = \frac{5}{\varepsilon^3} - \frac{163}{12\varepsilon^2} + O(\varepsilon^{-1}), \quad I(2) = \frac{130}{27\varepsilon^3} - \frac{695}{54\varepsilon^2} + O(\varepsilon^{-1}), \quad I(3) = \frac{169}{36\varepsilon^3} - \frac{395}{32\varepsilon^2} + O(\varepsilon^{-1})$$

+

$$\begin{aligned} & - 2(N+1)(N+2)(2+\varepsilon+N)I(N) \\ & \quad - (N+2)(-32-7\varepsilon+2\varepsilon^2-28N-5\varepsilon N-6N^2)I(N+1) \\ & - (120+3\varepsilon-14\varepsilon^2-\varepsilon^3+136N+13\varepsilon N-4\varepsilon^2 N+50N^2+4\varepsilon N^2+6N^3)I(N+2) \\ & \quad + (2-\varepsilon+N)(4+\varepsilon+N)(8+\varepsilon+2N)I(N+3) \\ & = \frac{1}{\varepsilon^3} \frac{-4(N+2)}{3(N+3)} + \frac{1}{\varepsilon^2} \left[-\frac{2(2N+7)S_1}{3(N+3)} - \frac{2(4N^4+35N^3+101N^2+105N+25)}{3(N+1)(N+2)(N+3)^2} \right] + O(\varepsilon^{-1}) \end{aligned}$$

↓

$$\begin{aligned} I(N) &= \varepsilon^{-3} \left[\frac{59N^2+120N+49}{9(N+1)^2} - \frac{2(N+3)}{3(N+1)} S_1(N) \right] \\ &+ \varepsilon^{-2} \left[\frac{-2(20N^3+58N^2+57N+22)}{3(N+1)^3} + \frac{2(N+2)(2N-1)S_1(N)}{3(N+1)^2} - \frac{S_1(N)^2}{N+1} - \frac{S_2(N)}{N+1} \right] + O(\varepsilon^{-1}) \end{aligned}$$

Solving a coupled recurrence system

$$A_0 \begin{pmatrix} I_1(N) \\ I_2(N) \\ I_3(N) \end{pmatrix} + A_1 \begin{pmatrix} I_1(N+1) \\ I_2(N+1) \\ I_3(N+1) \end{pmatrix} = \begin{pmatrix} r_1(N) \\ r_2(N) \\ r_3(N) \end{pmatrix}$$

uncoupling algorithms \downarrow Gerhold's OreSys.m (using Zürcher's algorithm)

$$I_1(N) = \varepsilon^{-3} \left[\frac{59N^2 + 120N + 49}{9(N+1)^2} - \frac{2(N+3)}{3(N+1)} S_1(N) \right] \\ + \varepsilon^{-2} \left[\frac{-2(20N^3 + 58N^2 + 57N + 22)}{3(N+1)^3} + \frac{2(N+2)(2N-1)S_1(N)}{3(N+1)^2} - \frac{S_1(N)^2}{N+1} - \frac{S_2(N)}{N+1} \right] + O(\varepsilon^{-1})$$

\downarrow insert

$$I_2(N) = \text{LinearCombination}(I_1(N), I_1(N+1), I_1(N+2))$$

$$I_3(N) = \text{LinearCombination}(I_1(N), I_1(N+1), I_1(N+2))$$

Solving a coupled recurrence system

```
In[9]:= << OreSys.m
```

```
OreSys by Stefan Gerhold (optimized by C. Schneider) © RISC-Linz
```

```
In[10]:= << SolveCoupledSystem.m
```

```
SolveCoupledSystem by Carsten Schneider © RISC-Linz
```

```
In[11]:= coupledSys =
```

```
 $\mathbf{A}_0 \cdot \{I_1[N], I_2[N], I_3[N]\} + \mathbf{A}_1 \cdot \{I_1[N], I_2[N], I_3[N]\} - \{r_1[n], r_2[n], r_3[N]\};$ 
```

Solving a coupled recurrence system

```
In[9]:= << OreSys.m
```

```
OreSys by Stefan Gerhold (optimized by C. Schneider) © RISC-Linz
```

```
In[10]:= << SolveCoupledSystem.m
```

```
SolveCoupledSystem by Carsten Schneider © RISC-Linz
```

```
In[11]:= coupledSys =
```

```
A0.{I1[N], I2[N], I3[N]} + A1.{I1[N], I2[N], I3[N]} - {r1[n], r2[n], r3[N]};
```

```
In[12]:= SolveCoupledRecSystem[coupledSys, {I1[N], I2[N], I3[N]}, ε, -3,  
{-2, -2, -2}, {I1[N], 1, { $\frac{5}{\epsilon^3} - \frac{163}{12\epsilon^2}, \frac{130}{27\epsilon^3} - \frac{695}{54\epsilon^2}, \frac{169}{36\epsilon^3} - \frac{395}{32\epsilon^2}$ }}}]
```

Solving a coupled recurrence system

In[9]:= << OreSys.m

OreSys by Stefan Gerhold (optimized by C. Schneider) © RISC-Linz

In[10]:= << SolveCoupledSystem.m

SolveCoupledSystem by Carsten Schneider © RISC-Linz

In[11]:= coupledSys =

$\mathbf{A}_0 \cdot \{I_1[N], I_2[N], I_3[N]\} + \mathbf{A}_1 \cdot \{I_1[N], I_2[N], I_3[N]\} - \{r_1[n], r_2[n], r_3[N]\};$

In[12]:= SolveCoupledRecSystem[coupledSys, {I1[N], I2[N], I3[N]}, ε, -3,
{-2, -2, -2}, {I1[N], 1, { $\frac{5}{\epsilon^3} - \frac{163}{12\epsilon^2}, \frac{130}{27\epsilon^3} - \frac{695}{54\epsilon^2}, \frac{169}{36\epsilon^3} - \frac{395}{32\epsilon^2}$ }}}]

Out[12]=
$$\left\{ \frac{1}{\epsilon^3} \left(\frac{4(3N^2 + 6N + 4)}{3(N+1)^2} + \frac{4S_1[N]}{3(N+1)} \right) + \frac{1}{\epsilon^2} \left(-\frac{2(20N^3 + 58N^2 + 57N + 22)}{3(N+1)^3} + \frac{2(N+2)(2N-1)S_1[N]}{3(N+1)^2} - \frac{S_1[N]^2}{N+1} - \frac{S_2[N]}{N+1} \right), \right.$$
$$\left. \frac{4}{3\epsilon^3} - \frac{2}{\epsilon^2}, \frac{8}{3\epsilon^3} + \frac{1}{\epsilon^2} \left(-\frac{4(4N^2 + 7N + 2)}{3(N+1)^2} + \frac{4(N+2)S_1[N]}{3(N+1)} \right) \right\}$$

Theorem 3. Given $\langle I_1(N) \rangle_{N \geq 0}, \dots, \langle I_n(N) \rangle_{N \geq 0}$ with

$$I_k(N) = I_{k,-3}(N)\varepsilon^{-3} + \dots + I_{k,l}(N)\varepsilon^l + O(\varepsilon^{l+1}), \quad k = 1, \dots, n$$

specified by the initial values $I_k(0), \dots, I_k(\lambda)$ with $k = 1, \dots, n$ and the recurrence system

$$\begin{aligned} A_0 \begin{pmatrix} I_1(N) \\ \vdots \\ I_n(N) \end{pmatrix} + A_1 \begin{pmatrix} I_1(N+1) \\ \vdots \\ I_n(N+1) \end{pmatrix} + \dots + A_d \begin{pmatrix} I_1(N+d) \\ \vdots \\ I_n(N+d) \end{pmatrix} \\ = \begin{pmatrix} \varepsilon^{-3}r_{1,-3}(N) + \dots + \varepsilon^{l_1}r_{1,l_1}(N) + O(\varepsilon^{l_1+1}) \\ \vdots \\ \varepsilon^{-3}r_{n,-3}(N) + \dots + \varepsilon^{l_n}r_{n,l_n}(N) + O(\varepsilon^{l_n+1}) \end{pmatrix} \end{aligned}$$

A_0, \dots, A_d : $n \times n$ matrices with entries from $\mathbb{K}(\varepsilon, N)$;

$r_{i,j}(N)$: nested sums and products.

Sigma computes, if possible, a representation of $I_{k,-3}(N), \dots, I_{k,l}(N)$ in terms of nested sums and products.

Theorem 3. Given $\langle I_1(N) \rangle_{N \geq 0}, \dots, \langle I_n(N) \rangle_{N \geq 0}$ with

$$I_k(N) = I_{k,-3}(N)\varepsilon^{-3} + \dots + I_{k,l}(N)\varepsilon^l + O(\varepsilon^{l+1}), \quad k = 1, \dots, n$$

specified by the initial values $I_k(0), \dots, I_k(\lambda)$ with $k = 1, \dots, n$ and the recurrence system

$$\begin{aligned} A_0 \begin{pmatrix} I_1(N) \\ \vdots \\ I_n(N) \end{pmatrix} + A_1 \begin{pmatrix} I_1(N+1) \\ \vdots \\ I_n(N+1) \end{pmatrix} + \dots + A_d \begin{pmatrix} I_1(N+d) \\ \vdots \\ I_n(N+d) \end{pmatrix} \\ = \begin{pmatrix} \varepsilon^{-3}r_{1,-3}(N) + \dots + \varepsilon^{l_1}r_{1,l_1}(N) + O(\varepsilon^{l_1+1}) \\ \vdots \\ \varepsilon^{-3}r_{n,-3}(N) + \dots + \varepsilon^{l_n}r_{n,l_n}(N) + O(\varepsilon^{l_n+1}) \end{pmatrix} \end{aligned}$$

A_0, \dots, A_d : $n \times n$ matrices with entries from $\mathbb{K}(\varepsilon, N)$;

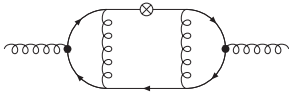
$r_{i,j}(N)$: nested sums and products.

Sigma computes, if possible, a representation of $I_{k,-3}(N), \dots, I_{k,l}(N)$ in terms of nested sums and products.

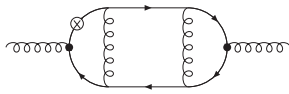
Problem 4: Solving a coupled differential system

Completed project: 3-loop massive ladder and V-diagrams

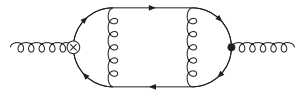
Ablinger, Behring, Blümlein, De Freitas, von Manteuffel, CS; Comput. Phys. Comm. 202 [arXiv:1509.08324 [hep-ph]]



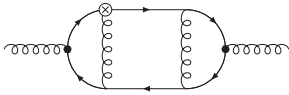
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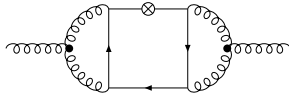
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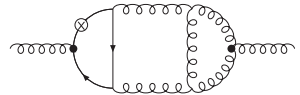
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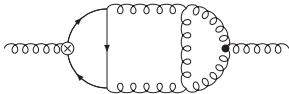
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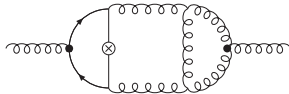
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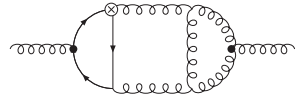
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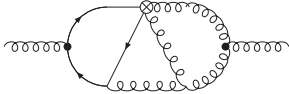
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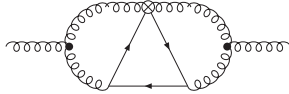
8



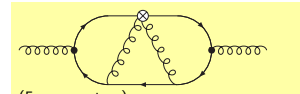
9



10



11



(5 propagators)

12

All diagrams are produced with axodraw (J. Vermaseren)

A coupled differential system for $\hat{I}_1(x)$, $\hat{I}_2(x)$, $\hat{I}_3(x)$

(produced by IBP [extension of REDUZE_2, A.v. Manteuffel])

$$D_x \begin{pmatrix} \hat{I}_1(x) \\ \hat{I}_2(x) \\ \hat{I}_3(x) \end{pmatrix} = \begin{pmatrix} -\frac{-1-\varepsilon+x}{(x-1)x} & -\frac{2}{(x-1)x} & 0 \\ \frac{\varepsilon(3\varepsilon+2)}{4(x-1)} & -\frac{-2-\varepsilon+3x+3\varepsilon x}{2(x-1)x} & -\frac{\varepsilon+1}{2(x-1)} \\ -\frac{\varepsilon(3\varepsilon+2)(x-2)}{4(x-1)x} & \frac{-2-5\varepsilon+x+3\varepsilon x}{2(x-1)x} & \frac{(-2\varepsilon-x+\varepsilon x)}{2(x-1)x} \end{pmatrix} \begin{pmatrix} \hat{I}_1(x) \\ \hat{I}_2(x) \\ \hat{I}_3(x) \end{pmatrix} + \begin{pmatrix} \hat{R}_1(x) \\ \hat{R}_2(x) \\ -\hat{R}_2(x) \end{pmatrix}$$

where

$$\hat{R}_1(x) = \frac{\hat{B}_4(x)}{(x-1)x},$$

$$\begin{aligned} \hat{R}_2(x) &= \frac{-(\varepsilon+2)^3}{16(\varepsilon+1)(x-1)x} \hat{B}_1(x) + \frac{(\varepsilon+2)(3\varepsilon+4)(19\varepsilon^2+36\varepsilon+16)}{16\varepsilon(5\varepsilon+6)(x-1)x} \hat{B}_2(x) \\ &+ \frac{(\varepsilon+1)^2(3\varepsilon+4)^2}{2\varepsilon(5\varepsilon+6)x} \hat{B}_3(x) + \frac{-24-50\varepsilon-25\varepsilon^2+8x+14\varepsilon x+6\varepsilon^2 x}{4(5\varepsilon+6)(x-1)x} \hat{B}_4(x) \end{aligned}$$

$\hat{B}_1(x)$, $\hat{B}_2(x)$, $\hat{B}_3(x)$ have been solved with symbolic summation.

Exploit their series representation

$$\hat{I}_1(x) = \sum_{N=0}^{\infty} I_1(N)x^N$$

$$\hat{I}_2(x) = \sum_{N=0}^{\infty} I_2(N)x^N$$

$$\hat{I}_3(x) = \sum_{N=0}^{\infty} I_3(N)x^N$$

DE system \rightarrow REC system

$$D_x \hat{I}_1(x) = - \frac{(-\varepsilon+x-1)}{(x-1)x} \hat{I}_1(x) - \frac{2}{(x-1)x} \hat{I}_2(x) \\ + \frac{1}{(x-1)x} \hat{B}_1(x) + \dots$$

$$D_x \hat{I}_2(x) = \frac{(3(\varepsilon+4)^2 - 22(\varepsilon+4) + 40)}{4(x-1)} \hat{I}_1(x) \\ + \frac{(-(\varepsilon+4)(3x-1) + 9x-2)}{2(x-1)x} \hat{I}_2(x) - \frac{(\varepsilon+1)}{2(x-1)} \hat{I}_3(x) \\ + \frac{1}{4} \frac{((\varepsilon+4)^2(6x-25) - 2(\varepsilon+4)(17x-75) + 48x-224)}{(5\varepsilon+6)(x-1)x} \hat{B}_1(x) + \dots$$

$$D_x \hat{I}_3(x) = - \frac{(3(\varepsilon+4)^2(x-2) - 22(\varepsilon+4)(x-2) + 40x-80)}{4(x-1)x} \hat{I}_1(x) \\ + \frac{((\varepsilon+4)(3x-5) - 11x+18)}{2(x-1)x} \hat{I}_2(x) - \frac{(-(\varepsilon+4)(x-2) + 5x-8)}{2(x-1)x} \hat{I}_3(x) \\ - \frac{1}{4} \frac{((\varepsilon+4)^2(6x-25) - 2(\varepsilon+4)(17x-75) + 48x-224)}{(5\varepsilon+6)(x-1)x} \hat{B}_1(x) + \dots$$

DE system \rightarrow REC system

$$D_x \hat{I}_1(x) = - \frac{(-\varepsilon+x-1)}{(x-1)x} \hat{I}_1(x) - \frac{2}{(x-1)x} \hat{I}_2(x) \\ + \frac{1}{(x-1)x} \hat{B}_1(x) + \dots$$

$$D_x \hat{I}_2(x) = \frac{(3(\varepsilon+4)^2 - 22(\varepsilon+4) + 40)}{4(x-1)} \hat{I}_1(x) \\ + \frac{(-(\varepsilon+4)(3x-1) + 9x-2)}{2(x-1)x} \hat{I}_2(x) - \frac{(\varepsilon+1)}{2(x-1)} \hat{I}_3(x) \\ + \frac{1}{4} \frac{((\varepsilon+4)^2(6x-25) - 2(\varepsilon+4)(17x-75) + 48x-224)}{(5\varepsilon+6)(x-1)x} \hat{B}_1(x) + \dots$$

$$D_x \hat{I}_3(x) = - \frac{(3(\varepsilon+4)^2(x-2) - 22(\varepsilon+4)(x-2) + 40x-80)}{4(x-1)x} \hat{I}_1(x) \\ + \frac{((\varepsilon+4)(3x-5) - 11x+18)}{2(x-1)x} \hat{I}_2(x) - \frac{(-(\varepsilon+4)(x-2) + 5x-8)}{2(x-1)x} \hat{I}_3(x) \\ - \frac{1}{4} \frac{((\varepsilon+4)^2(6x-25) - 2(\varepsilon+4)(17x-75) + 48x-224)}{(5\varepsilon+6)(x-1)x} \hat{B}_1(x) + \dots$$

DE system \rightarrow REC system

$$\begin{aligned}D_x \hat{I}_1(x) &= -\frac{(-\varepsilon+x-1)}{(x-1)x} \hat{I}_1(x) \\ &\quad - \frac{2}{(x-1)x} \hat{I}_2(x) \\ &\quad + \frac{1}{(x-1)x} \hat{B}_1(x) + \dots\end{aligned}$$

DE system \rightarrow REC system

$$\begin{aligned}(x-1)x D_x \hat{I}_1(x) &= -(-\varepsilon + x - 1) \hat{I}_1(x) \\ &\quad - 2 \hat{I}_2(x) \\ &\quad + \hat{B}_1(x) + \dots\end{aligned}$$

DE system \rightarrow REC system

$$\begin{aligned}(x-1)x D_x \sum_{N=0}^{\infty} I_1(N)x^N &= -(-\varepsilon + x - 1) \sum_{N=0}^{\infty} I_1(N)x^N \\ &\quad - 2 \sum_{N=0}^{\infty} I_2(N)x^N \\ &\quad + \sum_{N=0}^{\infty} B_1(N)x^N + \dots\end{aligned}$$

DE system \rightarrow REC system

$$\begin{aligned}(x-1)x \sum_{N=1}^{\infty} I_1(N)N x^{N-1} &= -(-\varepsilon + x - 1) \sum_{N=0}^{\infty} I_1(N)x^N \\ &\quad - 2 \sum_{N=0}^{\infty} I_2(N)x^N \\ &\quad + \sum_{N=0}^{\infty} B_1(N)x^N + \dots\end{aligned}$$

DE system \rightarrow REC system

$$\begin{aligned}(x-1)x \sum_{N=1}^{\infty} I_1(N)N x^{N-1} &= -(-\varepsilon + x - 1) \sum_{N=0}^{\infty} I_1(N)x^N \\ &\quad - 2 \sum_{N=0}^{\infty} I_2(N)x^N \\ &\quad + \sum_{N=0}^{\infty} B_1(N)x^N + \dots\end{aligned}$$

\downarrow N th coefficient

$$NI_1(N-1) - (\varepsilon + N + 1)I_1(N) + 2I_2(N) = B_1(N) + \dots$$

DE system \rightarrow REC system

$$NI_1(N-1) - (\varepsilon + N + 1)I_1(N) + 2I_2(N) \\ = B_1(N) + \dots$$

$$D_x \hat{I}_2(x) = \frac{(3(\varepsilon+4)^2 - 22(\varepsilon+4) + 40)}{4(x-1)} \hat{I}_1(x) \\ + \frac{(-(\varepsilon+4)(3x-1) + 9x-2)}{2(x-1)x} \hat{I}_2(x) - \frac{(\varepsilon+1)}{2(x-1)} \hat{I}_3(x) \\ + \frac{1}{4} \frac{((\varepsilon+4)^2(6x-25) - 2(\varepsilon+4)(17x-75) + 48x-224)}{(5\varepsilon+6)(x-1)x} \hat{B}_1(x) + \dots$$

$$D_x \hat{I}_3(x) = - \frac{(3(\varepsilon+4)^2(x-2) - 22(\varepsilon+4)(x-2) + 40x-80)}{4(x-1)x} \hat{I}_1(x) \\ + \frac{((\varepsilon+4)(3x-5) - 11x+18)}{2(x-1)x} \hat{I}_2(x) - \frac{(-(\varepsilon+4)(x-2) + 5x-8)}{2(x-1)x} \hat{I}_3(x) \\ - \frac{1}{4} \frac{((\varepsilon+4)^2(6x-25) - 2(\varepsilon+4)(17x-75) + 48x-224)}{(5\varepsilon+6)(x-1)x} \hat{B}_1(x) + \dots$$

A coupled system of difference equations

$$\begin{aligned} NI_1(N-1) - (\varepsilon + N + 1)I_1(N) + 2I_2(N) \\ = B_1(N) + \dots \end{aligned}$$

$$\begin{aligned} 2(\varepsilon + 2N + 2)I_2(N) - 2(3\varepsilon + 2N + 1)I_2(N-1) \\ + \varepsilon(3\varepsilon + 2)I_1(N-1) - 2(\varepsilon + 1)I_3(N-1) \\ = (5\varepsilon + 4)B_1(N) - \frac{2(\varepsilon + 1)(3\varepsilon + 4)}{5\varepsilon + 6}B_1(N-1) + \dots \end{aligned}$$

$$\begin{aligned} 4(\varepsilon - N)I_3(N) - 2\varepsilon(3\varepsilon + 2)I_1(N) + \varepsilon(3\varepsilon + 2)I_1(N-1) \\ - 2(3\varepsilon + 1)I_2(N-1) + 2(5\varepsilon + 2)I_2(N) \\ - 2(\varepsilon - 2N + 1)I_3(N-1) \\ = -\frac{2(\varepsilon + 1)(3\varepsilon + 4)}{5\varepsilon + 6}B_1(N-1) + (5\varepsilon + 4)B_1(N) + \dots \end{aligned}$$

A coupled system of difference equations

$$\begin{aligned} NI_1(N-1) - (\varepsilon + N + 1)I_1(N) + 2I_2(N) \\ = + \frac{4(N+2)}{3(N+1)}\varepsilon^{-3} + \left(\frac{2(2N+1)}{3(N+1)}S_1(N) - \frac{2(6N^2+13N+8)}{3(N+1)^2} \right)\varepsilon^{-2} + \dots \end{aligned}$$

$$\begin{aligned} 2(\varepsilon + 2N + 2)I_2(N) - 2(3\varepsilon + 2N + 1)I_2(N-1) \\ + \varepsilon(3\varepsilon + 2)I_1(N-1) - 2(\varepsilon + 1)I_3(N-1) \\ = \frac{8}{3}\varepsilon^{-3} + \left(\frac{4}{3}S_1(N)^2 - \frac{4(N+1)}{N}S_1(N) + \frac{4}{3}S_2(N) + \zeta_2 + 6 \right)\varepsilon^{-1} + \dots \end{aligned}$$

$$\begin{aligned} 4(\varepsilon - N)I_3(N) - 2\varepsilon(3\varepsilon + 2)I_1(N) + \varepsilon(3\varepsilon + 2)I_1(N-1) \\ - 2(3\varepsilon + 1)I_2(N-1) + 2(5\varepsilon + 2)I_2(N) \\ - 2(\varepsilon - 2N + 1)I_3(N-1) \\ = - \frac{8}{3}\varepsilon^{-3} - \left(\frac{8}{3}S_1(N) - 4 \right)\varepsilon^{-2} \\ - \left(\frac{4}{3}S_1(N)^2 - \frac{4(N+1)}{N}S_1(N) + \frac{4}{3}S_2(N) + \zeta_2 + 6 \right)\varepsilon^{-1} + \dots \end{aligned}$$

Solving a coupled recurrence system

$$A_0 \begin{pmatrix} I_1(N) \\ I_2(N) \\ I_3(N) \end{pmatrix} + A_1 \begin{pmatrix} I_1(N+1) \\ I_2(N+1) \\ I_3(N+1) \end{pmatrix} = \begin{pmatrix} r_1(N) \\ r_2(N) \\ r_3(N) \end{pmatrix}$$

with

$$A_0 = \begin{pmatrix} N+1 & 0 & 0 \\ \varepsilon(3\varepsilon+2) & -2(3\varepsilon+1) & -2(-1+\varepsilon-2N) \\ -\varepsilon(3\varepsilon+2) & 2(3+3\varepsilon+2N) & 2(\varepsilon+1) \end{pmatrix},$$

$$A_1 = \begin{pmatrix} -2-\varepsilon-N & 2 & 0 \\ -2\varepsilon(3\varepsilon+2) & 2(5\varepsilon+2) & 4(-1+\varepsilon-N) \\ 0 & -2(4+\varepsilon+2N) & 0 \end{pmatrix}$$

and

$$r_1(N) = \left(-\frac{4(N+3)}{3(N+2)}\varepsilon^{-3} + \left(\frac{2}{3} \frac{6N^3+29N^2+45N+21}{(N+1)(N+2)^2} - \frac{2(2N+3)S_1(N)}{3(N+2)}\right)\varepsilon^{-2} + O(\varepsilon^{-1})\right)$$

$$r_2(N) = -\frac{8}{3}\varepsilon^{-3} + \left(\frac{4(3N+1)}{3(N+1)} - \frac{8S_1(N)}{3}\right)\varepsilon^{-2} + O(\varepsilon^{-1}),$$

$$r_3(N) = \frac{8}{3}\varepsilon^{-3} + \left(\frac{-4(3N+1)}{3(N+1)} + \frac{8S_1(N)}{3}\right)\varepsilon^{-2} + O(\varepsilon^{-1})$$

Solving a coupled recurrence system

$$A_0 \begin{pmatrix} I_1(N) \\ I_2(N) \\ I_3(N) \end{pmatrix} + A_1 \begin{pmatrix} I_1(N+1) \\ I_2(N+1) \\ I_3(N+1) \end{pmatrix} = \begin{pmatrix} r_1(N) \\ r_2(N) \\ r_3(N) \end{pmatrix}$$

uncoupling algorithms \downarrow Gerhold's OreSys.m (using Zürcher's algorithm)

$$I_1(N) = \varepsilon^{-3} \left[\frac{59N^2 + 120N + 49}{9(N+1)^2} - \frac{2(N+3)}{3(N+1)} S_1(N) \right] \\ + \varepsilon^{-2} \left[\frac{-2(20N^3 + 58N^2 + 57N + 22)}{3(N+1)^3} + \frac{2(N+2)(2N-1)S_1(N)}{3(N+1)^2} - \frac{S_1(N)^2}{N+1} - \frac{S_2(N)}{N+1} \right] + O(\varepsilon^{-1})$$

\downarrow insert

$$I_2(N) = \text{LinearCombination}(I_1(N), I_1(N+1), I_1(N+2))$$

$$I_3(N) = \text{LinearCombination}(I_1(N), I_1(N+1), I_1(N+2))$$

Solving a coupled differential system

In[13]:= **coupledDESys** = **D**[{ $\hat{I}_1(x)$, $\hat{I}_2(x)$, $\hat{I}_3(x)$ }, **x**] - **A**.{ $\hat{I}_1(x)$, $\hat{I}_2(x)$, $\hat{I}_3(x)$ }];

In[14]:= **rhs** = { $\hat{R}_1(x)$, $\hat{R}_2(x)$, $-\hat{R}_2(x)$ } in power series representation;

In[15]:= **SolveCoupledDESystem**[**coupledDESys**, {**I**₁[**x**], **I**₂[**x**], **I**₃[**x**]}, ϵ , -3,
{-2, -2, -2}, **rhs**, ...]

Solving a coupled differential system

In[13]:= `coupledDESys = D[{I1(x), I2(x), I3(x)}, x] - A.{I1(x), I2(x), I3(x)};`

In[14]:= `rhs = {R1(x), R2(x), -R2(x)}` in power series representation;

In[15]:= `SolveCoupledDESystem[coupledDESys, {I1[x], I2[x], I3[x]}, ε, -3, {-2, -2, -2}, rhs, ...]`

$$\text{Out[15]= } \left\{ \frac{1}{\epsilon^3} \left(\frac{4(3N^2 + 6N + 4)}{3(N+1)^2} + \frac{4S_1[N]}{3(N+1)} \right) + \frac{1}{\epsilon^2} \left(-\frac{2(20N^3 + 58N^2 + 57N + 22)}{3(N+1)^3} + \frac{2(N+2)(2N-1)S_1[N]}{3(N+1)^2} - \frac{S_1[N]^2}{N+1} - \frac{S_2[N]}{N+1} \right), \right. \\ \left. \frac{4}{3\epsilon^3} - \frac{2}{\epsilon^2}, \frac{8}{3\epsilon^3} + \frac{1}{\epsilon^2} \left(-\frac{4(4N^2 + 7N + 2)}{3(N+1)^2} + \frac{4(N+2)S_1[N]}{3(N+1)} \right) \right\}$$

Theorem 4. Given power series $\hat{I}_1(x), \dots, \hat{I}_n(x)$ with

$$\hat{I}_k(x) = \sum_{N=0}^{\infty} \left(I_{k,-3}(N) \varepsilon^{-3} + \dots + I_{k,l}(N) \varepsilon^l + O(\varepsilon^{l+1}) \right) x^N$$

specified by the initial values $I_{k,j}(0)$ and the differential system

$$\begin{aligned} & A_0 \begin{pmatrix} \hat{I}_1(x) \\ \vdots \\ \hat{I}_n(x) \end{pmatrix} + A_1 D_x \begin{pmatrix} \hat{I}_1(x) \\ \vdots \\ \hat{I}_n(x) \end{pmatrix} \cdots + A_d D_x^d \begin{pmatrix} \hat{I}_1(x) \\ \vdots \\ \hat{I}_n(x) \end{pmatrix} \\ &= \begin{pmatrix} \sum_{N=0}^{\infty} \left(r_{1,-3}(N) \varepsilon^{-3} + \dots + r_{1,l_1}(N) \varepsilon^{l_1} + O(\varepsilon^{l_1+1}) \right) x^N \\ \vdots \\ \sum_{N=0}^{\infty} \left(r_{n,-3}(N) \varepsilon^{-3} + \dots + r_{n,l_n}(N) \varepsilon^{l_n} + O(\varepsilon^{l_n+1}) \right) x^N \end{pmatrix} \end{aligned}$$

A_0, \dots, A_d : $n \times n$ matrices with entries from $\mathbb{K}(\varepsilon, x)$;

$r_{i,j}(N)$: nested sums and products.

Sigma computes, if possible, a representation of $I_{k,-3}(N), \dots, I_{k,l}(N)$ in

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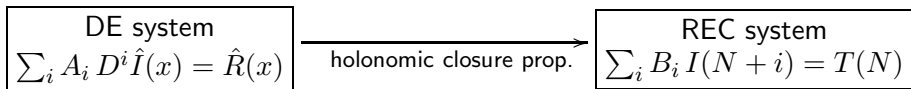
Sigma computes, if possible, a representation of $I_{k,-3}(N), \dots, I_{k,l}(N)$ in terms of nested sums and products.

SUMMARY: the pure REC approach

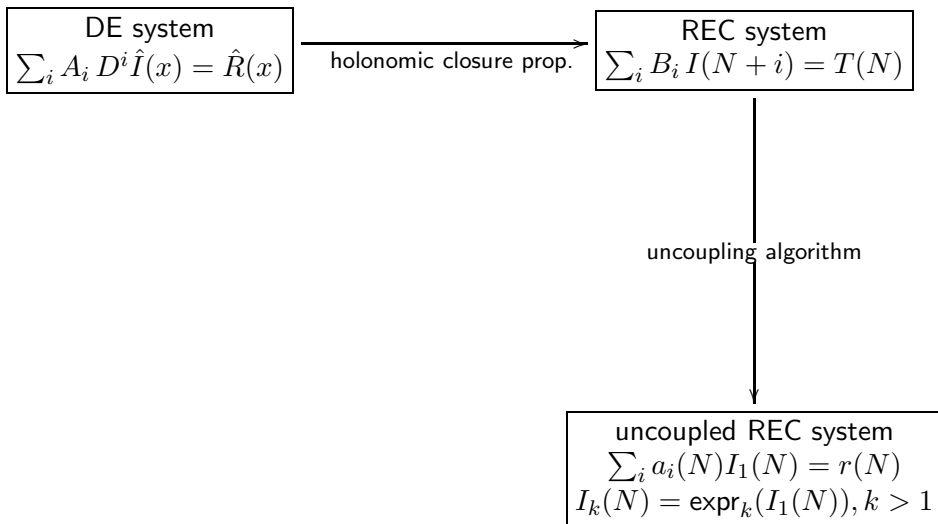
DE system

$$\sum_i A_i D^i \hat{I}(x) = \hat{R}(x)$$

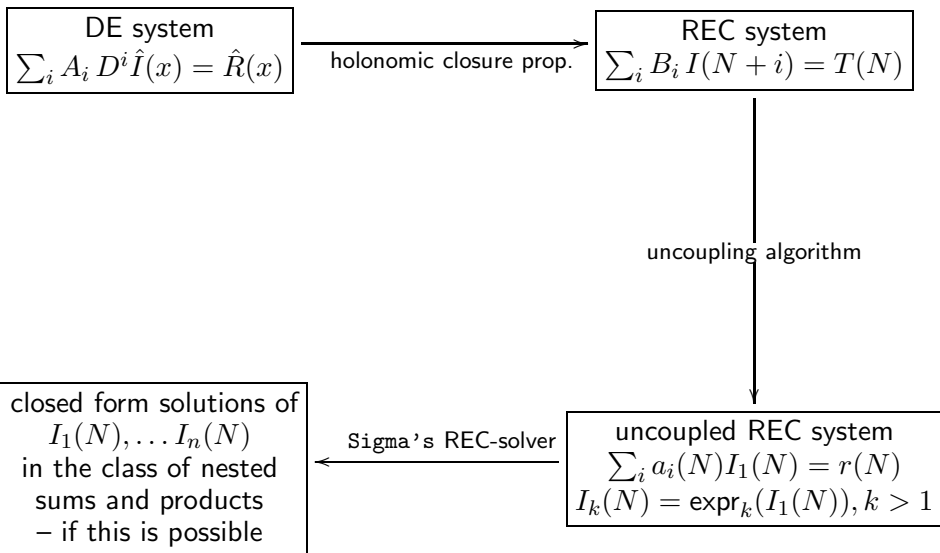
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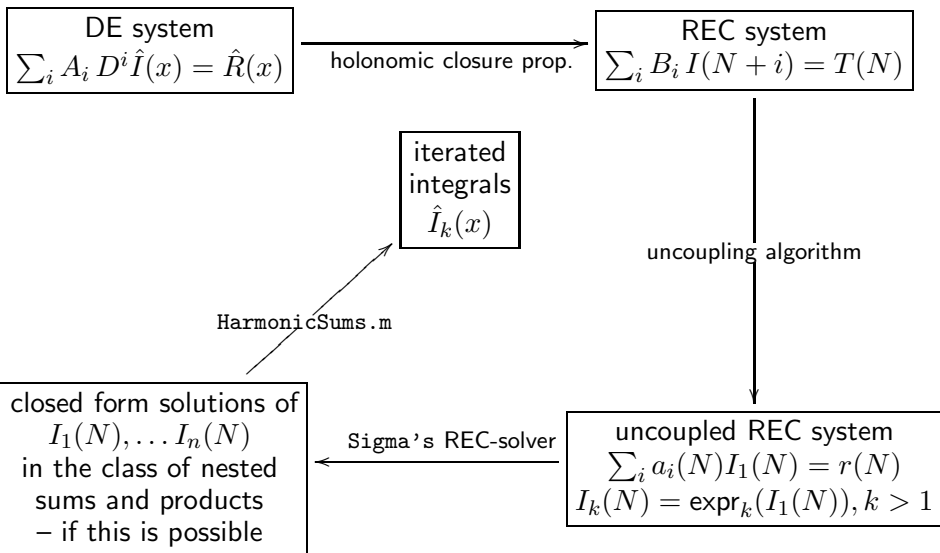
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DE system \rightarrow REC system

$$\begin{aligned}D_x \hat{I}_1(x) &= -\frac{(-\varepsilon+x-1)}{(x-1)x} \hat{I}_1(x) \\ &\quad - \frac{2}{(x-1)x} \hat{I}_2(x) \\ &\quad + \frac{1}{(x-1)x} \hat{B}_1(x) + \dots\end{aligned}$$

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\downarrow N th coefficient

$$NI_1(N-1) - (\varepsilon + N + 1)I_1(N) + 2I_2(N) = B_1(N) + \dots$$

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Exploit nice output of IBP!

A coupled differential system for $\hat{I}_1(x)$, $\hat{I}_2(x)$, $\hat{I}_3(x)$

(produced by IBP [extension of REDUZE_2, A.v. Manteuffel])

$$D_x \begin{pmatrix} \hat{I}_1(x) \\ \hat{I}_2(x) \\ \hat{I}_3(x) \end{pmatrix} = \begin{pmatrix} -\frac{-1-\varepsilon+x}{(x-1)x} & -\frac{2}{(x-1)x} & 0 \\ \frac{\varepsilon(3\varepsilon+2)}{4(x-1)} & -\frac{-2-\varepsilon+3x+3\varepsilon x}{2(x-1)x} & -\frac{\varepsilon+1}{2(x-1)} \\ -\frac{\varepsilon(3\varepsilon+2)(x-2)}{4(x-1)x} & \frac{-2-5\varepsilon+x+3\varepsilon x}{2(x-1)x} & \frac{(-2\varepsilon-x+\varepsilon x)}{2(x-1)x} \end{pmatrix} \begin{pmatrix} \hat{I}_1(x) \\ \hat{I}_2(x) \\ \hat{I}_3(x) \end{pmatrix} + \begin{pmatrix} \hat{R}_1(x) \\ \hat{R}_2(x) \\ -\hat{R}_2(x) \end{pmatrix}$$

where

$$\hat{R}_1(x) = \frac{\hat{B}_4(x)}{(x-1)x},$$

$$\begin{aligned} \hat{R}_2(x) &= \frac{-(\varepsilon+2)^3}{16(\varepsilon+1)(x-1)x} \hat{B}_1(x) + \frac{(\varepsilon+2)(3\varepsilon+4)(19\varepsilon^2+36\varepsilon+16)}{16\varepsilon(5\varepsilon+6)(x-1)x} \hat{B}_2(x) \\ &+ \frac{(\varepsilon+1)^2(3\varepsilon+4)^2}{2\varepsilon(5\varepsilon+6)x} \hat{B}_3(x) + \frac{-24-50\varepsilon-25\varepsilon^2+8x+14\varepsilon x+6\varepsilon^2 x}{4(5\varepsilon+6)(x-1)x} \hat{B}_4(x) \end{aligned}$$

$\hat{B}_1(x)$, $\hat{B}_2(x)$, $\hat{B}_3(x)$ have been solved with symbolic summation.

SUMMARY: the DE-REC approach

DE system

$$D\hat{I}(x) = A\hat{I}(x) + \hat{R}(x)$$

SUMMARY: the DE-REC approach

$$\begin{array}{c} \text{DE system} \\ D\hat{I}(x) = A\hat{I}(x) + \hat{R}(x) \end{array}$$

uncoupling algorithm

$$\begin{array}{c} \text{uncoupled DE system} \\ \sum_i a_i(x) D^i \hat{I}_1(x) = r(x) \\ \hat{I}_k(x) = \text{expr}_k(\hat{I}_1(x)), k > 1 \end{array}$$

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holonomic closure prop.

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closed form solutions of $I_1(N)$
in the class of nested
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– if this is possible

Sigma's REC-solver

linear recurrence

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extract coefficient

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closed form solutions of $I_1(N), I_2(N), \dots, I_n(N)$ in the class of nested sums and products – if this is possible

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Problem 4: Extract the N th coefficient

Extract the N th coefficient (example 1)

$$\frac{1}{(1-x)x} \hat{F}(x)$$

Extract the N th coefficient (example 1)

$$\frac{1}{(1-x)x} \hat{F}(x) = \frac{1}{(1-x)x} \sum_{N=2}^{\infty} \frac{x^N S_{2,1,1}(N)}{(1+N)^2(2+N)}$$

Extract the N th coefficient (example 1)

$$\begin{aligned}\frac{1}{(1-x)x}\hat{F}(x) &= \frac{1}{(1-x)x} \sum_{N=2}^{\infty} \frac{x^N S_{2,1,1}(N)}{(1+N)^2(2+N)} \\ &= \frac{1}{x} \left(\sum_{N=0}^{\infty} x^N \right) \left(\sum_{N=2}^{\infty} \frac{x^N S_{2,1,1}(N)}{(1+N)^2(2+N)} \right)\end{aligned}$$

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↓ Cauchy product

$$\begin{aligned}f(N) &= \sum_{k=2}^N \frac{S_{2,1,1}(k)}{(1+k)^2(2+k)} + \frac{S_{2,1,1}(N)}{(N+2)^2(N+3)} + \frac{S_1^2(N)}{2(N+1)^2(N+2)^2(N+3)} \\ &+ \frac{S_2(N)}{2(N+1)^2(N+2)^2(N+3)} + \frac{S_1(N)}{(N+1)^3(N+2)^2(N+3)} + \frac{1}{(N+1)^4(N+2)^2(N+3)}\end{aligned}$$

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↓ symbolic summation

harmonic sum expression

Extract the N th coefficient (example 2)

$$\frac{1}{x^2 - 2x - 1} \sum_{k=2}^{\infty} \frac{x^k S_{3,1}(k)}{(1+k)^2(2+k)}$$

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$$\frac{1}{2\sqrt{2}} \sum_{j=2}^{\infty} x^j \sum_{k=2}^j \left[(1 - \sqrt{2})^{-j+k-1} - (1 + \sqrt{2})^{-j+k-1} \right] \frac{S_{3,1}(k)}{(1+k)^2(2+k)}$$

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||

generalized harmonic sums with algebraic numbers

Alien sums, which do not fit to the physics!

Extract the N th coefficient (example 2)

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Alien sums, which do not fit to the physics!

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Method implemented in the package `SumProduction`

(using `Sigma`, `HarmonicSums`, `EvaluateMultiSums`)

[arXiv:1509.08324]

SUMMARY: the DE-REC approach

$$\text{DE system} \\ D\hat{I}(x) = A\hat{I}(x) + \hat{R}(x)$$

uncoupling algorithm

$$\text{uncoupled DE system} \\ \sum_i a_i(x) D^i \hat{I}_1(x) = r(x) \\ \hat{\mathbf{I}}_k(\mathbf{x}) = \text{expr}_k(\hat{\mathbf{I}}_1(\mathbf{x})), k > 1$$

extract coefficient

holonomic closure prop.

closed form solutions of $I_1(N), I_2(N), \dots, I_n(N)$ in the class of nested sums and products – if this is possible

Sigma's REC-solver

$$\text{linear recurrence} \\ \sum_i a'_i(N) I_1(N) = r'(N)$$

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