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Symbolic summation packages for elementary particle physics

Carsten Schneider

RISC, J. Kepler University Linz, Austria

A general tactic

Feynman integrals

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$\downarrow$ non-trivial transformations (DESY)

multiple sums

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multiple sums

$\downarrow$ symbolic summation

compact expression in terms of special functions

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↓ non-trivial transformations (DESY)

multiple sums

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Tactic 1: Expand the summand and simplify

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$$\text{GIVEN } F(N)=\sum_{k=1}^{N}(-1)^k e^{-\frac{3\varepsilon\gamma}{2}}\Big(-2-\frac{3\varepsilon}{2}\Big)!\times \underbrace{\times B\Big(2+k,\frac{\varepsilon}{2}\Big)B(-\varepsilon+k,-\varepsilon)B\Big(1-\frac{\varepsilon}{2}+k,1+\frac{\varepsilon}{2}\Big)\binom{N}{k}}_{f(N,k)}$$

where

$$B(a,b)=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

Tactic 1: Expand and simplify

GIVEN $$F(N)=\sum_{k=1}^{N}(-1)^k e^{-\frac{3\varepsilon\gamma}{2}}\big(-2-\frac{3\varepsilon}{2}\big)!\times \underbrace{\times B\big(2+k,\frac{\varepsilon}{2}\big)B(-\varepsilon+k,-\varepsilon)B\big(1-\frac{\varepsilon}{2}+k,1+\frac{\varepsilon}{2}\big)\binom{N}{k}}_{f(N,k)}$$

FIND the first coefficients of the ϵ-expansion

$$F(N)\overset{?}{=}F_{-3}(N)\varepsilon^{-3}+F_{-2}(N)\varepsilon^{-2}+F_{-1}(N)\varepsilon^{-1}+\dots$$

Tactic 1: Expand and simplify

$$\text{GIVEN } F(N) = \sum_{k=1}^{N} (-1)^k e^{-\frac{3\varepsilon\gamma}{2}} \Big(-2 - \frac{3\varepsilon}{2}\Big)! \times \underbrace{\times B\Big(2+k, \frac{\varepsilon}{2}\Big) B(-\varepsilon+k, -\varepsilon) B\Big(1-\frac{\varepsilon}{2}+k, 1+\frac{\varepsilon}{2}\Big)\binom{N}{k}}_{f(N,k)}$$

Step 1: Compute the first coefficients of the ϵ-expansion

$$f(N,k) = \quad f_{-3}(N,k)\ \varepsilon^{-3} + \quad f_{-2}(N,k)\ \varepsilon^{-2} + \quad f_{-1}(N,k)\ \varepsilon^{-1} +$$

Tactic 1: Expand and simplify

$$\text{GIVEN } F(N)=\sum_{k=1}^{N}(-1)^k e^{-\frac{3\varepsilon\gamma}{2}}\Big(-2-\frac{3\varepsilon}{2}\Big)!\times$$
$$\times\underbrace{B\Big(2+k,\frac{\varepsilon}{2}\Big)B(-\varepsilon+k,-\varepsilon)B\Big(1-\frac{\varepsilon}{2}+k,1+\frac{\varepsilon}{2}\Big)\binom{N}{k}}_{f(N,k)}$$

Step 2: Simplify the sums in

$$\sum_{k=1}^{N}f(N,k)=\Big(\sum_{k=1}^{N}f_{-3}(N,k)\Big)\varepsilon^{-3}+\Big(\sum_{k=1}^{N}f_{-2}(N,k)\Big)\varepsilon^{-2}+\Big(\sum_{k=1}^{N}f_{-1}(N,k)\Big)\varepsilon^{-1}+$$

Simplify

$$F_{-1}(N)=\sum_{k=1}^{N}(-1)^{k+1}\binom{N}{k}\Big(\frac{(2+3k)\big(-2+3k+7k^2+3k^3\big)}{3k^2(1+k)^3}+\frac{2S_2(k)}{1+k}+\frac{\zeta(2)}{2(1+k)}\Big)$$

where

$$S_a(N)=\sum_{i=1}^{N}\frac{\text{sign}(a)^i}{i^a}\text{ and }\zeta(a)=\sum_{i=1}^{\infty}\frac{1}{i^a}$$

Simplify

$$F_{-1}(N)=\sum_{k=1}^{N}(-1)^{k+1}\binom{N}{k}\Big(\frac{(2+3k)\big(-2+3k+7k^2+3k^3\big)}{3k^2(1+k)^3}+\frac{2S_2(k)}{1+k}+\frac{\zeta(2)}{2(1+k)}\Big)$$

$\downarrow$ (summation package `Sigma.m`)

$$\begin{aligned}\big(16N^3+144N^2+413N+384\big)(N+1)^2F_{-1}(N)\\ -(N+2)(2N+5)\big(16N^3+112N^2+221N+113\big)F_{-1}(N+1)\\ +(N+3)^2\big(16N^3+96N^2+173N+99\big)F_{-1}(N+2)\\ =\frac{1}{2}\big(4N^2+21N+29\big)\zeta(2)+\frac{-64N^5-500N^4-1133N^3+203N^2+3516N+3090}{3(N+2)(N+3)}\end{aligned}$$

Simplify

$$F_{-1}(N)=\sum_{k=1}^{N}(-1)^{k+1}\binom{N}{k}\Big(\frac{(2+3k)\big(-2+3k+7k^2+3k^3\big)}{3k^2(1+k)^3}+\frac{2S_2(k)}{1+k}+\frac{\zeta(2)}{2(1+k)}\Big)$$

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$$\begin{aligned}&\Big\{c_1\frac{1-4N}{N+1}+c_2\frac{-14N-13}{(N+1)^2}\\&\quad+\frac{(4N-1)S_1(N)}{N+1}+\frac{(1-4N)S_1(N)^2}{6(N+1)}+\frac{(14N+13)S_1(N)}{3(N+1)^2}\\&\quad+\frac{175N^2+334N+155}{12(N+1)^3}+\frac{(1-4N)S_2(N)}{6(N+1)}+\frac{\zeta(2)}{8(N+1)}\Big|c_1,c_2\in\mathbb{Q}\Big\}\end{aligned}$$

Simplify

$$F_{-1}(N)=\sum_{k=1}^{N}(-1)^{k+1}\binom{N}{k}\Big(\frac{(2+3k)\big(-2+3k+7k^2+3k^3\big)}{3k^2(1+k)^3}+\frac{2S_2(k)}{1+k}+\frac{\zeta(2)}{2(1+k)}\Big)$$

$$\bigcap$$

$$\Big\{c_1\,\frac{1-4N}{N+1}+c_2\,\frac{-14N-13}{(N+1)^2}$$
$$+\frac{(4N-1)S_1(N)}{N+1}+\frac{(1-4N)S_1(N)^2}{6(N+1)}+\frac{(14N+13)S_1(N)}{3(N+1)^2}$$
$$+\frac{175N^2+334N+155}{12(N+1)^3}+\frac{(1-4N)S_2(N)}{6(N+1)}+\frac{\zeta(2)}{8(N+1)}\Big|c_1,c_2\in\mathbb{Q}\Big\}$$

Simplify

$$F_{-1}(N)=\sum_{k=1}^{N}(-1)^{k+1}\binom{N}{k}\Big(\frac{(2+3k)\big(-2+3k+7k^2+3k^3\big)}{3k^2(1+k)^3}+\frac{2S_2(k)}{1+k}+\frac{\zeta(2)}{2(1+k)}\Big)$$

$$||$$

$$\Big(\frac{1}{12}-\frac{1}{8}\zeta(2)\Big)\frac{1-4N}{N+1}+1\,\frac{-14N-13}{(N+1)^2}$$
$$+\frac{(4N-1)S_1(N)}{N+1}+\frac{(1-4N)S_1(N)^2}{6(N+1)}+\frac{(14N+13)S_1(N)}{3(N+1)^2}$$
$$+\frac{175N^2+334N+155}{12(N+1)^3}+\frac{(1-4N)S_2(N)}{6(N+1)}+\frac{\zeta(2)}{8(N+1)}$$

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$F(N)=\sum_{k=0}^{N} f(N,k);$$

$f(N,k)$: indefinite nested product-sum in k;
N: extra parameter

FIND a recurrence for $F(N)$

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2. Recurrence solving

GIVEN a recurrence

$a_0(N),\dots,a_d(N),h(N)$:
indefinite nested product-sum expressions.

$$a_0(N)F(N)+\dots+a_d(N)F(N+d)=h(N);$$

FIND all solutions expressible by indefinite nested products/sums
(Abramov/Bronstein/Petkovšek/CS, in preparation)

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FIND all solutions expressible by indefinite nested products/sums
(Abramov/Bronstein/Petkovšek/CS, in preparation)

3. Find a "closed form"

F(N)=combined solutions in terms of indefinite nested sums.

In[1]:= $<<$ **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= $<<$ **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[3]:= $<<$ **EvaluateMultiSums.m**

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EvaluateMultiSums by Carsten Schneider © RISC-Linz

In[4]:= $\mathbf{mySum} =$

$$\sum_{k=1}^{N}(-1)^{k}e^{-\frac{3\varepsilon\gamma}{2}}\Big(-2-\frac{3\varepsilon}{2}\Big)!B\Big(2+k,\frac{\varepsilon}{2}\Big)B(-\varepsilon+k,-\varepsilon)B\Big(1-\frac{\varepsilon}{2}+k,1+\frac{\varepsilon}{2}\Big)\binom{N}{k};$$

In[5]:= $\mathbf{EvaluateMultiSum[mySum,\{\},\{N\},\{1\},ExpandIn\to\{\varepsilon,-3,\,-3\}]}$

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In[5]:= $\mathbf{EvaluateMultiSum[mySum,\{\},\{N\},\{1\},ExpandIn\to\{\varepsilon,-3,\,-3\}]}$

Out[5]= $\left\{\frac{59N^2+120N+49}{9(N+1)^2}-\frac{2(N+3)S_1[N]}{3(N+1)}\right\}$

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$$\sum_{k=1}^{N}(-1)^{k}e^{-\frac{3\varepsilon\gamma}{2}}\left(-2-\frac{3\varepsilon}{2}\right)!\,B\left(2+k,\frac{\varepsilon}{2}\right)B(-\varepsilon+k,-\varepsilon)B\left(1-\frac{\varepsilon}{2}+k,1+\frac{\varepsilon}{2}\right)\binom{N}{k};$$

In[5]:= $\mathbf{EvaluateMultiSum[mySum,\{\},\{N\},\{1\},ExpandIn\to\{\varepsilon,-3,\,-2\}]}$

Out[5]=
$$\left\{\frac{59N^2+120N+49}{9(N+1)^2}-\frac{2(N+3)S_1[N]}{3(N+1)},\right.$$
$$\left.-\frac{2(20N^3+58N^2+57N+22)}{3(N+1)^3}+\frac{2(N+2)(2N-1)S_1[N]}{3(N+1)^2}-\frac{S_1[N]^2}{N+1}-\frac{S_2[N]}{N+1}\right\}$$

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In[5]:= **EvaluateMultiSum[mySum, {}, {N}, {1}, ExpandIn → {ε, −3, − 1}]**

Out[5]=

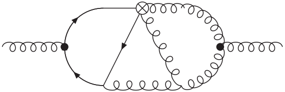
$$\Big\{\frac{59N^2+120N+49}{9(N+1)^2}-\frac{2(N+3)S_1[N]}{3(N+1)},$$

$$-\frac{2(20N^3+58N^2+57N+22)}{3(N+1)^3}+\frac{2(N+2)(2N-1)S_1[N]}{3(N+1)^2}-\frac{S_1[N]^2}{N+1}-\frac{S_2[N]}{N+1},$$

$$\Big(\frac{1}{12}-\frac{1}{8}\zeta(2)\Big)\frac{1-4N}{N+1}+\frac{-14N-13}{(N+1)^2}+\frac{(4N-1)S_1(N)}{N+1}+\frac{(1-4N)S_1(N)^2}{6(N+1)}+$$

$$\frac{(14N+13)S_1(N)}{3(N+1)^2}+\frac{175N^2+334N+155}{12(N+1)^3}+\frac{(1-4N)S_2(N)}{6(N+1)}+\frac{\zeta(2)}{8(N+1)}\Big\}$$

Consider a massive 3–loop ladder graph (Ablinger, Blümlein,Hasselhuhn,Klein, CS,Wissbrock, 2012)



$$= F_{-3}(N)\varepsilon^{-3} + F_{-2}(N)\varepsilon^{-2} + F_{-1}(N)\varepsilon^{-1} + \boxed{F_0(N)}$$

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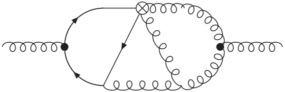

$$= F_{-3}(N)\varepsilon^{-3} + F_{-2}(N)\varepsilon^{-2} + F_{-1}(N)\varepsilon^{-1} + \boxed{F_0(N)}$$

Simplify

$$||$$

$$\sum_{j=0}^{N-3}\sum_{k=0}^{j}\sum_{l=0}^{k}\sum_{q=0}^{-j+N-3}\sum_{s=1}^{-l+N-q-3}\sum_{r=0}^{-l+N-q-s-3}(-1)^{-j+k-l+N-q-3}\times$$

$$\times\frac{\binom{j+1}{k+1}\binom{k}{l}\binom{N-1}{j+2}\binom{-j+N-3}{q}\binom{-l+N-q-3}{s}\binom{-l+N-q-s-3}{r}r!(-l+N-q-r-s-3)!(s-1)!}{(-l+N-q-2)!(-j+N-1)(N-q-r-s-2)(q+s+1)}$$

$$\Big[4S_1(-j+N-1)-4S_1(-j+N-2)-2S_1(k)$$

$$-\left(S_1(-l+N-q-2)+S_1(-l+N-q-r-s-3)-2S_1(r+s)\right)$$

$$+2S_1(s-1)-2S_1(r+s)\Big]$$ + **3 further 6–fold sums**

$$\boxed{F_0(N)} =$$

$$\frac{7}{12}S_1(N)^4+\frac{(17N+5)S_1(N)^3}{3N(N+1)}+\big(\frac{35N^2-2N-5}{2N^2(N+1)^2}+\frac{13S_2(N)}{2}+\frac{5(-1)^N}{2N^2}\big)S_1(N)^2$$
$$+\big(-\frac{4(13N+5)}{N^2(N+1)^2}+\big(\frac{4(-1)^N(2N+1)}{N(N+1)}-\frac{13}{N}\big)S_2(N)+\big(\frac{29}{3}-(-1)^N\big)S_3(N)$$
$$+\big(2+2(-1)^N\big)S_{2,1}(N)-28S_{-2,1}(N)+\frac{20(-1)^N}{N^2(N+1)}\big)S_1(N)+\big(\frac{3}{4}+(-1)^N\big)S_2(N)^2$$
$$-2(-1)^N S_{-2}(N)^2+S_{-3}(N)\big(\frac{2(3N-5)}{N(N+1)}+\big(26+4(-1)^N\big)S_1(N)+\frac{4(-1)^N}{N+1}\big)$$
$$+\big(\frac{(-1)^N(5-3N)}{2N^2(N+1)}-\frac{5}{2N^2}\big)S_2(N)+S_{-2}(N)\big(10S_1(N)^2+\big(\frac{8(-1)^N(2N+1)}{N(N+1)}$$
$$+\frac{4(3N-1)}{N(N+1)}\big)S_1(N)+\frac{8(-1)^N(3N+1)}{N(N+1)^2}+\big(-22+6(-1)^N\big)S_2(N)-\frac{16}{N(N+1)}\big)$$
$$+\big(\frac{(-1)^N(9N+5)}{N(N+1)}-\frac{29}{3N}\big)S_3(N)+\big(\frac{19}{2}-2(-1)^N\big)S_4(N)+\big(-6+5(-1)^N\big)S_{-4}(N)$$
$$+\big(-\frac{2(-1)^N(9N+5)}{N(N+1)}-\frac{2}{N}\big)S_{2,1}(N)+\big(20+2(-1)^N\big)S_{2,-2}(N)+\big(-17+13(-1)^N\big)S_{3,1}(N)$$
$$-\frac{8(-1)^N(2N+1)+4(9N+1)}{N(N+1)}S_{-2,1}(N)-\big(24+4(-1)^N\big)S_{-3,1}(N)+\big(3-5(-1)^N\big)S_{2,1,1}(N)$$
$$+32S_{-2,1,1}(N)+\left(\frac{3}{2}S_1(N)^2-\frac{3S_1(N)}{N}+\frac{3}{2}(-1)^N S_{-2}(N)\right)\zeta(2)$$

$$\boxed{F_0(N)} =$$

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$$S_1(N) = \sum_{i=1}^{N} \frac{1}{i}$$

$$+ \Big(- \ldots 1)^N(2N+1) / N(N+1) - \frac{13}{N}\Big)S_2(N) + \Big(\frac{29}{3} - (-1)^N\Big)S_3(N)$$

$$+ \Big(2 + \ldots 28S_{-2,1}(N) + \frac{20(-1)^N}{N^2(N+1)}\Big)S_1(N) + \Big(\frac{3}{4} + (-1)^N\Big)S_2(N)^2$$

$$- 2(-1)^N S_{-2}(N)^2 + S_{-3}(N)\Big(\frac{2(3N-5)}{N(N+1)} + \big(26 + 4(-1)^N\big)S_1(N) + \frac{4(-1)^N}{N+1}\Big)$$

$$+ \Big(\frac{(-1)^N(5-3N)}{2N^2(N+1)} - \frac{5}{2N^2}\Big)S_2(N) + S_{-2}(N)\Big(10S_1(N)^2 + \Big(\frac{8(-1)^N(2N+1)}{N(N+1)}$$

$$+ \frac{4(3N-1)}{N(N+1)}\Big)S_1(N) + \frac{8(-1)^N(3N+1)}{N(N+1)^2} + \big(-22 + 6(-1)^N\big)S_2(N) - \frac{16}{N(N+1)}\Big)$$

$$+ \Big(\frac{(-1)^N(9N+5)}{N(N+1)} - \frac{29}{3N}\Big)S_3(N) + \Big(\frac{19}{2} - 2(-1)^N\Big)S_4(N) + \big(-6 + 5(-1)^N\big)S_{-4}(N)$$

$$+ \Big(-\frac{2(-1)^N(9N+5)}{N(N+1)} - \frac{2}{N}\Big)S_{2,1}(N) + \big(20 + 2(-1)^N\big)S_{2,-2}(N) + \big(-17 + 13(-1)^N\big)S_{3,1}(N)$$

$$- \frac{8(-1)^N(2N+1) + 4(9N+1)}{N(N+1)}S_{-2,1}(N) - \big(24 + 4(-1)^N\big)S_{-3,1}(N) + \big(3 - 5(-1)^N\big)S_{2,1,1}(N)$$

$$+ 32S_{-2,1,1}(N) + \left(\frac{3}{2}S_1(N)^2 - \frac{3S_1(N)}{N} + \frac{3}{2}(-1)^N S_{-2}(N)\right)\zeta(2)$$

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$$+ \Big(2 + \cdots 28S_{-2,1}(N) + \frac{20(-1)^N}{N^2(N+1\cdots} \cdots 3 \cdots)^N\Big)S_2(N)^2$$

$$- 2(-1)^N S_{-2}(N)^2 + S_{-3}(N)\Big(\frac{2(3N-5)}{N(N+1)} + (26 + 4 \cdots \frac{\cdots)^N}{\cdots 1}\Big)$$

$$+ \Big(\frac{(-1)^N(5-3N)}{2N^2(N+1)} - \frac{5}{2N^2}\Big)S_2(N) + S_{-2}(N)\Big(10S_1(N)^2 + \Big(\frac{8(-1)^N(2N+1)}{N(N+1)}$$

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$$+ \Big(-\frac{2(-1)^N(9N+5)}{N(N+1)} - \frac{2}{N}\Big)S_{2,1}(N) + \big(20 + 2(-1)^N\big)S_{2,-2}(N) + \big(-17 + 13(-1)^N\big)S_{3,1}(N)$$

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$$+ 32S_{-2,1,1}(N) + \left(\frac{3}{2}S_1(N)^2 - \frac{3S_1(N)}{N} + \frac{3}{2}(-1)^N S_{-2}(N)\right)\zeta(2)$$

$$S_1(N) = \sum_{i=1}^{N}\frac{1}{i}$$

$$S_2(N) = \sum_{i=1}^{N}\frac{1}{i^2}$$

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$$+ \big(2 + \dots 28S_{-2,1}(N) + \frac{20(-1)^N}{N^2(N+1 \dots} \dots \frac{3}{\dots} \dots)^N\big)S_2(N)^2$$

$$-2(-1)^N S_{-2}(N)^2 + S_{-3}(N)\Big(\frac{2(3N-5)}{N(N+1)} + (26 + 4 \dots \frac{\dots)^N}{\dots 1}\Big)$$

$$+ \Big(\frac{(-1)^N(5-3N)}{2N^2(N+1)} - \frac{5}{2N^2}\Big)S_2(N) + S_{-2}(N)\Big(10S_1(N)^2 + \Big(\frac{8(-1)^N(2N+1)}{N(N+1)}$$

$$+ \frac{4(3N \dots}{N(N+ \dots} \dots -1)^N\big)S_2(N) - \frac{16}{N(N+1)}\Big)$$

$$+ \Big(\frac{(-1)^N \dots}{N(\dots} \dots (N) + \big(-6 + 5(-1)^N\big)S_{-4}(N)$$

$$+ \Big(- \frac{2(- \dots}{\dots} \dots\Big)S_{2,-2}(N) + \big(-17 + 13(-1)^N\big)S_{3,1}(N)$$

$$- \frac{8(-1)^N \dots}{N(N+1)} S_{-2,1}(N) - \big(24 + 4(-1)^N\big)S_{-3,1}(N) + \big(3 - 5(-1)^N\big)S_{2,1,1}(N)$$

$$+ 32S_{-2,1,1}(N) + \left(\frac{3}{2}S_1(N)^2 - \frac{3S_1(N)}{N} + \frac{3}{2}(-1)^N S_{-2}(N)\right)\zeta(2)$$

$$S_1(N) = \sum_{i=1}^{N}\frac{1}{i}$$

$$S_2(N) = \sum_{i=1}^{N}\frac{1}{i^2}$$

$$S_{-2,1,1}(N) = \sum_{i=1}^{N}\frac{(-1)^i\sum_{j=1}^{i}\frac{\sum_{k=1}^{j}\frac{1}{k}}{j}}{i^2}$$

A general tactic

Feynman integrals

↓ non-trivial transformations (DESY)

multiple sums

↓ symbolic summation

compact expression in terms of special functions

Tactic 2: Expand a recurrence in ε

$$F(N)=\sum_{k=1}^{N}(-1)^k e^{-\frac{3\varepsilon\gamma}{2}}\Big(-2-\frac{3\varepsilon}{2}\Big)!B\Big(2+k,\frac{\varepsilon}{2}\Big)B(-\varepsilon+k,-\varepsilon)B\Big(1-\frac{\varepsilon}{2}+k,1+\frac{\varepsilon}{2}\Big)\binom{N}{k}$$

$\downarrow$ (summation package `Sigma.m`)

$$2(N+1)^2F(N)+\big(3\varepsilon^2+3\varepsilon N+9\varepsilon-4N^2-12N-8\big)F(N+1)$$
$$-(2\varepsilon-N-1)(\varepsilon+2N+6)F(N+2)=0\varepsilon^{-3}-\frac{16}{3}\varepsilon^{-2}+\frac{40}{3}\varepsilon^{-1}-\big(2\zeta_2-\frac{68}{3}\big)\varepsilon^0+\dots$$

$$F(N)=\sum_{k=1}^{N}(-1)^k e^{-\frac{3\varepsilon\gamma}{2}}\big(-2-\frac{3\varepsilon}{2}\big)!B\big(2+k,\frac{\varepsilon}{2}\big)B(-\varepsilon+k,-\varepsilon)B\big(1-\frac{\varepsilon}{2}+k,1+\frac{\varepsilon}{2}\big)\binom{N}{k}$$

↓ (summation package `Sigma.m`)

$$2(N+1)^2F(N)+\big(3\varepsilon^2+3\varepsilon N+9\varepsilon-4N^2-12N-8\big)F(N+1)$$
$$-(2\varepsilon-N-1)(\varepsilon+2N+6)F(N+2)=0\varepsilon^{-3}-\frac{16}{3}\varepsilon^{-2}+\frac{40}{3}\varepsilon^{-1}-\big(2\zeta_2-\frac{68}{3}\big)\varepsilon^0+\dots$$

$$F(1)=\frac{2}{3}\varepsilon^{-3}-\frac{11}{6}\varepsilon^{-2}+\big(\frac{\zeta_2}{4}+\frac{79}{24}\big)\varepsilon^{-1}+\dots$$
$$F(2)=\frac{8}{9}\varepsilon^{-3}-\frac{73}{27}\varepsilon^{-2}+\big(\frac{\zeta_2}{3}+\frac{1415}{324}\big)\varepsilon^{-1}+\dots$$

Computing the ε-expansion from a recurrence relation

$$
\begin{aligned}
& a_0(\varepsilon,N)\Big[I(N)\Big]\\
&+a_1(\varepsilon,N)\Big[I(N+1)\Big]\\
&+\\
&\ \vdots\\
&+a_d(\varepsilon,N)\Big[I(N+d)\Big]\\
&\qquad = h_{-3}(N)\varepsilon^{-3}+h_{-2}(N)\varepsilon^{-2}+h_{-1}(N)\varepsilon^{-1}+\dots
\end{aligned}
$$

given (in terms of indefinite nested sums and products)

Computing the ε-expansion from a recurrence relation

$$a_0(\varepsilon,N)\Big[I_{-3}(N)\varepsilon^{-3}+I_{-2}(N)\varepsilon^{-2}+I_{-1}(N)\varepsilon^{-1}+\dots\Big]$$

$$+a_1(\varepsilon,N)\Big[I(N+1)\Big]$$

$$+$$

$$\vdots$$

$$+a_d(\varepsilon,N)\Big[I(N+d)\Big]$$

$$=h_{-3}(N)\varepsilon^{-3}+h_{-2}(N)\varepsilon^{-2}+h_{-1}(N)\varepsilon^{-1}+\dots$$

given (in terms of indefinite nested sums and products)

Computing the ε-expansion from a recurrence relation

$$a_0(\varepsilon, N)\Big[I_{-3}(N)\varepsilon^{-3} + I_{-2}(N)\varepsilon^{-2} + I_{-1}(N)\varepsilon^{-1} + \dots\Big]$$

$$+a_1(\varepsilon, N)\Big[I_{-3}(N+1)\varepsilon^{-3} + I_{-2}(N+1)\varepsilon^{-2} + I_{-1}(N+1)\varepsilon^{-1} + \dots\Big]$$

$$+$$

$$\vdots$$

$$+a_d(\varepsilon, N)\Big[I(N+d)\Big]$$

$$= h_{-3}(N)\varepsilon^{-3} + h_{-2}(N)\varepsilon^{-2} + h_{-1}(N)\varepsilon^{-1} + \dots$$

given (in terms of indefinite nested sums and products)

Computing the ε-expansion from a recurrence relation

$$
\begin{aligned}
&a_0(\varepsilon,N)\Big[I_{-3}(N)\varepsilon^{-3}+I_{-2}(N)\varepsilon^{-2}+I_{-1}(N)\varepsilon^{-1}+\dots\Big]\\
+&a_1(\varepsilon,N)\Big[I_{-3}(N+1)\varepsilon^{-3}+I_{-2}(N+1)\varepsilon^{-2}+I_{-1}(N+1)\varepsilon^{-1}+\dots\Big]\\
+&\\
&\vdots\\
+&a_d(\varepsilon,N)\Big[I_{-3}(N+d)\varepsilon^{-3}+I_{-2}(N+d)\varepsilon^{-2}+I_{-1}(N+d)\varepsilon^{-1}+\dots\Big]\\
&\qquad=h_{-3}(N)\varepsilon^{-3}+h_{-2}(N)\varepsilon^{-2}+h_{-1}(N)\varepsilon^{-1}+\dots
\end{aligned}
$$

given (in terms of indefinite nested sums and products)

Computing the ε-expansion from a recurrence relation

$$
\begin{aligned}
&a_0(\varepsilon,N)\Big[I_{-3}(N)\varepsilon^{-3}+I_{-2}(N)\varepsilon^{-2}+I_{-1}(N)\varepsilon^{-1}+\dots\Big]\\
+&a_1(\varepsilon,N)\Big[I_{-3}(N+1)\varepsilon^{-3}+I_{-2}(N+1)\varepsilon^{-2}+I_{-1}(N+1)\varepsilon^{-1}+\dots\Big]\\
+&\\
&\vdots\\
+&a_d(\varepsilon,N)\Big[I_{-3}(N+d)\varepsilon^{-3}+I_{-2}(N+d)\varepsilon^{-2}+I_{-1}(N+d)\varepsilon^{-1}+\dots\Big]\\
&\qquad\qquad=h_{-3}(N)\varepsilon^{-3}+h_{-2}(N)\varepsilon^{-2}+h_{-1}(N)\varepsilon^{-1}+\dots
\end{aligned}
$$

$\Downarrow$ lowest terms must agree

$$
\boxed{a_0(0,N)I_{-3}(N)+a_1(0,N)I_{-3}(N+1)+\dots+a_d(0,N)I_{-3}(N+d)=h_{-3}(N)}
$$

Computing the ε-expansion from a recurrence relation

$$
\begin{aligned}
&a_0(\varepsilon,N)\Big[I_{-3}(N)\varepsilon^{-3}+I_{-2}(N)\varepsilon^{-2}+I_{-1}(N)\varepsilon^{-1}+\dots\Big]\\
+&a_1(\varepsilon,N)\Big[I_{-3}(N+1)\varepsilon^{-3}+I_{-2}(N+1)\varepsilon^{-2}+I_{-1}(N+1)\varepsilon^{-1}+\dots\Big]\\
+&\\
&\vdots\\
+&a_d(\varepsilon,N)\Big[I_{-3}(N+d)\varepsilon^{-3}+I_{-2}(N+d)\varepsilon^{-2}+I_{-1}(N+d)\varepsilon^{-1}+\dots\Big]\\
&\qquad\qquad=h_{-3}(N)\varepsilon^{-3}+h_{-2}(N)\varepsilon^{-2}+h_{-1}(N)\varepsilon^{-1}+\dots
\end{aligned}
$$

$\Downarrow$ lowest terms must agree

$$
\boxed{a_0(0,N)I_{-3}(N)+a_1(0,N)I_{-3}(N+1)+\dots+a_d(0,N)I_{-3}(N+d)=h_{-3}(N)}
$$

REC solver: Given the initial values $I_{-3}(1), I_{-3}(2), \dots, I_{-3}(d)$, **decide** if $I_{-3}(N)$ can be written in terms of indefinite nested sums and products.

Computing the ε-expansion from a recurrence relation

$$
\begin{aligned}
&a_0(\varepsilon,N)\Big[I_{-3}(N)\varepsilon^{-3}+I_{-2}(N)\varepsilon^{-2}+I_{-1}(N)\varepsilon^{-1}+\dots\Big]\\
&+a_1(\varepsilon,N)\Big[I_{-3}(N+1)\varepsilon^{-3}+I_{-2}(N+1)\varepsilon^{-2}+I_{-1}(N+1)\varepsilon^{-1}+\dots\Big]\\
&+\\
&\ \vdots\\
&+a_d(\varepsilon,N)\Big[I_{-3}(N+d)\varepsilon^{-3}+I_{-2}(N+d)\varepsilon^{-2}+I_{-1}(N+d)\varepsilon^{-1}+\dots\Big]\\
&\qquad\qquad=h_{-3}(N)\varepsilon^{-3}+h_{-2}(N)\varepsilon^{-2}+h_{-1}(N)\varepsilon^{-1}+\dots
\end{aligned}
$$

$\Downarrow$ lowest terms must agree

$$\boxed{a_0(0,N)I_{-3}(N)+a_1(0,N)I_{-3}(N+1)+\dots+a_d(0,N)I_{-3}(N+d)=h_{-3}(N)}$$

Computing the ε-expansion from a recurrence relation

$$
\begin{aligned}
&a_0(\varepsilon,N)\Big[I_{-3}(N)\varepsilon^{-3}+I_{-2}(N)\varepsilon^{-2}+I_{-1}(N)\varepsilon^{-1}+\dots\Big]\\
&+a_1(\varepsilon,N)\Big[I_{-3}(N+1)\varepsilon^{-3}+I_{-2}(N+1)\varepsilon^{-2}+I_{-1}(N+1)\varepsilon^{-1}+\dots\Big]\\
&+\\
&\vdots\\
&+a_d(\varepsilon,N)\Big[I_{-3}(N+d)\varepsilon^{-3}+I_{-2}(N+d)\varepsilon^{-2}+I_{-1}(N+d)\varepsilon^{-1}+\dots\Big]\\
&\qquad\qquad=h_{-3}(N)\varepsilon^{-3}+h_{-2}(N)\varepsilon^{-2}+h_{-1}(N)\varepsilon^{-1}+\dots
\end{aligned}
$$

$\Downarrow$ lowest terms must agree

$$\boxed{a_0(0,N)I_{-3}(N)+a_1(0,N)I_{-3}(N+1)+\dots+a_d(0,N)I_{-3}(N+d)=h_{-3}(N)}$$

Computing the ε-expansion from a recurrence relation

$$\begin{aligned}
&\quad a_0(\varepsilon,N)\Big[I_{-2}(N)\varepsilon^{-2}+I_{-1}(N)\varepsilon^{-1}+\dots\Big]\\
&+a_1(\varepsilon,N)\Big[I_{-2}(N+1)\varepsilon^{-2}+I_{-1}(N+1)\varepsilon^{-1}+\dots\Big]\\
&+\\
&\ \vdots\\
&+a_d(\varepsilon,N)\Big[I_{-2}(N+d)\varepsilon^{-2}+I_{-1}(N+d)\varepsilon^{-1}+\dots\Big]\\
&\qquad\qquad\qquad=h'_{-3}(N)\varepsilon^{-3}+h'_{-2}(N)\varepsilon^{-2}+h'_{-1}(N)\varepsilon^{-1}+\dots
\end{aligned}$$

Computing the ε-expansion from a recurrence relation

$$
\begin{aligned}
&a_0(\varepsilon,N)\Big[I_{-2}(N)\varepsilon^{-2}+I_{-1}(N)\varepsilon^{-1}+\dots\Big]\\
+&a_1(\varepsilon,N)\Big[I_{-2}(N+1)\varepsilon^{-2}+I_{-1}(N+1)\varepsilon^{-1}+\dots\Big]\\
+&\\
&\vdots\\
+&a_d(\varepsilon,N)\Big[I_{-2}(N+d)\varepsilon^{-2}+I_{-1}(N+d)\varepsilon^{-1}+\dots\Big]\\
&\qquad=\underbrace{h'_{-3}(N)}_{\mathbf{=0}}\varepsilon^{-3}+h'_{-2}(N)\varepsilon^{-2}+h'_{-1}(N)\varepsilon^{-1}+\dots
\end{aligned}
$$

Computing the ε-expansion from a recurrence relation

$$
\begin{aligned}
&a_0(\varepsilon,N)\Big[I_{-2}(N)\varepsilon^{-2}+I_{-1}(N)\varepsilon^{-1}+\dots\Big]\\
&+a_1(\varepsilon,N)\Big[I_{-2}(N+1)\varepsilon^{-2}+I_{-1}(N+1)\varepsilon^{-1}+\dots\Big]\\
&+\\
&\;\vdots\\
&+a_d(\varepsilon,N)\Big[I_{-2}(N+d)\varepsilon^{-2}+I_{-1}(N+d)\varepsilon^{-1}+\dots\Big]\\
&\qquad\qquad\qquad\qquad\qquad\qquad=h'_{-2}(N)\varepsilon^{-2}+h'_{-1}(N)\varepsilon^{-1}+\dots
\end{aligned}
$$

Now repeat for $I_{-2}(N), I_{-1}(N), \dots$

Blümlein, Klein, CS, Stan, J. Symbol. Comput. 2012; arXiv:1011.2656[cs.SC]
Ablinger, Blümlein, Round, CS, LL2012, arXiv:1210.1685 [cs.SC]

$$F(N)=\sum_{k=1}^{N}(-1)^k e^{-\frac{3\varepsilon\gamma}{2}}\big(-2-\frac{3\varepsilon}{2}\big)!B\big(2+k,\frac{\varepsilon}{2}\big)B(-\varepsilon+k,-\varepsilon)B\big(1-\frac{\varepsilon}{2}+k,1+\frac{\varepsilon}{2}\big)\binom{N}{k}$$

$\downarrow$ (summation package `Sigma.m`)

$$2(N+1)^2F(N)+\big(3\varepsilon^2+3\varepsilon N+9\varepsilon-4N^2-12N-8\big)F(N+1)$$
$$-(2\varepsilon-N-1)(\varepsilon+2N+6)F(N+2)=0\varepsilon^{-3}-\frac{16}{3}\varepsilon^{-2}+\frac{40}{3}\varepsilon^{-1}-\big(2\zeta_2-\frac{68}{3}\big)\varepsilon^0+\dots$$

$$F(1)=\frac{2}{3}\varepsilon^{-3}-\frac{11}{6}\varepsilon^{-2}+\big(\frac{\zeta_2}{4}+\frac{79}{24}\big)\varepsilon^{-1}+\dots$$
$$F(2)=\frac{8}{9}\varepsilon^{-3}-\frac{73}{27}\varepsilon^{-2}+\big(\frac{\zeta_2}{3}+\frac{1415}{324}\big)\varepsilon^{-1}+\dots$$

$$F(N)=\sum_{k=1}^{N}(-1)^k e^{-\frac{3\varepsilon\gamma}{2}}\big(-2-\frac{3\varepsilon}{2}\big)!B\big(2+k,\frac{\varepsilon}{2}\big)B(-\varepsilon+k,-\varepsilon)B\big(1-\frac{\varepsilon}{2}+k,1+\frac{\varepsilon}{2}\big)\binom{N}{k}$$

↓ (summation package Sigma.m)

$$2(N+1)^2F(N)+\big(3\varepsilon^2+3\varepsilon N+9\varepsilon-4N^2-12N-8\big)F(N+1)$$
$$-(2\varepsilon-N-1)(\varepsilon+2N+6)F(N+2)=0\varepsilon^{-3}-\frac{16}{3}\varepsilon^{-2}+\frac{40}{3}\varepsilon^{-1}-\big(2\zeta_2-\frac{68}{3}\big)\varepsilon^{0}+\dots$$

$$F(1)=\frac{2}{3}\varepsilon^{-3}-\frac{11}{6}\varepsilon^{-2}+\big(\frac{\zeta_2}{4}+\frac{79}{24}\big)\varepsilon^{-1}+\dots$$
$$F(2)=\frac{8}{9}\varepsilon^{-3}-\frac{73}{27}\varepsilon^{-2}+\big(\frac{\zeta_2}{3}+\frac{1415}{324}\big)\varepsilon^{-1}+\dots$$

↓ (summation package Sigma.m)

$$F(N)=\tfrac{4N}{3(N+1)}\varepsilon^{-3}-\Big(\tfrac{2(2N+1)}{3(N+1)}S_1(N)+\tfrac{2N(2N+3)}{3(N+1)^2}\Big)\varepsilon^{-2}$$
$$\Big(\tfrac{(1-4N)}{6(N+1)}S_1(N)^2-\tfrac{N\left(N^2-2\right)}{3(N+1)^3}+\tfrac{(3N+2)(4N+5)}{3(N+1)^2}S_1(N)+\tfrac{(1-4N)}{6(N+1)}S_2(N)+\tfrac{N\zeta_2}{2(N+1)}\Big)\varepsilon^{-1}+\dots$$

Computing the ε-expansion from a recurrence relation

$$I(1)=\frac{5}{\varepsilon^3}-\frac{163}{12\varepsilon^2}+O(\varepsilon^{-1}),\ I(2)=\frac{130}{27\varepsilon^3}-\frac{695}{54\varepsilon^2}+O(\varepsilon^{-1}),\ I(3)=\frac{169}{36\varepsilon^3}-\frac{395}{32\varepsilon^2}+O(\varepsilon^{-1})$$

In[6]:= $\mathbf{recEp} = -2(N+1)(N+2)(2+\varepsilon+N)I[N]$

$$-(N+2)\big(-32-7\varepsilon+2\varepsilon^2-28N-5\varepsilon N-6N^2\big)I[N+1]$$
$$-\big(120+3\varepsilon-14\varepsilon^2-\varepsilon^3+136N+13\varepsilon N-4\varepsilon^2N+50N^2+4\varepsilon N^2+6N^3\big)I[N+2]$$
$$+(2-\varepsilon+N)(4+\varepsilon+N)(8+\varepsilon+2N)I[N+3]$$
$$==\frac{1}{\varepsilon^3}\frac{-4(N+2)}{3(N+3)}+\frac{1}{\varepsilon^2}\Big[-\frac{2(2N+7)S_1}{3(N+3)}-\frac{2\big(4N^4+35N^3+101N^2+105N+25\big)}{3(N+1)(N+2)(N+3)^2}\Big]+O(\varepsilon^{-1}).$$

Computing the ε-expansion from a recurrence relation

$$I(1)=\frac{5}{\varepsilon^3}-\frac{163}{12\varepsilon^2}+O(\varepsilon^{-1}),\ I(2)=\frac{130}{27\varepsilon^3}-\frac{695}{54\varepsilon^2}+O(\varepsilon^{-1}),\ I(3)=\frac{169}{36\varepsilon^3}-\frac{395}{32\varepsilon^2}+O(\varepsilon^{-1})$$

In[6]:= $\mathbf{recEp = -2(N+1)(N+2)(2+\varepsilon+N)I[N] - (N+2)\big(-32-7\varepsilon+2\varepsilon^2-28N-5\varepsilon N-6N^2\big)I[N+1] - \big(120+3\varepsilon-14\varepsilon^2-\varepsilon^3+136N+13\varepsilon N-4\varepsilon^2 N+50N^2+4\varepsilon N^2+6N^3\big)I[N+2] + (2-\varepsilon+N)(4+\varepsilon+N)(8+\varepsilon+2N)I[N+3] == \frac{1}{\varepsilon^3}\frac{-4(N+2)}{3(N+3)} + \frac{1}{\varepsilon^2}\Big[-\frac{2(2N+7)S_1}{3(N+3)} - \frac{2\big(4N^4+35N^3+101N^2+105N+25\big)}{3(N+1)(N+2)(N+3)^2}\Big] + O(\varepsilon^{-1}).}$

In[7]:= $\mathbf{GenerateExpansion[recEp[[1]], \{Coefficient[recEp[[2]],\varepsilon^{-3}], Coefficient[recEp[[2]],\varepsilon^{-2}]\}, I[N], \{\varepsilon,-3,-2\}, \{\{5,\frac{130}{27},\frac{169}{36}\},\{-\frac{163}{12},-\frac{695}{54},-\frac{395}{32}\}\}, MinInitialValue \to 1]}$

Computing the ε-expansion from a recurrence relation

$$I(1)=\frac{5}{\varepsilon^3}-\frac{163}{12\varepsilon^2}+O(\varepsilon^{-1}),\ I(2)=\frac{130}{27\varepsilon^3}-\frac{695}{54\varepsilon^2}+O(\varepsilon^{-1}),\ I(3)=\frac{169}{36\varepsilon^3}-\frac{395}{32\varepsilon^2}+O(\varepsilon^{-1})$$

In[6]:=
$$\mathbf{recEp} = -2(N+1)(N+2)(2+\varepsilon+N)I[N] - (N+2)\big(-32-7\varepsilon+2\varepsilon^2-28N-5\varepsilon N-6N^2\big)I[N+1] - \big(120+3\varepsilon-14\varepsilon^2-\varepsilon^3+136N+13\varepsilon N-4\varepsilon^2N+50N^2+4\varepsilon N^2+6N^3\big)I[N+2] + (2-\varepsilon+N)(4+\varepsilon+N)(8+\varepsilon+2N)I[N+3] == \frac{1}{\varepsilon^3}\frac{-4(N+2)}{3(N+3)}+\frac{1}{\varepsilon^2}\Big[-\frac{2(2N+7)S_1}{3(N+3)}-\frac{2\big(4N^4+35N^3+101N^2+105N+25\big)}{3(N+1)(N+2)(N+3)^2}\Big]+O(\varepsilon^{-1}).$$

In[7]:=
$$\mathbf{GenerateExpansion}[\mathbf{recEp}[[1]], \{\mathrm{Coefficient}[\mathbf{recEp}[[2]],\varepsilon^{-3}], \mathrm{Coefficient}[\mathbf{recEp}[[2]],\varepsilon^{-2}]\}, I[N], \{\varepsilon,-3,-2\}, \{\{5,\frac{130}{27},\frac{169}{36}\},\{-\frac{163}{12},-\frac{695}{54},-\frac{395}{32}\}\}, \mathrm{MinInitialValue}\to 1]$$

Out[7]=
$$\Big\{\frac{59N^2+120N+49}{9(N+1)^2}-\frac{2(N+3)S_1[N]}{3(N+1)},\ -\frac{2(20N^3+58N^2+57N+22)}{3(N+1)^3}+\frac{2(N+2)(2N-1)S_1[N]}{3(N+1)^2}-\frac{S_1[N]^2}{N+1}-\frac{S_2[N]}{N+1}\Big\}$$

Find a recurrence for the integral/sum

$$F(N)=\int_0^1\cdots\int_0^1\Phi(\varepsilon,N,x_1,x_2,\dots,x_7)dx_1dx_2\dots dx_7$$
$$\stackrel{?}{=}F_{-3}(N)\varepsilon^{-3}+F_{-2}(N)\varepsilon^{-2}+F_{-1}(N)\varepsilon^{-1}+\dots$$

MultiIntegrate package
(Jakob Ablinger)

$$a_0(\varepsilon,N)F(N)+\dots+a_d(\varepsilon,N)F(N+d)=h(\varepsilon,N)$$

Find a recurrence for the integral/sum

$$F(N) = \int_0^1 \cdots \int_0^1 \Phi(\varepsilon, N, x_1, x_2, \dots, x_7) dx_1 dx_2 \dots dx_7$$

$$\stackrel{?}{=} F_{-3}(N)\varepsilon^{-3} + F_{-2}(N)\varepsilon^{-2} + F_{-1}(N)\varepsilon^{-1} + \dots$$

MultiIntegrate package
(Jakob Ablinger)

$$\sum_{i_1} \cdots \sum_{i_7} f(\varepsilon, N, i_1, i_2, \dots, i_7)$$

MultiSum Package
(Wegschaider)

$$a_0(\varepsilon, N)F(N) + \dots + a_d(\varepsilon, N)F(N+d) = h(\varepsilon, N)$$

Find a recurrence for the integral/sum

$$F(N) = \int_0^1 \cdots \int_0^1 \Phi(\varepsilon, N, x_1, x_2, \dots, x_7) dx_1 dx_2 \dots dx_7$$
$$\stackrel{?}{=} F_{-3}(N)\varepsilon^{-3} + F_{-2}(N)\varepsilon^{-2} + F_{-1}(N)\varepsilon^{-1} + \dots$$

MultiIntegrate package
(Jakob Ablinger)

$$\sum_{i_1} \cdots \sum_{i_7} f(\varepsilon, N, i_1, i_2, \dots, i_7)$$

MultiSum Package
(Wegschaider)

RhoSum package (Mark Round)
(holonomic/difference field approach)

$$a_0(\varepsilon, N)F(N) + \dots + a_d(\varepsilon, N)F(N + d) = h(\varepsilon, N)$$

Find a recurrence for the integral/sum

$$F(N) = \int_0^1 \cdots \int_0^1 \Phi(\varepsilon, N, x_1, x_2, \dots, x_7) dx_1 dx_2 \dots dx_7$$

$$\stackrel{?}{=} F_{-3}(N)\varepsilon^{-3} + F_{-2}(N)\varepsilon^{-2} + F_{-1}(N)\varepsilon^{-1} + \dots$$

ε-recurrence solver

MultiIntegrate package
(Jakob Ablinger)

$$\sum_{i_1} \cdots \sum_{i_7} f(\varepsilon, N, i_1, i_2, \dots, i_7)$$

MultiSum Package
(Wegschaider)

RhoSum package (Mark Round)
(holonomic/difference field approach)

$$a_0(\varepsilon, N)F(N) + \dots + a_d(\varepsilon, N)F(N+d) = h(\varepsilon, N)$$

Tactic 3: Solve coupled systems of differential equations

A coupled differential system for $\hat{I}_1(x), \hat{I}_2(x), \hat{I}_3(x)$

(produced by IBP [extension of REDUZE_2, A.v. Manteuffel])

$$D_x \begin{pmatrix} \hat{I}_1(x) \\ \hat{I}_2(x) \\ \hat{I}_3(x) \end{pmatrix} = \begin{pmatrix} -\frac{-1-\varepsilon+x}{(x-1)x} & -\frac{2}{(x-1)x} & 0 \\ \frac{\varepsilon(3\varepsilon+2)}{4(x-1)} & -\frac{-2-\varepsilon+3x+3\varepsilon x}{2(x-1)x} & -\frac{\varepsilon+1}{2(x-1)} \\ -\frac{\varepsilon(3\varepsilon+2)(x-2)}{4(x-1)x} & \frac{-2-5\varepsilon+x+3\varepsilon x}{2(x-1)x} & \frac{(-2\varepsilon-x+\varepsilon x)}{2(x-1)x} \end{pmatrix} \begin{pmatrix} \hat{I}_1(x) \\ \hat{I}_2(x) \\ \hat{I}_3(x) \end{pmatrix} + \begin{pmatrix} \hat{R}_1(x) \\ \hat{R}_2(x) \\ -\hat{R}_2(x) \end{pmatrix}$$

where

$$\hat{R}_1(x) = \frac{\hat{B}_4(x)}{(x-1)x},$$

$$\hat{R}_2(x) = \frac{-(\varepsilon+2)^3}{16(\varepsilon+1)(x-1)x}\hat{B}_1(x) + \frac{(\varepsilon+2)(3\varepsilon+4)\left(19\varepsilon^2+36\varepsilon+16\right)}{16\varepsilon(5\varepsilon+6)(x-1)x}\hat{B}_2(x)$$
$$+ \frac{(\varepsilon+1)^2(3\varepsilon+4)^2}{2\varepsilon(5\varepsilon+6)x}\hat{B}_3(x) + \frac{-24-50\varepsilon-25\varepsilon^2+8x+14\varepsilon x+6\varepsilon^2 x}{4(5\varepsilon+6)(x-1)x}\hat{B}_4(x)$$

$\hat{B}_1(x), \hat{B}_2(x), \hat{B}_3(x)$ have been solved with symbolic summation.

Tactic 3: the DE-REC approach

DE system

$$D\hat{I}(x) = A\,\hat{I}(x) + \hat{R}(x)$$

Tactic 3: the DE-REC approach

DE system	OreSys package (S. Gerhold) → uncoupling algorithm	uncoupled DE system
$D\hat{I}(x) = A\,\hat{I}(x) + \hat{R}(x)$		$\sum_i a_i(x) D^i \hat{I}_1(x) = r(x)$ $\hat{I}_k(x) = \mathsf{expr}_k(\hat{I}_1(x)), k > 1$

Tactic 3: the DE-REC approach

DE system

$D\hat{I}(x) = A\,\hat{I}(x) + \hat{R}(x)$

$\xrightarrow[\text{uncoupling algorithm}]{\text{OreSys package (S. Gerhold)}}$

uncoupled DE system

$\sum_i a_i(x) D^i \hat{I}_1(x) = r(x)$

$\hat{I}_k(x) = \text{expr}_k(\hat{I}_1(x)), k > 1$

$$\hat{I}_1(x) = \sum_{N=0}^{\infty} I_1(N) x^N$$

Tactic 3: the DE-REC approach

DE system

$$D\hat{I}(x) = A\,\hat{I}(x) + \hat{R}(x)$$

OreSys package (S. Gerhold)

uncoupling algorithm

uncoupled DE system

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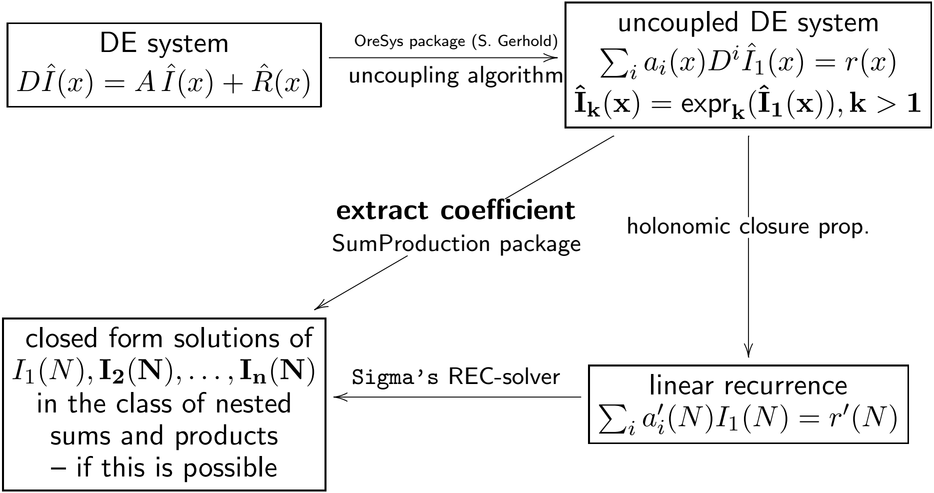
← Sigma's REC-solver ←

closed form solutions of $I_1(N)$ in the class of nested sums and products – if this is possible

Tactic 3: the DE-REC approach (SolveCoupledSystem package)



J. Ablinger, A. Behring, J. Blmlein, A. De Freitas, A. von Manteuffel, CS. Comput. Phys. Comm. 202, pp. 33-112.

Solving a coupled differential system

In[8]:= $<<$ **OreSys.m**

> OreSys by Stefan Gerhold (optimized by C. Schneider) © RISC-Linz

In[9]:= $<<$ **SolveCoupledSystem.m**

> SolveCoupledSystem by Carsten Schneider © RISC-Linz

In[10]:= $\mathbf{coupledDESys = D[\{\hat{I}_1(x), \hat{I}_2(x), \hat{I}_3(x)\}, x] - A.\{\hat{I}_1(x), \hat{I}_2(x), \hat{I}_3(x)\};}$

In[11]:= $\mathbf{rhs = \{\hat{R}_1(x), \hat{R}_2(x), -\hat{R}_2(x)\}}$ in power series representation;

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Out[12]=
$$\Big\{\frac{1}{\varepsilon^3}\Big(\frac{4\big(3N^2+6N+4\big)}{3(N+1)^2}+\frac{4S_1[N]}{3(N+1)}\Big)+\frac{1}{\varepsilon^2}\Big(-\frac{2(20N^3+58N^2+57N+22)}{3(N+1)^3}+\frac{2(N+2)(2N-1)S_1[N]}{3(N+1)^2}-\frac{S_1[N]^2}{N+1}-\frac{S_2[N]}{N+1}\Big),$$
$$\frac{4}{3\varepsilon^3}-\frac{2}{\varepsilon^2}, \frac{8}{3\varepsilon^3}+\frac{1}{\varepsilon^2}\Big(-\frac{4\big(4N^2+7N+2\big)}{3(N+1)^2}+\frac{4(N+2)S_1[N]}{3(N+1)}\Big)\Big\}$$

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