Symbolic summation packages for elementary particle physics

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A general tactic

Feynman integrals
A general tactic

Feynman integrals

↓ non-trivial transformations (DESY)

multiple sums
A general tactic

Feynman integrals

↓ non-trivial transformations (DESY)

multiple sums

↓ symbolic summation

compact expression in terms of special functions
A general tactic

Feynman integrals

\[ \downarrow \text{non-trivial transformations (DESY)} \]

Multiple sums

\[ \downarrow \text{symbolic summation} \]

Compact expression in terms of special functions

Tactic 1: Expand the summand and simplify
Tactic 1: Expand and simplify

\[ F(N) = \sum_{k=1}^{N} (-1)^k e^{-\frac{3\varepsilon \gamma}{2}} \left( -2 - \frac{3\varepsilon}{2} \right)! \times \]
\[ \times B \left( 2 + k, \frac{\varepsilon}{2} \right) B \left( -\varepsilon + k, -\varepsilon \right) B \left( 1 - \frac{\varepsilon}{2} + k, 1 + \frac{\varepsilon}{2} \right) \binom{N}{k} \]

where

\[ B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a + b)}. \]
Tactic 1: Expand and simplify

GIVEN

\[ F(N) = \sum_{k=1}^{N} (-1)^k e^{-\frac{3\varepsilon \gamma}{2}} \left( -2 - \frac{3\varepsilon}{2} \right)! \times \]
\[ \times B\left(2 + k, \frac{\varepsilon}{2}\right) B\left(-\varepsilon + k, -\varepsilon\right) B\left(1 - \frac{\varepsilon}{2} + k, 1 + \frac{\varepsilon}{2}\right) \binom{N}{k} \]
\[ f(N, k) \]

FIND the first coefficients of the \( \varepsilon \)-expansion

\[ F(N) \equiv F_{-3}(N)\varepsilon^{-3} + F_{-2}(N)\varepsilon^{-2} + F_{-1}(N)\varepsilon^{-1} + \ldots \]
Tactic 1: Expand and simplify

Given

\[ F(N) = \sum_{k=1}^{N} (-1)^k e^{-\frac{3\varepsilon \gamma}{2}} \left( -2 - \frac{3\varepsilon}{2} \right)! \times \]
\[ \times B\left(2 + k, \frac{\varepsilon}{2}\right) B\left(-\varepsilon + k, -\varepsilon\right) B\left(1 - \frac{\varepsilon}{2} + k, 1 + \frac{\varepsilon}{2}\right) \binom{N}{k} \]

\[ f(N, k) \]

Step 1: Compute the first coefficients of the \( \varepsilon \)-expansion

\[ f(N, k) = f_{-3}(N, k) \varepsilon^{-3} + f_{-2}(N, k) \varepsilon^{-2} + f_{-1}(N, k) \varepsilon^{-1} + \]
Tactic 1: Expand and simplify

\textbf{Step 2: Simplify} the sums in

\[ \sum_{k=1}^{N} f(N, k) = \left( \sum_{k=1}^{N} f_{-3}(N, k) \right) \varepsilon^{-3} + \left( \sum_{k=1}^{N} f_{-2}(N, k) \right) \varepsilon^{-2} + \left( \sum_{k=1}^{N} f_{-1}(N, k) \right) \varepsilon^{-1} + \text{...} \]
Simplify

\[ F_{-1}(N) = \sum_{k=1}^{N} (-1)^{k+1} \binom{N}{k} \left( \frac{(2 + 3k)(-2 + 3k + 7k^2 + 3k^3)}{3k^2(1 + k)^3} + \frac{2S_2(k)}{1 + k} + \frac{\zeta(2)}{2(1 + k)} \right) \]

where

\[ S_a(N) = \sum_{i=1}^{N} \frac{\text{sign}(a)^i}{i^a} \text{ and } \zeta(a) = \sum_{i=1}^{\infty} \frac{1}{i^a} \]
Simplify

\[ F_{-1}(N) = \sum_{k=1}^{N} (-1)^{k+1} \binom{N}{k} \left( \frac{2 + 3k}{3k^2(1 + k)^3} \left( -2 + 3k + 7k^2 + 3k^3 \right) \right) + \frac{2S_2(k)}{1 + k} + \frac{\zeta(2)}{2(1 + k)} \]

\[ \downarrow \text{ (summation package Sigma.m)} \]

\[ (16N^3 + 144N^2 + 413N + 384)(N + 1)^2 F_{-1}(N) \]

\[ - (N + 2)(2N + 5)(16N^3 + 112N^2 + 221N + 113) F_{-1}(N + 1) \]

\[ + (N + 3)^2 (16N^3 + 96N^2 + 173N + 99) F_{-1}(N + 2) \]

\[ = \frac{1}{2} (4N^2 + 21N + 29) \zeta(2) + \frac{-64N^5 - 500N^4 - 1133N^3 + 203N^2 + 3516N + 3090}{3(N+2)(N+3)} \]
Simplify

\[F_{-1}(N) = \sum_{k=1}^{N} (-1)^{k+1} \binom{N}{k} \left( \frac{(2 + 3k)(-2 + 3k + 7k^2 + 3k^3)}{3k^2(1 + k)^3} + \frac{2S_2(k)}{1 + k} + \frac{\zeta(2)}{2(1 + k)} \right) \]

\[\downarrow \text{(summation package Sigma.m)}\]

\[(16N^3 + 144N^2 + 413N + 384)(N+1)^2F_{-1}(N) \]
\[- (N + 2)(2N + 5)(16N^3 + 112N^2 + 221N + 113)F_{-1}(N + 1) \]
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\[= \frac{1}{2}(4N^2 + 21N + 29)\zeta(2) + \frac{-64N^5 - 500N^4 - 1133N^3 + 203N^2 + 3516N + 3090}{3(N+2)(N+3)} \]

\[\downarrow \text{(summation package Sigma.m)}\]

\[\begin{cases} 
  c_1 \frac{1 - 4N}{N + 1} + c_2 \frac{-14N - 13}{(N + 1)^2} \\
  + \frac{(4N - 1)S_1(N)}{N + 1} + \frac{(1 - 4N)S_1(N)^2}{6(N + 1)} + \frac{(14N + 13)S_1(N)}{3(N + 1)^2} \\
  + \frac{175N^2 + 334N + 155}{12(N + 1)^3} + \frac{(1 - 4N)S_2(N)}{6(N + 1)} + \frac{\zeta(2)}{8(N + 1)} \end{cases} \quad |c_1, c_2 \in \mathbb{Q}|\]
Simplify

\[ F_{-1}(N) = \sum_{k=1}^{N} (-1)^{k+1} \binom{N}{k} \left( \frac{2 + 3k}{3k^2(1 + k)^3} - \frac{2 + 3k + 7k^2 + 3k^3}{3k^2(1 + k)^3} \right) + \frac{2S_2(k)}{1 + k} + \frac{\zeta(2)}{2(1 + k)} \]

\[ \{ c_1 \frac{1 - 4N}{N + 1} + c_2 \frac{-14N - 13}{(N + 1)^2} \]

\[ + \frac{(4N - 1)S_1(N)}{N + 1} + \frac{(1 - 4N)S_1(N)^2}{6(N + 1)} + \frac{(14N + 13)S_1(N)}{3(N + 1)^2} \]

\[ + \frac{175N^2 + 334N + 155}{12(N + 1)^3} + \frac{(1 - 4N)S_2(N)}{6(N + 1)} + \frac{\zeta(2)}{8(N + 1)} \mid c_1, c_2 \in \mathbb{Q} \]
Simplify

\[ F_{-1}(N) = \sum_{k=1}^{N} (-1)^{k+1} \binom{N}{k} \left( \frac{2 + 3k}{3k^2(1 + k)^3} \right) \left( -2 + 3k + 7k^2 + 3k^3 \right) + \frac{2S_2(k)}{1 + k} + \frac{\zeta(2)}{2(1 + k)} \]

\[
\left( \frac{1}{12} - \frac{1}{8} \zeta(2) \right) \frac{1 - 4N}{N + 1} + 1 \frac{-14N - 13}{(N + 1)^2} \\
+ \frac{(4N - 1)S_1(N)}{N + 1} + \frac{(1 - 4N)S_1(N)^2}{6(N + 1)} + \frac{(14N + 13)S_1(N)}{3(N + 1)^2} \\
+ \frac{175N^2 + 334N + 155}{12(N + 1)^3} + \frac{(1 - 4N)S_2(N)}{6(N + 1)} + \frac{\zeta(2)}{8(N + 1)}
\]
1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger’s algorithm (1991))

GIVEN a definite sum

\[ F(N) = \sum_{k=0}^{N} f(N, k); \]

\( f(N, k) \): indefinite nested product-sum in \( k \);
\( N \): extra parameter

FIND a recurrence for \( F(N) \)
1. **Creative telescoping** (for the special case of hypergeometric terms see Zeilberger’s algorithm (1991))

   **GIVEN** a definite sum

   \[ F(N) = \sum_{k=0}^{N} f(N, k); \]

   \( f(N, k) \): indefinite nested product-sum in \( k \);

   \( N \): extra parameter

   **FIND** a recurrence for \( F(N) \)

2. **Recurrence solving**

   **GIVEN** a recurrence

   \[ a_0(N), \ldots, a_d(N), h(N): \]

   indefinite nested product-sum expressions.

   \[ a_0(N)F(N) + \cdots + a_d(N)F(N + d) = h(N); \]

   **FIND** all solutions expressible by indefinite nested products/sums

   (Abramov/Bronstein/Petkovšek/CS, in preparation)
Tactic 1: Expand and simplify

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger’s algorithm (1991))

GIVEN a definite sum

\[ F(N) = \sum_{k=0}^{N} f(N, k); \]

\( f(N, k) \): indefinite nested product-sum in \( k \);
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FIND a recurrence for \( F(N) \)

2. Recurrence solving

GIVEN a recurrence

\[ a_0(N), \ldots, a_d(N), h(N): \]

indefinite nested product-sum expressions.

\[ a_0(N)F(N) + \cdots + a_d(N)F(N + d) = h(N); \]

FIND all solutions expressible by indefinite nested products/sums

(Abramov/Bronstein/Petkovšek/CS, in preparation)

3. Find a “closed form”

\[ F(N) = \text{combined solutions in terms of indefinite nested sums.} \]
Tactic 1: Expand and simplify

In[1]:= << Sigma.m
Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= << HarmonicSums.m
HarmonicSums by Jakob Ablinger © RISC-Linz

In[3]:= << EvaluateMultiSums.m
EvaluateMultiSums by Carsten Schneider © RISC-Linz
Tactic 1: Expand and simplify

```plaintext
In[1]:= << Sigma.m
Sigma - A summation package by Carsten Schneider © RISC-Linz

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HarmonicSums by Jakob Ablinger © RISC-Linz

In[3]:= << EvaluateMultiSums.m
EvaluateMultiSums by Carsten Schneider © RISC-Linz

In[4]:= mySum =
\[\sum_{k=1}^{N} (-1)^{k} e^{-\frac{3\varepsilon\gamma}{2}} \left(-2 - \frac{3\varepsilon}{2}\right)! B\left(2 + k, \frac{\varepsilon}{2}\right) B\left(-\varepsilon + k, -\varepsilon\right) B\left(1 - \frac{\varepsilon}{2} + k, 1 + \frac{\varepsilon}{2}\right) \binom{N}{k}\];

In[5]:= EvaluateMultiSum[mySum, {}, {N}, {1}, ExpandIn -> {\varepsilon, -3, -3}]
```
Tactic 1: Expand and simplify

```
In[1]:= << Sigma.m
Sigma - A summation package by Carsten Schneider © RISC-Linz

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In[4]:= mySum =
\[\sum_{k=1}^{N} (-1)^k e^{-\frac{3\varepsilon\gamma}{2}} \left(-2 - \frac{3\varepsilon}{2}\right)! B\left(2 + k, \frac{\varepsilon}{2}\right) B(-\varepsilon + k, -\varepsilon) B\left(1 - \frac{\varepsilon}{2} + k, 1 + \frac{\varepsilon}{2}\right) \binom{N}{k};\]

In[5]:= EvaluateMultiSum[mySum, {}, {N}, {1}, ExpandIn \rightarrow \{\varepsilon, -3, -3\}]

Out[5]= \left\{ \frac{59N^2 + 120N + 49}{9(N + 1)^2} - \frac{2(N + 3)S_1[N]}{3(N + 1)} \right\}
```
Tactic 1: Expand and simplify

In[1]:= << Sigma.m
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In[4]:= mySum =
\[\sum_{k=1}^{N} (-1)^k e^{-\frac{3\epsilon\gamma}{2}} (-2 - \frac{3\epsilon}{2})!B\left(2 + k, \frac{\epsilon}{2}\right)B(-\epsilon + k, -\epsilon)B\left(1 - \frac{\epsilon}{2} + k, 1 + \frac{\epsilon}{2}\right)\left(\begin{array}{c}N \\ k\end{array}\right)\];

In[5]:= EvaluateMultiSum[mySum, {}, {N}, {1}, ExpandIn \rightarrow \{\epsilon, -3, -2\}]

Out[5]= \\begin{align*}
{\frac{59N^2 + 120N + 49}{9(N + 1)^2}} & - \frac{2(N + 3)S_1[N]}{3(N + 1)} \\
- \frac{2(20N^3 + 58N^2 + 57N + 22)}{3(N + 1)^3} + \frac{2(N + 2)(2N - 1)S_1[N]}{3(N + 1)^2} & - \frac{S_1[N]^2}{N + 1} - \frac{S_2[N]}{N + 1}
\end{align*}
Tactic 1: Expand and simplify

In[1]:= << Sigma.m
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In[4]:= mySum = 
\[ \sum_{k=1}^{N} (-1)^k e^{-\frac{3\varepsilon \gamma}{2}} \left(-2 - \frac{3\varepsilon}{2}\right)!B(2 + k, \frac{\varepsilon}{2})B(-\varepsilon + k, -\varepsilon)B\left(1 - \frac{\varepsilon}{2} + k, 1 + \frac{\varepsilon}{2}\right) \left(\frac{N}{k}\right) \]

In[5]:= EvaluateMultiSum[mySum, {}, {N}, {1}, ExpandIn \rightarrow \{\varepsilon, -3, -1\}]

Out[5]= \{ \frac{59N^2 + 120N + 49}{9(N + 1)^2} - \frac{2(N + 3)S_1[N]}{3(N + 1)}, \\
\frac{2(20N^3 + 58N^2 + 57N + 22)}{3(N + 1)^3} + \frac{2(N + 2)(2N - 1)S_1[N]}{3(N + 1)^2} - \frac{S_1[N]^2}{N + 1} - \frac{S_2[N]}{N + 1}, \\
\frac{1}{12} - \frac{1}{8}\zeta(2) - \frac{1 - 4N}{N + 1} - \frac{14N - 13}{(N + 1)^2} + \frac{(4N - 1)S_1(N)}{N + 1} + \frac{(1 - 4N)S_1(N)^2}{6(N + 1)} + \\
\frac{14N + 13)S_1(N)}{3(N + 1)^2} + \frac{175N^2 + 334N + 155}{12(N + 1)^3} + \frac{(1 - 4N)S_2(N)}{6(N + 1)} + \frac{\zeta(2)}{8(N + 1)} \}
Consider a massive 3–loop ladder graph (Ablinger, Blümlein, Hasselhuhn, Klein, CS, Wissbrock, 2012)

\[ F_{-3}(N) \varepsilon^{-3} + F_{-2}(N) \varepsilon^{-2} + F_{-1}(N) \varepsilon^{-1} + [F_0(N)] \]
Consider a massive 3–loop ladder graph (Ablinger, Blümlein, Hasselhuhn, Klein, CS, Wissbrock, 2012)

\[ = F_{-3}(N)\varepsilon^{-3} + F_{-2}(N)\varepsilon^{-2} + F_{-1}(N)\varepsilon^{-1} + F_0(N) \]

\[ \sum_{j=0}^{N-3} \sum_{k=0}^{j+N-3} \sum_{l=0}^{-j+N-3} \sum_{q=0}^{l+N-q-3} \sum_{s=1}^{-l+N-q-s-3} \sum_{r=0}^{l+N-q-s-3} (-1)^{j+k+l+N-q-3} \times \]

\[ \frac{(j+1)(k)(N-1)}{(k+1)(l)(j+2)(q)} \frac{(-j+N-3)}{(q)} \frac{(-l+N-q-3)}{s} \frac{(-l+N-q-s-3)}{r} r!(l+N-q-r-s-3)!(s-1)! \]

\[ (-l+N-q-2)!(N-q-r-s-2)(q+s+1) \]

\[ 4S_1(-j+N-1) - 4S_1(-j+N-2) - 2S_1(k) \]

\[ - (S_1(-l+N-q-2) + S_1(-l+N-q-r-s-3) - 2S_1(r+s)) \]

\[ + 2S_1(s-1) - 2S_1(r+s) \]

+ 3 further 6–fold sums
$$F_0(N) = \frac{7}{12} S_1(N)^4 + \frac{(17N + 5)S_1(N)^3}{3N(N + 1)} + \frac{(35N^2 - 2N - 5)}{2N^2(N + 1)^2} + \frac{13S_2(N)}{2} + \frac{5(-1)^N}{2N^2} S_1(N)^2$$

$$+ \left( - \frac{4(13N + 5)}{N^2(N + 1)^2} + \left( \frac{4((-1)^N(2N + 1)}{N(N + 1)} - \frac{13}{N}\right) S_2(N) + \left( \frac{29}{3} - (-1)^N\right) S_3(N) \right)$$

$$+ \left( 2 + 2(-1)^N \right) S_{2,1}(N) - 28S_{-2,1}(N) + \frac{20(-1)^N}{N^2(N + 1)} S_1(N) + \left( \frac{3}{4} + (-1)^N \right) S_2(N)^2$$

$$- 2(-1)^N S_{-2}(N)^2 + S_{-3}(N) \left( \frac{2(3N - 5)}{N(N + 1)} + (26 + 4(-1)^N) S_1(N) + \frac{4(-1)^N}{N + 1} \right)$$

$$+ \left( \frac{(-1)^N(5 - 3N)}{2N^2(N + 1)} - \frac{5}{2N^2} \right) S_2(N) + S_{-2}(N) \left( 10S_1(N)^2 + \frac{8(-1)^N(2N + 1)}{N(N + 1)} \right)$$

$$+ \frac{4(3N - 1)}{N(N + 1)} S_1(N) + \frac{8(-1)^N(3N + 1)}{N(N + 1)^2} + \left( - 22 + 6(-1)^N \right) S_2(N) - \frac{16}{N(N + 1)}$$

$$+ \left( \frac{(-1)^N(9N + 5)}{N(N + 1)} - \frac{29}{3N} \right) S_3(N) + \left( \frac{19}{2} - 2(-1)^N \right) S_4(N) + \left( - 6 + 5(-1)^N \right) S_{-4}(N)$$

$$+ \left( - \frac{2(-1)^N(9N + 5)}{N(N + 1)} - \frac{2}{N} \right) S_{2,1}(N) + \left( 20 + 2(-1)^N \right) S_{2,-2}(N) + \left( - 17 + 13(-1)^N \right) S_{3,1}(N)$$

$$- \frac{8(-1)^N(2N + 1) + 4(9N + 1)}{N(N + 1)} S_{-2,1}(N) - \left( 24 + 4(-1)^N \right) S_{-3,1}(N) + \left( 3 - 5(-1)^N \right) S_{2,1,1}(N)$$

$$+ 32S_{-2,1,1}(N) + \left( \frac{3}{2} S_1(N)^2 - \frac{3S_1(N)}{N} + \frac{3}{2} (-1)^N S_{-2}(N) \right) \zeta(2)$$
\[
F_0(N) = \frac{7}{12} S_1(N)^4 + \frac{(17N + 5)S_1(N)^3}{3N(N + 1)} + \left(\frac{35N^2 - 2N - 5}{2N^2(N + 1)^2}\right) + \frac{13S_2(N)}{2} + \frac{5(-1)^N}{2N^2}S_1(N)^2
\]

\[
+ \left( \sum_{i=1}^{N} \frac{1}{i} \right) \left( \frac{1}{N(N + 1)} \right) - \frac{13}{N}S_2(N) + \left( \frac{29}{3} - (-1)^N \right) S_3(N)
\]

\[
+ (2 - 28S_{-2,1}(N) + \frac{20(-1)^N}{N^2(N + 1)}) S_1(N) + \left( \frac{3}{4} + (-1)^N \right) S_2(N)^2
\]

\[
- 2(-1)^N S_{-2}(N)^2 + S_{-3}(N) \left( \frac{2(3N - 5)}{N(N + 1)} \right) + (26 + 4(-1)^N) S_1(N) + \frac{4(-1)^N}{N + 1}
\]

\[
+ \left( \frac{(-1)^N(5 - 3N)}{2N^2(N + 1)} - \frac{5}{2N^2} \right) S_2(N) + S_{-2}(N) \left( 10S_1(N)^2 + \frac{8(-1)^N(2N + 1)}{N(N + 1)} \right)
\]

\[
+ \frac{4(3N - 1)}{N(N + 1)} S_1(N) + \frac{8(-1)^N(3N + 1)}{N(N + 1)^2} + \left( - 22 + 6(-1)^N \right) S_2(N) - \frac{16}{N(N + 1)}
\]

\[
+ \left( \frac{(-1)^N(9N + 5)}{N(N + 1)} - \frac{29}{3N} \right) S_3(N) + \left( \frac{19}{2} - 2(-1)^N \right) S_4(N) + \left( - 6 + 5(-1)^N \right) S_{-4}(N)
\]

\[
+ \left( - \frac{2(-1)^N(9N + 5)}{N(N + 1)} - \frac{2}{N} \right) S_{2,1}(N) + (20 + 2(-1)^N) S_{2,-2}(N) + \left( - 17 + 13(-1)^N \right) S_{3,1}(N)
\]

\[
- \frac{8(-1)^N(2N + 1) + 4(9N + 1)}{N(N + 1)} S_{-2,1}(N) - \left( 24 + 4(-1)^N \right) S_{-3,1}(N) + \left( 3 - 5(-1)^N \right) S_{2,1,1}(N)
\]

\[
+ 32S_{-2,1,1}(N) + \left( \frac{3}{2} S_1(N)^2 - \frac{3S_1(N)}{N} + \frac{3}{2} (-1)^N S_{-2}(N) \right) \zeta(2)
\]
Tactic 1: Expand and simplify

\[
F_0(N) = \frac{7}{12} S_1(N)^4 + \frac{(17N + 5)S_1(N)^3}{3N(N + 1)} + \left( \frac{35N^2 - 2N - 5}{2N^2(N + 1)^2} \right) + \frac{13S_2(N)}{2} + \frac{5(-1)^N}{2N^2} S_1(N)^2 \\
+ \left( \frac{(-1)^N(2N + 1)}{N(N + 1)} \right) S_2(N) + \left( \frac{29}{3} - (-1)^N \right) S_3(N) \\
+ (2 + \frac{1}{2} \sum_{i=1}^{N} \frac{1}{i^2}) S_2(N) + \frac{20(-1)^N}{N^2(N + 1)^2} S_2(N)^2 \\
- 2(-1)^N S_2(N)^2 + S_3(N) \left( \frac{2(3N - 5)}{N(N + 1)} + (26 + 4(-1)^N) \right) \\
+ \left( \frac{(-1)^N(5 - 3N)}{2N^2(N + 1)} - \frac{5}{2N^2} \right) S_2(N) + S_2(N) \left( 10S_1(N)^2 + \frac{20(-1)^N}{N(N + 1)} \right) \\
+ \left( \frac{4(3N - 1)}{N(N + 1)} \right) S_1(N) + \frac{8(-1)^N(3N + 1)}{N(N + 1)^2} + \left( -22 + 6(-1)^N \right) S_2(N) - \frac{16}{N(N + 1)} \\
+ \left( \frac{(-1)^N(9N + 5)}{N(N + 1)} - \frac{29}{3N} \right) S_3(N) + \left( \frac{19}{2} - 2(-1)^N \right) S_4(N) + \left( -6 + 5(-1)^N \right) S_4(N) \\
+ \left( - \frac{2(-1)^N(9N + 5)}{N(N + 1)} - \frac{2}{N} \right) S_{2,1}(N) + \left( 20 + 2(-1)^N \right) S_{2,-2}(N) + \left( -17 + 13(-1)^N \right) S_{3,1}(N) \\
- \frac{8(-1)^N(2N + 1) + 4(9N + 1)}{N(N + 1)} S_{-2,1}(N) - \left( 24 + 4(-1)^N \right) S_{-3,1}(N) + \left( 3 - 5(-1)^N \right) S_{2,1,1}(N) \\
+ 32S_{-2,1,1}(N) + \left( \frac{3}{2} S_1(N)^2 - \frac{3S_1(N)}{N} + \frac{3}{2}(-1)^N S_2(N) \right) \zeta(2)
\]
\[
F_0(N) = \\
\frac{7}{12} S_1(N)^4 + \frac{(17N + 5)S_1(N)^3}{3N(N + 1)} + \frac{35N^2 - 2N - 5}{2N^2(N + 1)^2} + \frac{13S_2(N)}{2} + \frac{5(-1)^N}{2N^2} S_1(N)^2 \\
+ (-1)^N \sum_{i=1}^{N} \frac{1}{i} + \left( -\frac{1}{N(N + 1)} \right) S_2(N) + \left( \frac{29}{3} - (-1)^N \right) S_3(N) \\
+ (2 - \frac{1}{N(N + 1)}) S_{2,1}(N) + \frac{20(-1)^N}{N^2(N + 1)^2} S_{2,1}(N)^2 \\
- 2(-1)^N S_{-2}(N)^2 + S_{-3}(N) \left( \frac{2(3N - 5)}{N(N + 1)} + (26 + 4(-1)^N) \right) S_2(N) + S_{-2}(N) \left( 10S_1(N)^2 + \left( \frac{8(1 - (2N + 1)}{N(N + 1)} \right) \right) \\
+ \left( \frac{(-1)^N(5 - 3N)}{2N^2(N + 1)^2} - \frac{5}{2N^2} \right) S_2(N) + S_{-2}(N) \left( 10S_1(N)^2 + \left( \frac{8(1 - (2N + 1)}{N(N + 1)} \right) \right) \\
+ \frac{4(3N - 5)}{N(N + 1)} S_{-2}(N) + \left( \frac{(-1)^N}{N(N + 1)} \right) S_{-2}(N) + \left( \frac{(-1)^N}{N(N + 1)} \right) S_{-2}(N) \\
+ \left( \frac{(-1)^N}{N(N + 1)} \right) S_{-2}(N) + \left( \frac{(-1)^N}{N(N + 1)} \right) S_{-2}(N) \\
- 8(-1)^N \left( \frac{32S_{-2,1,1}(N) + \left( \frac{3}{2} \right) S_1(N)^2 - \frac{3S_1(N)}{N} + \frac{3}{2} (-1)^N S_{-2}(N) \right) \right) \zeta(2)
\]
Tactic 2: Expand the recurrence

A general tactic

- Feynman integrals
  - non-trivial transformations (DESY)
  - multiple sums
  - symbolic summation
  - compact expression in terms of special functions

Tactic 2: Expand a recurrence in $\varepsilon$
Tactic 2: Expand the recurrence

\[ F(N) = \sum_{k=1}^{N} (-1)^k e^{-3\varepsilon \gamma \frac{N}{2}} \left( -2 - \frac{3\varepsilon}{2} \right)! B(2 + k, \frac{\varepsilon}{2}) B(-\varepsilon + k, -\varepsilon) B(1 - \frac{\varepsilon}{2} + k, 1 + \frac{\varepsilon}{2}) \binom{N}{k} \]

\[ \downarrow \text{(summation package Sigma.m)} \]

\[ 2(N + 1)^2 F(N) + (3\varepsilon^2 + 3\varepsilon N + 9\varepsilon - 4N^2 - 12N - 8) F(N + 1) \]
\[ - (2\varepsilon - N - 1)(\varepsilon + 2N + 6) F(N + 2) = 0\varepsilon^{-3} - \frac{16}{3}\varepsilon^{-2} + \frac{40}{3}\varepsilon^{-1} - (2\zeta_2 - \frac{68}{3})\varepsilon^0 + \ldots \]
Tactic 2: Expand the recurrence

\[ F(N) = \sum_{k=1}^{N} (-1)^k e^{-\frac{3\varepsilon\gamma}{2}} \left( -2 - \frac{3\varepsilon}{2} \right)! B\left(2 + k, \frac{\varepsilon}{2}\right) B(-\varepsilon + k, -\varepsilon) B\left(1 - \frac{\varepsilon}{2} + k, 1 + \frac{\varepsilon}{2}\right) \binom{N}{k} \]

\[ \downarrow \text{(summation package Sigma.m)} \]

\[ 2(N + 1)^2 F(N) + \left(3\varepsilon^2 + 3\varepsilon N + 9\varepsilon - 4N^2 - 12N - 8\right) F(N + 1) \\
-(2\varepsilon - N - 1)(\varepsilon + 2N + 6) F(N + 2) = 0\varepsilon^{-3} - \frac{16}{3}\varepsilon^{-2} + \frac{40}{3}\varepsilon^{-1} - \left(2\zeta_2 - \frac{68}{3}\right)\varepsilon^0 + \ldots \]

\[ F(1) = \frac{2}{3}\varepsilon^{-3} - \frac{11}{6}\varepsilon^{-2} + \left(\frac{\zeta_2}{4} + \frac{79}{24}\right)\varepsilon^{-1} + \ldots \]

\[ F(2) = \frac{8}{9}\varepsilon^{-3} - \frac{73}{27}\varepsilon^{-2} + \left(\frac{\zeta_2}{3} + \frac{1415}{324}\right)\varepsilon^{-1} + \ldots \]
Computing the $\varepsilon$-expansion from a recurrence relation

\[ a_0(\varepsilon, N) [I(N)] + a_1(\varepsilon, N) [I(N + 1)] + \cdots + a_d(\varepsilon, N) [I(N + d)] = h_{-3}(N) \varepsilon^{-3} + h_{-2}(N) \varepsilon^{-2} + h_{-1}(N) \varepsilon^{-1} + \ldots \]

given (in terms of indefinite nested sums and products)
Computing the $\varepsilon$-expansion from a recurrence relation

$$a_0(\varepsilon, N) \left[ I_{-3}(N)\varepsilon^{-3} + I_{-2}(N)\varepsilon^{-2} + I_{-1}(N)\varepsilon^{-1} + \ldots \right]$$

$$+ a_1(\varepsilon, N) \left[ I(N + 1) \right]$$

$$+ \ldots$$

$$+ a_d(\varepsilon, N) \left[ I(N + d) \right]$$

$$= h_{-3}(N)\varepsilon^{-3} + h_{-2}(N)\varepsilon^{-2} + h_{-1}(N)\varepsilon^{-1} + \ldots$$

given (in terms of indefinite nested sums and products)
Computing the $\varepsilon$-expansion from a recurrence relation

\[
\begin{align*}
& a_0(\varepsilon, N) \left[ I_{-3}(N)\varepsilon^{-3} + I_{-2}(N)\varepsilon^{-2} + I_{-1}(N)\varepsilon^{-1} + \ldots \right] \\
& + a_1(\varepsilon, N) \left[ I_{-3}(N + 1)\varepsilon^{-3} + I_{-2}(N + 1)\varepsilon^{-2} + I_{-1}(N + 1)\varepsilon^{-1} + \ldots \right] \\
& + \cdots \\
& + a_d(\varepsilon, N) \left[ I(N + d) \right] \\
& = h_{-3}(N)\varepsilon^{-3} + h_{-2}(N)\varepsilon^{-2} + h_{-1}(N)\varepsilon^{-1} + \ldots 
\end{align*}
\]

given (in terms of indefinite nested sums and products)
Tactic 2: Expand the recurrence

Computing the $\varepsilon$-expansion from a recurrence relation

$$a_0(\varepsilon, N) \left[ I_{-3}(N)\varepsilon^{-3} + I_{-2}(N)\varepsilon^{-2} + I_{-1}(N)\varepsilon^{-1} + \ldots \right]$$

$$+ a_1(\varepsilon, N) \left[ I_{-3}(N + 1)\varepsilon^{-3} + I_{-2}(N + 1)\varepsilon^{-2} + I_{-1}(N + 1)\varepsilon^{-1} + \ldots \right]$$

$$+ \ldots$$

$$+ a_d(\varepsilon, N) \left[ I_{-3}(N + d)\varepsilon^{-3} + I_{-2}(N + d)\varepsilon^{-2} + I_{-1}(N + d)\varepsilon^{-1} + \ldots \right]$$

$$= h_{-3}(N)\varepsilon^{-3} + h_{-2}(N)\varepsilon^{-2} + h_{-1}(N)\varepsilon^{-1} + \ldots$$

given (in terms of indefinite nested sums and products)
Computing the $\varepsilon$-expansion from a recurrence relation

\[
a_0(\varepsilon, N) \left[ I_{-3}(N)\varepsilon^{-3} + I_{-2}(N)\varepsilon^{-2} + I_{-1}(N)\varepsilon^{-1} + \ldots \right]
+ a_1(\varepsilon, N) \left[ I_{-3}(N + 1)\varepsilon^{-3} + I_{-2}(N + 1)\varepsilon^{-2} + I_{-1}(N + 1)\varepsilon^{-1} + \ldots \right]
+ \ldots
+ a_d(\varepsilon, N) \left[ I_{-3}(N + d)\varepsilon^{-3} + I_{-2}(N + d)\varepsilon^{-2} + I_{-1}(N + d)\varepsilon^{-1} + \ldots \right]
= h_{-3}(N)\varepsilon^{-3} + h_{-2}(N)\varepsilon^{-2} + h_{-1}(N)\varepsilon^{-1} + \ldots
\]

\[
\downarrow \text{lowest terms must agree}
\]

\[
a_0(0, N)I_{-3}(N) + a_1(0, N)I_{-3}(N + 1) + \cdots + a_d(0, N)I_{-3}(N + d) = h_{-3}(N)
\]
Computing the $\varepsilon$-expansion from a recurrence relation

$$a_0(\varepsilon, N) \left[ I_{-3}(N)\varepsilon^{-3} + I_{-2}(N)\varepsilon^{-2} + I_{-1}(N)\varepsilon^{-1} + \ldots \right]$$

$$+ a_1(\varepsilon, N) \left[ I_{-3}(N+1)\varepsilon^{-3} + I_{-2}(N+1)\varepsilon^{-2} + I_{-1}(N+1)\varepsilon^{-1} + \ldots \right]$$

$$+ \cdots$$

$$+ a_d(\varepsilon, N) \left[ I_{-3}(N+d)\varepsilon^{-3} + I_{-2}(N+d)\varepsilon^{-2} + I_{-1}(N+d)\varepsilon^{-1} + \ldots \right]$$

$$= h_{-3}(N)\varepsilon^{-3} + h_{-2}(N)\varepsilon^{-2} + h_{-1}(N)\varepsilon^{-1} + \ldots$$

$\downarrow$ lowest terms must agree

$$a_0(0,N)I_{-3}(N) + a_1(0,N)I_{-3}(N+1) + \cdots + a_d(0,N)I_{-3}(N+d) = h_{-3}(N)$$

REC solver: Given the initial values $I_{-3}(1), I_{-3}(2), \ldots, I_{-3}(d)$, decide if $I_{-3}(N)$ can be written in terms of indefinite nested sums and products.
Computing the $\varepsilon$-expansion from a recurrence relation

\[
a_0(\varepsilon, N) \left[ I_{-3}(N)\varepsilon^{-3} + I_{-2}(N)\varepsilon^{-2} + I_{-1}(N)\varepsilon^{-1} + \ldots \right] \\
+ a_1(\varepsilon, N) \left[ I_{-3}(N + 1)\varepsilon^{-3} + I_{-2}(N + 1)\varepsilon^{-2} + I_{-1}(N + 1)\varepsilon^{-1} + \ldots \right] \\
+ \ldots \\
+ a_d(\varepsilon, N) \left[ I_{-3}(N + d)\varepsilon^{-3} + I_{-2}(N + d)\varepsilon^{-2} + I_{-1}(N + d)\varepsilon^{-1} + \ldots \right]
\]

\[= \ h_{-3}(N)\varepsilon^{-3} + h_{-2}(N)\varepsilon^{-2} + h_{-1}(N)\varepsilon^{-1} + \ldots \]

\[\downarrow \] lowest terms must agree

\[
a_0(0, N)I_{-3}(N) + a_1(0, N)I_{-3}(N+1) + \cdots + a_d(0, N)I_{-3}(N+d) = h_{-3}(N)
\]
Computing the $\varepsilon$-expansion from a recurrence relation

\[
a_0(\varepsilon, N) \left[ I_{-3}(N)\varepsilon^{-3} + I_{-2}(N)\varepsilon^{-2} + I_{-1}(N)\varepsilon^{-1} + \ldots \right]
\]

\[
+a_1(\varepsilon, N) \left[ I_{-3}(N+1)\varepsilon^{-3} + I_{-2}(N+1)\varepsilon^{-2} + I_{-1}(N+1)\varepsilon^{-1} + \ldots \right]
\]

\[\vdots\]

\[
+a_d(\varepsilon, N) \left[ I_{-3}(N+d)\varepsilon^{-3} + I_{-2}(N+d)\varepsilon^{-2} + I_{-1}(N+d)\varepsilon^{-1} + \ldots \right]
\]

\[= h_{-3}(N)\varepsilon^{-3} + h_{-2}(N)\varepsilon^{-2} + h_{-1}(N)\varepsilon^{-1} + \ldots\]

\[
\Downarrow \text{lowest terms must agree}
\]

\[
a_0(0,N)I_{-3}(N) + a_1(0,N)I_{-3}(N+1) + \cdots + a_d(0,N)I_{-3}(N+d) = h_{-3}(N)
\]
Computing the $\varepsilon$-expansion from a recurrence relation

\[
\begin{align*}
  & a_0(\varepsilon, N) \left[ I_{-2}(N)\varepsilon^{-2} + I_{-1}(N)\varepsilon^{-1} + \ldots \right] \\
  & + a_1(\varepsilon, N) \left[ I_{-2}(N+1)\varepsilon^{-2} + I_{-1}(N+1)\varepsilon^{-1} + \ldots \right] \\
  & + \ldots \\
  & + a_d(\varepsilon, N) \left[ I_{-2}(N+d)\varepsilon^{-2} + I_{-1}(N+d)\varepsilon^{-1} + \ldots \right] \\
  & = h_{-3}'(N)\varepsilon^{-3} + h_{-2}'(N)\varepsilon^{-2} + h_{-1}'(N)\varepsilon^{-1} + \ldots
\end{align*}
\]
Computing the $\varepsilon$-expansion from a recurrence relation

$$a_0(\varepsilon, N) \left[ I_{-2}(N)\varepsilon^{-2} + I_{-1}(N)\varepsilon^{-1} + \ldots \right]$$
$$+ a_1(\varepsilon, N) \left[ I_{-2}(N+1)\varepsilon^{-2} + I_{-1}(N+1)\varepsilon^{-1} + \ldots \right]$$
$$+ \cdots$$
$$+ a_d(\varepsilon, N) \left[ I_{-2}(N+d)\varepsilon^{-2} + I_{-1}(N+d)\varepsilon^{-1} + \ldots \right]$$
$$= h'_{-3}(N)\varepsilon^{-3} + h'_{-2}(N)\varepsilon^{-2} + h'_{-1}(N)\varepsilon^{-1} + \ldots$$
$$= 0$$
Computing the $\varepsilon$-expansion from a recurrence relation

\[
\begin{align*}
  a_0(\varepsilon, N) \left[ I_{-2}(N)\varepsilon^{-2} + I_{-1}(N)\varepsilon^{-1} + \ldots \right] \\
  + a_1(\varepsilon, N) \left[ I_{-2}(N+1)\varepsilon^{-2} + I_{-1}(N+1)\varepsilon^{-1} + \ldots \right] \\
  + \ldots \\
  + a_d(\varepsilon, N) \left[ I_{-2}(N+d)\varepsilon^{-2} + I_{-1}(N+d)\varepsilon^{-1} + \ldots \right] \\
  = h'_{-2}(N)\varepsilon^{-2} + h'_{-1}(N)\varepsilon^{-1} + \ldots
\end{align*}
\]

Now repeat for $I_{-2}(N), I_{-1}(N), \ldots$
Tactic 2: Expand the recurrence

\[
F(N) = \sum_{k=1}^{N} (-1)^k e^{-\frac{3\varepsilon\gamma}{2}} \left( -2 - \frac{3\varepsilon}{2} \right)! B(2 + k, \frac{\varepsilon}{2}) B(-\varepsilon + k, -\varepsilon) B(1 - \frac{\varepsilon}{2} + k, 1 + \frac{\varepsilon}{2}) \binom{N}{k}
\]

\[
\downarrow \text{(summation package Sigma.m)}
\]

\[
2(N + 1)^2 F(N) + \left( 3\varepsilon^2 + 3\varepsilon N + 9\varepsilon - 4N^2 - 12N - 8 \right) F(N + 1)
- (2\varepsilon - N - 1)(\varepsilon + 2N + 6) F(N + 2) = 0\varepsilon^{-3} - \frac{16}{3} \varepsilon^{-2} + \frac{40}{3} \varepsilon^{-1} - \left( 2\zeta_2 - \frac{68}{3} \right) \varepsilon^0 + \ldots
\]

\[
F(1) = \frac{2}{3} \varepsilon^{-3} - \frac{11}{6} \varepsilon^{-2} + \left( \frac{\zeta_2}{4} + \frac{79}{24} \right) \varepsilon^{-1} + \ldots
\]

\[
F(2) = \frac{8}{9} \varepsilon^{-3} - \frac{73}{27} \varepsilon^{-2} + \left( \frac{\zeta_2}{3} + \frac{1415}{324} \right) \varepsilon^{-1} + \ldots
\]
\[ F(N) = \sum_{k=1}^{N} (-1)^k e^{-\frac{3\varepsilon\gamma}{2}} \left( -2 - \frac{3\varepsilon}{2} \right)! \binom{2k, \varepsilon}{2} B(-\varepsilon + k, -\varepsilon) B\left(1 - \frac{\varepsilon}{2} + k, 1 + \frac{\varepsilon}{2}\right) \binom{N}{k} \] 

\[ \downarrow \text{(summation package Sigma.m)} \]

\[ 2(N + 1)^2 F(N) + \left( 3\varepsilon^2 + 3\varepsilon N + 9\varepsilon - 4N^2 - 12N - 8 \right) F(N + 1) \]
\[ - (2\varepsilon - N - 1)(\varepsilon + 2N + 6) F(N + 2) = 0\varepsilon^{-3} - \frac{16}{3}\varepsilon^{-2} + \frac{40}{3}\varepsilon^{-1} - \left( 2\zeta_2 - \frac{68}{3} \right)\varepsilon^0 + \ldots \]

\[ F(1) = \frac{2}{3}\varepsilon^{-3} - \frac{11}{6}\varepsilon^{-2} + \left( \frac{\zeta_2}{4} + \frac{79}{24} \right)\varepsilon^{-1} + \ldots \]

\[ F(2) = \frac{8}{9}\varepsilon^{-3} - \frac{73}{27}\varepsilon^{-2} + \left( \frac{\zeta_2}{3} + \frac{1415}{324} \right)\varepsilon^{-1} + \ldots \]

\[ \downarrow \text{(summation package Sigma.m)} \]

\[ F(N) = \frac{4N}{3(N+1)}\varepsilon^{-3} - \left( \frac{2(2N+1)}{3(N+1)} S_1(N) + \frac{2N(2N+3)}{3(N+1)^2} \right)\varepsilon^{-2} \]
\[ \left( \frac{(1-4N)}{6(N+1)} \right) S_1(N)^2 - \frac{N(N^2-2)}{3(N+1)^3} + \frac{(3N+2)(4N+5)}{3(N+1)^2} S_1(N) + \frac{(1-4N)}{6(N+1)} S_2(N) + \frac{N\zeta_2}{2(N+1)} \right)\varepsilon^{-1} + \ldots \]
Tactic 2: Expand the recurrence

Computing the $\varepsilon$-expansion from a recurrence relation

\[ I(1) = \frac{5}{\varepsilon^3} - \frac{163}{12\varepsilon^2} + O(\varepsilon^{-1}), \quad I(2) = \frac{130}{27\varepsilon^3} - \frac{695}{54\varepsilon^2} + O(\varepsilon^{-1}), \quad I(3) = \frac{169}{36\varepsilon^3} - \frac{395}{32\varepsilon^2} + O(\varepsilon^{-1}) \]

\[
\text{In[6]} := \text{recEp} = -2(N + 1)(N + 2)(2 + \varepsilon + N)I[N] \\
- (N + 2)(-32 - 7\varepsilon + 2\varepsilon^2 - 28N - 5\varepsilon N - 6N^2)I[N + 1] \\
- (120 + 3\varepsilon - 14\varepsilon^2 - \varepsilon^3 + 136N + 13\varepsilon N - 4\varepsilon^2 N + 50N^2 + 4\varepsilon N^2 + 6N^3)I[N + 2] \\
+ (2 - \varepsilon + N)(4 + \varepsilon + N)(8 + \varepsilon + 2N)I[N + 3] \\
== \frac{1}{\varepsilon^3} - \frac{4(N + 2)}{3(N + 3)} + \frac{1}{\varepsilon^2} \left[ - \frac{2(2N + 7)S_1}{3(N + 3)} - \frac{2(4N^4 + 35N^3 + 101N^2 + 105N + 25)}{3(N + 1)(N + 2)(N + 3)^2} \right] + O(\varepsilon^{-1}).
\]
Computing the $\varepsilon$-expansion from a recurrence relation

\[ I(1) = \frac{5}{\varepsilon^3} - \frac{163}{12\varepsilon^2} + O(\varepsilon^{-1}), \quad I(2) = \frac{130}{27\varepsilon^3} - \frac{695}{54\varepsilon^2} + O(\varepsilon^{-1}), \quad I(3) = \frac{169}{36\varepsilon^3} - \frac{395}{32\varepsilon^2} + O(\varepsilon^{-1}) \]

In[6]:= \text{recEp} = -2(N + 1)(N + 2)(2 + \varepsilon + N)I[N] - (N + 2)\left(-32 - 7\varepsilon + 2\varepsilon^2 - 28N - 5\varepsilon N - 6N^2\right)I[N + 1] - \left(120 + 3\varepsilon - 14\varepsilon^2 - \varepsilon^3 + 136N + 13\varepsilon N - 4\varepsilon^2 N + 50N^2 + 4\varepsilon N^2 + 6N^3\right)I[N + 2] + (2 - \varepsilon + N)(4 + \varepsilon + N)(8 + \varepsilon + 2N)I[N + 3] == \frac{1}{\varepsilon^3}\left[-\frac{4(N + 2)}{3(N + 3)} + \frac{1}{\varepsilon^2}\left[-\frac{2(2N + 7)S_1}{3(N + 3)} - \frac{2(4N^4 + 35N^3 + 101N^2 + 105N + 25)}{3(N + 1)(N + 2)(N + 3)^2}\right]\right] + O(\varepsilon^{-1}).

In[7]:= \text{GenerateExpansion}[\text{recEp[[1]]}, \{\text{Coefficient[recEp[[2]], \varepsilon^{-3}], Coefficient[recEp[[2]], \varepsilon^{-2}]}\}, I[N], \{\varepsilon, -3, -2\}, \{\{5, \frac{130}{27}, \frac{169}{36}\}, \{-\frac{163}{12}, -\frac{695}{54}, -\frac{395}{32}\}\}, \text{MinInitialValue} \rightarrow 1]
Computing the $\varepsilon$-expansion from a recurrence relation

$$I(1) = \frac{5}{\varepsilon^3} - \frac{163}{12\varepsilon^2} + O(\varepsilon^{-1})$$

$$I(2) = \frac{130}{27\varepsilon^3} - \frac{695}{54\varepsilon^2} + O(\varepsilon^{-1})$$

$$I(3) = \frac{169}{36\varepsilon^3} - \frac{395}{32\varepsilon^2} + O(\varepsilon^{-1})$$

In[6]:= \text{recEp} = -2(N + 1)(N + 2)(2 + \varepsilon + N)I[N] - (N + 2)(-32 - 7\varepsilon + 2\varepsilon^2 - 28N - 5\varepsilon N - 6N^2)I[N + 1] - (120 + 3\varepsilon - 14\varepsilon^2 - \varepsilon^3 + 136N + 13\varepsilon N - 4\varepsilon^2 N + 50N^2 + 4\varepsilon N^2 + 6N^3)I[N + 2] + (2 - \varepsilon + N)(4 + \varepsilon + N)(8 + \varepsilon + 2N)I[N + 3] =\frac{1}{\varepsilon^3} -\frac{4(N + 2)}{3(N + 3)} + \frac{1}{\varepsilon^2} \left[ -\frac{2(2N + 7)S_1}{3(N + 3)} - \frac{2(4N^4 + 35N^3 + 101N^2 + 105N + 25)}{3(N + 1)(N + 2)(N + 3)^2} \right] + O(\varepsilon^{-1}).$$

In[7]:= \text{GenerateExpansion}[\text{recEp}[1]], \{\text{Coefficient}[\text{recEp}[2], \varepsilon^{-3}], \text{Coefficient}[\text{recEp}[2], \varepsilon^{-2}]\}, I[N], \{\varepsilon, -3, -2\}, \{\{5, \frac{130}{27}, \frac{169}{36}\}, \{-\frac{163}{12}, -\frac{695}{54}, -\frac{395}{32}\}\}, \text{MinInitialValue} \rightarrow 1$

Out[7]= $$\left\{ \frac{59N^2 + 120N + 49}{9(N + 1)^2} - \frac{2(N + 3)S_1[N]}{3(N + 1)}, -\frac{2(20N^3 + 58N^2 + 57N + 22)}{3(N + 1)^3} + \frac{2(N + 2)(2N - 1)S_1[N]}{3(N + 1)^2} - \frac{S_1[N]^2}{N + 1} - \frac{S_2[N]}{N + 1} \right\}$$
Find a recurrence for the integral/sum

\[ F(N) = \int_0^1 \cdots \int_0^1 \Phi(\varepsilon, N, x_1, x_2, \ldots, x_7) \, dx_1 \, dx_2 \cdots dx_7 \]

\[ \overset{?}{=} F_{-3}(N)\varepsilon^{-3} + F_{-2}(N)\varepsilon^{-2} + F_{-1}(N)\varepsilon^{-1} + \ldots \]

MultIntegrate package
(Jakob Ablinger)

\[ a_0(\varepsilon, N)F(N) + \ldots + a_d(\varepsilon, N)F(N + d) = h(\varepsilon, N) \]
**Find a recurrence for the integral/sum**

\[ F(N) = \int_0^1 \cdots \int_0^1 \Phi(\varepsilon, N, x_1, x_2, \ldots, x_7) \, dx_1 \, dx_2 \ldots dx_7 \]

\[ \Rightarrow F_{-3}(N) \varepsilon^{-3} + F_{-2}(N) \varepsilon^{-2} + F_{-1}(N) \varepsilon^{-1} + \ldots \]

**MultiIntegrate package (Jakob Ablinger)**

\[ \sum_{i_1} \cdots \sum_{i_7} f(\varepsilon, N, i_1, i_2, \ldots, i_7) \]

**MultiSum Package (Wegschaider)**

\[ a_0(\varepsilon, N) F(N) + \ldots + a_d(\varepsilon, N) F(N + d) = h(\varepsilon, N) \]
Tactic 2: Expand the recurrence

Find a recurrence for the integral/sum

\[ F(N) = \int_0^1 \cdots \int_0^1 \Phi(\varepsilon, N, x_1, x_2, \ldots, x_7) \, dx_1 \, dx_2 \cdots dx_7 \]

\( \overset{?}{=} F_{-3}(N) \varepsilon^{-3} + F_{-2}(N) \varepsilon^{-2} + F_{-1}(N) \varepsilon^{-1} + \ldots \)

MultiIntegrate package (Jakob Ablinger)

MultiSum Package (Wegschaider)

RhoSum package (Mark Round) (holonomic/difference field approach)

\[ \sum_{i_1} \cdots \sum_{i_7} f(\varepsilon, N, i_1, i_2, \ldots, i_7) \]

\[ a_0(\varepsilon, N) F(N) + \ldots + a_d(\varepsilon, N) F(N + d) = h(\varepsilon, N) \]
Find a recurrence for the integral/sum

\[ F(N) = \int_0^1 \cdots \int_0^1 \Phi(\varepsilon, N, x_1, x_2, \ldots, x_7) \, dx_1 \, dx_2 \cdots dx_7 \]

\[ \overset{?}{=} F_{-3}(N)\varepsilon^{-3} + F_{-2}(N)\varepsilon^{-2} + F_{-1}(N)\varepsilon^{-1} + \ldots \]

\[ \sum_{i_1} \cdots \sum_{i_7} f(\varepsilon, N, i_1, i_2, \ldots, i_7) \]

\[ a_0(\varepsilon, N)F(N) + \ldots + a_d(\varepsilon, N)F(N + d) = h(\varepsilon, N) \]

\( \varepsilon \)-recurrence solver

MultIntegrate package (Jakob Ablinger)

MultiSum Package (Wegschaider)

RhoSum package (Mark Round) (holonomic/difference field approach)

RISC, J. Kepler University Linz

Carsten Schneider
Tactic 3: Solve coupled systems of differential equations
A coupled differential system for $\hat{I}_1(x), \hat{I}_2(x), \hat{I}_3(x)$

(produced by IBP [extension of REDUCE_2, A.v. Manteuffel])

$$
D_x \begin{pmatrix}
\hat{I}_1(x) \\
\hat{I}_2(x) \\
\hat{I}_3(x)
\end{pmatrix} =
\begin{pmatrix}
-\frac{1-\varepsilon+x}{(x-1)x} & -\frac{2}{(x-1)x} & 0 \\
\frac{\varepsilon(3\varepsilon+2)}{4(x-1)} & -\frac{2-\varepsilon+3x+3\varepsilon x}{2(x-1)x} & -\frac{\varepsilon+1}{2(x-1)} \\
-\frac{\varepsilon(3\varepsilon+2)(x-2)}{4(x-1)x} & -\frac{2-5\varepsilon+x+3\varepsilon x}{2(x-1)x} & \frac{(-2\varepsilon-x+\varepsilon x)}{2(x-1)x}
\end{pmatrix}
\begin{pmatrix}
\hat{I}_1(x) \\
\hat{I}_2(x) \\
\hat{I}_3(x)
\end{pmatrix} +
\begin{pmatrix}
\hat{R}_1(x) \\
\hat{R}_2(x) \\
-\hat{R}_2(x)
\end{pmatrix}
$$

where

$$
\hat{R}_1(x) = \frac{\hat{B}_4(x)}{(x-1)x},
$$

$$
\hat{R}_2(x) = \frac{-(\varepsilon+2)^3}{16(\varepsilon+1)(x-1)x} \hat{B}_1(x) + \frac{(\varepsilon+2)(3\varepsilon+4)(19\varepsilon^2 + 36\varepsilon + 16)}{16\varepsilon(5\varepsilon+6)(x-1)x} \hat{B}_2(x)
+ \frac{(\varepsilon+1)^2(3\varepsilon+4)^2}{2\varepsilon(5\varepsilon+6)x} \hat{B}_3(x) + \frac{-24 - 50\varepsilon - 25\varepsilon^2 + 8x + 14\varepsilon x + 6\varepsilon^2 x}{4(5\varepsilon+6)(x-1)x} \hat{B}_4(x)
$$

$\hat{B}_1(x), \hat{B}_2(x), \hat{B}_3(x)$ have been solved with symbolic summation.

RISC, J. Kepler University Linz

Carsten Schneider
Tactic 3: Solve coupled systems

Tactic 3: the DE-REC approach

DE system

\[ D\hat{I}(x) = A\hat{I}(x) + \hat{R}(x) \]
Tactic 3: the DE-REC approach

DE system
\[ D\hat{I}(x) = A\hat{I}(x) + \hat{R}(x) \]

OreSys package (S. Gerhold)
uncoupling algorithm

uncoupled DE system
\[ \sum_i a_i(x) D^i \hat{I}_1(x) = r(x) \]
\[ \hat{I}_k(x) = \text{expr}_k(\hat{I}_1(x)), \ k > 1 \]
Tactic 3: Solve coupled systems

\begin{align*}
\text{DE system} & \\
D \hat{I}(x) &= A \hat{I}(x) + \hat{R}(x) \\
\text{uncoupled DE system} & \\
\sum_i a_i(x) D^i \hat{I}_1(x) &= r(x) \\
\hat{I}_k(x) &= \text{expr}_k(\hat{I}_1(x)), k > 1 \\
\hat{I}_1(x) &= \sum_{N=0}^{\infty} I_1(N) x^N \\
\end{align*}
Tactic 3: Solve coupled systems

**Tactic 3: the DE-REC approach**

**DE system**

\[ D\hat{I}(x) = A\hat{I}(x) + \hat{R}(x) \]

OreSys package (S. Gerhold)

Uncoupling algorithm

**Uncoupled DE system**

\[ \sum_i a_i(x) D^i \hat{I}_1(x) = r(x) \]

\[ \hat{I}_k(x) = \text{expr}_k(\hat{I}_1(x)), k > 1 \]

Holonomic closure prop.

**Linear recurrence**

\[ \sum_i a'_i(N) I_1(N) = r'(N) \]
Tactic 3: the DE-REC approach

DE system

\[ D\hat{I}(x) = A\hat{I}(x) + \hat{R}(x) \]

OreSys package (S. Gerhold)

uncoupling algorithm

uncoupled DE system

\[ \sum_i a_i(x) D^i \hat{I}_1(x) = r(x) \]

\[ \hat{I}_k(x) = \text{expr}_k(\hat{I}_1(x)), k > 1 \]

holonomic closure prop.

closed form solutions of

\( I_1(N) \)

in the class of nested sums and products – if this is possible

Sigma’s REC-solver

linear recurrence

\[ \sum_i a'_i(N) I_1(N) = r'(N) \]
Tactic 3: the DE-REC approach (SolveCoupledSystem package)

DE system

\[ D \hat{I}(x) = A \hat{I}(x) + \hat{R}(x) \]

uncoupled DE system

\[ \sum_i a_i(x) D^i \hat{I}_1(x) = r(x) \]

\[ \hat{I}_k(x) = \text{expr}_k(\hat{I}_1(x)), \ k > 1 \]

extract coefficient

SumProduction package

holonomic closure prop.

closed form solutions of

\[ I_1(N), I_2(N), \ldots, I_n(N) \]

in the class of nested sums and products

– if this is possible

linear recurrence

\[ \sum_i a'_i(N) I_1(N) = r'(N) \]

Solving a coupled differential system

In[8]:= << OreSys.m

OreSys by Stefan Gerhold (optimized by C. Schneider) © RISC-Linz

In[9]:= << SolveCoupledSystem.m

SolveCoupledSystem by Carsten Schneider © RISC-Linz

In[10]:= coupledDESys = D[{\hat{I}_1(x), \hat{I}_2(x), \hat{I}_3(x)}, x] - A.{\hat{I}_1(x), \hat{I}_2(x), \hat{I}_3(x)};

In[11]:= rhs = {\hat{R}_1(x), \hat{R}_2(x), -\hat{R}_2(x)} in power series representation;

In[12]:= SolveCoupledDESystem[coupledDESys, {I_1[x], I_2[x], I_3[x]}, \varepsilon, -3, 
{\{-2, -2, -2\}, rhs, \ldots}]}
Tactic 3: Solve coupled systems

Solving a coupled differential system

\[ \text{In[8]} := \text{<< OreSys.m}} \]

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\[ \text{In[9]} := \text{<< SolveCoupledSystem.m}} \]

SolveCoupledSystem by Carsten Schneider © RISC-Linz

\[ \text{In[10]} := \text{coupledDESys} = D\{\{\hat{I}_1(x), \hat{I}_2(x), \hat{I}_3(x)\}, x\} - A\{\hat{I}_1(x), \hat{I}_2(x), \hat{I}_3(x)\}; \]

\[ \text{In[11]} := \text{rhs} = \{\hat{R}_1(x), \hat{R}_2(x), -\hat{R}_2(x)\} \text{ in power series representation;} \]

\[ \text{In[12]} := \text{SolveCoupledDESystem[coupledDESys, \{I_1[x], I_2[x], I_3[x]\}, \epsilon, -3, \{-2, -2, -2\}, rhs, \ldots \}} \]

\[ \text{Out[12]} = \left\{ \frac{1}{\epsilon^3} \left( \frac{4(3N^2 + 6N + 4)}{3(N + 1)^2} \cdot \frac{4S_1[N]}{3(N + 1)} \right) + \frac{1}{\epsilon^2} \left( - \frac{2(20N^3 + 58N^2 + 57N + 22)}{3(N + 1)^3} \right) + \frac{4}{3\epsilon^3} - \frac{2}{\epsilon^2}, \frac{8}{3\epsilon^3} + \frac{1}{\epsilon^2} \left( - \frac{4(4N^2 + 7N + 2)}{3(N + 1)^2} \right) + \frac{4(N + 2)S_1[N]}{3(N + 1)} \right\} \]
SUMMARY (of RISC packages)

Backbone: a new difference ring/field approach implemented in Sigma.m
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  - CS’s EvaluateMultiSum.m
  - M. Round’s RhoSum.m (difference ring/holonomic approach)
  - K. Wegschaider’s MultiSum.m (enhanced Sister Celine/WZ approach, utilized by F. Stan)
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  - J. Ablinger’s HarmonicSums.m