Odd Collatz Sequence and Binary Representations

Jose Capco
Email: jose.capco@fh-wels.at

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Abstract

Here we investigate the odd numbers in Collatz sequences (sequences arising from the $3n+1$ problem). We are especially interested in methods in binary number representations of the numbers in the sequence. In the first section, we show some results for odd Collatz sequences using mostly binary arithmetics. We see how some results become more obvious in binary arithmetic than in usual method of computing the Collatz sequence. In the second section of this paper we deal with some known results and show how we can use binary representation and OCS from the first section to prove some known results. We give a generalization of a result by Andaloro [2] and show a generalized sufficient condition for the Collatz conjecture to be true: If for a fixed natural number n the Collatz conjecture holds for numbers congruent to 1 modulo 2^n then the Collatz conjecture is true.

1 Introduction

Some of the observations here about odd Collatz sequences in binary is not entirely new. Odd Collatz sequences in binary is more common in computer and technical literatures, though it is very hard to pinpoint an original reference. One literature with a description most similar to what we will introduce in this paper is written by N. Mondal and P.P. Ghosh (see [8]). Wikipedia describes it under *abstract machine automation* [9].

The aim in this paper is to present a sequence of sufficiency sets for the Collatz or the $3n+1$ problem whose set-theoretic limit approaches the set containing only 1. These sets can be chosen to have a natural density that is arbitrarily small. This is an intuitive extension of the result of Andaloro [2], where he showed sufficiency for residue class 1 mod 2^n for $n = 2, 3$ and 4 (i.e. If n is $2, 3$ or 4 then the Collatz conjecture is true iff it holds for numbers congruent to 1 modulo 2^n). In the past years, sufficiency sets were provided that had similar properties but they usually have a more complex structure (see [5]). The nearest result was shown around 30 years ago by Korec and Znám (see [7]) who reduced the Collatz conjecture to residue class sufficiency sets that were dependent on primitive roots modulo an arbitrary power of an odd prime. In the second section a much simpler sequence of sufficiency sets will be presented.

To arrive to our main result we will give in this section a description of the odd Collatz sequence in binary, include some useful terminologies and follow them with new observations and theories using mostly binary computations. In the next section we will show how some known results can easily be proven using these techniques and we will finally show this generalisation of the result of Andaloro.

To avoid any misinterpretation, the natural numbers $\mathbb{N}$ is regarded without 0. We also use the following symbols and terminologies:

- By *OCS* or *odd Collatz sequence* of an odd number we mean the sequence of odd numbers in the $3n + 1$ problem starting from the given odd number, which we will call the *seed* of the OCS, and ending with the first occurrence of 1 if the sequence reaches 1 otherwise the sequence is infinite (it may be cyclic and infinite). So for instance the OCS of 11 would be

$$\{11, 17, 13, 5, 1\}$$

We just write *OCS* if we mean an arbitrary odd Collatz sequence or if the seed is known and in plural form we write *OCS's*. Obviously $3n+1$ (i.e. the Collatz conjecture) is solved if we prove that the OCS of any odd number is finite.

- The OCS of a number x is *cyclic* in the same way that a Collatz sequence is cyclic, i.e. there exists a number $y \in 2\mathbb{N}+1$ such that y occurs twice in the OCS. In this case, the OCS is obviously also infinite. As of date, it is not known whether one can have a cyclic OCS. If this were the case, then obviously the Collatz conjecture would be false.
- At an early stage we adapt the binary representation of positive integers. Unless otherwise stated or obvious we often use this convention for numbers. So by 1101 we mean the binary representation of the decimal 13.
- We use shortcut and *block representation of binary numbers*: Given any integer $n \in \mathbb{N}$ by 0_n (resp. 1_n) we mean the *binary block* having the digit 0 n-times (resp. 1 n-times). The blocks 1_0 and 0_0 are just *empty* binary blocks.
- We reserve the capital letters for arbitrary binary blocks (of arbitrary length). By X_1 we mean a binary block having 1 in the most right side of it, by X^1 we mean a binary block having 1 in the most left side of it and it is also obvious what we mean by X_1^1 (similarly for other capital letters). It is important to note that we allow binary blocks for which there are some 0 written in the beginning (e.g. 0011 and 11 are different blocks)
- If X is a binary block, then for any $n \in \mathbb{N}$ by $(X)_n$ we mean n repetitions of X. So for example $10(011)_3$ is the same as 10011011011. In rare cases we may see $(X)_0$ which is just an empty block. So $10(X)_011$ is the same as 1011 for any binary block X.
- Given a number $n \in \mathbb{N}$ by $|n|_b \in \mathbb{N}$ we mean the bit length of n, we abuse this same notation for binary blocks. So $|211|_b$ for the decimal 211 is the same as $|11010011|_b$ for the binary representation of 211 and both would have value 8. Any binary block can be used with $|\cdot|_b$ in the same way. So if $X = 0001110$ then $|X|_b = 7$. We say that X has $|X|_b$ *number of bits* or has *bitlength* $|X|_b$ and sometimes (but rarely) we want to speak of the n-th bit of a block (unless otherwise stated, counting is from left to right).
- Sometimes we do not care about the block in a binary representation and so we just put in dots to show that there might be a binary block. So sometimes we just write $\cdots 101 \cdots$ or $\cdots 111$ or $110 \cdots$.
- Given an OCS of a binary number X, by $X \rightsquigarrow Y$ (we say X *leads to* Y) we mean that the OCS of X has the number Y in it. Also if Y is the $(n+1)$-th number in the OCS of X we write $X \overset{n}{\rightsquigarrow} Y$ and we say X *leads to* Y *after* n *iterations*. So for the decimal 11, we can write
$$1011 \rightsquigarrow 1101$$
or $1011 \overset{2}{\rightsquigarrow} 1101$ and
$$1011 \overset{1}{\rightsquigarrow} 10001$$
i.e. the decimal 11 leads to 13, and in an OCS if a decimal 11 occurs then 17 is the next number after it.
- For brevity we sometimes write $X = X^1$ to mean that we assume that the binary block X has a leading 1 analogously we write $X = X_1$, $X = X_1^1$ etc.

Example. So the decimal 1607 represented by the binary number 11001000111 can be represented as $1_20_210_31_3$ and it is of the forms:

$$X11, 1100X^111, 110X_10_3111, X_1^1000111, \ldots$$

Proposition 1. The following holds

a.) For all $n > 1$ we have $X^1 01_n \overset{1}{\rightsquigarrow} Y^1 01_{n-1}$ with $|X^1 01_n|_b \leq |Y^1 01_{n-1}|_b \leq |X^1 01_n|_b + 1$.

b.) $X^1 01 \overset{1}{\rightsquigarrow} Y^1$ with $|Y^1|_b \leq |X^1|_b$. Specifically one has $X^1 101 \overset{1}{\rightsquigarrow} Y^1$ with $|Y^1|_b < |X^1 101|_b$.

c.) For all $n \geq 2$, $1_n 0 X_1 \overset{1}{\rightsquigarrow} 101_{n-1} 0 Y_1$ if there is a carry on the $(n+1)$-th bit. If there is no carry on the $(n+1)$-th bit we get $1_n 0 X_1 \overset{1}{\rightsquigarrow} 101_{n-2} 01 Y_1$.

d.) $10X_1 \overset{1}{\rightsquigarrow} 100Y_1$ if there is a carry on the second bit otherwise $10X_1 \overset{1}{\rightsquigarrow} 11Y_1$.

e.) Let X^1 be a binary block, then $X^1 01, X^1 1 \overset{1}{\rightsquigarrow} Z_1$ for some odd number Z_1.

Proof. For brevity we drop most of the subscripts and superscripts for the blocks and the squiggly arrows.

(a) We compute the next number after $X01_n$ in the OCS

$$\begin{array}{rccc} X\overset{1}{0} & \overset{1}{1} & \overset{1}{1}_{n-1} & 1 \\ +X & 0 & 1_{n-1} & 1 \\ \hline Y & 0 & 1_{n-1} & \cancel{0} \end{array}$$

In the addition above, the first row is 2 times $X01_n$ plus 1 and the second row is just $X01_n$, thus the resulting sum is the next number in the Collatz sequence. We then trim away all the trailing 0 of the sum to get the next odd number in the Collatz sequence. This is a trick that we will use throughout. Since

$$X01_n < X01_n + X01_{n-1} + 1 \leq X01_n 0$$

note that the last inequality is just multiplying $X01_n$ by 2. So

$$|X01_n|_b \leq |Y01_{n-1}|_b \leq |X01_n|_b + 1$$

(b) We compute the next number after $X01$ in the OCS

$$\begin{array}{rcc} X\overset{1}{0} & \overset{1}{1} & 1 \\ +X & 0 & 1 \\ \hline Z & \cancel{0} & \cancel{0} \end{array}$$

The next number in the sequence we take the largest odd divisor of Z, this we write as Y. For the case $X101$ we have

$$\begin{array}{rccc} X\overset{1}{1} & \overset{1}{0} & \overset{1}{1} & 1 \\ +X & 1 & 0 & 1 \\ \hline Z & \cancel{0} & \cancel{0} & \cancel{0} \end{array}$$

Y will then be the largest odd divisor of Z. This also shows that $|X101|_b \leq |Z00|_b \leq |X101|_b + 1$ which implies that

$$|Y|_b \leq |Z|_b \leq |X101|_b + 1 - 2 < |X101|_b$$

(c) If there is a carry on the $(n-1)$-th bit we get

$$\begin{array}{cccc} \overset{1}{1} & \overset{1}{1}_{n-1} & \overset{1}{0} & X1 \\ + & 1_{n-1} & 1 & 0X \\ \hline 10 & 1_{n-1} & 0 & Y \end{array}$$

otherwise (if there is no carry)

$$\begin{array}{cccc} \overset{1}{1} & \overset{1}{1}_{n-1} & \overset{0}{0} & X1 \\ + & 1_{n-1} & 1 & 0X \\ \hline 10 & 1_{n-2}0 & 1 & Y \end{array}$$

Here if $n = 2$ then, as defined, we ignore 1_{n-2} and the result is $1001Y$.

(d) follow binary addition as in (c). We leave this as an easy exercise for the reader.

(e) We get

$$\begin{array}{rccc} X\overset{1}{1} & \overset{1}{0} & \overset{1}{1} & 1 \\ +X & 1 & 0 & 1 \\ \hline Y & \not{0} & \not{0} & \not{0} \end{array}$$

and

$$\begin{array}{rc} X\overset{1}{1} & 1 \\ +X & 1 \\ \hline Y & \not{0} \end{array}$$

with $Y = X1 + 1 + X$. So Z is just the largest odd divisor of Y. □

Applying Proposition 1 (e) recursively immediately leads to the following generalization

Corollary 2. Suppose X_1 and Y_1 are binary blocks such that X_1 represents an odd number that is not the decimal 1 and that $X_1 \overset{1}{\rightsquigarrow} Y_1$ then

$$X_1(01)_n \overset{1}{\rightsquigarrow} Y_1 \quad \forall n \in \mathbb{N}$$

Lemma 3. Let $n \geq 2$

(a) $X1_n \overset{1}{\rightsquigarrow} Y1_{n-1}$ with $Y = X1 + X + 1$.

(b) $X001_n \overset{1}{\rightsquigarrow} Y101_{n-1}$ with $Y = X0 + X$

(c) $X101_n \overset{1}{\rightsquigarrow} Y001_{n-1}$ with $Y = X1 + X + 1$.

Note: If x and y are the decimal of X and Y respectively then $Y = X1 + X + 1$ is just the same as $y = 3x + 2$ and $Y = X0 + X$ is the same as $y = 3x$

Proof. (a) We compute

$$\begin{array}{rcc} X\overset{1}{1} & \overset{1}{1}_{n-1} & 1 \\ +X & 1_{n-1} & 1 \\ \hline Y & 1_{n-1} & \not{0} \end{array}$$

with the leading block being $Y = X1 + X + 1$ (1 in the last summand is due to the carry).
(b) One computes

$$\begin{array}{rcccc} X0 & \overset{1}{0} & \overset{1}{1} & \overset{1}{1}_{n-1} & 1 \\ +X & 0 & 0 & 1_{n-1} & 1 \\ \hline Y & 1 & 0 & 1_{n-1} & \not{0} \end{array}$$

with the desired result.
(c) Similarly

$$\begin{array}{rcccc} X\overset{1}{1} & \overset{1}{0} & \overset{1}{1} & \overset{1}{1}_{n-1} & 1 \\ +X & 1 & 0 & 1_{n-1} & 1 \\ \hline Y & 0 & 0 & 1_{n-1} & \not{0} \end{array}$$

with the desired result □

Corollary 4. Suppose that $k, n \in \mathbb{N}$ with $n > 1$ and $k < n$ then for any binary block X we have

$$X001_n \overset{k}{\rightsquigarrow} \begin{cases} Y101_{n-k} & k \in 2\mathbb{N}+1 \\ Y001_{n-k} & k \in 2\mathbb{N} \end{cases}$$

for some binary block Y. We can make a similar statement for $X101_n$ with the odd and even results for k reversed, i.e.

$$X101_n \overset{k}{\rightsquigarrow} \begin{cases} Y101_{n-k} & k \in 2\mathbb{N} \\ Y001_{n-k} & k \in 2\mathbb{N}+1 \end{cases}$$

Proof. Follows immediately by iterating the result in Lemma 3(b) and (c). $\square$

Corollary 5. Let $n \in 2\mathbb{N}+1$ then there exists an $x \in 2\mathbb{N}+1$ such that $1_n \overset{n+1}{\rightsquigarrow} x$ and $1_{n+1} \overset{n+1}{\rightsquigarrow} x$ i.e. the OCS' of 1_n and 1_{n+1} *coalesces* (this is equivalent to claiming that the OCS of 2^n-1 and $2^{n+1}-1$ coalesces for any **odd** number n).

Proof. Because n is odd by Corollary 4, one has

$$\begin{aligned} &1_n \overset{n-1}{\rightsquigarrow} Y001 \\ &1_{n+1} \rightsquigarrow n-1Y0011 \end{aligned}$$

for some binary block Y. Now

$$\begin{array}{rcccc} Y0 & \overset{1}{0} & \overset{1}{1} & \overset{1}{1} & 1 \\ +Y & 0 & 0 & 1 & 1 \\ \hline Z & 1 & 0 & 1 & \cancel{0} \end{array}$$

and

$$\begin{array}{rccc} Y0 & \overset{1}{0} & \overset{1}{1} & 1 \\ +Y & 0 & 0 & 1 \\ \hline Z & 1 & \cancel{0} & \cancel{0} \end{array}$$

with $Z = Y0 + Y$ (so if z and y are the decimal of Z and Y respectively then $z = 3y$). So

$$\begin{aligned} &1_n \overset{n}{\rightsquigarrow} Z101 \\ &1_{n+1} \overset{n}{\rightsquigarrow} Z1 \end{aligned}$$

The result then follows from Proposition 1 (e). $\square$

Lemma 6. Let $n > 2$ then

$$X^1 0_n 1 \overset{1}{\rightsquigarrow} Y^1 0_{n-2} 1$$

with $Y^1 = X^1 0 + X^1$. And Y^1 has a trailing bit 1 if X^1 has a trailing bit 1 (i.e. Y is odd if X is odd).

Proof. For brevity we write X instead of X^1 taking note that the leading bit is 1.

$$\begin{array}{rcccc} X0 & 0_{n-2} & \overset{1}{0} & \overset{1}{1} & 1 \\ +X & 0_{n-2} & 0 & 0 & 1 \\ \hline Y & 0_{n-2} & 1 & \cancel{0} & \cancel{0} \end{array}$$

Since $Y = X0 + X$, Y has a trailing bit 1 if X has a trailing bit 1. $\square$

Proposition 7. Let X_1^1 be the binary representation of an odd positive number different from 1, then there exists $n \in \mathbb{N}$ such that $n \leq 4$ and one of the following holds

- $X_1^1 \overset{n}{\rightsquigarrow} 1$
- $X_1^1 \overset{n}{\rightsquigarrow} Y_1^1$ and either X_1 or Y_1^1 contains the binary block 101

Proof. For brevity we write X instead of X_1^1. We can easily assume that X is greater than 8 (we can check the proposition for odd numbers below 8 and see that this is true). Thus we know that X can have at least three bits. Now, consider the *first three leading bits* of X. If it is 101 then we are done, if it is 111 then one easily checks that $111\cdots \overset{1}{\rightsquigarrow} 101\cdots$ and we are done. If it is 100 then after one iteration it leads to a number with binary representation with three leading bits being 110 (if there is no carry on the third leading bit) or 111. We have already tackled leading bits 111 so we assume X has three leading bits 110 and show that after at most three iterations this leads to a number with a binary representation containing 101.

If there is a carry for the third leading bit of the first iteration we have

$$\begin{array}{ccccc} & \overset{1}{1} & \overset{1}{1} & \overset{1}{0} & \cdots\cdot \\ + & & 1 & 1 & 0\cdots \\ \hline 10 & & 1 & 0 & \cdots\cdot \end{array}$$

and so we have our result. We thus assume that there is no carry on the third leading bit for the first iteration and after the first iteration we get

$$\begin{array}{ccccc} & \overset{1}{1} & 1 & 0 & \cdots\cdot \\ + & & 1 & 1 & 0\cdots \\ \hline 10 & & 0 & 1 & \cdots\cdot \end{array}$$

If there is no carry at the fourth leading bit of the second iteration we have $1001\cdots \overset{1}{\rightsquigarrow} 1101\cdots$ and we are done. Thus we assume that the second iteration has a carry at the fourth leading bit, in this case we have

$$1001\cdots \overset{1}{\rightsquigarrow} 111\cdots \overset{1}{\rightsquigarrow} 101\cdots$$

Thus we have our result at most in the third iteration! $\square$

Proposition 8. Given an odd number x such that it does not leads to 1 after one or two iterations then x leads to a number with at most $|x|_b + 2$ bitlength after 3 iterations.

Proof. Let X with leading bit 1 be the binary representation of x. If the first iteration does not lead to an increase in bitlength, then we are done. If it does then there is a carry in the first leading bit of the first iteration. So we assume $X \overset{1}{\rightsquigarrow} 10\cdots$. If the second iteration does not increase the bitlength then we are done. Otherwise there is a carry in the second leading bit of the second iteration and so we have $X \overset{2}{\rightsquigarrow} 100\cdots$. But we know that $100\cdots$ after one iteration does not increase its bitlength (because there is no carry on the first or second leading bit). $\square$

Lemma 9. Let X^1 be a binary block with leading 1 and suppose that $k \in \mathbb{N}$, then there exists a binary block Y^1 such that for any $n > 2k$ (notice Y^1 is independent of the choice of n) one has

$$X^1 0_n 1 \overset{k}{\rightsquigarrow} Y^1 0_{n-2k} 1$$

Proof. Let $x \in \mathbb{N}$ be the decimal number that X^1 represents. We simply apply Lemma 6 k-times recursively and see that $y = 3^k x$, where y is the number that Y^1 represents. Y^1 and y are obviously independent of the choice of n. $\square$

Proposition 10. Let $k \in \mathbb{N}$ then for any $n > 2k$ we have

$$10_n 1 \overset{k}{\rightsquigarrow} \begin{cases} X0110_{n-2k}1 & k \in 2\mathbb{N}+1 \\ X0010_{n-2k}1 & k \in 2\mathbb{N} \end{cases}$$

where $X = X^1$ is a binary block dependent on k but independent of the choice of n.

Proof. One checks that for any $m > 2$ one has

$$\cdots 0010_m 1 \overset{1}{\rightsquigarrow} \cdots 0110_{m-2} 1$$

and

$$\cdots 0110_m 1 \overset{1}{\rightsquigarrow} \cdots 0010_{m-2} 1$$

this is just an application of Lemma 9 and its proof (where a block $\cdots 011$ is obtained by multiplying 3 with a block $\cdots 001$ and vice versa). If we apply this result k-times (if it makes it easier to understand, we can harmlessly regard $10_n 1$ initially as $0010_n 1$) we obtain our result. The last sentence that X is independent on the choice of n comes from Lemma 9. $\square$

We have already informally used the term *coalesce* in Corollary 5. Though it is easy to understand what we meant, it is best to define it more rigorously in order to avoid confusion.

Definition. Let $x, y \in 2\mathbb{N}+1$ (binary or decimal representation), then we say that the OCS's of x and y *coalesces* if there exists numbers $m, n \in \mathbb{N} \cup \{0\}$ and $z \in \mathbb{N}$ such that the OCS of x has more than m terms, the OCS of y has more than n terms and $x \overset{m}{\rightsquigarrow} z$ and $y \overset{n}{\rightsquigarrow} z$. To make it more precise we also write *(the OCS's of)* (x, y) *coalesces after* (m, n) *iterations*. Similar to coalescing pairs, we can define coalescing finite numbers $x_1, x_2, \ldots, x_k \in 2\mathbb{N}+1$.

So Corollary 5 says that for $n \in 2\mathbb{N}+1$, 1_n and 1_{n+1} coalesces or more specifically $(1_n, 1_{n+1})$ coalesces in $(n+1, n+1)$ iterations.

Corollary 11. Let $k \in 2\mathbb{N}$ then the binary representation pair

$$(10_{2k}1, 10_{2k+1}1)$$

coalesces after $(k+2, k+2)$ iterations.

Proof. Since $k > 1$ and $k-1$ is odd, we agree by Proposition 10 that after $k-1$ iterations there exists a binary block X representing an even number such that $10_{2k}1$ and $10_{2k+1}1$ leads to $X11001$ and $X110001$ respectively.

After k iterations, $10_{2k}1$ and $10_{2k+1}1$ will respectively lead to

$$\begin{array}{rccccc} X\overset{1}{1} & 1 & 0 & \overset{1}{0} & \overset{1}{1} & 1 \\ +X & 1 & 1 & 0 & 0 & 1 \\ \hline Y & 0 & 1 & 1 & \cancel{0} & \cancel{0} \end{array}$$

and

$$\begin{array}{rcccccc} X\overset{1}{1} & 1 & 0 & 0 & \overset{1}{0} & \overset{1}{1} & 1 \\ +X & 1 & 1 & 0 & 0 & 0 & 1 \\ \hline Y & 0 & 1 & 0 & 1 & \cancel{0} & \cancel{0} \end{array}$$

where $Y = X1 + X + 1$.

Clearly $Y \neq 0$ and Y represents an even number (since X does it too) and so after $k+1$ iterations $10_{2k}1$ will lead to the following

$$\begin{array}{rccc} Y\overset{1}{0} & \overset{1}{1} & \overset{1}{1} & 1 \\ +Y & 0 & 1 & 1 \\ \hline Z & 0 & 1 & \cancel{0} \end{array}$$

and $10_{2k+1}1$ will lead to

$$\begin{array}{rcccc} Y\overset{1}{0} & \overset{1}{1} & \overset{1}{0} & \overset{1}{1} & 1 \\ +Y & 0 & 1 & 0 & 1 \\ \hline Z & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \end{array}$$

with $Z = Y0 + Y + 1$. Notice that Z is in the OCS of $10_{2k+1}1$ because it represents an odd number (since Y represents an even number).

We observe that Z is neither 0 nor 1. So after $k+2$ iterations both $10_{2k}1$ and $10_{2k+1}1$ lead to the largest odd factor of the number $Z0 + Z + 1 = Z1 + Z$ $\square$

Definition. Let $x \in 2\mathbb{N}+1$ and $X = X_1^1$ be its binary representation. Then X can be written as follows

$$1_{n_k}0_{m_k}1_{n_{k-1}} \ldots 1_{n_1}0_{m_1}1_{n_0} \qquad k \geq 0$$

We now define the function $\mu : 2\mathbb{N}+1 \to \mathbb{N}$, the image of x or X is

$$\mu(x) = \mu(X) := \max\{n_0, n_1, \ldots, n_k, m_1, \ldots, m_k\}$$

and we can similarly define the image of any odd number from the decompositions of bunches of 1's and 0's in its binary representation. The number $2k$ is called the *number of alternations* and we are able in this way to define the function

$$\sigma : \mathbb{N} \to 2\mathbb{N} \qquad \sigma(x) = \sigma(X) := 2k$$

Proposition 12. Let $n \in \mathbb{N}$ and suppose that the OCS of $x \in 2\mathbb{N}+1$ be infinite and not cyclic. Then there exists $y \in 2\mathbb{N}+1$ in the OCS of x such that one of the following holds:

- $\sigma(y) \geq n$
- $\mu(y) \geq n$

Proof. For any number $k \in \mathbb{N}$ there exists a $y \in 2\mathbb{N}+1$ in the OCS of x such that $|y|_b > k$, otherwise: The OCS may contain only numbers with at most k bitlength and this set of numbers is finite. Thus, because we assumed that the OCS is infinite, the OCS will have repeating numbers making it cyclic which is a contradiction.

So for n^2 there is a $y \in 2\mathbb{N}+1$ in the OCS of x such that $|y|_b > n^2$. Suppose now that $\sigma(y) < n$ (so $\sigma(y)+1 \leq n$) and $\mu(y) < n$, then we get

$$|y|_b \leq (\sigma(y)+1)\mu(y) < n^2$$

and this is a contradiction. □

Intuitively, one may think that the number of alternations and the number of the largest bit blocks consisting of only 1's or of only 0's can help in the understanding of development of the sequence of the odd Collatz sequences of some numbers. This is what we aim to do in the future.

2 Application

We use techniques from the previous section to show that some known results can sometimes be very easily proven. We also show new results extending known results that will help us understand the Collatz problem much better.

Remark 13. We express some of the results in the previous section in a more standardly used decimal form

a.) Proposition 1 (e) and Corollary 2: For any $x \in 2\mathbb{N}+1$ such that $x \neq 1$ if $x \overset{1}{\rightsquigarrow} y$ then all numbers of the form

$$4^n x + 4^{n-1} + \cdots + 4 + 1 \quad (n \in \mathbb{N})$$

leads to y after 1 iteration.

b.) Lemma 3: Let $n \geq 2$ be an integer then

$$\forall x \in \mathbb{N} \quad 2^n x + 2^n - 1 \overset{1}{\rightsquigarrow} 2^{n-1} y + 2^{n-1} - 1 \text{ where } y = 3x+2$$
$$\forall x \in \mathbb{N} \quad 2^{n+2} x + 2^n - 1 \overset{1}{\rightsquigarrow} 2^{n+1} y + 2^n + 2^{n-1} - 1 \text{ where } y = 3x$$
$$\forall x \in \mathbb{N} \quad 2^{n+2} x + 2^{n+1} + 2^n - 1 \overset{1}{\rightsquigarrow} 2^{n+1} y + 2^{n-1} - 1 \text{ where } y = 3x+2$$

c.) Lemma 9: Let $x \in 2\mathbb{N}+1$ and $n, k \in \mathbb{N}$ be such that $n > 2k$ then

$$2^{n+1} x + 1 \overset{k}{\rightsquigarrow} 2^{n-2k+1} 3^k x + 1$$

Lemma 4.1 of [4] discusses a result that is easy to prove using binary representations. It basically implies the following

Proposition 14. (see [4] Lemma 4.1) Collatz conjecture holds for numbers of the form

$$\frac{2^{2t}-1}{3} \qquad t \in \mathbb{N}$$

Proof. We can assume that $t > 2$ because we know that the Collatz conjecture holds for 1 (trivially) and 5. A number having the above mentioned form is a geometric series and can be written as

$$\frac{2^{2t}-1}{3} = \sum_{i=1}^{t} 2^{2i-2} = \sum_{i=1}^{t} 4^{i-1}$$

The above is very easy to represent in binary form, it is an odd number of the form $X(01)_n$ with $n = t-2$ and $X = 101$ (decimal 5). By Corollary 2 (or see also Remark 13 (a)) this number leads to 1 after only one iteration (because the binary $X = 101$ leads to 1 after one iteration). □

Another known result is Corollary 5. I am not the first to prove this result. The result has been known much earlier, though I am not sure who first tried to prove it. For instance, Theorem 2 of [6] (a short paper in Chinese) has a proof of this. But Corollary 5 has a proof that shows how easy it is to use OCS and binary arithmetic to prove this result.

We now use OCS and binary arithmetic to show that the results by Andaloro (see [2]) can be extended. To do this, we will first use a specific form of the proposition (for the case $p = 3$) below that I first discussed with Henning Makholm. He proved the generalised proposition presented below. I present his proof with some modifications for clarity and exactness.

Proposition 15. Let $x \in \mathbb{N}$, p be an odd prime number and $n \in \mathbb{N}$ then there exists a number $m \in \mathbb{N} \cup \{0\}$ such that

$$p^n \mid (p+1)^m x + \frac{(p+1)^m - 1}{p}$$

Proof. (Henning Makholm) Observe by the binomial expansion that $p \mid (p+1)^m - 1$ so $\frac{(p+1)^m-1}{p}$ is an integer. For $m \in \mathbb{N} \cup \{0\}$ denote

$$r_m := (p+1)^m x + \frac{(p+1)^m - 1}{p}$$

It suffices to show that there exist an $m \in \mathbb{N} \cup \{0\}$ such that p^{n+1} divides $pr_m = (p+1)^m(px+1) - 1$ In other words we need to prove that there is an m such that

$$(p+1)^m(px+1) \equiv 1 \bmod p^{n+1}$$

For simplicity we denote the ring $\mathbb{Z}/p^{n+1}\mathbb{Z}$ by A. Now define the set

$$G := \{py + 1 \bmod p^{n+1} \,:\, y = 0, 1, \ldots, p^n - 1\} \subset A$$

All elements in G are units in A because for all $y \in \mathbb{N}$ we know that $py + 1$ is relatively prime to p. Now if $y_1 \neq y_2$ for

$$y_1, y_2 \in \{0, 1, \ldots, p^n - 1\}$$

and if $py_1 + 1 \equiv py_2 + 1 \bmod p^{n+1}$ then it follows that

$$p^n \mid (y_2 - y_1)$$

But since the difference between y_1 and y_2 cannot exceed $p^n - 1$, this can happen only if $y_1 = y_2$ which is a contradiction. Thus G has exactly p^n distinct elements. Since p is an odd prime, the multiplicative group A^* (i.e. group of units of A) is cyclic and therefore any subgroup is cyclic. So G is cyclic and we have a canonical group isomorphism

$$(G, *) \xrightarrow{\sim} (\mathbb{Z}_{p^n}, +)$$

It suffices to prove that $p+1$ is the generator of G. The generators of $(\mathbb{Z}_{p^n},+)$ are exactly the numbers congruent module p^n that are relatively prime to p^n equivalently non-generators in $(\mathbb{Z}_{p^n},+)$ are numbers that are multiples of p modulo p^n. From the above isomorphism, a non-generator of G is an element that is of the form h^p for some $h \in G$. Now suppose that $p+1 \equiv (py+1)^p \bmod p^{n+1}$ for some $y \in \{0,1,\ldots,p^n-1\}$. If we expand $(py+1)^p$ we get

$$(py+1)^p = 1 + \binom{p}{1}py + \cdots = 1 + p^2 N$$

for some $N \in \mathbb{N}$. So there is a $k \in \mathbb{Z}$ such that

$$kp^{n+1} = p^2N - p \Rightarrow (pN - kp^n) = p(N - kp^{n-1}) = 1$$

this is a contradiction because $p > 1$ and $N - kp^{n-1}$ is an integer. Thus $p+1$ is indeed a generator of the multiplicative subgroup G and there is indeed an $m \in \mathbb{N} \cup \{0\}$ such that

$$(p+1)^m(px+1) \equiv 1 \bmod p^{n+1}$$

□

Remark 16. We can interpret Proposition 15 in binary for $p = 3$. In this case there exists an $m \in \mathbb{N} \cup \{0\}$ such that 3^n divides

$$4^m x + \frac{4^m - 1}{3} = 4^m x + 4^{m-1} + \ldots 4 + 1$$

If $X = X^1$ is the binary number representing x then this would mean that 3^n divides the binary number $X(01)_m$ (note that m can be 0, in which case 3^n divides x). This compact representation is sometimes easier to understand when computing in binary.

Let $x \in 2\mathbb{N}+1$ such that $x \neq 1$ and is represented by a binary number $X = X_1$ then Proposition 1 (e) says that $X01$ and X lead to the same number after one iteration. This is also stated in [2] (in Remark after Theorem 1): If x is an odd number then $4x+1$ and x lead to the same number after one iteration (in OCS). So in fact it suffices to show that the Collatz conjecture holds for all numbers congruent to 1 modulo 4 in order to prove the Collatz conjecture. Andaloro showed an even more general case. In [2] he showed that the Collatz conjecture is true iff the conjecture holds for all numbers congruent to 1 modulo 8 (see [2] Lemma 5) iff the conjecture holds for all numbers congruent to 1 modulo 16 (see [2] Theorem 2).

We will generalize this further using our observation from the odd collatz sequence with binary numbers and conclude that the conjecture is true iff the conjecture holds for all numbers congruent to 1 modulo 2^n for arbitrary $n \in \mathbb{N}$. We have so far seen that this is true for $n = 1,2,3,4$ ($n = 1$ being our interpretation of the conjecture itself using only odd numbers!).

Theorem 17. Let $n \in \mathbb{N}$ then the Collatz conjecture is true iff the Collatz conjecture holds for all x such that $x \equiv 1 \bmod 2^n$

Proof. The sufficient condition is obvious so we will prove the necessary condition. Suppose $y \in \mathbb{N}$ is odd and not equal to 1 and let $n \in \mathbb{N}$ be the whole number in our Theorem. Because of Lemma 5 and Theorem 2 of [2] (see paragraph before this Theorem), we may assume that $n > 2$. Furthermore, without loss of generality we can even assume that $n \in 2\mathbb{N}+1$ (otherwise we can prove the above Theorem for $n+1$ and the result with n follows). Let $Y = Y_1^1$ be the binary number that represents y and set $k = \frac{n-1}{2}$, then by Proposition 15 (with $p = 3$, see also Remark 16) there is an $m \in \mathbb{N} \cup \{0\}$ such that 3^k divides $Y(01)_m$. Let $X = X^1$ be the binary number representing a number $x \in \mathbb{N}$ such that

$$3^k x = 4^m y + \frac{4^m - 1}{3}$$

By Lemma 9 we have $X0_n1 \overset{k}{\rightsquigarrow} Y(01)_m01$. Let Z be the binary number such that $Y \overset{1}{\rightsquigarrow} Z$ (recall that we assumed Y is not 1). Then by Corollary 2 (the second squiggly arrow below)

$$X0_n1 \overset{k}{\rightsquigarrow} Y(01)_{m+1} \overset{1}{\rightsquigarrow} Z$$

We are proving the necessary condition, so $X0_n1$ (which is a number that is congruent to 1 modulo 2^n) eventually leads to 1. Thus $Y \rightsquigarrow 1$ as well since it leads to Z which is in the OCS of $X0_n1$. $\square$

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