Denominator Bounds for Higher Order Systems of Linear Recurrence Equations

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Overview

Let $\mathbb{F}$ be a field and $\sigma\colon \mathbb{F} \to \mathbb{F}$ be an automorphism. We extend σ to the rational functions $\mathbb{F}(t)$ by

Σ^* case: either $\sigma(t) = t + c$ for some $c \in \mathbb{F}^*$,

Π case: or $\sigma(t) = qt$ for some $q \in \mathbb{F}^*$.

If there is no $p \in \mathbb{F}(t) \setminus \mathbb{F}$ such that $\sigma(p) = p$, then we call $(\mathbb{F}(t), \sigma)$ a *$\Pi\Sigma^*$-extension* of $(\mathbb{F}, \sigma)$ (see, for example, [6, 7]).

We consider systems of the form

$$A_\ell \sigma^\ell(y) + \ldots + A_1 \sigma(y) + A_0 y = b \qquad \text{(SYS)}$$

where

- $A_0, \ldots, A_\ell \in \mathbb{F}[t]^{m \times n}$ are polynomial matrices, and
- $b \in \mathbb{F}[t]^m$ is a polynomial vector.

We are interested in finding rational solutions $y \in \mathbb{F}(t)^n$.

In this work, we concentrate on finding a *denominator bound*, that is, a nonzero polynomial $d \in \mathbb{F}[t]$ such that $dy \in \mathbb{F}[t]^n$ is a polynomial vector for all possible rational solutions $y \in \mathbb{F}(t)^n$.

Previous Work

Most existing algorithms as for instance [3, 1] work by translating the higher order system to a first order system. We only know of one method, [4], dealing directly with higher order systems. Our algorithm is similar to that later work; however, we expand it in several points:

- Most importantly, we address the problem for general $\Pi\Sigma^*$ extensions instead of concentrating on the case that $\mathbb{F}$ is a constant field and σ is the shift operator $t \mapsto t+1$.
- In addition our method does not require the system matrices to be square or their rows to be linearly independent.

Head & Tail Regularity

We call the system (SYS) *head regular* if $m = n$ and $\det A_\ell \neq 0$; and we call it *tail regular* if $m = n$ and $\det A_0 \neq 0$.

We can transform a system to head or tail regular form in the following way: We consider $A = A_\ell \sigma + \ldots + A_1 \sigma + A_0$ as an operator matrix over the ring $\mathbb{F}(t)[\sigma, \sigma^{-1}]$ of *Ore Laurent polynomials*. Applying *row-/column-reduction* (see, for example, [5]) with respect to σ we obtain unimodular operator matrices S and T such that

$$SAT = \begin{pmatrix} \tilde{A}_{\tilde{\ell}} \sigma^{\tilde{\ell}} + \ldots + \tilde{A}_1 \sigma + \tilde{A}_0 & 0 \\ 0 & 0 \end{pmatrix}$$

where $\tilde{A}_{\tilde{\ell}}$ is a regular polynomial matrix; that is, the corresponding subsystem is head regular. Similarly row-/column-reduction with respect to σ^{-1} produces a tail regular system. If the original system is already head regular, then the tail regular system will have the same number of nonzero rows and moreover using row operations will be sufficient.

Because the transformation matrices are unimodular, the solutions of the transformed systems (if any exist) are in one-to-one correspondence to the solutions of the original system.

Spread & Dispersion

The *spread* of two nonzero polynomials f and $g \in \mathbb{F}[t] \setminus \{0\}$ is defined as

$$\operatorname{spread}(f, g) = \{k \geq 0 : \gcd(f, \sigma^k(g)) \notin \mathbb{F}\}.$$

The *dispersion* of f and g is

$$\operatorname{disp}(f, g) = \max \operatorname{spread}(f, g)$$

with the conventions that $\max \emptyset = -\infty$ and $\max S = \infty$ whenever S is infinite.

In the Σ^* case, the spread is always finite. In the Π case, there is a problem when $t \mid f$ and $t \mid g$. However, that is the only problematic case (see, for example, [7]). Thus, below we will simply assume that in the Π case t does not occur as a divisor.

Main Result

Assume now that the system (SYS) has already been brought into head regular form

$$A = A_\ell \sigma^\ell + \ldots + A_1 \sigma + A_0$$

and that its related tail regular system is

$$PA = \tilde{A}_{\tilde{\ell}} \sigma^{\tilde{\ell}} + \ldots + \tilde{A}_1 \sigma + \tilde{A}_0$$

where P is a unimodular operator matrix. Let m be a common denominator of A_ℓ^{-1} and p be a common denominator of $\tilde{A}_0^{-1}$.

If $y = d^{-1} z$ is a rational solution with $\gcd(z_1, \ldots, z_n, d) = 1$, then

$$\operatorname{disp}(d, d) \leq \operatorname{disp}(\sigma^{-\ell}(m), p) = D$$

and

$$d \,\Big|\, \gcd\Big(\prod_{j=0}^{D} \sigma^{-\ell-j}(m), \prod_{j=0}^{D} \sigma^{j}(p)\Big).$$

The proof is very similar to the proof of the analogous result for scalar equations in [7]. The main idea is to isolate the $\sigma^\ell(y)$ term in the equation (SYS) and then to apply shifts and linear combinations to the remaining terms. At some point they will be further apart from $\sigma^\ell(y)$ then the dispersion of the denominator d. When this happens, the denominator can only cancel with the coefficients of the system.

Example

We do an example for the case $\mathbb{F} =$ and $\sigma(t) = 2t$. Consider the system

$$\begin{pmatrix} -2(4t+1)(2t+1)^2 & 0 \\ 2(t+1)(2t+1)(t+2) & -9(t+1)(2t+1)(t+2) \end{pmatrix} y(2t) + \begin{pmatrix} 3(2t+1)(t+2)(4t+1) & 3(2t+1)(t+2)(4t+1) \\ 0 & 3(t+2)(t^2+3t+2) \end{pmatrix} y(t) = 0.$$

This system is already both head and tail regular. We obtain

$$m = 18(4t+1)(2t+1)^2(t+1)(t+2) \qquad \text{and} \qquad p = 3(4t+1)(2t+1)(t+2)^2(t+1).$$

The dispersion is 2. This yields a denominator bound of

$$(2t+1)(t+1)^3(t+2)^2$$

which matches the solutions

$$\begin{pmatrix} -\frac{1}{t+1} \\ \frac{t+4}{3(t+1)(t+2)} \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} -\frac{1}{t+2} \\ \frac{1}{3(t+1)} \end{pmatrix}.$$

References

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