Representation of hypergeometric products in difference rings*

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Abstract

In his pioneering work [1, 2], Michael Karr introduced $\Pi\Sigma$-fields which provide a rather general framework for symbolic summation. He worked out the first algorithmic steps to represent indefinite nested sums and products as transcendental extensions over a computable ground field $\mathbb{K}$ called the field of constants. Furthermore, he presented an algorithm that solves the parameterized telescoping problem, and as special case the telescoping and creative telescoping problem [3] within a given $\Pi\Sigma$-field.

Over the recent years, Karr's $\Pi\Sigma$-field approach has been extended to a difference ring approach [4, 5]. In this new framework also algebraic products like α^n with α being a primitive root of unity can be tackled. More precisely, a general machinery is obtained in which indefinite nested sums defined over hypergeometric products can be rephrased in the setting of the so-called R$\Pi\Sigma^*$-ring extensions. Moreover, general algorithms have been developed that solve the parameterized telescoping equations within such a constructed difference ring.

This machinery implemented within the summation package `Sigma` [14] works in general if one knows in advance how the arising hypergeometric products can be rephrased within an R$\Pi\Sigma^*$-ring extension. In particular, restricting to a field $\mathbb{K}$ that is given as the quotient field of a unique factorization domain $\mathbb{U}$, such a construction can be done automatically [6, 4]: given a finite set of hypergeometric products that evaluate to elements in $\mathbb{K}$, one can construct an R$\Pi\Sigma^*$-extension with constant field $\mathbb{K}$ in which the products can be rephrased. This is especially the case, if $\mathbb{U}$ forms a polynomial ring over $\mathbb{Q}$ or over $\mathbb{Q}(\mathbb{i})$. However, when one tries to extend the constant field to include other number fields, one looses the unique factorization property inherent in the constant field $\mathbb{K}$.

In a nutshell, in order to obtain a complete summation machinery in which one can represent hypergeometric products defined over $\mathbb{K}$ where $\mathbb{K}$ is a rational function field over an algebraic number field, and more generally, in which one can rephrase indefinite nested sums over such hypergeometric products, we will solve the following missing building block that can be formulated as follows.

Product-representation in RΠ*-extensions:*
Given a finite number of hypergeometric products that evaluate to our general field $\mathbb{K}$ (see

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above); *construct* an RΠ-extension (where the constant field $\mathbb{K}$ might be extended by algebraic number extensions) and compute elements in the obtained difference ring such that they represent the given input products.

With the obtained algorithm, we can rephrase, for instance, the hypergeometric product expression

$$P(n)=\prod_{k=1}^{n}\frac{-945\sqrt{-5}}{189\,k}+\prod_{k=1}^{n}\frac{-7056\,k}{45\sqrt{-5}\,(\mathbb{i}+\sqrt{3})^4\,(k+2)^2}+\prod_{k=1}^{n}\frac{-51631104\,k}{15\sqrt{-5}\,(\mathbb{i}+\sqrt{3})^{10}\,(k+2)^5} \tag{1}$$

in a difference ring with constant field $\mathbb{K}=\mathbb{Q}\big((-1)^{\frac{1}{6}},\sqrt{-5}\big)$ where $(-1)^{\frac{1}{6}}=\frac{\sqrt{3}+\mathbb{i}}{2}$. We omit the explicit construction in this summary, but present the output reinterpreted again as hypergeometric products. More precisely, we will find the following identity:

$$P(n)=\frac{\Big(\big((-1)^{\frac{1}{6}}\big)^n\Big)^9\Big(\big(\sqrt{5}\big)^n\Big)^3}{n!}+\frac{4\Big(\big((-1)^{\frac{1}{6}}\big)^n\Big)^{11}\big(7^n\big)^2}{\Big(\big(\sqrt{5}\big)^n\Big)^3\,(n+1)^2\,(n+2)^2\,n!}+\frac{32\Big(\big((-1)^{\frac{1}{6}}\big)^n\Big)^5\big(7^n\big)^5}{\Big(\big(\sqrt{5}\big)^n\Big)^3\,(n+1)^5\,(n+2)^5\,(n!)^4}. \tag{2}$$

Besides this algorithmic representation within the setting of difference rings, we will emphasize the following aspects:

1. Based on difference ring theory [7, 8] the produced representation is given in terms of hypergeometric products that are algebraically independent among each other. E.g., the right hand side of (2) can be given within a Laurent polynomial ring. More precisely, they can be given within the ring

$$\mathbb{K}(n)\big[\big((-1)^{\frac{1}{6}}\big)^n\big]\Big[\big(\sqrt{5}\big)^n,\tfrac{1}{(\sqrt{5})^n}\Big]\big[7^n,\tfrac{1}{7^n}\big]\big[n!,\tfrac{1}{n!}\big] \tag{3}$$

   defined over the ring $\mathbb{K}(n)\big[\big((-1)^{\frac{1}{6}}\big)^n\big]$ where the generators $\big(\sqrt{5}\big)^n$, 7^n and $n!$ are transcendental.

2. Adjoining the first two products in (1) to $\mathbb{K}(n)$ with $\mathbb{K}=\mathbb{Q}\big(\mathbb{i}+\sqrt{3},\sqrt{-5}\big)$ would lead to a Π-extension. However, the third product cannot be handled this way since there is the relation

$$\left(\prod_{k=1}^{n}\frac{-51631104\,k}{15\sqrt{-5}\,(\mathbb{i}+\sqrt{3})^{10}\,(k+2)^5}\right)^2=\left(\prod_{k=1}^{n}\frac{-945\sqrt{-5}}{189\,k}\right)^3\left(\prod_{k=1}^{n}\frac{-7056\,k}{45\sqrt{-5}\,(\mathbb{i}+\sqrt{3})^4\,(k+2)^2}\right)^5.$$

   In general, obtaining such relations can be non-trivial and one will have to resort to our algorithm to construct the ring (3) to represent such hypergeometric products.

3. Further, we solve the zero-recognition problem for the class of expressions in terms of hypergeometric products: a polynomial expression in terms of hypergeometric products evaluates to the zero-sequence (from a certain point on) if and only if the expression is simplified with our algorithm to the 0 element. This important feature is exploited, e.g., in [8, 9] to solve the zero-recognition problem for nested sums over such hypergeometric products.

4. Moreover, our produced result produces hypergeometric products where the degrees in the numerators and denominators of the multiplicands have minimal degrees. Restricting to just one hypergeometric product, this problem has been treated also in [10].

The outline of the poster is as follows. We first begin with a detailed problem specification within the setting of sequences and the underlying computer algebra formulation in the setting of difference rings. Then we will work out in detail how one can represent hypergeometric products in the setting of RΠ-extensions. Here we generalize non-trivially the techniques presented in [6, 4] which are based on sub-algorithms of [1]. In addition, we utilize Ge's algorithm [11] (see also [12, 6]) which solves the orbit problem: given $f_1, \dots, f_d \in \mathbb{K}^*$, his algorithm computes a basis of the $\mathbb{Z}$-module $\{(z_1, \dots, z_d) \in \mathbb{Z}^d \mid f_1^{z_1} \cdots f_d^{z_d} = 1\}$. Finally, we observe that the presented algorithm can be extended further to the class of q-hypergeometric, multibasic and mixed hypergeometric products [13, 15] and give some further comments how this can be accomplished.

References

[1] Michael Karr. Summation in Finite Terms. *Journal of the ACM (JACM)*,28(2):305–350, 1981.

[2] Michael Karr. Theory of Summation in Finite Terms, *J. Symbolic Comput.*,1:303–315, 1985.

[3] Doron Zeilberger. The Method of Creative Telescoping. *J. Symbolic Comput.*, 11:195–204, 1991.

[4] Carsten Schneider. A Streamlined Difference Ring Theory: Indefinite Nested Sums, the Alternating Sign, and the Parameterized Telescoping Problem, *2014 16th International Symposium on Symbolic and Numeric Algorithms for Scientific Computing (SYNASC)*, 26–33, 2014.

[5] Carsten Schneider. A Difference Ring Theory for Symbolic Summation, *Journal of Symbolic Computation*,72, 82–127, 2016.

[6] Carsten Schneider. Product Representations in ΠΣ-Fields, *Annals of Combinatorics*, 9(1):75–99, 2005.

[7] Marius van der Put and Michael F. Singer. *Galois Theory of Difference Equations,* volume 1666 of *Lecture Notes in Mathematics.*, Springer-Verlag, Berlin,1997.

[8] Carsten Schneider. Summation Theory II: Characterizations of RΠΣ-Extensions and Algorithmic Aspects. pages 1–54, 2016. arXiv:1603.04285 [cs.SC].

[9] Carsten Schneider. Symbolic Summation in Difference Rings and Applications. In M. Rosenkranz, editor *Proc. ISSAC 2016*, 2016.

[10] Sergei A. Abramov and Marko Petkovšek. Polynomial Ring Automorphisms, Rational (w, σ)-Canonical Forms, and the Assignment Problem, *J. Symbolic Comput.*, 45(6):684–708, 2010.

[11] Guoqiang Ge. *Algorithms Related to the Multiplicative Representation of Algebraic Numbers*, PhD thesis, Univeristy of California at Berkeley, 1993.

[12] Manuel Kauers. *Algorithms for Nonlinear Higher Order Difference Equations*. PhD thesis, RISC-Linz, October 2005.

[13] Andrej Bauer and Marko Petkovšek. Multibasic and Mixed Hypergeometric Gosper-Type Algorithms. *Journal of Symbolic Computation*, 28:(4-5):711–736, October, 1999.

[14] Carsten Schneider. Symbolic Summation Assists Combinatorics. *Sém. Lothar. Combin.*, 56:1–36, 2007. Article B56b.

[15] Shaoshi Chen and Frédéric Chyzak and Ruyong Feng and Guofeng Fu and Ziming Li. On the Existence of Telescopers for Mixed Hypergeometric Terms. *Journal of Symbolic Computation*, 68:1–26, 2015.