

# Workshop on Symbolic Combinatorics and Algorithmic Differential Algebra

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The workshop on **Symbolic Combinatorics and Computational Differential Algebra** was held from September 14 to September 20, 2015 as part of the **Thematic Program on Computer Algebra** at the Fields Institute in Toronto, Canada. The workshop was devoted to algorithmic developments in Combinatorics and Differential Algebra with a particular focus on the interaction of these two areas.

*Symbolic Combinatorics:* Symbolic algorithms and software have recently been developed that allow researchers to discover and prove combinatorial identities as well as understand analytic and algebraic properties of generating functions. These functions seldom have closed form solutions and even when they do, direct evaluation may be intractable. Recent work in Symbolic Combinatorics has focused on representing such functions by annihilating differential or difference operators and then using techniques from differential and difference algebra, as well as analysis, to analyze these equations and their solutions. Besides methods relating to creative telescoping and verification of function identities, other areas of emphasis will include effective methods in difference algebra, the Galois theories of difference equations, and the interaction of differential and difference algebra in combinatorics.

*Computational Differential Algebra* is an approach to nonlinear or linear differential equations and differential structures, focusing not only on finding explicit closed form solution, but also on simplifying such equations to yield preconditioning for subsequent numerical solution, as well clearer understanding of the qualitative behavior of solutions. Highlighted aspects will include differential-elimination completion algorithms; geometric differential algebra, moving frames and differential invariants; differential Galois theory and integrability; complexity and open problems. We have greatly enjoyed the diverse and inspiring contributions from the speakers who contributed to the program. Their abstracts are quoted below.

*Carlos Arreche, North Carolina State University*

## **On the computation of the difference-differential Galois group for a second-order linear difference equation**

Abstract. Given a linear difference equation, there is a difference-differential Galois group that encodes the differential-algebraic dependencies among the solutions of the equation. After giving a brief introduction to this theory, I will describe algorithms to compute the Galois group associated to a second-order linear difference equation over  $C(x)$ , the field of rational functions over a computable field  $C$  of characteristic zero, with respect to the  $C$ -linear shift automorphism that sends  $x$  to  $x + 1$ . I will also discuss some concrete examples to illustrate these algorithms, and show explicitly in the examples how to derive the differential-algebraic dependencies among the solutions from the knowledge of the defining equations for the Galois group.

*Cyril Banderier, Institut Galilée – Université Paris-Nord*

## **From algebraic to differentially-algebraic functions in combinatorics: impact of positive (integer) coefficients**

Abstract. Asymptotics of recurrences is the key to get the typical properties of combinatorial structures, and thus the complexity of many algorithms relying on these structures. The associated generating function often follows a linear differential equation: we are here in the so-called “D-finite” world. For the matters of asymptotics, this case of

linear recurrences (with polynomial coefficients) is well covered by the “Analytic Combinatorics” book of Flajolet and Sedgewick (though the computations of constants is still a challenge, related to the theory of Kontsevich-Zagier periods and evaluation of G-functions and E-functions). At the border of this D-finite world, lies “algebraic-differential functions”. The terminology is not yet fixed and similar terms are used, up to a permutation, by several authors: let  $dz^m$  be the  $m$ -th derivative of  $F(z)$ , the function is said “algebraic-differential” if there exists a polynomial  $P$  such that  $P(z, F, F', \dots, dz^m F) = 0$ . For all these worlds, having some positive (integer) coefficients leads to some strong constraints on the asymptotics (these is now well understood for algebraic function), and we try to see what happens in a more general setting.

We will give examples of such functions (motivated by some combinatorial problems), and show how a symbolic combinatorics approach can help for automatic asymptotics of their coefficients, and some open related open questions/challenges for computer algebra (joint works with Michael Drmota and Hsien-Kuei Hwang)

*Moulay A. Barkatou, Université de Limoges*

### **On Apparent Singularities of Systems of Linear Differential Equations with Rational Function Coefficients**

Abstract. Let  $(S) \frac{dY}{dz} = A(z)Y$  be a system of first order linear differential equations with rational function coefficients. The (finite) singularities of  $(S)$  are the poles of the entries of the matrix  $A(z)$ . A singular point  $z_0$  of  $(S)$  is called an apparent singularity for  $(S)$  if there is a basis of solutions of  $(S)$  which are holomorphic in a neighborhood of  $z_0$ . In this talk we shall present a new algorithm which, given a system of the form  $(S)$ , detects apparent singularities and constructs an equivalent system  $(S')$  with rational coefficients the singularities of which coincide with the non apparent singularities of  $(S)$ . Our method can, in particular, be applied to the companion system of any linear differential equation with arbitrary order  $n$ . We thus have an alternative method to the standard methods for removing apparent singularities of linear differential operators. We shall compare our method to the one designed for operators and we shall show some applications and examples of computation. This talk is based on a joint work, with S. Maddah, recently presented at ISSAC 2015.

*Alin Bostan, INRIA*

### **Quasi-optimal computation of the $p$ -curvature**

Abstract. The  $p$ -curvature of a system of linear differential equations in characteristic  $p$  is a matrix that measures to what extent the system is close to having a fundamental matrix of rational function solutions. This notion, originally introduced in the arithmetic theory of differential equations, has been recently used as an effective tool in computer algebra and in combinatorial applications. We describe a recent algorithm for computing the  $p$ -curvature, whose complexity is almost optimal with respect to the size of the output. The new algorithm performs remarkably well in practice. Its design relies on the existence of a well-suited ring, of so-called Hurwitz series, for which an analogue of the Cauchy–Lipschitz Theorem holds, and on a FFT-like method in which the “evaluation points” are Hurwitz series. Joint work with Xavier Caruso and Éric Schost.

*Frédéric Chyzak, INRIA Saclay Île-de-France*

### **Explicit generating series for small-step walks in the quarter plane**

Abstract. Lattice walks occur frequently in discrete mathematics, statistical physics, probability theory, and operational research. The algebraic properties of their enumeration generating series vary greatly according to the family of admissible steps chosen to define them: their generating series are sometimes rational, algebraic, or D-finite, or sometimes they possess no apparent equation. This has recently motivated a large classification effort. Interestingly, the equations involved often have degrees, orders, and sizes, making calculations an interesting challenge for computer algebra.

In this talk, we study nearest-neighbours walks on the square lattice, that is, models of walks on the square lattice, defined by a fixed step set that is a subset of the 8 non-zero vectors with coordinates 0 or  $\pm 1$ . We concern ourselves with the counting of walks constrained to remain in the quarter plane, counted by length. In the past, Bousquet-Mélou and Mishna showed that only 19 essentially different models of walks possess a non-algebraic D-finite generating series; the linear differential equations have then been guessed by Bostan and Kauers. In this work in progress, we give the first proof that these equations are satisfied by the corresponding generating series. This allows to derive nice formulas for the generating series, expressed in terms of Gauss’ hypergeometric series, to decide their algebraicity or transcendence. This also gives hope to extract asymptotic formulas for the number of walks counted by lengths.

(Based on work in progress with Alin Bostan, Mark van Hoeij, Manuel Kauers, and Lucien Pech.)

*Shaoshi Chen, Chinese Academy of Sciences*

**Proof of the Wilf-Zeilberger Conjecture**

Abstract. In 1992, Wilf and Zeilberger conjectured that a hypergeometric term in several discrete and continuous variables is holonomic if and only if it is proper. Strictly speaking the conjecture does not hold, but it is true when reformulated properly: Payne proved a piecewise interpretation in 1997, and independently, Abramov and Petkovsek in 2002 proved a conjugate interpretation. Both results address the pure discrete case of the conjecture. In this paper we extend their work to hypergeometric terms in several discrete and continuous variables and prove the conjugate interpretation of the Wilf-Zeilberger conjecture in this mixed setting. This is a joint work with Christoph Koutschan.

*Evelyne Hubert, INRIA Méditerranée; Coauthor: Mathieu Collowald*

**A moment matrix approach to symmetric cubatures**

Abstract. A quadrature is an approximation of the definite integral of a function by a weighted sum of function values at specified points, or nodes, within the domain of integration. Gaussian quadratures are constructed to yield exact results for any polynomials of degree  $2r - 1$  or less by a suitable choice of  $r$  nodes and weights. Cubature is a generalization of quadrature in higher dimension.

Constructing a cubature amounts to find a linear form  $p \rightarrow a_1p(x_1) + \dots + a_r p(x_r)$  from the knowledge of its restriction to polynomials of degree  $d$  or less. The unknowns are the weights  $a_j$  and the nodes  $x_j$ . An approach based on moment matrices was proposed in [2, 4]. We give a basis-free version in terms of the Hankel operator  $H$  associated to a linear form. The existence of a cubature of degree  $d$  with  $r$  nodes boils down to conditions of ranks and positive semidefiniteness on  $H$ . We then recognize the nodes as the solutions of a generalized eigenvalue problem. Standard domains of integration are symmetric under the action of a finite group. It is natural to look for cubatures that respect this symmetry [1, 3]. Introducing adapted bases obtained from representation theory, the symmetry constraint allows to block diagonalize the Hankel operator  $H$ . The size of the blocks is explicitly related to the orbit types of the nodes. From the computational point of view, we then deal with smaller-sized matrices both for securing the existence of the cubature and computing the nodes.

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*Manuel Kauers, Johannes Kepler University*

**Creative Telescoping via Hermite Reduction**

Abstract. Creative telescoping is a key tool in symbolic summation and integration. It is used for constructing for a given definite sum or integral an associated linear recurrence or differential equation, which can then be used by other algorithms for finding out all sorts of interesting facts about the quantity in question. Four generations of creative telescoping algorithms can be distinguished: the first was based on elimination in ideals of operator algebras. The second is the classical Zeilberger algorithm and its variants. The third goes back to an idea of Apagodu and Zeilberger. These algorithms are particularly easy to implement and to analyze, but may not find optimal solutions. The fourth and final (so far) generation of creative telescoping algorithms is based on Hermite reduction. This idea was first worked out for definite integrals of multivariate rational functions by Bostan, Chen and Chyzak. It has since been extended to more general classes of sums and integrals by these and other authors. In the talk, we will explain the idea of this approach and a striking advantage over earlier algorithms. We will also present a Hermite-reduction based algorithm applicable to definite hypergeometric sums, published this year in a joint ISSAC paper with Chen, Huang, and Li.

*Christian Krattenthaler, Universitaet Wien*

**Two problems at the interface of combinatorics and symbolic computation**

Abstract. I shall present two (unrelated) problems which arise in work on congruences for combinatorial sequences that I have been pursuing with Thomas Müller. The first one concerns the formal power series  $\sum_n z^{p^n}$ , where  $p$  is

a given prime number. The question is to determine the minimal degree of a polynomial relation satisfied by this series modulo some fixed power of  $p$ . The second problem arises in combinatorial group theory and concerns the (Hadamard) product of P-recursive sequences. The question is what one can say about the leading coefficient of the recurrence satisfied by the product sequence. In both cases, I shall explain the context, and I will formulate precise conjectures (that may sometimes rather be speculations).

*George Labahn, University of Waterloo*

### **Rational Invariants of Finite Abelian Groups and their use in solving polynomial systems**

Abstract. We give a constructive procedure for determining a generating set of rational invariants of the linear action of a finite abelian group in the non-modular case and investigate its use in the symmetry reductions of polynomial systems. Finite abelian subgroups of  $GL(n, K)$  can be diagonalized which allows the group action to be accurately described by an integer matrix of exponents. We can make use of integer linear algebra to construct both a minimal generating set of invariants and the substitution to rewrite any invariant in terms of this generating set. The set of invariants provide a symmetry reduction scheme for polynomial systems whose solution set is invariant by a finite abelian group action.

This is joint work with Evelyne Hubert (INRIA Méditerranée).

*Sumayya Suzy Maddah, XLIM*

### **Formal Reduction of Singularly-Perturbed Linear Differential Systems**

Abstract. In this talk, we are interested in the local analysis of singularly-perturbed linear differential systems. Such systems have a vast literature and arise profoundly in applications. However, their symbolic resolution is still open to investigation. The complications arise mainly from the phenomenon of turning points. We extend notions introduced for the treatment of their unperturbed counterparts and generalize corresponding algorithms to construct formal solutions. The underlying components of the formal reduction proposed are stand-alone algorithms as well and serve different purposes (e.g. rank reduction, classification of singularities, computation of the restraining index). We present our proposed algorithm and associated packages written in Maple. This is a joint work with Molay A. Barkatou.

*Elizabeth Mansfield, University of Kent; Joint work with Gloria Mari Beffa*

### **Lattice based multispace and applications to moving frames**

Abstract. We propose a generalisation of the jet bundle based on equivalence classes of functions in a neighbourhood of a point, in which a function is equivalent to its Lagrange interpolation on diffeomorphic images of corner lattices. Under coalescence of the lattice, this interpolation becomes the Taylor polynomial, and thus the jet bundle is an embedded sub manifold. The construction relies on the multivariate approximation results of de Boor and Ron. We use our construction to provide a space in which smooth and discrete moving frames can be treated by a uniform construction, and in which certain convergence results can be assumed. In particular, we will have limits of discrete to smooth invariants as well as showing the recurrence relations of the discrete invariants will limit to the differential syzygies of the smooth invariants. Other possible uses of our construction will be discussed. However, the difference operators associated to the Lagrange interpolation have some interesting properties which could make them hard to work with.

*Stephen Melczer, University of Waterloo & ENS Lyon*

### **Effective Analytic Combinatorics in Several Variables, With Applications to Lattice Path Enumeration**

Abstract. Recent work in the study of analytic combinatorics in several variables has shown how to derive asymptotics for the coefficients of certain families of D-finite functions by representing them as diagonals of multivariate rational functions. Although a theoretical framework has been laid over the last two decades (and before), there are still obstacles which has made the ultimate dream of an effective implementation of these techniques elusive. We begin by surveying the field of analytic combinatorics and outline the difficulties mentioned above. This will be followed by a more hopeful look at the problem when restricted to the so-called “combinatorial” case. Previous work by Melczer and Mishna on the enumeration of lattice paths whose defining sets of steps are highly symmetric can be viewed in this context, and it will be shown how the results they obtained relate to more general structure theorems. Finally, new work on lattice paths whose sets of steps are missing some symmetries – whose analysis relies on rational functions that are not combinatorial – will be outlined.

This is joint work with Mark Wilson, Marni Mishna, George Labahn, and Bruno Salvy.

*Johannes Middeke, Johannes Kepler University*

### **Uncoupling of Linear Difference/Differential Systems of Higher Order**

Abstract. We consider linear systems of difference or differential equations of higher order. Modelling them as matrices of Ore polynomials, we will explore the use of matrix normal forms for reducing the system to independent scalar equations.

*Marni Mishna, Simon Fraser University*

### **The combinatorics behind lattice path asymptotics**

Abstract. Lattice path walk models are straightforward combinatorial classes that are well suited to a systematic analysis. Indeed many of the enumerative results from the past 20 years or so have had a computer algebra component to the argument. In this talk I will first survey some of these techniques and results (with a focus on asymptotic enumeration), and then discuss the underlying combinatorial structure which guides these results. The combinatorial interpretations are key to various applications such as uniform random generation of lattice walks in the quarter plane, notably for models with strong drift.

Joint work with Samuel Johnson, Jeremie Lumbroso, Yann Ponty and Karen Yeats.

*Peter J. Olver, University of Minnesota*

### **Algebras of Differential Invariants**

Abstract. A classical theorem of Lie and Tresse states that the algebra of differential invariants of a Lie group or (suitable) Lie pseudo-group action is finitely generated. I will present a fully constructive algorithm, based on the equivariant method of moving frames, that reveals the full structure of such non-commutative differential algebras, and, in particular, pinpoints generating sets of differential invariants as well as their differential syzygies. A variety of applications and outstanding issues will be discussed, including equivalence and symmetry detection in image processing, and some surprising results in classical surface geometries.

*Alexey Ovchinnikov, CUNY Queens College*

### **Algorithms for computing Galois groups of parameterized differential equations**

Abstract. Galois groups of parameterized linear differential equations are linear differential algebraic groups, that is, groups of matrices whose entries satisfy polynomial differential equations. The theory was initiated by Cassidy, Landesman, and Singer. Arreche and Dreyfus have developed algorithms for computing such groups of second order differential equations, and Arreche has applied this to study properties of special functions.

We will discuss how to calculate the unipotent radicals of parameterized Galois groups if the differential operator is a composition of two completely reducible differential operators. We will also discuss how to use this to calculate the Galois group of a third order linear differential equation with parameters and to illustrate this by a concrete example of the Lommel differential equation.

This is joint work with Charlotte Hardouin.

*Peter Paule, Research Institute for Symbolic Computation (RISC), Johannes Kepler University Linz*

### **Computer Algebra Algorithms for Modular Functions**

Abstract. The talk reports on a joint project with Silviu Radu. Its goal is to develop computer algebra algorithms to assist mathematical research in connection with modular functions. A major application is a new unified approach to the celebrated Ramanujan congruences for partition numbers modulo powers of 5, 7, and 11. The algorithms presented in the talk are also relevant outside this application domain, for instance, to decide subalgebra membership or to construct algebraic relations for functions on Riemann surfaces.

*Greg Reid, University of Western Ontario*

### **Symbolic-Numeric Differential Algebra and Applications**

Abstract. Geometric involutivity is a property which first arose in the geometric analysis of systems of analytic nonlinear PDE for compatibility and formal existence of solutions. Cartan conjectured a finite prolongation (differentiation) procedure to makes systems geometrically involutive and this finiteness was eventually proved under certain regularity assumptions by Kuranishi. The geometric invariance of geometric involution, yields favourable properties for symbolic-numeric implementations, as shown in joint work with Wittkopf. In the special case of PDE

with constant coefficients and their naturally associated polynomial systems, geometric involutivity yields Geometric Involutive Bases. Such bases are a geometric cousin of Pommaret Involutive Bases, which are a type of Groebner Basis. Computation and exploitation of Geometric Involutive Bases for systems of polynomial equations were further developed in joint work with Zhi, Scott, Wu and others.

I will discuss some recent theoretical developments and applications of symbolic-numeric algorithms to determine geometric involutivity. Specifically some recent progress in determination of approximate symmetry Lie algebras and related structures of differential equations is discussed. This is joint work with Ian Lisle, Tracy Huang and Shenghao Yang. I will also discuss some applications of involutivity to semi-definite programming methods for real solutions of polynomial systems which has played a significant role in our recent work, and also those of Lasserre, Zhi and their respective collaborators.

*Georg Regensburger, Austrian Academy of Sciences*

### **Computational and algebraic aspects of integro-differential operators**

Abstract. Differential operators with polynomial coefficients provide a rich algebraic structure with a wealth of results and algorithmic methods. Adding integral operators and evaluations (that is, multiplicative functionals), many new phenomena appear, including zero divisors and non-finitely generated ideals. The resulting algebra of integro-differential operators allows us in particular to study initial value and boundary problems for linear ODEs from an algebraic point of view.

In this talk, we will first describe a construction and normal forms of integro-differential operators with polynomial coefficients and discuss some basic algebraic properties. Then we present an algorithmic approach for computing polynomial solutions and the index for a class of linear operators on the polynomial ring that includes integro-differential operators. A generating set for right annihilators can be constructed in terms of polynomial solutions. For initial value problems, an involution of the algebra of integro-differential operators also allows us to compute left annihilators, which can be interpreted as compatibility conditions of integro-differential equations. We illustrate our approach with an implementation in the computer algebra system Maple. This is joint work with Alban Quadrat.

Finally, we will outline a recent generalization of integro-differential operators that allows integral operators acting on functions with singularities and arbitrary linear functionals. This is joint work with Clemens G. Raab.

*Austin Roche, Maplesoft*

### **Targeted overdetermined functional decomposition and applications to differential equations**

Abstract. The equivalence method for ordinary differential equations (ODEs) involves finding a transformation mapping a given equation into a target equation with known solution. Such equivalence transformations are found by solving systems relating the differential invariants of the input equation to those of the target equation. Standard solution techniques, applicable when the invariants are rational functions, use algebraic elimination and can solve complete families of equations characterized by many parameters. However, the complexity of such methods increases exponentially with the number of parameters, and can become impractical in interesting cases. A technique is presented for overdetermined rational function decomposition, specifically tailored to systems of invariants. Its complexity is effectively independent of the number of parameters defining the system. The use of this technique in the ODE equivalence method is described, including the use of minimal invariants which ensures the solution of all rational coefficient equations in the target class. Recognizing that related invariants tend to be composed as products of powers of a set of common polynomials, and furthermore that the pattern of these polynomials is invariant under composition by rational functions, we can compute these component polynomials invariantly. Using the target system as a model, a sequence of invariant computations is built that successively simplifies the system, leading eventually to the determination of the parameters and transformation function. The resulting algorithm mimics a formula in its specificity but lacks the associated expression growth. Application to solving the first order Abel ODE will be shown.

*Markus Rosenkranz, University of Kent*

### **Partial Integral Operators and the Hopf Algebra of Linear Substitutions**

Abstract. In contrast to their differential analogue, partial integral operators exhibit a rather subtle interaction with the action of  $GL(n)$  and its singular shell. We propose a new algebraic approach that reduces the matrix action on coefficient functions to the following three ingredients: (i) coalgebra structure on “univariate” coefficient functions, (ii) permutations, (iii) scaling of independent variables. Having these ingredients and a suitable univariate integral

operator, the tensor algebra is endowed with a linear action and a hierarchy of integral operators compatible with the substitutions.

The resulting multivariate hierarchy can then be used as coefficient functions in an extensive operator ring that contains additionally the partial integral operators as well as all linear substitutions. We propose a rewrite system for doing computations in this operator ring. The rewrite system is terminating, and we present a natural system of normal forms for it. It is conjectured that the system is moreover confluent, and hence a noncommutative Groebner basis for the ideal it generates. Applications to a simple class of linear partial differential equations are presented.

*Ekaterina Shemyakova, SUNY at New Paltz*

**Darboux transformations for differential operators on the superline**

Abstract. I shall present our very general result on a full description of Darboux transformations of any order for arbitrary (nondegenerate) differential operators on the superline. We show that every Darboux transformation of such operators factorizes into elementary Darboux transformations of order one. Similar statement holds for operators on the ordinary line. This is a joint work with S. Hill and Th. Voronov.

*Rika Yatchak, Johannes Kepler University*

**A primer on lattice path combinatorics**

Abstract. Lattice paths are fundamental combinatorial objects that appear in many different areas of mathematics. We provide an overview of classical techniques for enumerating lattice paths and finding expressions for their generating functions.

*Joachim von zur Gathen, University of Bonn*

**Combinatorics on polynomial equations: do they describe nice varieties?**

Abstract. We consider natural combinatorial questions about systems of multivariate polynomials over a finite field and the variety  $V$  that they define over an algebraic closure. Fixing the number of variables, the number of polynomials and the sequence of degrees, there are finitely many such systems. We ask: for how many systems is  $V$  nice? Is that usually the case?

“Nice” can refer to various properties:

- The system is regular, so that no polynomial is a zero divisor modulo the previous ones.
- $V$  is a set-theoretic complete intersection.
- $V$  is an ideal-theoretic complete intersection.
- $V$  is absolutely irreducible.
- $V$  is nonsingular.
- $V$  is non-degenerate (not contained in a hyperplane).

All but the last property usually hold. More precisely, for each of them we present a nonzero obstruction polynomial in the coefficients of the system so that the property holds when the obstruction does not vanish. The obstructions come with explicit bounds on their degrees. They yield estimates on the probability for the properties to hold. These probabilities tend rapidly to 1 with growing field size. Somewhat surprisingly, the last property behaves differently. Fixing the degree of  $V$ , most systems (with at least two polynomials) describe varieties that are hypersurfaces in some proper linear subspace.

Joint work with Guillermo Matera.

*Doron Zeilberger, Rutgers University*

**To Think in a Symbolic-Computational Way...**

Abstract. Currently, symbolic computation is still a small, fringe, discipline, at the borderline of math and computer science, with relatively little prestige. But this will change very soon, since once people (including those working in this area) would learn how to FULLY think in a symbolic-computational way, mathematics would grow exponentially, both in quantity, and especially in quality.

*Yang Zhang, University of Manitoba*

**The generalized inverses of Ore matrices and applications**

Abstract. Ore matrices are matrices over Ore algebras (including differential operators and difference operators), which have a long research history, at least dated back to Jacobson's seminal work in 1940s. In past two decades, Ore matrices have attracted more and more people in computer algebra area, and many important properties of Ore matrices have been discussed by using symbolic computation methods.

It is well-known that the generalized inverse of matrices over commutative rings (or fields), especially the Moore-Penrose inverse, play important roles in matrix theory and have applications in many areas: solving matrix equations, statistics, engineering, etc. This motivates us to consider the generalized inverse of Ore matrices.

First we define the generalized inverse and the Moore-Penrose inverse for Ore matrices, and prove some basic properties including uniqueness. Unlike the commutative case, the generalized inverse for a given Ore matrix may not exist. We use blocked matrices and greatest common right (left) divisor computations to give some sufficient and necessary conditions for Ore matrices to have the generalized inverses and the Moore-Penrose inverses. Moreover when these inverses exist, we construct algorithms to compute them.