An Algebraic-Geometric Method for Computing Zolotarev Polynomials

Georg Grasegger* N. Thieu Vo† ‡

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In this paper we study a differential equation which arises from finding Zolotarev polynomials. Using modifications of recently developed algorithms for finding rational solutions of algebraic ordinary differential equations, we construct a method for computing explicit expressions for Zolotarev polynomials.

1 Introduction

Zolotarev polynomials are the solution of a problem from approximation theory. They are defined to be those polynomials of a given degree which deviate least from zero in a certain interval. Since the first statement of the problem by Zolotarev the polynomials have been intensively studied (see for instance [2]). However, it seems there are still very few of them computed explicitly.

Explicit expressions for Zolotarev polynomials of degree up to four have been known for long time (compare [15, 20]). One approach to compute Zolotarev polynomials is by quantifier elimination with cylindrical algebraic decomposition. In [5] this is used for computing Zolotarev polynomials up to degree 5. Challenged by Collins and Kaltofen [10] this was extended later up to degree 12 in [12]. Analytic ideas and relations to applications have been used in [23, 24, 13, 4]. In these papers a differential equation is used which is solved by the polynomials. Algebraic solutions can be found in [14, 21, 18].

Here we use a different approach from a symbolic differential point of view. It is known, that Zolotarev polynomials fulfill a first-order algebraic ODE (c. f. [15]). Symbolic methods for computing rational solutions of algebraic differential equations have been recently

*Johann Radon Institute for Computational and Applied Mathematics (RICAM), Austrian Academy of Sciences (ÖAW)

†RISC, Johannes Kepler University Linz

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developed (see [6, 16, 8, 3, 25, 9]). Since, the ODE for the Zolotarev polynomials contains some parameters, which have to be chosen in a suitable way, these methods cannot be applied directly. In this paper we show, which modifications are needed and we give an algorithm on how to symbolically compute Zolotarev polynomials for any degree. We give explicit expressions up to degree 6.

2 Preliminaries

Zolotarev polynomials are considered in approximation theory as those polynomials which deviate least from zero in a certain interval. For a precise definition we refer to [2, 15, 20]. In this paper we study a property of Zolotarev polynomials which is related to ordinary differential equations. According to [15, 20] the (proper) Zolotarev polynomials (up to multiplication by a constant) on the interval $[-1, 1]$ have to fulfill the first-order AODE

$$n^2(x-\beta)^2(1-y^2) - (1-x^2)(x-\gamma)(x-\delta)y'^2 = 0\,, \tag{1}$$

where $n \in \mathbb{N}$ for some β, γ, δ. We call ODE (1) the *Zolotarev ODE*. In the following we consider the problem only from the point of view of the differential equation (1). According to [20] there are three types of Zolotarev polynomials:

- Chebyshev polynomials
- Stretched Chebyshev polynomials
- Proper Zolotarev polynomials

The proper Zolotarev polynomials form a one-parameter class of solutions of Equation (1), i. e. $y(x) = Z_n(x) = Z_n(x,t)$. To obtain proper Zolotarev polynomials Z_n we additionally require the condition

$$Z_n'(\beta) = 0\,, \qquad\qquad Z_n(\gamma) = -Z_n(\delta) = \pm 1\,, \tag{2}$$

Note, that by coefficient comparison in (1) we get that a non-constant solution of the ODE has to be of degree n. However, there is one exception, if we consider proper Zolotarev polynomials, then the coefficients depend on a parameter, say t. In case we evaluate for some t such that the leading coefficient vanishes we still have a solution of lower degree. This can happen, because the parameters β, γ, δ will also depend on t.

Algorithms for computing rational solutions of algebraic ordinary differential equations (AODEs) have already been developed. The algorithms we use take advantage of tools from algebraic geometry in particular parametrizations of algebraic curves. In [3, 25] algorithms are presented which find rational general solutions, i. e. solutions which contain an arbitrary constant. They are based on a combination of approaches by Fuchs [7] and Kovacic [11]. Using [7, 11] one can find all rational solutions of a given AODE (compare [9]). The method proposed in this paper is based on Algorithm 4 in [9].

By $\mathbb{K}$, we denote an algebraically closed field, e. g. in practice we might think of algebraic numbers $\overline{\mathbb{Q}}$ or the field of complex numbers $\mathbb{C}$. In this paper we fix $\mathbb{K} = \mathbb{C}$. Furthermore, we assume that $\mathbb{K}$ is equipped with a trivial derivation. The trivial derivation on $\mathbb{K}$ is naturally extended to a derivation on the field $\mathbb{K}(x)$ of rational functions for which the derivation of x is equal to 1. We always denote the derivation by $\frac{d}{dx}$, or $'$ for short.

A first-order AODE is a differential equation of the form $F(x, y, y') = 0$, where $F \in \mathbb{K}[x, y, z] \setminus \mathbb{K}[x, y]$. We assume that F is an irreducible polynomial. Hence, the Zolotarev ODE fulfills these conditions. A rational solution of a differential equation is a rational function $y(x) \in \mathbb{K}(x)$ such that $F(x, y(x), y'(x)) = 0$.

Let $F \in \mathbb{K}[x, y, z] \setminus \mathbb{K}[x, y]$ define a first-order AODE, $F(x, y, y') = 0$. We consider the equation to be algebraic by replacing the derivative by an independent variable, i. e. $F(x, y, z) = 0$. An algebraic curve $\mathcal{C}$ is the zero-set of an irreducible polynomial, in our case F, i. e. $\mathcal{C} = \left\{(a, b) \in \mathbb{A}^2(\overline{\mathbb{K}(x)}) \mid F(x, a, b) = 0\right\}$. This curve we call the *corresponding curve* of the differential equation $F(x, y, y') = 0$. The main tool from algebraic geometry used in this paper is parametrization which is defined as follows.

Definition 2.1.
Let $\mathcal{C}$ be an irreducible algebraic curve in $\mathbb{A}^2(\overline{\mathbb{K}(x)})$.

- *A rational parametrization, or briefly a parametrization, of the curve $\mathcal{C}$ is a rational map $\mathcal{P} : \mathbb{A}^1(\overline{\mathbb{K}(x)}) \to \mathcal{C} \subseteq \mathbb{A}^2(\overline{\mathbb{K}(x)})$ such that the image of $\mathcal{P}$ is dense in $\mathcal{C}$ w. r. t. Zariski topology.*
- *If furthermore $\mathcal{P}$ is a birational equivalence, it is called a proper parametrization.*
- *If $\mathcal{P}$ has the form $\mathcal{P}(x, t) = (p_1(x, t), p_2(x, t))$ for some $p_1, p_2 \in \mathbb{K}(x, t)$, $\mathcal{P}$ is called a strong parametrization. An algebraic curve having a strong parametrization is called strongly parametrizable.*

Similarly, we call a first-order AODE, $F(x, y, y') = 0$, strongly parametrizable if its corresponding curve is strongly parametrizable.

Note, that not all algebraic curves admit a rational parametrization. Such a parametrization exists if and only if its genus is zero (see for instance [19] for further details).

3 General Computations

We first might try to apply algorithms from [3, 25] to find a rational general solution of the Zolotarev ODE. However, it turns out, that such a solution does not exist. Note, that we can still find a one-parameter class of solutions, as long as the coefficients β, γ, δ also depend on this parameter. Hence, we apply a modification of Algorithm 4 of [9] to find all rational solutions. This algorithm is basically consists of approaches by Fuchs [7] and Kovacic [11]. For this we compute a strong rational parametrization $\mathcal{P} = (p_1, p_2)$

with

$$\mathcal{P} = \left(\frac{4n^2t^2 + (x^2-1)\,(x-\gamma)(x-\delta)}{4n^2t^2 - (x^2-1)\,(x-\gamma)(x-\delta)}, \frac{4n^2t(x-\beta)}{4n^2t^2 - (x^2-1)\,(x-\gamma)(x-\delta)} \right) .$$

Note, that the associated algebraic system has only the zero solution. The associated ODE (compare [7, 9]) is

$$\omega' = \frac{x-\beta}{4} + \frac{1}{2}\left(\frac{2x}{x^2-1} + \frac{1}{x-\gamma} + \frac{1}{x-\delta} \right)\omega + \frac{n^2(\beta - x)}{(x^2-1)\,(x-\gamma)(x-\delta)}\omega^2 .$$

Let b_i be the coefficients w. r. t. ω of the right hand side of this associated ODE. Then we can transform this ODE to a Riccati equation in rational normal form

$$y' + y^2 = a = \frac{1}{4}\left(\frac{b_2'}{b_2} + b_1\right)^2 - \frac{1}{2}\left(\frac{b_2'}{b_2} + b_1\right)' - b_0 b_2 .$$

This ODE is now rationally solvable with Kovacic's algorithm [11]. We find out that a has five double poles $x_1 = -1$, $x_2 = 1$, $x_3 = \beta$, $x_4 = \gamma$, $x_5 = \delta$. For each pole x_i we get two possible values $c_{i,1}$.

$$c_{1,1} \in \left\{\frac{1}{4}, \frac{3}{4}\right\}, \quad c_{2,1} \in \left\{\frac{1}{4}, \frac{3}{4}\right\}, \quad c_{3,1} \in \left\{-\frac{1}{2}, \frac{3}{2}\right\}, \quad c_{4,1} \in \left\{\frac{1}{4}, \frac{3}{4}\right\}, \quad c_{5,1} \in \left\{\frac{1}{4}, \frac{3}{4}\right\} .$$

Note, that these values do not depend on n nor on β, γ, δ. Similarly there are two possible values for d_{-1}, i. e. $d_{-1} \in \left\{\frac{1-n}{2}, \frac{n+1}{2}\right\}$.

In general one needs to check all possible combinations of these values and test whether $m = d_{-1} - \sum_{i=1}^{n} c_{i,1}$ is a non-negative integer. In the following, however, we focus on choices which lead to a solution.

So far we only applied the general algorithm for solving the associated ODE. Since we know that we are looking for a one parameter family of polynomial solutions, we also know that there has to be a relation of the parameters β, γ, δ in order to get such a solution. Hence, we do a slight modification of Kovacic's algorithm. The relation between the parameters can be found while solving the second-order ODE of the algorithm, i. e.

$$P'' + 2\overline{y}(x)P' + (\overline{y}'(x) + \overline{y}(x)^2 - a(x))P = 0 , \tag{3}$$

where P is an undetermined polynomial of degree m and $\overline{y}$ is a rational function depending on the poles of a, in our case

$$\overline{y} = \frac{c_{1,1}}{x+1} + \frac{c_{2,1}}{x-1} + \frac{c_{3,1}}{x-\beta} + \frac{c_{4,1}}{x-\gamma} + \frac{c_{5,1}}{x-\delta} .$$

Algorithm 1 is a combination of the approach by Fuchs [7] (compare Algorithm 4 in [9]) and a modification of Kovacic's algorithm [11].

Algorithm 1 Rational Solutions of the Zolotarev ODE

Input: A degree n

Output: Rational solutions of the Zolotarev ODE and corresponding relations between β, γ, δ

1: $Sol = \emptyset$, $Rel = \emptyset$
2: Let $\mathcal{P}, b_0, b_1, b_2, a$ be as defined above
3: Let $x_1 = -1, x_2 = 1, x_3 = \beta, x_4 = \gamma, x_5 = \delta$ be the poles of $a(x)$ in $\mathbb{K}$
4: **for all** possible combinations of $c_{i,1}$ and d_{-1} **do**
5:     Compute $m = d_{-1} - \sum_{i=1}^{5} c_{i,1}$.
6:     **if** m is a non-negative integer **then**
7:         Denote $\overline{y}(x) = \sum_{i=1}^{n} \frac{c_{i,1}}{(x-x_i)}$.
8:         $P = x^m + \sum_{i=0}^{m-1} c_i x^i$
9:         Compute the set of solutions $CSol$ of equation (3) for variables $\beta, \gamma, \delta, c_0, \ldots, c_{m-1}$.
10:         **for all** R in $CSol$ **do**
11:             Compute $\omega = \frac{-1}{b_2}\left(\overline{y} + \frac{P'}{P} + \frac{b_2'}{2b_2} + \frac{b_1}{2}\right)$
12:             Compute $y(x) = p_1(x, \omega(x))$
13:             Substitute R in $y(x)$ and append result to Sol.
14:             Substitute R in (β, γ, δ) and append result to Rel.
15:         **end for**
16:     **end if**
17: **end for**
18: **return** (Sol, Rel).

Note, that Step 9 is not yet specifying on how to obtain these solutions. By coefficient comparison in the second-order ODE we get a set of algebraic equations defining an algebraic space curve. We call this curve the *relation curve*. In many cases one can try to just solve the system of algebraic equations. However, this might not always yield all possibilities and it might be computationally expensive. So we propose the following. By eliminating some of the variables in the algebraic equations we project the relation curve to a plane curve, which we call the *reduced relation curve*.

A parametrization of the reduced relation curve yields finally a solution for the other parameters. Note, that the reduced relation curve might have non-zero genus. In this case no rational parametrizations exist. However, radical solutions might help. In any way we have an algebraic parametrization.

Theorem 3.1.
Algorithm 1 computes all rational solutions of the Zolotarev ODE. Hence, it also computes all polynomial solutions.

Using Algorithm 1 we can compute the proper Zolotarev polynomials, but also Chebyshev polynomials and polynomial which do not fulfill Condition (2). Proper Zolotarev polynomials are considered in Section 4, and Chebyshev polynomials are treated in Section 5. The polynomials which are a solution of the Zolotarev equation but do not fulfill Condition 2 are not investigated further in this paper.

As described above in general we would need to check all possible combinations of choices of $c_{i,1}$ and d_{-1}. However, we show now, that we can restrict to some special choices, when we are looking for proper Zolotarev polynomials. Note, that the choice $d_{-1} = \frac{1-n}{2}$ never leads to a non-negative m, hence we can ignore it in general and assume from now on, that $d_{-1} = \frac{1+n}{2}$.

In Table 2 and 1 we show all choices of $c_{i,1}$ and decide whether they possibly lead to proper Zolotarev polynomials. For this we sort the 32 choices and give them an ID number (first column). Note, that the IDs are distributed in a lexicographical ordering, but in the tables we collect them by their suitability for even or odd n. The next five columns tell the choices of the $c_{i,1}$. One column is devoted to the degree m. Then we show the results of Condition (2) and the last column gives a short indicator on whether it is fulfilled (✓) or not (✗). Note, that for small n some of the m are in fact negative and therefore not relevant in these cases.

One can see that only those choices with $c_{4,1} \neq c_{5,1}$ are relevant considering solutions which fulfill Condition (2). Furthermore, let us consider a choice where $c_{4,1} = \lambda$ and $c_{5,1} = \mu$ for $\lambda \neq \mu$, then the choice where $c_{4,1} = \mu$ and $c_{5,1} = \lambda$ and all other $c_{i,1}$ are the same, results in solutions where the roles of γ and δ are just switched. This can be seen form the elementary symmetric role of γ and δ in the Zolotarev ODE.

ID	$c_{1,1}$	$c_{2,1}$	$c_{3,1}$	$c_{4,1}$	$c_{5,1}$	m	$y'(\beta)$	$y(\gamma)$	$y(\delta)$	✓/✗
1	$\frac{1}{4}$	$\frac{1}{4}$	$-\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{n}{2}$	0	-1	-1	✗
4	$\frac{1}{4}$	$\frac{1}{4}$	$-\frac{1}{2}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{n-2}{2}$	0	1	1	✗
5	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{3}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{n-4}{2}$	0	-1	-1	✗
8	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{3}{2}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{n-6}{2}$	0	1	1	✗
10	$\frac{1}{4}$	$\frac{3}{4}$	$-\frac{1}{2}$	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{n-2}{2}$	0	-1	1	✓
11	$\frac{1}{4}$	$\frac{3}{4}$	$-\frac{1}{2}$	$\frac{3}{4}$	$\frac{1}{4}$	$\frac{n-2}{2}$	0	1	-1	✓
14	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{3}{2}$	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{n-6}{2}$	0	-1	1	✓
15	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{3}{2}$	$\frac{3}{4}$	$\frac{1}{4}$	$\frac{n-6}{2}$	0	1	-1	✓
18	$\frac{3}{4}$	$\frac{1}{4}$	$-\frac{1}{2}$	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{n-2}{2}$	0	-1	1	✓
19	$\frac{3}{4}$	$\frac{1}{4}$	$-\frac{1}{2}$	$\frac{3}{4}$	$\frac{1}{4}$	$\frac{n-2}{2}$	0	1	-1	✓
22	$\frac{3}{4}$	$\frac{1}{4}$	$\frac{3}{2}$	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{n-6}{2}$	0	-1	1	✓
23	$\frac{3}{4}$	$\frac{1}{4}$	$\frac{3}{2}$	$\frac{3}{4}$	$\frac{1}{4}$	$\frac{n-6}{2}$	0	1	-1	✓
25	$\frac{3}{4}$	$\frac{3}{4}$	$-\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{n-2}{2}$	0	-1	-1	✗
28	$\frac{3}{4}$	$\frac{3}{4}$	$-\frac{1}{2}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{n-4}{2}$	0	1	1	✗
29	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{3}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{n-6}{2}$	0	-1	-1	✗
32	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{3}{2}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{n-8}{2}$	0	1	1	✗

Table 1: Possible choices for the coefficients $c_{i,1}$ for even n

ID	$c_{1,1}$	$c_{2,1}$	$c_{3,1}$	$c_{4,1}$	$c_{5,1}$	m	$y'(\beta)$	$y(\gamma)$	$y(\delta)$	✓/✗
2	$\frac{1}{4}$	$\frac{1}{4}$	$-\frac{1}{2}$	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{n-1}{2}$	0	-1	1	✓
3	$\frac{1}{4}$	$\frac{1}{4}$	$-\frac{1}{2}$	$\frac{3}{4}$	$\frac{1}{4}$	$\frac{n-1}{2}$	0	1	-1	✓
6	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{3}{2}$	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{n-5}{2}$	0	-1	1	✓
7	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{3}{2}$	$\frac{3}{4}$	$\frac{1}{4}$	$\frac{n-5}{2}$	0	1	-1	✓
9	$\frac{1}{4}$	$\frac{3}{4}$	$-\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{n-1}{2}$	0	-1	-1	✗
12	$\frac{1}{4}$	$\frac{3}{4}$	$-\frac{1}{2}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{n-3}{2}$	0	1	1	✗
13	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{3}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{n-5}{2}$	0	-1	-1	✗
16	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{3}{2}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{n-7}{2}$	0	1	1	✗
17	$\frac{3}{4}$	$\frac{1}{4}$	$-\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{n-1}{2}$	0	-1	-1	✗
20	$\frac{3}{4}$	$\frac{1}{4}$	$-\frac{1}{2}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{n-3}{2}$	0	1	1	✗
21	$\frac{3}{4}$	$\frac{1}{4}$	$\frac{3}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{n-5}{2}$	0	-1	-1	✗
24	$\frac{3}{4}$	$\frac{1}{4}$	$\frac{3}{2}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{n-7}{2}$	0	1	1	✗
26	$\frac{3}{4}$	$\frac{3}{4}$	$-\frac{1}{2}$	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{n-3}{2}$	0	-1	1	✓
27	$\frac{3}{4}$	$\frac{3}{4}$	$-\frac{1}{2}$	$\frac{3}{4}$	$\frac{1}{4}$	$\frac{n-3}{2}$	0	1	-1	✓
30	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{3}{2}$	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{n-7}{2}$	0	-1	1	✓
31	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{3}{2}$	$\frac{3}{4}$	$\frac{1}{4}$	$\frac{n-7}{2}$	0	1	-1	✓

Table 2: Possible choices for the coefficients $c_{i,1}$ for odd n

Hence, if we look for proper Zolotarev polynomials it suffices to check 4 of the 32 cases.

In order to check whether Condition (2) is satisfied, we need to compute the solution y. We can do so without knowing the relations between β, γ, δ in a general way. From this general form, the results can be deduced. We focus here on those choices which actually fulfill Condition 2. For the other choices this can be done as well, but the form is different. In our case y is of the form

$$y(x) = \frac{N(x)}{D(x)} = \frac{n^2(x-\beta)^\kappa(x-\alpha)Y_1P^2 + (x-\bar{\alpha})Y_2(Y_3P + 2(x-\alpha)Y_4P')^2}{-n^2(x-\beta)^\kappa(x-\alpha)Y_1P^2 + (x-\bar{\alpha})Y_2(Y_3P + 2(x-\alpha)Y_4P')^2}, \tag{4}$$

where $gcd(N, D) = 1$, $Y_i \in \mathbb{K}[x]$ and

$$\kappa = \begin{cases} 2 & \text{if } 2 \mid \lfloor \frac{\text{ID}}{4} \rfloor \\ 4 & \text{if } 2 \nmid \lfloor \frac{\text{ID}}{4} \rfloor \end{cases} \qquad \alpha = \begin{cases} \delta & \text{if } 2 \mid \text{ID} \\ \gamma & \text{if } 2 \nmid \text{ID} \end{cases} \qquad \bar{\alpha} = \begin{cases} \gamma & \text{if } 2 \mid \text{ID} \\ \delta & \text{if } 2 \nmid \text{ID} \end{cases}$$

ID	Y_1	Y_2	Y_3	Y_4
2	1	(x^2-1)	1	1
3	1	(x^2-1)	1	1
6	1	(x^2-1)	$-\beta-4\delta+5x$	$(x-\beta)$
7	1	(x^2-1)	$-\beta-4\gamma+5x$	$(x-\beta)$
10	$(x-1)$	$(x+1)$	$-\delta+2x-1$	$(x-1)$
11	$(x-1)$	$(x+1)$	$-\gamma+2x-1$	$(x-1)$
14	$(x-1)$	$(x+1)$	$(\beta+4)\delta+\beta+6x^2-x(2\beta+5\delta+5)$	$(x-1)(x-\beta)$
15	$(x-1)$	$(x+1)$	$(\beta+4)\gamma+\beta+6x^2-x(2\beta+5\gamma+5)$	$(x-1)(x-\beta)$
18	$(x+1)$	$(x-1)$	$-\delta+2x+1$	$(x+1)$
19	$(x+1)$	$(x-1)$	$-\gamma+2x+1$	$(x+1)$
22	$(x+1)$	$(x-1)$	$\beta(\delta-1)-4\delta+6x^2+x(-2\beta-5\delta+5)$	$(x+1)(x-\beta)$
23	$(x+1)$	$(x-1)$	$\beta(\gamma-1)-4\gamma+6x^2+x(-2\beta-5\gamma+5)$	$(x+1)(x-\beta)$
26	(x^2-1)	1	$3x^2-2\delta x-1$	(x^2-1)
27	(x^2-1)	1	$3x^2-2\gamma x-1$	(x^2-1)
30	(x^2-1)	1	$\beta+4\delta+x\left(2\beta\delta+7x^2-3x(\beta+2\delta)-5\right)$	$(x^2-1)\,(x-\beta)$
31	(x^2-1)	1	$\beta+4\gamma+x\left(2\beta\gamma+7x^2-3x(\beta+2\gamma)-5\right)$	$(x^2-1)\,(x-\beta)$

Table 3: Parts of y for different choices of $c_{i,1}$

According to [9, Thm. 3.4] a rational solution of the Zolotarev ODE can only have poles at the zeros of the coefficient of y'^2, i. e.

$$(1-x^2)(x-\gamma)(x-\delta)\,.$$

It is easy to see from Table 3 and partly from Tables 1 and 2, that for all IDs we have that $N(-1)=\pm D(-1)$, $N(1)=\pm D(1)$, $N(\gamma)=\pm D(\gamma)$ and $N(\delta)=\pm D(\delta)$. Hence, y is in any case a polynomial. Note, that this does not yet tell that a certain choice does lead to a Zolotarev polynomial, since we did not yet consider the algebraic equations for the parameters. The following theorems follow immediately.

Theorem 3.2.
If for a given choice of $c_{i,1}$ there is a polynomial solution of the second-order ODE (3) then (4) is a polynomial solution of the Zolotarev ODE.

Theorem 3.3.
All rational solutions of the Zolotarev ODE are in fact polynomial.

4 Proper Zolotarev Polynomials for Specific Degrees

In this section we use the general algorithm to find proper Zolotarev polynomials, i. e. those polynomials which fulfill the Zolotarev ODE and additionally Condition (2). The Zolotarev polynomials of degree two and three are easy to find. For degree four we need to parametrize the reduced relation curve, hence, we need elimination. For degree five the reduced relation curve is of genus 1 in all the possible cases, and hence, there is no explicit expression for Z_5 with rational coefficients.

4.1 Polynomials of Degree 2

This section describes the case of $n = 2$, where the resulting Zolotarev Polynomial has degree 2. We skip all the choices which do not yield a solution. We take the choice with ID 10. Then $m = 0$. Note, that this is not the only possible choice, which yields the solution. By definition

$$\bar{y} = \frac{1}{4(x+1)} + \frac{3}{4(x-1)} - \frac{1}{2(x-\beta)} + \frac{1}{4(x-\gamma)} + \frac{3}{4(x-\delta)}\,.$$

Using coefficient comparison for the Riccati equation in rational normal form, we get algebraic equations for β, γ, δ. Solving these equations yields $\beta = -1, \gamma = -1$ or $\gamma = 2\beta + 1, \delta = 2\beta - 1$. The first solution is not interesting for us, so we focus on the second. In this case we get $\omega = \frac{1}{4}(x+1)(-2\beta + x - 1)$ and the final solution of the Zolotarev ODE is

$$Z_2(x, \beta) = \frac{-x^2 + 2\beta x + 1}{2\beta}\,,$$

where β is chosen to be the free parameter. This solution fulfills Condition (2), hence it defines the one-parameter class of second-degree proper Zolotarev polynomials.

4.2 Polynomials of Degree 3

This section is devoted to the case of $n = 3$, i. e. Zolotarev Polynomials with degree 3. We skip all the choices which do not yield a solution. We take the choice with ID 26, for which we have $m = 0$. Note, that this is not the only possible choice, which yields the solution. By definition

$$\bar{y} = \frac{3}{4(x+1)} + \frac{3}{4(x-1)} - \frac{1}{2(x-\beta)} + \frac{1}{4(x-\gamma)} + \frac{3}{4(x-\delta)}\,.$$

Using coefficient comparison for the Riccati equation in rational normal form, we get algebraic equations for β, γ, δ. Solving these equations yields $\gamma = \frac{9\beta^2+1}{6\beta}$ and $\delta = \frac{3\beta^2-1}{2\beta}$. In this case we get $\omega = -\frac{(3\beta x+1)\left(9\beta^2-6\beta x+1\right)}{108\beta^2}$ and the final solution of the Zolotarev ODE is

$$Z_3(x, \beta) = \frac{81\beta^4\left(2x^2-1\right) - 108\beta^3 x\left(x^2-1\right) + \beta^2\left(36-54x^2\right) + 1}{\left(1-9\beta^2\right)^2}\,,$$

where β is chosen to be the free parameter. This solution fulfills Condition (2).

One gets a similar result choosing ID 2.In this case $m = 1$ and

$$\bar{y} = \frac{1}{4(x+1)} + \frac{1}{4(x-1)} - \frac{1}{2(x-\beta)} + \frac{1}{4(x-\gamma)} + \frac{3}{4(x-\delta)}\,,$$

where a term of the form $\frac{1}{c_0+x}$ is missing. Finally, the setting $\gamma = \frac{3\beta^2-1}{2\beta}$, $\delta = \frac{9\beta^2+1}{6\beta}$ and $c_0 = \frac{1}{3\beta}$ yields a solution which fulfill Condition 2.

4.3 Polynomials of Degree 4

In this section we describe the case of $n = 4$ which needs a special treatment compared to the previous cases. We choose ID 10. Then $m = 1$ and by definition

$$\bar{y} = \frac{1}{4(x+1)} + \frac{3}{4(x-1)} - \frac{1}{2(x-\beta)} + \frac{3}{4(x-\gamma)} + \frac{1}{4(x-\delta)}\,,$$

from which we know that a term of the form $\frac{1}{c_0+x}$ is missing for a solution of the Riccati equation. Using this we have to solve the following set of equations

$$\begin{aligned}
0 &= c_0\left(16\beta^3 + \beta(\gamma(-\delta) + \gamma + 3\delta + 1) - 2(\gamma+1)\delta\right) - 2(\beta(2\gamma\delta + \gamma + 3\delta) - 2\gamma\delta)\,,\\
0 &= 16\beta^3 + \beta(-9\gamma\delta + 5\gamma + 7\delta + 9) + c_0\left(2\beta(\gamma + 3\delta - 2) - 48\beta^2 - (\gamma+1)(\delta-1)\right)\\
&\qquad + 2\gamma(\delta - 1)\,,\\
0 &= -48\beta^2 + 4\beta(3\gamma + 5\delta - 2) + c_0(40\beta - 2\delta + 2) + 3\gamma\delta - 3\gamma - 5\delta - 3\,,\\
0 &= 6\left(4\beta - \gamma - 2c_0 - 2\delta + 1\right)\,.
\end{aligned}$$

These equations define the relation curve. Note, that the degrees of γ, δ and c_0 are all one. Eliminating c_0 and γ we get the following equation.

$$\begin{aligned}
0 = 256\beta^4 - 128\beta^3(3\delta - 1) + 64\beta^2\left(3\delta^2 - 2\delta - 1\right)\\
- 8\beta\left(5\delta^3 - 5\delta^2 - 9\delta + 1\right) + (\delta+1)^2\left(3\delta^2 - 18\delta + 11\right)\,.
\end{aligned}$$

This equation defines the reduced relation curve in the variables β, γ. We compute a rational parametrization $\mathcal{Q} = (q_1, q_2)$

$$\begin{aligned}
q_1 &= \frac{81t^4 + 29376t^3 + 3921408t^2 + 228114432t + 4817223680}{6291456t + 570425344}\,,\\
q_2 &= \frac{81t^4 + 32832t^3 + 4935168t^2 + 326713344t + 8032681984}{1572864t + 142606336}\,.
\end{aligned}$$

This yields

$$\begin{aligned}
c_0 &= \frac{-27t^3 - 7920t^2 - 782592t - 25710592}{524288}\,,\\
\gamma &= \frac{-81t^4 - 25920t^3 - 3055104t^2 - 156254208t - 2881028096}{-1572864t - 142606336}
\end{aligned}$$

The parametrization can be simplified a lot by using the linear transformation $t \to 16/3(4t-17)$.

$$\beta = \frac{(t^2-3)(t^2+1)}{8t}, \qquad \gamma = \frac{(t+2)t^3-1}{2t},$$
$$\delta = \frac{t^3(t-2)-1}{2t}, \qquad c_0 = \frac{1}{2}\left(1-t\left(t^2+t+1\right)\right).$$

Hence, we get the following solution.

$$\omega = -\frac{(x+1)(2x-t^3+t^2-t-1)(2xt-t^4-2t^3+1)}{16t(-2x+t^3+t^2+t-1)},$$
$$Z_4(x,t) = \frac{-8tx^4+a_3x^3+a_2x^2+a_1x+a_0}{(t^2-1)^3(t^2+1)^2},$$
$$a_0 = 2t\left(3t^6+t^4+t^2-1\right),$$
$$a_1 = \left(t^{10}-t^8-2t^6-10t^4-7t^2+3\right),$$
$$a_2 = -2t\left(3t^6+t^4+t^2-5\right),$$
$$a_3 = 4\left(3t^4+2t^2-1\right),$$

which is exactly the same as in [20].

4.4 Polynomials of Degree 5

In this section we aim for proper Zolotarev polynomials of degree $n=5$. Of course using the algorithm one finds Chebyshev polynomials (for the case $\beta=\gamma=\delta$) and polynomials which do not fulfill condition (2). For the other cases we proceed as in the case of $n=4$. We get a set of algebraic equations in β,γ,δ,c_i which define the relation curve. By elimination of γ and all c_i we get a the reduced relation curve in β,δ. In case of $n=5$ the reduced relation curve has always genus 1. Since the reduced relation curve cannot have higher genus than the relation curve itself, there is no rational parametrization of the latter either. Note, that this does not tell whether there is an explicit expression for Z_5 in terms of rational coefficients.

The only possibility for an explicit representation of the proper Zolotarev Polynomial is by using a radical parametrization of the resulting curve. Let us consider the case with ID 3. Then $m=2$. For the moment we skip the part where we find the relation of the parameters. Just from the choice taken so far we can conclude, that

$$\omega(x) = \frac{(x^2-1)(x-\delta)(c_1(3x-2\gamma)+c_0+x(5x-4\gamma))}{50(x(c_1+x)+c_0)(x-\beta)}.$$

Hence, we also know that, with $P=c_0+c_1x+x^2$,

$$y(x) = \frac{(x^2-1)(x-\delta)(c_1(3x-2\gamma)+c_0+x(5x-4\gamma))^2+25P^2(x-\beta)^2(x-\gamma)}{(x^2-1)(x-\delta)(c_1(3x-2\gamma)+c_0+x(5x-4\gamma))^2-25P^2(x-\beta)^2(x-\gamma)}$$

is a solution of the Zolotarev ODE and it is easy to check, that it fulfills Condition (2). So far we did not need the precise relation of the parameters. We compute it now. The reduced relation curve in this case is defined by

$$G = 5 + 43\beta^2 - 425\beta^4 + 625\beta^6 - 18\beta\delta + 620\beta^3\delta - 1250\beta^5\delta - 8\delta^2 - 308\beta^2\delta^2 \\ + 900\beta^4\delta^2 + 64\beta\delta^3 - 280\beta^3\delta^3 + 32\beta^2\delta^4 \,.$$

Since, it has genus 1 it is an elliptic curve which can be transformed into Weierstrass form (see [22] for an algorithm),

$$\bar{G} = x^3 - (67108864/1875)x - 755914244096/421875 + y^2 \,.$$

It is easy to compute a radical parametrization of $\bar{G} = 0$ which then by backward transformation yields a radical parametrization of the reduced relation curve. Simplifying the result we get

$$\mathcal{Q} = \left(\frac{\sqrt{2}t\sqrt{\frac{5t^3+5t^2-t-1}{t^3}}}{5(t+1)^2}, \frac{t\sqrt{\frac{5t^3+5t^2-t-1}{t^3}}\,(5t^3+13t^2-t-1)}{2\sqrt{2}(t+1)^2\,(5t^2-1)} \right) .$$

This yields (using a shorthand notation α)

$$\begin{aligned}
\beta &= \alpha\,, & \gamma &= \frac{5\alpha\,(11t^3+3t^2+t+1)}{4-20t^2}\,,\\
\delta &= \frac{5\alpha\,(5t^3+13t^2-t-1)}{20t^2-4}\,, & c_0 &= \frac{-11t^2+2t+1}{10t^2-2}\,,\\
c_1 &= \frac{5\alpha(t-1)^2t}{10t^2-2}\,, & \alpha &= \frac{\sqrt{2}t\sqrt{\frac{5t^3+5t^2-t-1}{t^3}}}{5(t+1)^2}\,.
\end{aligned}$$

From this we get an explicit expression for the solution

$$y(x) = \frac{1}{(t-1)^6(3t+1)^4}\sum_{i=0}^{5} a_i x^i \,,$$

where

$$\begin{aligned}
a_0 &= -2581t^{10} - 8122t^9 - 6221t^8 - 1128t^7 + 966t^6 + 500t^5 + 174t^4 + 56t^3 \\
&\quad - 17t^2 - 10t - 1\,,\\
a_1 &= -40\alpha t(t+1)^3\left(275t^6 + 60t^5 - 45t^4 - 32t^3 - 7t^2 + 4t + 1\right)\,,\\
a_2 &= 4\Big(1375t^{10} + 4125t^9 + 2850t^8 + 360t^7 - 210t^6 - 22t^5 - 188t^4 - 128t^3 \\
&\quad +11t^2 + 17t + 2\Big)\,,\\
a_3 &= 40\alpha t(t+1)^3\left(50t^7 + 425t^6 + 190t^5 - 55t^4 - 90t^3 - 21t^2 + 10t + 3\right)\,,\\
a_4 &= -8(t+1)^3\left(1-5t^2\right)^2\left(15t^3 - 5t^2 + 5t + 1\right)\,,\\
a_5 &= -80\alpha t(t+1)^6\left(1-5t^2\right)^2\,.
\end{aligned}$$

4.5 Polynomials of Degree 6

Surprisingly the Zolotarev polynomial of degree 6 is easier to compute than the one for degree 5. The equation of the reduced relation curve for the choice with ID 10 (Table 1) is factorizable and one of the factors is $-1+2\beta-\delta$. It is easy to find the parametrization $\mathcal{Q}=(\frac{t+1}{2},t)$. Hence, we get

$$\beta=\frac{t+1}{2}\,,\qquad \gamma=t+2\,,\qquad \delta=t\,,\qquad c_0=\frac{1}{2}(-t-3)\,,\qquad c_1=-t-1\,.$$

Using this parametrization we get the following solution.

$$\begin{aligned}\omega(x)&=-\frac{(x+1)(t-x+2)(t(2x-1)-2(x-1)x+1)}{12(2tx+t-2(x-1)x+3)}\,,\\ y(x)&=\frac{1}{(1+t)^3}\sum_{i=0}^{6}a_ix^i\,,\end{aligned}$$

with

$$\begin{aligned}&a_0=1-6t-3t^2\,, && a_2=3(1+10t+5t^2)\,, && a_4=-12(2t+t^2)\,, && a_6=-4\,,\\ &a_1=3(3+t)(1-t^2)\,, && a_3=4(1+t)(-5+2t+t^2)\,, && a_5=12(1+t)\,.\end{aligned}$$

5 Chebyshev Polynomials

In this section we deal with a special case of the parameters β,γ,δ. What we get are special solutions without arbitrary constants. In the case under consideration it is well known that the solutions are the Chebyshev polynomials. We show how this can be concluded from our idea. In case $\beta=\gamma=\delta$ we get a simpler parametrization:

$$\mathcal{P}(t)=\left(\frac{4n^2t^2+x^4-x^2}{4n^2t^2-x^4+x^2},\frac{4n^2tx}{4n^2t^2-x^4+x^2}\right)\,.$$

Hence,

$$a=-\frac{s^2+2-n^2\left(s^2-1\right)}{4\left(s^2-1\right)^2}\,,$$

which has the double poles ±1. As in the general case we get

$$c_{1,1}\in\left\{\frac{1}{4},\frac{3}{4}\right\}\,,\qquad c_{2,1}\in\left\{\frac{1}{4},\frac{3}{4}\right\}\,,\qquad d_{-1}\in\left\{\frac{1-n}{2},\frac{n+1}{2}\right\}\,.$$

In the algorithm we need to pick the non-negative integers in the set

$$\left\{-\frac{n}{2},-\frac{1}{2}(n+1),-\frac{1}{2}(n+1),-\frac{n}{2}-1,\frac{n}{2},\frac{n-1}{2},\frac{n-1}{2},\frac{n-2}{2}\right\}\,.$$

The first four elements are always negative since n is positive. We consider two cases

- $2 \mid n$: In this case $m \in \{\frac{n}{2}, \frac{n}{2} - 1\}$.
  - For the case $m = \frac{n}{2}$ the second-order ODE which is to be solved is of the form

$$(1-x^2)p'' - xp' + \left(\frac{n}{2}\right)^2 p = 0\,.$$

    This is the Chebyshev equation which has the m-th Chebyshev polynomial T_m as solution (see [1, Table 18.8.1]). Then $\omega(s) = \frac{s(s^2-1)U_{m-1}(s)}{2nT_m(s)}$. Substituting this ω to t in the first component of $\mathcal{P}$ we get $-T_n(x)$ as a solution of the original ODE.
  - For the other case we get in a similar way the second-order ODE

$$(1-x^2)p'' - 3xp' + \frac{n^2-4}{4}p = 0\,.$$

    This is the Chebyshev equation which has the m-th Chebyshev polynomial of second kind U_m as solution (see [1, Table 18.8.1]). Then $\omega(s) = \frac{sT_{m+1}(s)}{2nU_m(s)}$. Applying this to t in the first component of $\mathcal{P}$ we get $T_n(x)$ as a solution of the original ODE.
- $2 \nmid n$: In this case $m = \frac{n-1}{2}$. There are two choices in which m is of this form. In the first case we get the second-order ODE

$$(1-x^2)p'' + (1-2x)p' + \frac{1}{4}(n^2-1)p = 0\,.$$

  This is the Jacobi differential equation with parameters $-\frac{1}{2}, \frac{1}{2}$. Hence, the Jacobi polynomial

$$P_m^{(-\frac{1}{2},\frac{1}{2})}(s) = W_m(s)P_m^{(-\frac{1}{2},\frac{1}{2})}(1) = \frac{\left(\frac{1}{2}\right)_m \cos\left(\frac{1}{2}n\cos^{-1}(s)\right)}{m!\cos\left(\frac{1}{2}\cos^{-1}(s)\right)}$$

  is a solution (see [1, Table 18.8.1]). Here, W_m is the Chebyshev polynomial of fourth kind and $(a)_k$ denotes the Pochhammer symbol (rising factorial). For the equalities see Equations 18.7.6, 18.6.1 and 18.5.4 in [1]. Then $\omega(s) = -\frac{s\sqrt{1-s^2}\tan\left(\frac{1}{2}n\cos^{-1}(s)\right)}{2n}$. Applying this ω to t in the first component of $\mathcal{P}$ we get $-T_n(x)$ as a solution of the original ODE.

  In the second case were m is of this form we get the second order ODE

$$(1-x^2)p'' + (-1-2x)p' + \frac{1}{4}(n^2-1)p = 0\,,$$

  which is a Jacobi differential equation with parameters $\frac{1}{2}, -\frac{1}{2}$. Hence, the Jacobi polynomial

$$P_m^{(\frac{1}{2},-\frac{1}{2})}(s) = \frac{V_m(s)P_m^{(\frac{1}{2},-\frac{1}{2})}(1)}{2m+1} = \frac{\left(\frac{3}{2}\right)_m \sin\left(\frac{1}{2}n\cos^{-1}(s)\right)}{m!\,n\sin\left(\frac{1}{2}\cos^{-1}(s)\right)}$$

is a solution (see [1, Table 18.8.1]). Here, V_m is the Chebyshev polynomial of third kind and $(a)_k$ denotes the Pochhammer symbol (rising factorial). For the equalities see Equations 18.7.5, 18.6.1 and 18.5.3 in [1]. The $\omega(s) = \frac{s\sqrt{1-s^2}\cot\left(\frac{1}{2}n\cos^{-1}(s)\right)}{2n}$. Substitution of ω to t in the first component of $\mathcal{P}$ yields $T_n(x)$ as a solution of the original ODE.

6 Conclusion

We present a general algorithm for symbolic computation of rational solutions of the Zolotarev ODE. It is shown, that the rational solutions are indeed polynomial, i. e. Zolotarev polynomials. Furthermore, we give explicit expressions for proper Zolotarev polynomials up to degree 6.

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