

# Anti-Unification of Concepts in Description Logic $\mathcal{EL}$

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## Abstract

We study anti-unification for the description logic  $\mathcal{EL}$  and introduce the notion of least general generalisation, which generalises simultaneously least common subsumer and concept matching. The idea of generalisation of two concepts is to detect maximal similarities between them, and to abstract over their differences uniformly. We demonstrate that a finite minimal complete set of generalisations for  $\mathcal{EL}$  concepts always exists and establish complexity bounds for computing them. We present an anti-unification algorithm that computes generalisations with a fixed skeleton, study its properties and report on preliminary experimental evaluation.

## Introduction

Description Logics as a knowledge representation formalism gained particular prominence in recent years due to widespread adoption of the web ontology language OWL as a W3C web standard (2012). Not only does the strong link between the direct model theoretic semantics of OWL 2 and the semantics of description logics provide OWL ontologies with an unambiguous meaning but it also enables one to harvest the power of logical reasoning in various ontology application scenarios, see e.g. (Baader et al. 2003; Yu 2014; Domingue, Fensel, and Hendler 2011) for more details.

Capturing expert knowledge and representing it in the form of concept descriptions and axioms is a laborious and time consuming task, which is further hindered by the fact that domain experts may disagree on basic definitions, knowledge engineers may choose to describe different concepts at different levels of granularity, different names can denote semantically equivalent concepts, concept description can be machine learned etc., which leads to the need to be able to consolidate and unify different concept descriptions into one, best suitable for a particular application.

The notion of a *least common subsumer* (*lcs* for short) has been introduced in (Cohen, Borgida, and Hirsh 1992) precisely to capture ‘the largest set of commonalities’ between concepts. An *lcs* of two concepts is a concept  $C$  that subsumes both of them and such that no other common subsumer of the given concepts is strictly subsumed by  $C$ . It can be seen that while such a concept indeed captures the commonalities between the given concepts, it does not highlight

differences between them nor suggest a way to consolidate such differences into a new concept description. A survey of results on *lcs* can be found in (Baader and Küsters 2006).

The problem of identifying and consolidating differences between concepts has been addressed in the context of concept matching and concept unification. For example, Baader and Morawska (2010) give the following example of the use of unification to eliminate redundancies from ontologies: concept descriptions  $\text{Human} \sqcap \text{Male} \sqcap \exists \text{loves.Sports\_car}$  and  $\text{Man} \sqcap \exists \text{loves.}(\text{Car} \sqcap \text{Fast})$  intuitively refer to the same concept of a ‘man loving fast cars’, yet they are clearly not equivalent. Differences in the representation can be resolved by treating  $\text{Man}$  and  $\text{Sports\_car}$  as *concept variables* and unifying the two concept descriptions with the substitution  $\{\text{Man} \mapsto \text{Human} \sqcap \text{Male}, \text{Sports\_car} \mapsto \text{Car} \sqcap \text{Fast}\}$ . While powerful, this approach requires ontology engineers to identify which concept names should be treated as constants and which should be treated as variables, which may not always be obvious.

In this paper we propose a new way of identifying and consolidating differences between concept descriptions based on the notion of *concept generalisation* by *anti-unification*. Speaking abstractly, a generalisation of two terms  $s_1$  and  $s_2$  is a term  $t$  such that  $s_i$  can be ‘obtained’ from  $t$  by applying some substitution  $\tau_i$  to  $t$ ,  $i = 1, 2$ . Notice that every two terms always have a generalisation  $t = X$ , where  $X$  is a variable, known as the *most* general generalisation. Interesting cases are *least* general generalisations (*lgg* for short), which retain the common parts of the input terms as much as possible, and abstract with the help of variables over the differences in the input uniformly. Anti-unification is a technique that has been successfully used to compute *lggs* in various theories, starting from the pioneering works by Plotkin (1970) and Reynolds (1970).

We study the anti-unification problem for concepts in description logic  $\mathcal{EL}$ , which underpins the OWL 2 EL profile (Baader, Brandt, and Lutz 2005). In the DL context terms are concepts and the informal notion of ‘obtaining the original concepts from the generalisation by substitutions’ can be specialised in two ways:  $\tau_i(t)$  is equivalent to  $s_i$  (generalisation modulo equivalence), or  $\tau_i(t)$  subsumes  $s_i$  (generalisation modulo subsumption). These notions are motivated by matching modulo equivalence and subsumption, respectively (Baader and Küsters 2006).

Note that conjunction is idempotent. Interestingly, it was shown by Pottier (1989) that anti-unification in equational theories with *two* idempotent function symbols is infinitary. In contrast, we show that generalisation modulo subsumption coincides with the *lcs*, and a finite minimal complete set of generalisations modulo equivalence always exists but can be non-elementary in the size of the given concepts.

In the course of looking for lower complexity variants, we define fixed-skeleton exact generalisations, where we assume that the underlying tree structure of the generalisation (its skeleton) is fixed and contains only ‘essential’ nodes, and the goal is to minimise the ‘variable part’. We design an algorithm that solves this problem when the homomorphisms from the skeleton to the original concepts are also given, and prove that it is terminating, sound, and complete.

One advantage of our approach is that not only does it compute generalisations, but also provides information how the variables can be instantiated to obtain the original concepts. This is given in the form of so called anti-unification triples (AUTs in short) of the form  $X : [L_1, U_1] \triangleq [L_2, U_2]$ . Such a triple indicates that the original concepts  $C_1$  and  $C_2$  differ at the nodes where  $X$  occurs and tells us that replacing  $X$  by any concept between (in the sense of subsumption) the lower bound  $L_i$  and the upper bound  $U_i$  will give the corresponding node in  $C_i$ , for  $i = 1, 2$ .

Knowledge engineer might find the information provided by the computed generalisation and the AUTs useful: She can clearly see (in the generalisation) where the concepts in question agree, and observe (in the AUTs) how those concepts differ. Moreover, if an upper bound in an AUT, e.g.,  $U_1$ , is a concept name, it can be treated as a unification variable with a possible instantiation by  $U_2$ . For instance, in the ‘man loving fast cars’ example above, our algorithm returns the generalisation  $X \sqcap \exists \text{loves}.Y$  and two AUTs:  $X : [\dots, \text{Human} \sqcap \text{Male}] \triangleq [\dots, \text{Man}]$  and  $Y : [\dots, \text{SportsCar}] \triangleq [\dots, \text{Car} \sqcap \text{Fast}]$  (we omit the lower bounds for brevity). From here one can conclude that if  $\text{Man}$  is defined as  $\text{Human} \sqcap \text{Male}$  and  $\text{SportsCar}$  is defined as  $\text{Car} \sqcap \text{Fast}$ , the original concepts become equivalent.

In our preliminary experiments we have successfully computed generalisations of 49 675 528 pairs from definitions of fully defined concepts of the  $\mathcal{EL}$  fragment of the GALEN ontology (Rector et al. 2003) revealing commonalities between concepts.

Due to space restriction, some technical proofs are deferred to the full version published as RISC Technical Report, <http://www.risc.jku.at/publications/>.

## Preliminaries

Let  $\mathbb{N}_C$  and  $\mathbb{N}_R$  be countably infinite and disjoint sets of *concept names* and *role names*, respectively. In the description logic  $\mathcal{EL}$ , *concepts*  $C$  are built according to the syntax rule

$$C ::= A \mid \exists r.C \mid \prod_{C \in \mathcal{C}} C,$$

where  $A$  ranges over  $\mathbb{N}_C$ ,  $r$  ranges over  $\mathbb{N}_R$ , and the expression  $\prod_{C \in \mathcal{C}} C$  is a *conjunction* over the (multi)set of concepts  $\mathcal{C}$ , where no  $C \in \mathcal{C}$  is a conjunction in turn. When

$\mathcal{C} = \{C_1, \dots, C_n\}$ , for  $n \geq 2$ , we write  $C_1 \sqcap \dots \sqcap C_n$ . The conjunction over the empty set is abbreviated as  $\top$ .

The semantics of concepts is defined by means of *interpretations*  $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ , where the interpretation *domain*  $\Delta^{\mathcal{I}}$  is a non-empty set, and  $\cdot^{\mathcal{I}}$  is a function mapping each concept name  $A$  to a subset  $A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$  and each role name  $r$  to a binary relation  $r^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$ . The function  $\cdot^{\mathcal{I}}$  is inductively extended to arbitrary concepts by setting  $(\prod_{C \in \mathcal{C}} C)^{\mathcal{I}} := \bigcap_{C \in \mathcal{C}} C^{\mathcal{I}}$ , and  $(\exists r.C)^{\mathcal{I}} := \{d \in \Delta^{\mathcal{I}} \mid \exists e \in C^{\mathcal{I}} : (d, e) \in r^{\mathcal{I}}\}$ . For  $\top$  we have  $\top^{\mathcal{I}} := \Delta^{\mathcal{I}}$ . Concept  $D$  *subsumes* concept  $C$ , in symbols  $C \sqsubseteq D$ , if for every interpretation  $\mathcal{I}$  we have  $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ . Concept  $C$  is *equivalent* to concept  $D$ , in symbols  $C \equiv D$ , if both  $C \sqsubseteq D$  and  $D \sqsubseteq C$ . It is easy to see that thus defined concepts and interpretations are equivalent to the standard definition with binary conjunction (Baader, Brandt, and Lutz 2005).

A concept  $C$  is a *least common subsumer (lcs)* of concepts  $C_1$  and  $C_2$  if  $C_i \sqsubseteq C$ , for  $i = 1, 2$ , and for any other concept  $D$  such that  $C_i \sqsubseteq D$ , for  $i = 1, 2$ , we have  $C \sqsubseteq D$ . For a set of concepts  $S = \{C_1, \dots, C_n\}$  we define  $\text{lcs}(S)$  as  $\text{lcs}(C_1, \text{lcs}(C_2, \dots, C_n) \dots)$ . It is known that any  $\mathcal{EL}$  concepts  $C_1, C_2$  always have a unique least common subsumer, and *lcs* is commutative and associative, so  $\text{lcs}(S)$  is correctly defined (Baader, Küsters, and Molitor 1999).

## $\mathcal{EL}$ Anti-Unification

To define the anti-unification problem, we partition the set of concept names  $\mathbb{N}_C$  into *concept constants*  $\mathbb{N}_{C_a}$  and *concept variables*  $\mathbb{N}_{C_v}$ . A substitution  $\sigma$  is a mapping from the set of concept variables to the set of  $\mathcal{EL}$  concepts such that  $\sigma(X) \neq X$  for finitely many  $X \in \mathbb{N}_{C_v}$ . The expression  $\sigma : \{X_1 \mapsto C_1, \dots, X_n \mapsto C_n\}$  denotes that  $\sigma(X_i) = C_i$ , for  $i = 1, \dots, n$  and implicitly  $\sigma(Y) = Y$  for all  $Y \in \mathbb{N}_{C_v} \setminus \{X_1, \dots, X_n\}$ . Substitutions are extended to  $\mathcal{EL}$  concepts in the usual way:  $\sigma(A) = A$ , for  $A \in \mathbb{N}_{C_a}$ ;  $\sigma(\prod_{C \in \mathcal{C}} C) = \prod_{C \in \mathcal{C}} \sigma(C)$ ;  $\sigma(\exists r.C) = \exists r.\sigma(C)$ .

We say that a concept  $D$  is *more general* than a concept  $C$  *modulo equivalence*, denoted  $D \preceq_{\equiv} C$  (more general *modulo subsumption*, denoted  $D \preceq_{\sqsubseteq} C$ ) if there exists a substitution  $\sigma$  such that  $C \equiv \sigma(D)$  (or  $C \sqsubseteq \sigma(D)$ , respectively). A concept  $G$  is a *generalisation* of concepts  $C_1$  and  $C_2$  *modulo equivalence* (a generalisation *modulo subsumption*) if it is more general than both  $C_1$  and  $C_2$ , that is, if there exist substitutions  $\tau_1$  and  $\tau_2$  such that  $C_i \equiv \tau_i(G)$  (or  $C_i \sqsubseteq \tau_i(G)$ , respectively), for  $i = 1, 2$ . Notice that  $G$  is a generalisation of  $C_1, C_2$  modulo equivalence (or subsumption) iff there exists matchers (Baader and Küsters 2006) for the matching problems  $C_i \equiv^? G$  (or  $C_i \sqsubseteq^? G$ , respectively), for  $i = 1, 2$ .

Any concepts  $C_1, C_2$  always have some generalisation: consider  $G = X$ , where  $X \in \mathbb{N}_{C_v}$  is fresh, and  $\tau_i(X) = C_i$ , for  $i = 1, 2$ . Then  $G$  is a generalisation of  $C_1, C_2$  both modulo equivalence and subsumption. Obviously, such a generalisation is too crude as often less general generalisations exist. For example, for  $C_1 = A \sqcap B$  and  $C_2 = A \sqcap B'$  the generalisation  $G' = A \sqcap Y$  is less general than  $G = X$ , as  $G'$  can be obtained from  $G$  by substituting  $A \sqcap Y$  into  $X$ . This leads us to the following definition.

If  $C \preceq_{\equiv} D$  and  $D \preceq_{\equiv} C$  then we say that  $C$  and  $D$  are *equi-general* modulo equivalence, denoted  $C \approx_{\equiv} D$  (equi-generality modulo subsumption is defined similarly). Then a generalisation  $G$  of concepts  $C_1$  and  $C_2$  is *least general* (abbreviated as *lgg*) if whenever a generalisation  $G'$  of  $C_1$  and  $C_2$  is less general than  $G$  then  $G'$  is equi-general to  $G$ .

It turns out that least general generalisations modulo subsumption coincide with the least common subsumer and so can be computed in polynomial time.

**Proposition 1** *For any  $\mathcal{EL}$  concepts  $C_1$  and  $C_2$  a least general generalisation modulo subsumption always exists and is equi-general to  $lcs(C_1, C_2)$ .*

**Proof.** First notice that  $lcs(C_1, C_2)$  is a generalisation of  $C_1$  and  $C_2$  modulo subsumption as by definition of the least common subsumer we have  $C_i \sqsubseteq \sigma_{id}(lcs(C_1, C_2))$ , where  $\sigma_{id}$  is the identity substitution.

Let  $G$  be an arbitrary generalisation of  $C_1, C_2$  modulo subsumption. Then  $C_i \sqsubseteq \tau_i(G)$ , for some  $\tau_i$  and  $i = 1, 2$ . But then by (Baader and Küsters 2006) we have  $C_i \sqsubseteq \sigma_{\top}(G)$ , where  $\sigma_{\top}$  is the substitution that replaces every variable in  $G$  with  $\top$ . By the properties of the *lcs* we have  $lcs(C_1, C_2) \sqsubseteq \sigma_{\top}(G)$ . So  $G \preceq_{\sqsubseteq} lcs(C_1, C_2)$ .

Thus  $lcs(C_1, C_2)$  is a least general generalisation of  $C_1$  and  $C_2$  and any other least general generalisation of  $C_1$  and  $C_2$  is equi-general to  $lcs(C_1, C_2)$ .  $\square$

In the view of Proposition 1, from now on we only consider generalisations modulo equivalence and use  $\preceq$  and  $\approx$  without any indices. Unlike the modulo subsumption case, there exist incomparable *lggs*. For example, for  $C_1 = \exists r.(A \sqcap B) \sqcap \exists r.(A' \sqcap B')$  and  $C_2 = \exists r.(A \sqcap A') \sqcap \exists r.(B \sqcap B')$  both  $\exists r.(A \sqcap X) \sqcap \exists r.(B' \sqcap Y)$  and  $\exists r.(B \sqcap X) \sqcap \exists r.(A' \sqcap Y)$  are (incomparable) *lggs*.

We say that a set of generalisations  $\mathcal{G}$  of concepts  $C_1$  and  $C_2$  is *complete* (for  $C_1$  and  $C_2$ ) if for any generalisation  $G'$  of  $C_1$  and  $C_2$  there exists  $G \in \mathcal{G}$  such that  $G \preceq G'$ . The set  $\mathcal{G}$  is a *minimal complete set* of generalisations of  $C_1$  and  $C_2$  (written  $mcs\mathcal{G}(C_1, C_2)$ ) if it, in addition, satisfies the minimality property: For no two distinct  $G_1, G_2 \in \mathcal{G}$ ,  $G_1 \preceq G_2$  holds. Hence, the elements of  $mcs\mathcal{G}(C_1, C_2)$  are all the *lggs* of  $C_1$  and  $C_2$ .

**Theorem 2** *For every  $\mathcal{EL}$  concepts  $C_1, C_2$ , a finite minimal complete set of generalisations exists.*

**Proof.** Let  $G$  be a generalisation of  $C_1$  and  $C_2$ . It follows from Lemma 6.3.1 in (Küsters 2001) that there exist substitutions  $\tau_1$  and  $\tau_2$  such that  $C_i \equiv \tau_i(G)$ , for  $i = 1, 2$ , and for every concept variable  $X$  occurring in  $G$  its image  $\tau_i(X)$  is equivalent to the conjunction of some elements of  $sc(C_i)$ , where  $sc(C)$  is the set of subconcepts of a concept  $C$  defined recursively as:  $sc(\top) = \{\top\}$ ,  $sc(A) = \{\top, A\}$ ,  $sc(\exists r.C) = \{\top\} \cup \{\exists r.C' \mid C' \in sc(C)\}$ ,  $sc(C \sqcap D) = \{C' \sqcap D' \mid C' \in sc(C), D' \in sc(D)\}$ . Let  $N = 2^{|sc(C_1)|} \times 2^{|sc(C_2)|}$ . Notice that if  $G$  contains more than  $N$  different concept variables, then for some  $X$  and  $Y$  we have  $\tau_i(X) \equiv \tau_i(Y)$ , for  $i = 1, 2$ . Then a substitution  $\sigma : X \mapsto Y$  maps  $G$  into a generalisation  $G'$  containing fewer variables than  $G$ .

Let  $\mathcal{S}$  be the set of all generalisations of  $C_1, C_2$  such that every  $G \in \mathcal{S}$  contains at most  $N$  different concept variables. We can assume w.l.o.g. for every  $G \in \mathcal{S}$  that  $G$  only uses variable from  $\{X_1, \dots, X_N\}$  (this can be achieved by renaming variables). It should be obvious that the role depth of every generalisation of  $C_1, C_2$  does not exceed the maximal role depth of  $C_1, C_2$ . But then the number of different, up to equivalence, concepts in  $\mathcal{S}$  is finite.

So, one can select a finite subset  $\mathcal{S}' \in \mathcal{S}$  such that for every generalisation  $G$  of  $C_1$  and  $C_2$  there exists  $G' \in \mathcal{S}'$  with  $G \preceq G'$  and for no distinct  $G_1, G_2 \in \mathcal{S}'$  we have  $G_1 \preceq G_2$ . Then for every *lgg*  $G$  of  $C_1$  and  $C_2$  there exists an *lgg*  $G' \in \mathcal{S}'$  equi-general to  $G$ , that is,  $\mathcal{S}'$  is a finite complete set of *lggs* as required.  $\square$

The proof of Theorem 2 gives a non-elementary upper bound on the size of *lggs*. As the following example shows this upper bound can be reached.

**Example 3** *Let  $n > 0$  be even. Consider concepts*

$$C_1 := \underbrace{\exists r \dots \exists r}_n . \left( \prod_{i=1}^n (A_i^1 \sqcap A_i^2) \right) \sqcap \prod_{i=1}^n (\exists s_i^1 . A_i^1 \sqcap \exists s_i^2 . A_i^2)$$

and

$$C_2 := \underbrace{\exists r \dots \exists r}_n . \left( \prod_{i=1}^n (A_i^1 \sqcap A_i^2) \right) \sqcap \prod_{i=1}^n (\exists s_i^1 . A_i^2 \sqcap \exists s_i^2 . A_i^1).$$

Let the set of concepts  $\mathcal{C}_0$  be defined as

$$\left\{ \prod_{i \in S} (A_i^1 \sqcap A_i^2) \sqcap \prod_{i \notin S} (X_i^1 \sqcap X_i^2) \mid \begin{array}{l} S \subseteq \{1, \dots, n\}, \\ |S| = n/2 \end{array} \right\}.$$

We define for every  $0 \leq i < n$

$$\mathcal{C}_{i+1} := \left\{ \exists r . \left( \prod_{C \in \mathcal{C}_i} C \right) \mid \mathcal{C} \subseteq \mathcal{C}_i, |\mathcal{C}| = \frac{1}{2} |\mathcal{C}_i| \right\}.$$

One can see that

$$G := \prod_{C \in \mathcal{C}_n} C \sqcap \prod_{i=1}^n (\exists s_i^1 . X_i^1 \sqcap \exists s_i^2 . X_i^2).$$

is an *lgg* of  $C_1$  and  $C_2$ . Indeed, for any substitutions  $\tau_1, \tau_2$  such that  $\tau_i(G) \equiv C_i$ , for  $i = 1, 2$ , we have  $\tau_1(X_i^1) = A_i^1$  and  $\tau_1(X_i^2) = A_i^2$ , while  $\tau_2(X_i^1) = A_i^2$  and  $\tau_2(X_i^2) = A_i^1$ . Thus, any substitution  $\sigma$  such that  $\sigma(G)$  is also generalisation is essentially a variable renaming. There are  $c^n$  elements in  $\mathcal{C}_0$ , for some  $c > 1$ ,  $c^{c^n}$  elements in  $\mathcal{C}_1$ ,  $c^{c^{c^n}}$  elements in  $\mathcal{C}_3$  etc. Thus, the size of  $G$  is non-elementary in terms of  $n$ . There are  $c^n$  elements in  $\mathcal{C}_0$ , for some  $c > 1$ ,  $c^{c^n}$  elements in  $\mathcal{C}_1$ ,  $c^{c^{c^n}}$  elements in  $\mathcal{C}_3$  etc. Thus, the size of  $G$  is non-elementary in terms of  $n$ .

Notice, however, that the concept

$$\exists r \dots \exists r . \left( \prod_{i=1}^n (A_i^1 \sqcap A_i^2) \right) \sqcap \prod_{i=1}^n (\exists s_i^1 . X_i^1 \sqcap \exists s_i^2 . X_i^2)$$

is a polynomial size *lgg* of  $C_1, C_2$ , incomparable with  $G$ .

## Fixed skeleton generalisation

Example 3 demonstrates that without restraints least general generalisation can be of size non-elementary in the size of given concepts. Moreover, the notion of an *lgg* introduced in the previous section may not always be intuitive as it is not ‘monotone’ regarding modifications to the given concepts.

Consider, for example,  $C_1 = C_2 = \exists r.A \sqcap \exists r.B$ . Then, as one would expect,  $G = \exists r.A \sqcap \exists r.B$  is an *lgg* of  $C_1$  and  $C_2$ ;  $X \sqcap G$  is an *lgg* of  $D \sqcap C_1$  and  $E \sqcap C_2$ ; and  $\exists s.Y \sqcap G$  is an *lgg* of  $\exists s.A \sqcap C_1$  and  $\exists s.B \sqcap C_2$ . However, for  $C'_1 = \exists s.A \sqcap \exists t.(D \sqcap C_1)$  and  $C'_2 = \exists s.B \sqcap \exists t.(E \sqcap C_2)$ , concept  $G' = \exists s.Y \sqcap \exists t.(X \sqcap G)$ , counter to expectations, is not an *lgg* of  $C'_1$  and  $C'_2$  as  $\{X \mapsto Z \sqcap \exists r.Y\}$  maps  $G'$  into  $G'' = \exists s.Y \sqcap \exists t.(Z \sqcap \exists r.A \sqcap \exists r.B \sqcap \exists r.Y)$ , which is a generalisation of  $C'_1, C'_2$  strictly less general than  $G'$ .

More control can be gained by restricting generalisations to have a fixed tree structure or *skeleton*. Formally the skeleton  $\text{skel}(C)$  of a concept  $C$  is the concept obtained from  $C$  by removing all occurrences of variables.

We say that a concept  $G$  is a *generalisation* of concepts  $C_1, C_2$  with a fixed skeleton  $G_{\text{sk}}$  iff  $G$  is a generalisation of  $C_1, C_2$  and  $\text{skel}(G) = G_{\text{sk}}$ . We say that  $G$  is an *lgg* of concepts  $C_1, C_2$  with a fixed skeleton  $G_{\text{sk}}$  if  $G$  is a generalisation of  $C_1, C_2$  with a fixed skeleton  $G_{\text{sk}}$  and whenever a generalisation  $G'$  of  $C_1$  and  $C_2$  with the same skeleton  $G_{\text{sk}}$  is less general than  $G$  then  $G'$  is equi-general to  $G$ .

It can be readily checked that results of Theorem 2 transfer to the fixed skeleton case. Hence, for every  $\mathcal{EL}$  concepts  $C_1, C_2$  a finite minimal complete set of generalisations with a fixed skeleton  $G_{\text{sk}}$  always exists (possibly empty if  $C_1$  and  $C_2$  do not have generalisations with skeleton  $G_{\text{sk}}$ ).

To develop an algorithm computing fixed skeleton *lggs*, following (Baader and Küsters 2006), we use a structural characterisation of subsumption. We identify each  $\mathcal{EL}$  concept  $C$  with a finite *description tree*  $\mathfrak{T}_C$  whose nodes are labelled with sets of concept names and whose edges are labelled with role names. In detail, if  $C$  is a concept name  $A$  or  $\top$ , then  $\mathfrak{T}_C$  has a single node  $\mathbf{d}_C$  with label  $l(\mathbf{d}_C) = \{A\}$  if  $C = A$ , or  $l(\mathbf{d}_C) = \emptyset$  if  $C = \top$ ; if  $C = \exists r.D$ , then  $\mathfrak{T}_C$  is obtained from  $\mathfrak{T}_D$  by adding a new root  $\mathbf{d}_C$  and an edge from  $\mathbf{d}_C$  to the root  $\mathbf{d}_D$  of  $\mathfrak{T}_D$  with the label  $l(\mathbf{d}_C, \mathbf{d}_D) = r$  (we then call  $\mathbf{d}_D$  an immediate  $r$ -successor of  $\mathbf{d}_C$ ); if  $C = \prod_{i=1}^n C_i$ , for  $n > 0$ , then  $\mathfrak{T}_C$  is obtained by identifying the roots  $\mathbf{d}_{C_i}$  of all  $\mathfrak{T}_{C_i}$ ,  $1 \leq i \leq n$ , into  $\mathbf{d}_C$  and setting  $l(\mathbf{d}_C) = \bigcup_{i=1}^n l(\mathbf{d}_{C_i})$ . We write  $\text{root}(C)$  for the root node of  $\mathfrak{T}_C$ . Conversely, every tree  $\mathfrak{T}$  of the described form gives rise to an  $\mathcal{EL}$  concept  $C_{\mathfrak{T}}$  in the obvious way. A concept  $D$  subsumes a concept  $C$  iff there exists a *homomorphism* from  $\mathfrak{T}_D$  to  $\mathfrak{T}_C$  defined as a function  $\varphi$  from the nodes of  $\mathfrak{T}_D$  to the nodes of  $\mathfrak{T}_C$  satisfying the following properties (Baader and Küsters 2006):

1.  $\varphi(\text{root}(D)) = \text{root}(C)$ ;
2. for all  $\mathbf{d}_1, \mathbf{d}_2$  nodes of  $\mathfrak{T}_D$  and  $r \in \mathbb{N}_R$  such that  $\mathbf{d}_2$  is an  $r$ -successor of  $\mathbf{d}_1$  in  $\mathfrak{T}_D$ , we have that  $\varphi(\mathbf{d}_2)$  is an  $r$ -successor of  $\varphi(\mathbf{d}_1)$ ;
3. for every node  $\mathbf{d}$  of  $\mathfrak{T}_D$ , we have  $l(\mathbf{d}) \subseteq l(\varphi(\mathbf{d}))$ .

We say that a function  $\varphi$  is a *variable ignoring* homomorphism from  $D$  to  $C$  if condition 3 above is replaced with

- 3'. for every node  $\mathbf{d}$  of  $\mathfrak{T}_D$ , we have  $(l(\mathbf{d}) \cap \mathbb{N}_{C_a}) \subseteq l(\varphi(\mathbf{d}))$ .

Homomorphisms are extended from nodes to sets of nodes in the usual way:  $\varphi(S) := \bigcup_{\mathbf{d} \in S} \{\varphi(\mathbf{d})\}$ .

We do not always distinguish explicitly between a concept and its tree representation and between nodes and subtrees rooted at the nodes, which allows us to speak, for example, about the nodes and subtrees of an  $\mathcal{EL}$  concept, apply substitutions to nodes and consider homomorphisms to be functions between concepts. We also treat a variable ignoring homomorphism from  $D$  to  $C$  as a homomorphism from  $\text{skel}(D)$  to  $C$  and vice versa.

While our methods can be applied to the general case, for the sake of presentation in this paper we restrict our consideration to *exact generalisations*. Intuitively, exactness requires the skeletons to contain only essential nodes that match the corresponding nodes in the input concepts entirely. We say that a concept  $G$  is an exact generalisation of concepts  $C_1$  and  $C_2$  if there exist substitutions  $\tau_1$  and  $\tau_2$  and variable ignoring homomorphisms  $\varphi_1$  and  $\varphi_2$  from  $G$  to  $C_1, C_2$ , respectively, such that for every node  $\mathbf{d}$  of  $G$  we have  $\tau_i(\mathbf{d}) \equiv \varphi_i(\mathbf{d})$ , for  $i = 1, 2$ .

Since  $\text{root}(G)$  is mapped by  $\varphi_i$  into  $\text{root}(C_i)$ , for  $i = 1, 2$ , we have  $\tau_i(G) \equiv C_i$ , so every exact generalisation is a generalisation. For example, for  $C_1 = \exists r.(A \sqcap B)$  and  $C_2 = \exists r.A \sqcap \exists r.B$  both  $G_1 = \exists r.(A \sqcap X) \sqcap \exists r.(B \sqcap X)$  and  $G_2 = \exists r.A \sqcap \exists r.(B \sqcap Y)$  are least general generalisations with the same skeleton  $G_{\text{sk}} = \exists r.A \sqcap \exists r.B$  and homomorphisms  $\varphi_i : G_{\text{sk}} \rightarrow C_i$ , for  $i = 1, 2$ , are uniquely determined; however,  $G_1$  is exact while  $G_2$  is not. The notions of an exact *lgg* and of an exact (least general) generalisation with a fixed skeleton are defined in the obvious way.

Looking back at the notions of generalisation we use in this paper, one can see that we started with unrestricted generalisation and then tried to make it more specific by introducing fixed skeletons. Further, we defined a variant that we called exact generalisation, and its version with a fixed skeleton. In the definition of the latter (that has not been explicitly spelled), the existence of the corresponding homomorphisms are asserted. We now make a step further and define exact generalisations with a fixed skeleton when the homomorphisms are given.

We say that  $\mathfrak{F} = (G_{\text{sk}}, \varphi_1, \varphi_2)$  is a *fixed skeleton exact generalisation framework* (or simply *generalisation framework* for short) for concepts  $C_1$  and  $C_2$  if  $\text{lcs}(C_1, C_2) \sqsubseteq G_{\text{sk}}$  and  $\varphi_i : G_{\text{sk}} \rightarrow C_i$ , for  $i = 1, 2$ , are homomorphisms. We say that a concept  $G$  is an *exact generalisation* of concepts  $C_1$  and  $C_2$  w.r.t. a *generalisation framework*  $\mathfrak{F}$  if  $\text{skel}(G) = G_{\text{sk}}$ , and there exist substitutions  $\tau_1$  and  $\tau_2$  (called *witness substitutions*) such that for every node  $\mathbf{d}$  of  $G$  we have  $\varphi_i(\mathbf{d}) \equiv \tau_i(\mathbf{d})$ . An exact *lgg* w.r.t.  $\mathfrak{F}$  is defined in the obvious way.

In what follows we develop a non-deterministic polynomial time algorithm that given concepts  $C_1, C_2$  and a generalisation framework  $\mathfrak{F}$  computes an exact *lgg*  $G$  w.r.t.  $\mathfrak{F}$ . To achieve that, we characterise substitutions  $\tau_i$  that witness  $G$  being an exact *lgg* w.r.t.  $\mathfrak{F}$  in terms of *anti-unification triples*,

**(m): Merge**

$$\frac{X : [\mathbf{L}_1^X, \mathbf{U}_1^X] \triangleq [\mathbf{L}_2^X, \mathbf{U}_2^X], \quad Y : [\mathbf{L}_1^Y, \mathbf{U}_1^Y] \triangleq [\mathbf{L}_2^Y, \mathbf{U}_2^Y]}{Z : [\text{lbs}(\mathbf{L}_1^X, \mathbf{L}_1^Y), \mathbf{U}_1^X \cap \mathbf{U}_1^Y] \triangleq [\text{lbs}(\mathbf{L}_2^X, \mathbf{L}_2^Y), \mathbf{U}_2^X \cap \mathbf{U}_2^Y]} \{X \mapsto Z, Y \mapsto Z\}$$

**(sm): Split-merge**

$$\frac{X : [\mathbf{L}_1^X, \mathbf{U}_1^X \cap \mathbf{U}'_1^X] \triangleq [\mathbf{L}_2^X, \mathbf{U}_2^X \cap \mathbf{U}'_2^X], \quad Y : [\mathbf{L}_1^Y, \mathbf{U}_1^Y] \triangleq [\mathbf{L}_2^Y, \mathbf{U}_2^Y]}{Z : [\text{lbs}(\mathbf{L}_1^X, \mathbf{L}_1^Y), \mathbf{U}_1^X \cap \mathbf{U}_1^Y] \triangleq [\text{lbs}(\mathbf{L}_2^X, \mathbf{L}_2^Y), \mathbf{U}_2^X \cap \mathbf{U}_2^Y]} \{X \mapsto Z \cap X', Y \mapsto Z\}$$

$$X' : [\mathbf{L}_1^X, \mathbf{U}'_1^X] \triangleq [\mathbf{L}_2^X, \mathbf{U}'_2^X]$$

**(ssm): Split-split-merge**

$$\frac{X : [\mathbf{L}_1^X, \mathbf{U}_1^X \cap \mathbf{U}'_1^X] \triangleq [\mathbf{L}_2^X, \mathbf{U}_2^X \cap \mathbf{U}'_2^X], \quad Y : [\mathbf{L}_1^Y, \mathbf{U}_1^Y \cap \mathbf{U}'_1^Y] \triangleq [\mathbf{L}_2^Y, \mathbf{U}_2^Y \cap \mathbf{U}'_2^Y]}{Z : [\text{lbs}(\mathbf{L}_1^X, \mathbf{L}_1^Y), \mathbf{U}_1^X \cap \mathbf{U}_1^Y] \triangleq [\text{lbs}(\mathbf{L}_2^X, \mathbf{L}_2^Y), \mathbf{U}_2^X \cap \mathbf{U}_2^Y]} \{X \mapsto Z \cap X', Y \mapsto Z \cap Y'\}$$

$$X' : [\mathbf{L}_1^X, \mathbf{U}'_1^X] \triangleq [\mathbf{L}_2^X, \mathbf{U}'_2^X], Y' : [\mathbf{L}_1^Y, \mathbf{U}'_1^Y] \triangleq [\mathbf{L}_2^Y, \mathbf{U}'_2^Y]$$

Where

- (i)  $\text{lbs}(\mathbf{L}_1^X, \mathbf{L}_1^Y) \subseteq \mathbf{U}_1^X \cap \mathbf{U}_1^Y$ ,  $\text{lbs}(\mathbf{L}_2^X, \mathbf{L}_2^Y) \subseteq \mathbf{U}_2^X \cap \mathbf{U}_2^Y$ ; for no conjunct  $C \neq \top$  of  $\mathbf{U}'_i^X$  we have  $\text{lbs}(\mathbf{L}_1^X, \mathbf{L}_1^Y) \subseteq C$ ;  
for no conjunct  $D \neq \top$  of  $\mathbf{U}'_i^Y$  we have  $\text{lbs}(\mathbf{L}_2^X, \mathbf{L}_2^Y) \subseteq D$ ;
- (ii)  $X \cap Y$  is not equivalent to a subconcept of  $G$ ;
- (iii) in the **(ssm)** rule for  $\mathbf{d}, \mathbf{e}$  nodes of  $G$  such that  $X \in l(\mathbf{d})$  and  $Y \in l(\mathbf{e})$  we have  $\mathbf{V}_G(\mathbf{d}) \cap \mathbf{V}_G(\mathbf{e}) = \emptyset$ .

Figure 1: Minimisation rules

AUTs for short, which are tuples of the form

$$X : [\mathbf{L}_1^X, \mathbf{U}_1^X] \triangleq [\mathbf{L}_2^X, \mathbf{U}_2^X],$$

where  $X$  is a concept variable and  $\mathbf{L}_1^X, \mathbf{U}_1^X, \mathbf{L}_2^X$  and  $\mathbf{U}_2^X$  are  $\mathcal{EL}$  concepts such that  $\mathbf{L}_i^X \subseteq \mathbf{U}_i^X$  for  $i = 1, 2$ . Intuitively, every substitution that replaces a variable  $X$  in  $G$  with a concept  $C$  ‘between the lower and upper bounds for  $X$ ’, that is, such that  $\mathbf{L}_i^X \subseteq C \subseteq \mathbf{U}_i^X$ , for  $i = 1, 2$  is a witness to  $G$  being an exact *lgg* w.r.t.  $\mathfrak{F}$ . We then use this characterisation to demonstrate that every exact *lgg* w.r.t.  $\mathfrak{F}$  can be obtained with a substitution from the most general exact generalisation w.r.t.  $\mathfrak{F}$ , in which every node of  $G_{\text{sk}}$  contains a unique variable. Then a complete set of exact *lggs* with skeleton  $G_{\text{sk}}$  for concepts  $C_1, C_2$  can be computed by minimising the set of all exact *lggs* of  $C_1, C_2$  w.r.t. framework  $(G_{\text{sk}}, \varphi_1, \varphi_2)$ , for all possible choices of homomorphisms  $\varphi_i : G_{\text{sk}} \rightarrow C_i$ .

Let  $C$  and  $D$  be  $\mathcal{EL}$  concepts,  $\varphi$  be a variable ignoring homomorphism from  $D$  to  $C$  and  $\mathbf{d}$  be a node of  $D$ . The *difference at node  $\mathbf{d}$  w.r.t.  $\varphi$*  between concepts  $C$  and  $D$  is the concept

$$C \ominus_{\varphi}^{\mathbf{d}} D := \prod_{A \in l(\varphi(\mathbf{d})) \setminus l(\mathbf{d})} A \cap \prod_{i=1}^n \exists s_i . C_i,$$

where  $\{\mathbf{c}_1, \dots, \mathbf{c}_n\}$ ,  $n \geq 0$ , is the set of all immediate successors of  $\varphi(\mathbf{d})$  in  $C$  such that for all  $1 \leq i \leq n$ ,

- $s_i = l(\varphi(\mathbf{d}), \mathbf{c}_i)$ ,
- $\mathbf{c}_i = \text{root}(C_i)$ , and
- there exists no node  $\mathbf{d}' \in D$  such that  $l(\mathbf{d}, \mathbf{d}') = s_i$  and  $\varphi(\mathbf{d}') = \mathbf{c}_i$ ,

For example, for  $C = \exists r.(A \cap B) \cap \exists r.(A \cap B') \cap \exists s.A$ ,  $D = \exists r.(A \cap Y)$ , let  $\mathbf{d} = \text{root}(D)$ ,  $\mathbf{c} = \text{root}(C)$ ,  $\varphi$  be the variable ignoring homomorphism from  $D$  to  $C$  that maps the  $r$ -successor of  $\mathbf{d}$  (denoted  $\mathbf{d}_r$ ) into the first  $r$ -successor of  $\mathbf{c}$  (denoted  $\mathbf{c}_r^1$ ), i.e.,  $\varphi(\mathbf{d}) = \mathbf{c}$  and  $\varphi(\mathbf{d}_r) = \mathbf{c}_r^1$ . Then we have  $l(\mathbf{d}) = l(\mathbf{c}) = \emptyset$ ,  $l(\mathbf{d}_r) = \{A, Y\}$ ,  $l(\mathbf{c}_r^1) = \{A, B\}$ , and, thus,  $C \ominus_{\varphi}^{\mathbf{d}} D = \exists r.(A \cap B') \cap \exists s.A$  and  $C \ominus_{\varphi}^{\mathbf{d}_r} D = B$ .

Let  $X$  be a concept variable,  $C$  be a concept and  $\mathbf{d}$  be a node of  $C$ . We denote by  $\mathbf{N}_C(X)$  the set of all nodes of  $C$  with  $X$  in their label, and by  $\mathbf{V}_C(\mathbf{d})$  the set of all variables in the label of  $\mathbf{d}$ . Let  $\mathfrak{F} = (G_{\text{sk}}, \varphi_1, \varphi_2)$  be a generalisation framework for concepts  $C_1$  and  $C_2$  and  $G$  be a concept with  $\text{skel}(G) = G_{\text{sk}}$ . We say that a set of AUTs  $S$  is *compatible* with  $G, C_1, C_2$  w.r.t.  $\mathfrak{F}$  iff the following conditions hold:

- (c1)** For every variable  $X$  of  $G$ , the set  $S$  contains exactly one AUT  $X : [\mathbf{L}_1^X, \mathbf{U}_1^X] \triangleq [\mathbf{L}_2^X, \mathbf{U}_2^X]$ .
- (c2)** For every AUT  $X : [\mathbf{L}_1^X, \mathbf{U}_1^X] \triangleq [\mathbf{L}_2^X, \mathbf{U}_2^X] \in S$ , we have  $\mathbf{L}_i^X \equiv \text{lbs}(\varphi_i(\mathbf{N}_G(X)))$ ,  $i = 1, 2$ ;
- (c3)** For every node  $\mathbf{d}$  of  $G_{\text{sk}}$ , we have

$$\prod_{X \in \mathbf{V}_G(\mathbf{d})} \mathbf{U}_i^X \subseteq (C_i \ominus_{\varphi_i}^{\mathbf{d}} G_{\text{sk}}), \quad i = 1, 2.$$

When  $C_1, C_2$  and  $\mathfrak{F}$  are clear from the context, we simply talk about  $S$  being compatible with  $G$ .

The following lemma is proved by induction on the role depth of  $C_1$  and  $C_2$ .

**Lemma 4** *Let  $C_1, C_2$  be concepts,  $\mathfrak{F} = (G_{\text{sk}}, \varphi_1, \varphi_2)$  be a generalisation framework, and  $G$  be a concept such that  $\text{skel}(G) = G_{\text{sk}}$ . Let  $S$  be a set of AUTs such that for every variable  $X$  of  $G$ , it contains exactly one AUT*

$X : [L_1^X, U_1^X] \triangleq [L_2^X, U_2^X]$ . Assume that the substitutions  $\tau_i$ , for  $i = 1, 2$ , are defined as follows:

$$\tau_i := \{X \mapsto U_i^X \mid X : [L_1^X, U_1^X] \triangleq [L_2^X, U_2^X] \in S\}.$$

Then  $S$  is compatible with  $G, C_1, C_2$  w.r.t.  $\mathfrak{F}$  iff  $G$  is an exact generalisation of  $C_1, C_2$  w.r.t.  $\mathfrak{F}$  witnessed by  $\tau_1, \tau_2$ .

Notice that for any substitutions  $\sigma, \sigma'$  and concept  $C$  such that  $\sigma(X) \sqsubseteq \sigma'(X)$  for every  $X$  occurring in  $C$  we have  $\sigma(C) \sqsubseteq \sigma'(C)$ . We use this fact to prove the following.

**Corollary 5** *If  $\tau_i$  are such that for every variable of  $G$  we have  $L_i^X \sqsubseteq \tau_i(X) \sqsubseteq U_i^X$ , for  $i = 1, 2$ , then  $\tau(G) \equiv C_i$ .*

Given concepts  $C_1$  and  $C_2$  and a generalisation framework  $\mathfrak{F} = (G_{\text{sk}}, \varphi_1, \varphi_2)$  we construct

- the concept  $G_{\mathfrak{F}}$  by adding a fresh variable to the label of its every node, and
- the set of AUTs  $S_{\mathfrak{F}}$ , which for every variable  $X$  of  $G_{\mathfrak{F}}$  contains the AUT  $X : [L_1^X, U_1^X] \triangleq [L_2^X, U_2^X]$ , where  $L_i^X = \varphi_i(\mathbf{d})$  and  $U_i^X = C_i \ominus_{\varphi_i} G$ , where  $\mathbf{d}$  is the unique node of  $G_{\mathfrak{F}}$  containing variable  $X$ .

It should be obvious that  $G_{\mathfrak{F}}$  is an exact generalisation of  $C_1, C_2$  w.r.t.  $\mathfrak{F}$  and  $S_{\mathfrak{F}}$  is compatible with  $G_{\mathfrak{F}}, C_1, C_2$  w.r.t.  $\mathfrak{F}$  (in fact  $G_{\mathfrak{F}}$  is a *most* general exact generalisation of  $C_1, C_2$  w.r.t.  $\mathfrak{F}$ ).

We say that a concept  $G'$  and a set of AUTs  $S'$  is obtained from a concept  $G$  and a set of AUTs  $S$  by an application of a *minimisation rule*  $\alpha$  with a *side substitution*  $\sigma$ , given in Figure 1, in symbols  $(G, S) \rightsquigarrow_{\alpha}^{\sigma} (G', S')$ , if  $\alpha$  is of the form

$$\frac{S_1}{S_2} \sigma,$$

and if (up to variable renaming)  $S$  can be represented as  $S = S_1 \cup S'_1$  and  $S' = S_2 \cup S'_1$ , where  $\cup$  denotes the disjoint union, and  $G' = \sigma(G)$ . We write  $(G, S) \rightsquigarrow (G', S')$  if  $S \rightsquigarrow_{\alpha}^{\sigma} S'$  for some  $\alpha$  and  $\sigma$ . We denote by  $\rightsquigarrow^*$  the reflexive transitive closure of  $\rightsquigarrow$ .

**Example 6** *Consider concepts*

$$C_1 = \exists r.(A \sqcap B) \sqcap \exists s.(B \sqcap C) \sqcap \exists t.(A \sqcap C) \text{ and} \\ C_2 = \exists r.(A' \sqcap B') \sqcap \exists s.(B' \sqcap C') \sqcap \exists t.(A' \sqcap C').$$

Let  $G_{\text{sk}} = \exists r.\top \sqcap \exists s.\top \sqcap \exists t.\top$ . Notice that homomorphisms  $\varphi_i : G_{\text{sk}} \rightarrow C_i$ , and hence generalisation framework  $\mathfrak{F} = (G_{\text{sk}}, \varphi_1, \varphi_2)$ , are uniquely determined.

Then  $G_{\mathfrak{F}} = \exists r.X \sqcap \exists s.Y \sqcap \exists t.Z \sqcap W$  and  $S_{\mathfrak{F}}$  is the following set of AUTs.

$$X : [A \sqcap B, A \sqcap B] \triangleq [A' \sqcap B', A' \sqcap B'] \\ Y : [B \sqcap C, B \sqcap C] \triangleq [B' \sqcap C', B' \sqcap C'] \\ Z : [A \sqcap C, A \sqcap C] \triangleq [A' \sqcap C', A' \sqcap C'] \\ W : [C_1, \top] \triangleq [C_2, \top].$$

Notice that  $\text{lcs}(A \sqcap B, B \sqcap C)$  is  $B$ , and conjunctions  $A \sqcap B$  and  $B \sqcap C$  can be represented as  $U_1^X \sqcap U_1^X$  and  $U_1^Y \sqcap U_1^Y$ , respectively, where  $U_1^X = U_1^Y = B$ ,  $U_1^X = A$

and  $U_1^Y = C$ . Thus, the **(ssm)** rule is applicable to  $X$  and  $Y$  with the side substitution  $\{X \mapsto X_1 \sqcap V_1, Y \mapsto Y_1 \sqcap V_1\}$  and the set of AUTs

$$X_1 : [A \sqcap B, A] \triangleq [A' \sqcap B', A'] \\ Y_1 : [B \sqcap C, C] \triangleq [B' \sqcap C', C'] \\ V_1 : [B, B] \triangleq [B', B'].$$

Similarly, the **(sm)** rule is applicable to  $X_1$  and  $Z$  with the side substitution  $\{X_1 \mapsto V_2, Z \mapsto Z_1 \sqcap V_2\}$  and the set of AUTs

$$V_2 : [A, A] \triangleq [A', A'], \quad Z_1 : [A \sqcap C, C] \triangleq [A' \sqcap C', C']$$

Finally, the **(m)** rule applies to  $Y_1$  and  $Z_1$  giving  $\{Z_1 \mapsto V_3, Y_1 \mapsto V_3\}$  and

$$V_3 : [C, C] \triangleq [C', C'].$$

Putting it all together an application of the substitution

$$\{X \mapsto V_2 \sqcap V_1, Y \mapsto V_3 \sqcap V_1, Z \mapsto V_3 \sqcap V_2\}$$

to  $G$  produces a generalisation

$$G' = \exists r.(V_1 \sqcap V_2) \sqcap \exists s.(V_3 \sqcap V_1) \sqcap \exists t.(V_3 \sqcap V_2) \sqcap W.$$

It can be seen that the lgg of  $C_1$  and  $C_2$  w.r.t.  $\mathfrak{F}$

$$G = \exists r.(V_1 \sqcap V_2) \sqcap \exists s.(V_3 \sqcap V_1) \sqcap \exists t.(V_3 \sqcap V_2)$$

can be obtained from  $G'$  by eliminating variable  $W$  with a substitution  $W \mapsto \top$ .

In what follows we prove that every generalisation can be obtained by applying minimisation rules to  $(G_{\mathfrak{F}}, S_{\mathfrak{F}})$  and then simplifying the result, that is, that our procedure is complete. We illustrate the main ideas of the completeness proof here and defer the technical details to the full version.

The outline of our approach is as follows. Given a generalisation  $G$  and a compatible set of AUTs  $S$  we construct a generalisation  $G'$  and a set of AUTs  $S'$  such that  $(G, S)$  can be obtained from  $(G', S')$  by an application of one of the minimisation rules and  $(G', S')$  is in some sense closer to  $(G_{\mathfrak{F}}, S_{\mathfrak{F}})$ . By inductive reasoning, we end up with a sequence of rule applications such that  $(G, S)$  is obtained starting from some  $(G_0, S_0)$  such that  $(G_0, S_0)$  cannot be obtained by any rule application. We then apply the same sequence of rules to  $(G_{\mathfrak{F}}, S_{\mathfrak{F}})$  aiming to produce  $(G, S)$ .

Notice that not every syntactic form of every generalisation can be obtained by applying minimisation rules to  $(F_{\mathfrak{F}}, S_{\mathfrak{F}})$ , as the following example demonstrates.

**Example 7** *Consider  $C_1 = A_1 \sqcap A_2, C_2 = B_1 \sqcap B_2$  and  $G = X_1 \sqcap X_2$ . Then  $G_{\text{sk}} = \top$  and the homomorphisms  $\varphi_1, \varphi_2$  are trivial. Notice that the set of AUTs  $S$  defined as*

$$\{X_i : [A_1 \sqcap A_2, A_i] \triangleq [B_1 \sqcap B_2, B_i] \mid 1 \leq i \leq 2\}$$

is compatible with  $G, C_1, C_2$  w.r.t.  $\mathfrak{F} = (G_{\text{sk}}, \varphi_1, \varphi_2)$ , yet  $(G, S)$  cannot be obtained by applying minimisation rules to  $G_{\mathfrak{F}} = X$  and  $S_{\mathfrak{F}} = \{X : [A_1 \sqcap A_2, A_1 \sqcap A_2] \triangleq [B_1 \sqcap B_2, B_1 \sqcap B_2]\}$ .

To address this issue, suppose that  $S$  is a set of AUTs compatible with the concept  $G$  w.r.t. some generalisation framework  $\mathfrak{F}$  and let  $\mathcal{V}$  be a set of variables of  $G$  such that every  $\{X, Y\} \subseteq \mathcal{V}$  we have  $\mathbf{N}_G(X) = \mathbf{N}_G(Y)$ . Notice that since all variables of  $\mathcal{V}$  only occur in the same nodes of  $G$ , by definition of compatibility, we have  $L_i^X \equiv L_i^Y$ , for  $i = 1, 2$ .

We say that  $G'$  and  $S'$  are obtained by *reducing repeated variables*  $\mathcal{V}$  into  $Z$ , in symbols  $(G, S)^\mathcal{V} \Rightarrow_Z (G', S')$  if

- $G'$  is obtained from  $G$  by replacing in the label of every node  $\mathbf{d}$  all  $X \in \mathcal{V}$  with  $Z$ , and
- $S'$  is obtained from the set  $S$  by replacing the AUTs  $X : [L_1^X, U_1^X] \triangleq [L_2^X, U_2^X]$ , for each  $X \in \mathcal{V}$ , with  $Z : [L_1, \prod_{X \in \mathcal{V}} U_1^X] \triangleq [L_2, \prod_{X \in \mathcal{V}} U_2^X]$ , where  $L_i \equiv L_i^X$ , for  $i = 1, 2$  and  $X \in \mathcal{V}$ .

We write  $(G, S) \Rightarrow (G', S')$  if  $(G, S)^\mathcal{V} \Rightarrow_Z (G', S')$  for some  $\mathcal{V}$  and  $Z$ . The relation  $\Rightarrow^*$  is the reflexive transitive closure of  $\Rightarrow$ . It should be clear that  $G'$  is equi-general to  $G$  and  $S'$  is compatible with  $G'$  w.r.t.  $\mathfrak{F}$ .

Another problem is that backward applications of minimisation rules starting with  $(G, S)$  not always result in exactly  $(G_{\mathfrak{F}}, S_{\mathfrak{F}})$ .

**Example 8** Let  $C_1 = \exists r. A \sqcap \exists r. B$ ,  $C_2 = \exists r. (A \sqcap B)$  and  $G_{\text{sk}} = \exists r. A \sqcap \exists r. B$ . The homomorphisms  $\varphi_i : G_{\text{sk}} \rightarrow C_i$ , for  $i = 1, 2$ , are uniquely determined by the shape of  $G_{\text{sk}}$  and  $C_1, C_2$ .

Consider a concept  $G = \exists r. (A \sqcap X) \sqcap \exists r. (B \sqcap X)$ . It is easy to see that  $G$  is an exact generalisation of  $C_1, C_2$  w.r.t.  $\mathfrak{F} = (G_{\text{sk}}, \varphi_1, \varphi_2)$  and the following set of AUTs  $S$  is compatible with  $G$  w.r.t.  $\varphi_1$  and  $\varphi_2$ :

$$S := \{X : [\top, \top] \triangleq [A \sqcap B, A \sqcap B]\}.$$

Then  $(G_1, S_1) \rightsquigarrow_{(m)}^\sigma (G, S)$ , where  $G_1 = \exists r. (A \sqcap X_1) \sqcap \exists r. (B \sqcap X_2)$  and

$$S_1 := \{X_1 : [A, \top] \triangleq [A \sqcap B, A \sqcap B], \\ X_2 : [B, \top] \triangleq [A \sqcap B, A \sqcap B]\}.$$

Notice that  $S_1$  differs from

$$S_{\mathfrak{F}} := \{X_1 : [A, \top] \triangleq [A \sqcap B, B], \\ X_2 : [B, \top] \triangleq [A \sqcap B, A]\}$$

However  $G_{\mathfrak{F}} = G_1$  and  $(G_{\mathfrak{F}}, S_{\mathfrak{F}}) \rightsquigarrow^* (G, S)$ .

For two sets of AUTs  $S$  and  $S'$ , compatible with  $G$  w.r.t.  $\mathfrak{F}$ ,  $S$  is *weaker* than  $S'$ , in symbols  $(G, S') \gg (G, S)$ , if for every variable  $X$  that occurs in  $G$  and the corresponding AUTs  $X : [L_1, U_1] \triangleq [L_2, U_2] \in S$  and  $X : [L'_1, U'_1] \triangleq [L'_2, U'_2] \in S'$  we have  $L_i \equiv L'_i$  and  $U'_i \sqsubseteq U_i$ , for  $i = 1, 2$ . In Example 8 above, we have  $(G_{\mathfrak{F}}, S_1) \gg (G_{\mathfrak{F}}, S_{\mathfrak{F}})$ .

The following lemma is proved by induction on the number of variables that have more than one occurrence in the generalisation.

**Lemma 9** Let  $C_1, C_2$  and  $G$  be  $\mathcal{EL}$  concepts such that the label of every node of  $G$  contains a variable. Let  $S$  be a set of AUTs compatible with  $G$  w.r.t. some generalisation

framework  $\mathfrak{F}$ . Then there exist a concept  $G^*$  and sets of AUTs  $S^*$  and  $S'$  such that  $(G_{\mathfrak{F}}, S_{\mathfrak{F}}) \rightsquigarrow^* (G^*, S^*)$ , and  $(G, S) \Rightarrow^* (G^*, S') \gg (G^*, S^*)$ .

Lemma 9 requires every node of  $G$  to contain a variable, which does not always hold. We say that a concept  $G$  and a set of AUTs  $S$  are obtained by *variable elimination* from  $(G', S')$  if  $G = \{X \mapsto \top\}(G')$  and  $S$  is obtained from  $S'$  by removing the AUT  $X : [L_1^X, U_1^X] \triangleq [L_2^X, U_2^X]$  such that  $U_1^X \sqcap U_2^X \equiv \top$ . Let  $\text{elim}_S(G')$  denote the result of exhaustive variable elimination. Then the following theorem immediately follows from Lemma 9.

**Theorem 10 (Completeness)** Let  $C_1, C_2$  and  $G$  be  $\mathcal{EL}$  concepts and  $\mathfrak{F}$  a generalisation framework. If  $G$  is an exact generalisation of  $C_1, C_2$  w.r.t.  $\mathfrak{F}$  then  $(G_{\mathfrak{F}}, S_{\mathfrak{F}}) \rightsquigarrow^* (G', S)$ , for some set of AUTs  $S$  and a generalisation  $G'$  of  $C_1, C_2$  such that  $\text{elim}_S(G')$  is less general than  $G$ .

The following statement is an immediate consequence of the shape of the minimisation rules.

**Theorem 11 (Soundness)** Let  $C_1, C_2$  be  $\mathcal{EL}$  concepts and  $\mathfrak{F}$  be a generalisation framework. Let a concept  $G$  and a set of AUTs  $S$  be such that  $(G_{\mathfrak{F}}, S_{\mathfrak{F}}) \rightsquigarrow^* (G, S)$ . Then  $\text{elim}_S(G)$  is an exact generalisation of  $C_1, C_2$  w.r.t.  $\mathfrak{F}$ .

Notice that every application of the (**ssm**) rule introduces a shared variable  $Z$  in the nodes where  $X$  and  $Y$  occur; every application of the (**sm**) rule introduces  $Z \sqcap X'$  into the nodes where  $X$  occurs. We use these properties to prove inductively that every computation terminates.

**Theorem 12 (Termination)** Let  $C_1, C_2$  be  $\mathcal{EL}$  concepts and  $\mathfrak{F}$  a generalisation framework. Then any sequence  $(G_{\mathfrak{F}}, S_{\mathfrak{F}}) = (G_0, S_0), (G_1, S_1), \dots, (G_m, S_m)$  such that  $(G_i, S_i) \rightsquigarrow (G_{i+1}, S_{i+1})$  and no minimisation rule applies to  $(G_m, S_m)$  contains polynomially many elements.

Finally we notice that every  $(G_i, S_i)$  can be computed in (non-deterministic) polynomial time. Indeed, even though minimisation rules are formulated in such a way that a repeated computation of *lcs* is needed, which can be exponential in the size of input, we only use *lcs* to check that  $U_i^X$  and  $U_i^Y$  are such that  $\text{lcs}(L_i^X, L_i^Y) \sqsubseteq U_i^X \sqcap U_i^Y$ , which is equivalent to  $L_i^X \sqsubseteq U_i^X \sqcap U_i^Y$  and  $L_i^Y \sqsubseteq U_i^X \sqcap U_i^Y$ .

**Complexity of fixed skeleton generalisations.** We use the machinery developed to analyse exact generalisations w.r.t. generalisation frameworks to establish complexity bounds for computing fixed skeleton generalisations. It follows from the NP-completeness of matching modulo equivalence (Baader and Küsters 2006) that the computational problem of checking if  $G$  is a generalisation of  $C_1$  and  $C_2$  is NP-complete. For least general generalisations, we rely on the fact that if  $G$  is a fixed skeleton *lgg* of concepts  $C_1, C_2$  with skeleton  $G_{\text{sk}}$  then there exist homomorphisms  $\varphi_i : G_{\text{sk}} \rightarrow C_i$ , for  $i = 1, 2$ , such that  $G$  is an *lgg* of  $C_1$  and  $C_2$  w.r.t.  $(G_{\text{sk}}, \varphi_1, \varphi_2)$ .

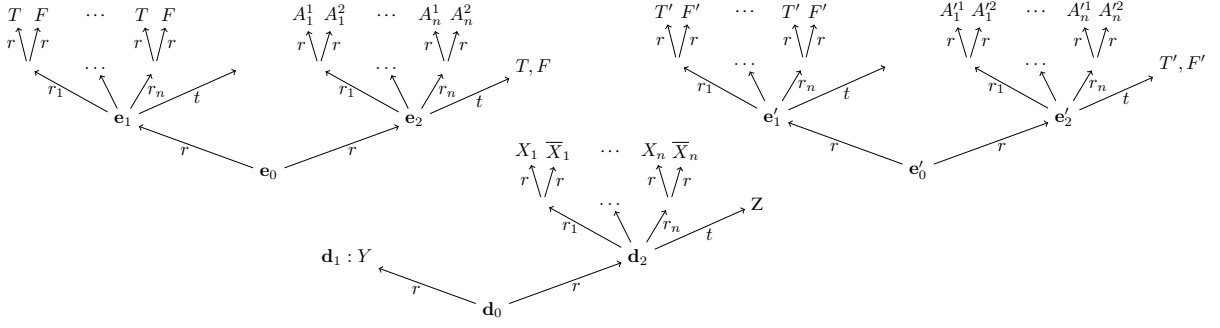


Figure 2: Concepts  $C'_1$  (left) and  $C'_2$  (right) and  $G'$  (bottom) for the proof of Theorem 13.

**Theorem 13** Let  $C_1, C_2, G_{\text{sk}}$  and  $G$  be  $\mathcal{EL}$  concepts. Then the computational problem of checking if  $G$  is a least general exact generalisation of  $C_1$  and  $C_2$  with skeleton  $G_{\text{sk}}$  is coNP-hard.

**Proof.** We proceed by reduction from unsatisfiability. Our proof consists for two steps. First we present concepts  $C'_1, C'_2$  and  $G'$  and analyse conditions under which  $G'$  is an exact lgg of  $C'_1, C'_2$  w.r.t.  $(G'_{\text{sk}}, \varphi_1, \varphi_2)$ , for some homomorphisms  $\varphi_i : G'_{\text{sk}} \rightarrow C'_i, i = 1, 2, G'_{\text{sk}} = \text{skel}(G')$ , and discuss implications of these considerations on the value that variables  $X_i$  and  $\bar{X}_i$ , which encode propositional interpretations, take under witness substitutions. In the second step, we extend  $C'_1, C'_2, G'$  into concepts  $C_1^\phi, C_2^\phi$  and  $G^\phi$  in such a way that  $G^\phi$  is an exact least general generalisation of  $C_1^\phi$  and  $C_2^\phi$  iff  $\phi$  is unsatisfiable.

Consider concepts  $C'_1, C'_2$  and  $G'$ .

$$\begin{aligned} C'_1 := & \exists r. (\exists r_1. (\exists r. T \sqcap \exists r. F) \sqcap \dots \sqcap \\ & \exists r_n. (\exists r. T \sqcap \exists r. F) \sqcap \exists t. \top) \sqcap \\ & \exists r. (\exists r_1. (\exists r. A_1^1 \sqcap \exists r. A_1^2) \sqcap \dots \sqcap \\ & \exists r_n. (\exists r. A_n^1 \sqcap \exists r. A_n^2) \sqcap \exists t. (T \sqcap F)), \end{aligned}$$

$$\begin{aligned} C'_2 := & \exists r. (\exists r_1. (\exists r. T' \sqcap \exists r. F') \sqcap \dots \sqcap \\ & \exists r_n. (\exists r. T' \sqcap \exists r. F') \sqcap \exists t. \top) \sqcap \\ & \exists r. (\exists r_1. (\exists r. A_1^1 \sqcap \exists r. A_1^2) \sqcap \dots \sqcap \\ & \exists r_n. (\exists r. A_n^1 \sqcap \exists r. A_n^2) \sqcap \exists t. (T' \sqcap F')). \end{aligned}$$

and

$$\begin{aligned} G' := & \exists r. Y \sqcap \exists r. (\exists r_1. (\exists r. X_1 \sqcap \exists r. \bar{X}_1) \sqcap \dots \sqcap \\ & \exists r_n. (\exists r. X_n \sqcap \exists r. \bar{X}_n) \sqcap \exists t. Z). \end{aligned}$$

We illustrate concepts  $C'_1$  and  $G'$  in Figure 2: Let  $G'_{\text{sk}} = \text{skel}(G')$  and let sets of homomorphisms  $\Phi_1^u$  and  $\Phi_1^x$  from  $G'_{\text{sk}}$  to  $C'_1$  be defined as  $\Phi_1^u = \{\varphi_1^u \mid \varphi_1^u(\mathbf{d}_1) = \mathbf{e}_1, \varphi_1^u(\mathbf{d}_2) = \mathbf{e}_2\}$  and  $\Phi_1^x = \{\varphi_1^x \mid \varphi_1^x(\mathbf{d}_1) = \mathbf{e}_2, \varphi_1^x(\mathbf{d}_2) = \mathbf{e}_1\}$ . Sets of homomorphisms  $\Phi_2^u$  and  $\Phi_2^x$  from  $G'_{\text{sk}}$  to  $C'_2$  are defined similarly.

It should be clear that for any  $\varphi_1^u \in \Phi_1^u$  and  $\varphi_2^u \in \Phi_2^u$ , the concept  $G'$  is an exact generalisation of  $C'_1$  and  $C'_2$  w.r.t.  $\mathfrak{F}^u = (G'_{\text{sk}}, \varphi_1^u, \varphi_2^u)$  with witness substitutions  $\sigma_1^u$  and  $\sigma_2^u$  such that

$$\begin{aligned} \sigma_1^u(Y) & \equiv \exists r_1. (\exists r. T \sqcap \exists r. F) \sqcap \dots \sqcap \\ & \exists r_n. (\exists r. T \sqcap \exists r. F) \sqcap \exists t. \top, \\ \sigma_1^u(Z) & \equiv T \sqcap F \text{ and } \{\sigma_1^u(X_i), \sigma_1^u(\bar{X}_i)\} \equiv \{A_i^1, A_i^2\}, \end{aligned}$$

for every  $i, 1 \leq i \leq n$ . For  $\sigma_2^u$  the condition is similar. By considering cases it can be proved that  $G'$  is lgg( $C'_1, C'_2$ ) w.r.t.  $\mathfrak{F}^u$ .

Concept  $G'$  is also an exact generalisation of  $C'_1$  and  $C'_2$  w.r.t.  $\mathfrak{F} = (G'_{\text{sk}}, \varphi_1^x, \varphi_2^x)$ , for any  $\varphi_1^x \in \Phi_1^x$  and  $\varphi_2^x \in \Phi_2^x \cup \Phi_2^x$ , with witness substitutions  $\sigma_1^x$

$$\begin{aligned} \sigma_1^x(Y) & \equiv \exists r_1. (\exists r. A_1^1 \sqcap \exists r. A_1^2) \sqcap \dots \sqcap \\ & \exists r_n. (\exists r. A_n^1 \sqcap \exists r. A_n^2) \sqcap \exists t. (T \sqcap F), \\ \sigma_1^x(Z) & \equiv \top \text{ and } \{\sigma_1^x(X_i), \sigma_1^x(\bar{X}_i)\} \equiv \{T, F\}, \end{aligned}$$

for every  $i, 1 \leq i \leq n$ . It is easy to see that, since for some  $i \neq j$  we have  $\sigma_1^x(W_i) = \sigma_1^x(W_j)$  where  $W_i$  is one of  $X_i, \bar{X}_i$  and  $W_j$  is one of  $X_j, \bar{X}_j$ , the (m) or (sm) minimisation rule always applies to the corresponding set of AUTs and so  $G'$  is not an lgg( $C'_1, C'_2$ ) w.r.t.  $\mathfrak{F}$ . For  $\mathfrak{F} = (G'_{\text{sk}}, \varphi_1, \varphi_2^x)$  the reasoning is similar.

Thus,  $G'$  is an exact least general generalisation of  $C'_1$  and  $C'_2$  w.r.t. some generalisation framework  $\mathfrak{F} = (G'_{\text{sk}}, \varphi_1, \varphi_2)$  iff  $\varphi_1 \in \Phi_1^u$  and  $\varphi_2 \in \Phi_2^u$ .

Consider now

$$\begin{aligned} C''_1 & \equiv \exists t_F. F \sqcap \prod_{i=1}^n (\exists t_i^1. A_i^1 \sqcap \exists t_i^2. A_i^2), \\ C''_2 & \equiv \exists t_F. F' \sqcap \prod_{i=1}^n (\exists t_i^1. A_i^1 \sqcap \exists t_i^2. A_i^2), \text{ and} \\ G'' & \equiv \exists t_F. Y_F \sqcap \prod_{i=1}^n (\exists t_i^1. Y_i^1 \sqcap \exists t_i^2. Y_i^2) \end{aligned}$$

It should be clear that  $G''$  is an exact lgg of  $C''_1$  and  $C''_2$  with a fixed skeleton  $\text{skel}(G'')$  and for every witness substitutions  $\tau_1$  and  $\tau_2$  we always have

$$\begin{aligned} \tau_1(Y_F) & \equiv F; \tau_1(Y_i^1) \equiv A_i^1; \tau_1(Y_i^2) \equiv A_i^2, \\ \tau_2(Y_F) & \equiv F'; \tau_2(Y_i^1) \equiv A_i^1; \tau_2(Y_i^2) \equiv A_i^2, i = 1, 2. \end{aligned}$$

Finally, given a 3CNF formula

$$\phi = (l_1^{(1)} \vee l_2^{(1)} \vee l_3^{(1)}) \wedge \dots \wedge (l_1^{(m)} \vee l_2^{(m)} \vee l_3^{(m)}),$$



where  $l_i^{(j)}$  is a literal, define a translation function  $f(p_i) = X_i$  and  $f(\neg p_i) = \bar{X}_i$  and concepts  $G^\phi$ ,  $C_1^\phi$ , and  $C_2^\phi$  as

$$\begin{aligned} C_1^\phi &:= C'_1 \sqcap C''_1 \sqcap \prod_{j=1}^m \exists s_j. (\mathbf{A} \sqcap T \sqcap F), \text{ and} \\ C_2^\phi &:= C'_2 \sqcap C''_2 \sqcap \prod_{j=1}^m \exists s_j. (\mathbf{A}' \sqcap T' \sqcap F'), \\ G^\phi &:= G' \sqcap G'' \sqcap \prod_{j=1}^m \exists s_j. \left( \mathbf{Y} \sqcap Y_F \sqcap \right. \\ &\quad \left. f(l_1^{(j)}) \sqcap f(l_2^{(j)}) \sqcap f(l_3^{(j)}) \sqcap Z \right), \end{aligned}$$

where  $\mathbf{Y} = Y_1^1 \sqcap Y_1^2 \sqcap \dots \sqcap Y_n^1 \sqcap Y_n^2$ ,  $\mathbf{A} = A_1^1 \sqcap A_1^2 \sqcap \dots \sqcap A_n^1 \sqcap A_n^2$ , and  $\mathbf{A}' = A_1'^1 \sqcap A_1'^2 \sqcap \dots \sqcap A_n'^1 \sqcap A_n'^2$ .

For every witness substitution  $\sigma_1^u$  (resp.  $\sigma_2^u$ ) we always have  $(\sigma_1^u \cup \tau_1)(G^\phi) \equiv C_1^\phi$  (resp.  $(\sigma_2^u \cup \tau_2)(G^\phi) \equiv C_2^\phi$ ); for  $\sigma_1^x$  we have  $(\sigma_1^x \cup \tau_1)(G^\phi) \equiv C_1^\phi$  iff  $I \models \phi$ , where the interpretation  $I$  is defined as  $I = \{p_i \mid \sigma_1^x(X_i) = T\}$ . Thus,  $G^\phi$  is an exact least general generalisation with a fixed skeleton of concepts  $C_1^\phi$ ,  $C_2^\phi$  iff  $\phi$  is unsatisfiable.  $\square$

### Generalisation variants

We have introduced the notion of a concept generalisation as capturing the commonalities and highlighting differences between given concepts. Notice that some generalisations do not fit this motivation. Let  $G$  be a generalisation of concepts  $C_1$ ,  $C_2$  witnessed by substitutions  $\tau_1$ ,  $\tau_2$ . Consider substitutions

- $\sigma = \{X_i \mapsto Y_i \sqcap Z_i \mid X_i \text{ occurs in } G, Y_i, Z_i \text{ are fresh}\}$ ,
- $\tau_1'(Y_i) = \tau_1(X_i)$  and  $\tau_1'(Z_i) = \top$ ,
- $\tau_2'(Y_i) = \top$  and  $\tau_2'(Z_i) = \tau_2(X_i)$ .

Then it is easy to see that  $\sigma(G)$  is a generalisation of  $C_1$ ,  $C_2$  with witness substitutions  $\tau_1'$ ,  $\tau_2'$  and  $\sigma(G)$  is equi-general to  $G$ . Notice that under substitutions  $\tau_i$  there is no ‘interaction’ between  $C_1$  and  $C_2$ .

**Top-Free Generalisations.** We say that a concept  $G$  is a *top-free* generalisation of  $C_1$ ,  $C_2$  if there exists witness substitutions  $\tau_1$ ,  $\tau_2$  such that  $\tau_i(G) \equiv C_i$ , for  $i = 1, 2$  and  $\tau_i(X) \not\equiv \top$  for all variables  $X$  occurring in  $G$ . We say that a generalisation is *top-forcing*, if it is not top-free.

**Example 14 (Top-free generalisation)** Let  $C_1 = A \sqcap \exists r.A$ ,  $C_2 = A \sqcap \exists r.B$ , and  $\mathfrak{F}$  be the framework with  $G_{\text{sk}} = A \sqcap \exists r.\top$  and the obvious homomorphisms. Then  $A \sqcap X \sqcap \exists r.(X \sqcap Y)$  is an lgg of  $C_1$  and  $C_2$  w.r.t  $\mathfrak{F}$ , which is top-forcing, because  $\sigma(X) \equiv \top$  for any  $\sigma$  that matches  $G$  to  $C_2$ . On the other hand,  $A \sqcap \exists r.X$  is a top-free lgg of  $C_1$  and  $C_2$ .

Notice that the construction in the proof of Theorem 13 uses top-free least general generalisation. Hence, the coNP hardness holds for checking top-free lgg as well.

The proof of Theorem 10 can be extended to show that every top-free lgg of concepts  $C_1$ ,  $C_2$  w.r.t. a generalisation framework  $\mathfrak{F}$  can be computed by our procedure with an additional requirement that  $lcs(L_1^X, L_1^Y) \not\equiv \top$  and  $lcs(L_2^X, L_2^Y) \not\equiv \top$  in the minimisation rules.

**Rigid Generalisations.** In the anti-unification literature, it is often the case that in the argument positions of a generalisation, ‘neighbour’ variables are forbidden. Such generalisations are called *rigid* ones. They provide a cheaper alternative to otherwise a pretty large number of not necessarily rigid lgg, see, e.g., (Kutsia, Levy, and Villaret 2014).

In our case, rigidity requires generalisations to contain at most one variable in each node. It has a drastic effect: The problem becomes unitary, i.e., there is a single rigid lgg, and it can be computed in polynomial time. The existence of a single rigid lgg follows from Lemma 9. Indeed, for every rigid generalisation  $G$  with a variable in every node, and a set of AUTs  $S$  compatible with  $C_1$ ,  $C_2$ ,  $G$  w.r.t.  $\mathfrak{F}$ , there exist a concept  $G^*$  and sets of AUTs  $S^*$  and  $S'$  such that  $(G_{\mathfrak{F}}, S_{\mathfrak{F}}) \rightsquigarrow^* (G^*, S^*)$ , and  $(G, S) \rightrightarrows^* (G^*, S') \gg (G^*, S^*)$ . Notice, however, that under these conditions  $G^*$  is necessarily rigid and thus the only minimisation rule used in  $\rightsquigarrow^*$  is **(m)** so variables do not split, which does not cause branching and the algorithm ends up with a single solution in polynomially many steps. Clearly the computation of  $\text{elim}_{S^*}(G^*)$  does not split any variables either.

### Experimental evaluation

We have implemented our anti-unification algorithm as a *Mathematica* package. To reduce the degree of non-determinism, in the **(sm)** and **(ssm)** rules we additionally require

- (iv)  $U_1^X \sqcap U_1^Y \sqcap U_2^X \sqcap U_2^Y \not\equiv \top$ ,
- (v)  $U_1^Y \sqcap U_2^Y \not\equiv \top$ , and  $U_1^Y \sqcap U_2^Y \not\equiv \top$

and use the following strategy:

- Repeat until no minimisation rule is applicable
  - Apply **(m)** exhaustively (does not generate branching)
  - Apply **(sm)** whenever applicable (causes branching)
  - Apply **(ssm)** whenever applicable (causes branching)
- If no rule is applicable, return  $\text{elim}_S(G)$ .

It is easy to see that this strategy leads to incompleteness, as the following modification of Example 6 demonstrates.

**Example 15** Consider concepts

$$\begin{aligned} C_1 = C_2 = & \\ & \exists r.(A \sqcap B) \sqcap \exists s.(B \sqcap C) \sqcap \exists t.(A \sqcap C) \sqcap \\ & \exists u_1.A \sqcap \exists u_2.B \sqcap \exists u_3.C \end{aligned}$$

Let the skeleton be

$$\begin{aligned} G_{\text{sk}} = & \exists r.(A \sqcap B) \sqcap \exists s.(B \sqcap C) \sqcap \exists t.(A \sqcap C) \sqcap \\ & \exists u_1.\top \sqcap \exists u_2.\top \sqcap \exists u_3.\top. \end{aligned}$$

Notice that homomorphisms  $\varphi_i : G_{\text{sk}} \rightarrow C_i$ , and hence generalisation framework  $\mathfrak{F} = (G_{\text{sk}}, \phi_1, \phi_2)$ , are uniquely determined. Then

$$\begin{aligned} G = & \exists r.(A \sqcap B \sqcap V_1 \sqcap V_2) \sqcap \\ & \exists s.(B \sqcap C \sqcap V_3 \sqcap V_1) \sqcap \\ & \exists t.(A \sqcap C \sqcap V_3 \sqcap V_2) \sqcap \\ & \exists u.V_1 \sqcap \exists u.V_2 \sqcap \exists u.V_3 \end{aligned}$$

is an LGG of  $C_1$  and  $C_2$  w.r.t  $\mathfrak{F}$ . However, to compute it with our minimisation rules, both conditions (iv) and (v) above have to be violated.

In many practical cases our strategy gives shorter and more meaningful generalisations. The implementation accepts a generalisation framework as input, but if such input is not given, it can also construct one based on the *lcs*.

To evaluate our procedure, we have computed generalisations of the right hand sides of concept equalities of the form  $A \equiv C$  of the GALEN-EL ontology (Kazakov and Klinov 2015; Rector et al. 2003). There are 9968 such definitions in the ontology. We experimented with various ways of computing *lcs*-based skeleton, from taking just the *lcs* to making it some smaller size concept that subsumes the *lcs* but still retains the common structure of the input concepts. The statistics reported here is for the latter.

We ran our test on all 9968 selected axioms. To avoid computing trivial or very simple generalisations, we restricted our consideration to cases when generalisations were either ground or had skeleton of depth at least 2.

It took 77.5 hours on Dell Linux Workstation with Intel Xeon E5-2680 v2 CPU and 384 GB RAM to anti-unify 49 675 528 concept pairs and compute 99 529 answers satisfying our selection criteria, among which 750 were ground, 42 258 showed that the anti-unified concepts differed from each other only by concept names (we call them *renaming generalisations*), and 56 521 showed non-renaming differences. To evaluate the scalability of our implementation, we also run our tool on 1 000 randomly picked axioms, which took it 47 minutes.

The generalisations can be also characterised by other criteria:

- Among nonground generalisations, 92 589 were linear (i.e., no variable occurred more than once) and 6 190 non-linear.
- 93 343 generalisations were rigid and 6 186 non-rigid.
- 93 418 generalisations were top-free and 6 111 top-forcing.

Our experiments revealed interesting insights into the structure of the ontology; we illustrate some of our findings below. A generalisation  $G$  being ground indicates that the input concepts are identical, i.e., some concept descriptions repeat in the ontology: There are axioms of the form  $A \equiv G$  and  $B \equiv G$ , which could have been better modelled as  $A \equiv G$  and  $B \equiv A$ . Examples of repeated definitions are AbductionOfGlenoHumeralJoin, Anomymous-469, and Anomymous-488, all defined as

Abduction  $\sqcap$   $\exists$ actsSpecificallyOn. GlenoHumeralJoint;  
Anomymous-109 and Anomymous-331 both are defined as  
Device  $\sqcap$   $\exists$ isSpecificPhysicalMeansOf.  
(NonDirectInspecting  $\sqcap$   $\exists$ actsSpecificallyOn.Eye);

etc.

Interesting relationships between the input concepts can be observed when generalisations are renamings. For instance, one can see that the definitions of AdenomaOfColon

and AdrenalPhaeochromocytoma<sup>1</sup> have quite a similar structure, since their generalisation is

BodyStructure  $\sqcap$   $\exists$ hasUniqueAssociatedProcess.  
(BenignNeoplasticProcess  $\sqcap$   $\exists$ actsSpecificallyOn.  
( $X_1 \sqcap$   $\exists$ isSpecificStructuralComponentOf.  $X_2$ )),

where the differences are only in concept names as the computed AUTs show

$X_1 : [\dots, \text{Adenocyte}] \triangleq [\dots, \text{ChromaffinCell}]$ ,  
 $X_2 : [\dots, \text{Colon}] \triangleq [\dots, \text{AdrenalMedulla}]$ .

(The lower bounds are omitted for brevity.) This generalisation is also rigid and top-free.

Nonlinear renaming generalisation of Anonymous-257 and Anonymous-529 shows that these definitions differ by concept names in two places, but in exactly the same way:

$X_1 \sqcap$   $\exists$ IsDivisionOf.  
( $X_1 \sqcap$   $\exists$ isPairedOrUnpaired. atLeastPaired),

where

$X_1 : [\dots, \text{GenericBodyStructure}] \triangleq [\dots, \text{BodyPart}]$ .

Like the previous one, this generalisation is also rigid and top-free.

An example of nonrigid, nonlinear, nonrenaming generalisation has been obtained, e.g., for Anonymous-158 and Anonymous-340:

absence  $\sqcap$   $\exists$ isExistenceOf. ( $X_1 \sqcap X_2 \sqcap$   
 $\exists$ isConsequenceOf. (GeneralisedProcess  $\sqcap X_1 \sqcap X_3 \sqcap$   
 $\exists$ LocativeAttribute. OrganicSolidStructure)),

where the AUTs are

$X_1 : [\dots, \exists$ hasIntrinsicAbnormalityStatus. normal]  $\triangleq$   
[ $\dots$ , GeneralisedProcess],  
 $X_2 : [\dots, \text{GeneralisedSubstance}] \triangleq [\dots, \top]$ ,  
 $X_3 : [\dots, \top] \triangleq$   
[ $\dots, \exists$ hasAbnormalityStatus.nonNormal].

From this generalisation one can see that the definitions of Anonymous-158 and Anonymous-340 are quite similar. As for the differences, the generalisation and the AUT for  $X_1$  show that the definition of Anonymous-158 contains the concept  $\exists$ hasIntrinsicAbnormalityStatus. normal in two places where the definition of Anonymous-340 has GeneralisedProcess. From the AUT for  $X_2$  we conclude that Anonymous-158 contains GeneralisedSubstance, which is not present in Anonymous-340. Similarly, the AUT for  $X_3$  says that  $\exists$ hasAbnormalityStatus.nonNormal occurs in Anonymous-340 but not in Anonymous-158.

Note that this generalisation is also top-free, because the lower bounds for  $X_2$  and  $X_3$  are not  $\top$ .

<sup>1</sup>Definitions of these and the other concepts used in this section can be found in Appendix.

## Conclusions

We have studied the anti-unification problem for the description logic  $\mathcal{EL}$ , proved that a finite minimal complete set of generalisations for  $\mathcal{EL}$  concepts always exists and established complexity bounds for computing them. We presented an anti-unification algorithm for a fixed generalisation framework case and proved that it always terminates with a generalisation of the given concepts and that every  $lgg$  can be computed by an algorithm run. We evaluated its performance in a case study. Investigating the existence of tractable strategies that guarantee that the output is always an  $lgg$  and computing  $lggs$  in presence of background knowledge (TBoxes) constitutes future work.

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## Proofs

We restate Lemma 4 for reader's convenience.

**Lemma 4** *Let  $C_1, C_2$  be concepts,  $\mathfrak{F} = (G_{\text{sk}}, \varphi_1, \varphi_2)$  be a generalisation framework, and  $G$  be a concept such that  $\text{skel}(G) = G_{\text{sk}}$ . Let  $S$  be a set of AUTs such that for every variable  $X$  of  $G$ , it contains exactly one AUT  $X : [\mathbb{L}_1^X, \mathbb{U}_1^X] \triangleq [\mathbb{L}_2^X, \mathbb{U}_2^X]$ . Assume that the substitutions  $\tau_i$ , for  $i = 1, 2$ , are defined as follows:*

$$\tau_i := \{X \mapsto \mathbb{U}_i^X \mid X : [\mathbb{L}_1^X, \mathbb{U}_1^X] \triangleq [\mathbb{L}_2^X, \mathbb{U}_2^X] \in S\}.$$

*Then  $S$  is compatible with  $G$  w.r.t.  $\mathfrak{F}$  iff  $G$  is an exact generalisation of  $C_1, C_2$  w.r.t.  $\mathfrak{F}$  witnessed by  $\tau_1, \tau_2$ .*

**Proof.**

( $\implies$ ) Suppose that  $S$  is compatible with  $G$  w.r.t.  $\mathfrak{F}$ . We prove that  $\tau_i(\mathbf{d}) \equiv \varphi_i(\mathbf{d})$ , for  $i = 1, 2$ , for every node  $\mathbf{d}$  of  $G$  by induction on the role depth of  $\mathbf{d}$ .

Suppose that the role depth of  $\mathbf{d}$  is 0. Then  $\mathbf{d} = A_1 \sqcap \dots \sqcap A_l \sqcap X_1 \dots \sqcap X_m$ . Thus,

$$\begin{aligned} \tau_i(\mathbf{d}) &= A_1 \sqcap \dots \sqcap A_l \sqcap \tau_i(X_1) \sqcap \dots \sqcap \tau_i(X_m) \\ &\equiv A_1 \sqcap \dots \sqcap A_l \sqcap \mathbb{U}_i^{X_1} \sqcap \dots \sqcap \mathbb{U}_i^{X_m}, \end{aligned}$$

for  $i = 1, 2$ .

By definition of a variable ignoring homomorphism, we have  $\varphi_i(\mathbf{d}) \sqsubseteq A_1 \sqcap \dots \sqcap A_l$ .

By definition of a compatible set of AUTs, for  $i = 1, 2$  and every  $X \in \{X_1, \dots, X_m\}$ , we have  $\mathbb{L}_i^X = \text{lcs}(\varphi_i(\mathbf{N}_G(X))) = \varphi_i(\mathbf{d})$ , and so  $\varphi_i(\mathbf{d}) = \mathbb{L}_i^X$ , for every  $X \in \{X_1, \dots, X_m\}$ , and thus  $\varphi_i(\mathbf{d}) = \mathbb{L}_i^{X_1} \sqcap \dots \sqcap \mathbb{L}_i^{X_m}$ , for  $i = 1, 2$ . As  $\mathbb{L}_i^{X_j} \sqsubseteq \mathbb{U}_i^{X_j}$ , for all  $i, j$ , we have  $\varphi_i(\mathbf{d}) \sqsubseteq \mathbb{U}_i^{X_1} \sqcap \dots \sqcap \mathbb{U}_i^{X_m}$ , and so  $\varphi_i(\mathbf{d}) \sqsubseteq \tau_i(\mathbf{d})$ , for  $i = 1, 2$ .

For the proof of  $\tau_i(\mathbf{d}) \sqsubseteq \varphi_i(\mathbf{d})$  notice that, by definition of a compatible set of AUTs,  $\prod_{j=1, \dots, m} \mathbb{U}_i^{X_j} \sqsubseteq (C_i \ominus_{\varphi_i}^{\mathbf{d}} G)$ . It should be clear then that  $\tau_i(\mathbf{d}) \sqsubseteq \varphi_i(\mathbf{d})$  and we are done.

Suppose that the Lemma is proved for all nodes  $\mathbf{d}$  with role depth smaller than  $n$  and let the role depth of a node  $\mathbf{d}$  be  $n$ .

To simplify notation, we prove the property for  $\tau_1$ ; for  $\tau_2$  the proof is similar.

Let

$$\mathbf{d} = A_1 \sqcap \dots \sqcap A_l \sqcap X_1 \sqcap \dots \sqcap X_{l'} \sqcap \exists r_1. C_1 \sqcap \dots \sqcap \exists r_m. C_m$$

and

$$\begin{aligned} \varphi_1(\mathbf{d}) &= B_1 \sqcap \dots \sqcap B_o \sqcap \\ &\quad \exists s_1. D_1 \sqcap \dots \sqcap \exists s_p. D_p \sqcap \\ &\quad \exists s_{p+1}. D_{p+1} \sqcap \dots \sqcap \exists s_q. D_q, \end{aligned}$$

where for every  $j \in \{1 \dots p\}$  there exists  $i \in \{1 \dots m\}$  such that  $s_j = r_i$  and  $\varphi(C_i) = D_j$ . By induction hypothesis,  $\tau_1(C_i) \equiv D_j$  and so

$$\tau_1(\mathbf{d}) = A_1 \sqcap \dots \sqcap A_l \sqcap$$

$$\begin{aligned} &\exists r_1. \tau_1(C_1) \sqcap \dots \sqcap \exists r_m. \tau_1(C_m) \sqcap \\ &\tau_1(X_1) \sqcap \dots \sqcap \tau_1(X_m) \\ &\equiv A_1 \sqcap \dots \sqcap A_l \sqcap \\ &\quad \exists s_1. D_1 \sqcap \dots \sqcap \exists s_p. D_p \sqcap \\ &\quad \tau_1(X_1) \sqcap \dots \sqcap \tau_1(X_m) \end{aligned}$$

Then, as in the case of role depth 0, we have  $\varphi_1(\mathbf{d}) \sqsubseteq \mathbb{U}_1^{X_1} \sqcap \dots \sqcap \mathbb{U}_1^{X_m}$  and thus  $\varphi_1(\mathbf{d}) \sqsubseteq \tau_1(\mathbf{d})$ .

For the other direction, notice that

$$\begin{aligned} C_1 \ominus_{\varphi_1}^{\mathbf{d}} G &= \prod_{A \in \{B_1, \dots, B_o\} \setminus \{A_1, \dots, A_l\}} A \sqcap \\ &\quad \exists s_{p+1}. D_{p+1} \sqcap \dots \sqcap \exists s_q. D_q \end{aligned}$$

Thus

$$\begin{aligned} \tau_1(\mathbf{d}) &\sqsubseteq A_1 \sqcap \dots \sqcap A_l \sqcap \\ &\quad \exists s_1. D_1 \sqcap \exists s_p. D_p \sqcap \\ &\quad \prod_{A \in \{B_1, \dots, B_o\} \setminus \{A_1, \dots, A_l\}} A \\ &\quad \sqcap \exists s_{p+1}. D_{p+1} \sqcap \dots \sqcap \exists s_q. D_q \\ &\equiv \varphi_1(\mathbf{d}) \end{aligned}$$

( $\impliedby$ ) Conversely, suppose that for every node  $\mathbf{d}$  of  $G$  we have  $\tau_i(\mathbf{d}) \equiv \varphi_i(\mathbf{d})$ , for  $i = 1, 2$ .

For every  $X$  occurring in  $G$  we define  $\mathbb{L}_i^X = \text{lgg}(\varphi_i(\mathbf{N}_G(X)))$  and  $\mathbb{U}_i^X = \tau_i(X)$ . It is straightforward to check that  $S$  is compatible with  $G$  w.r.t.  $\mathfrak{F}$ .  $\square$

**Lemma 9** *Let  $C_1, C_2$  and  $G$  be  $\mathcal{EL}$  concepts such that the label of every node of  $G$  contains a variable. Let  $S$  be a set of AUTs compatible with  $G$  w.r.t. some generalisation framework  $\mathfrak{F}$ . Then there exist a concept  $G^*$  and sets of AUTs  $S^*$  and  $S'$  such that  $(G_{\mathfrak{F}}, S_{\mathfrak{F}}) \rightsquigarrow^* (G^*, S^*)$ , and  $(G, S) \rightrightarrows^* (G^*, S') \gg (G^*, S^*)$ .*

**Proof.** We say that  $X$  is a *singleton* in  $C$  if  $|\mathbf{N}_C(X)| = 1$ . If  $|\mathbf{N}_C(X)| > 1$  we say that  $X$  has *multiple occurrences* in  $C$ .

We prove the lemma by induction on the number of occurrences of concept variables that have multiple occurrences in  $G$ . The base of induction is the case when  $G$  contains no multiple occurrences of concept variables. Suppose that  $S$  is compatible with  $G, C_1$  and  $C_2$  w.r.t.  $\mathfrak{F}$  and  $G$  only contains singletons. Let  $G_0$  be  $G$  and  $S_0$  be  $S$ . Suppose that the label of a node  $\mathbf{d}$  of  $G_0$  contains some concept variables  $X_1, \dots, X_{n_d}$ , for  $n_d > 1$ . Let  $G_1$  be obtained from  $G_0$  by replacing in the label of node  $\mathbf{d}$  all  $X_1, \dots, X_{n_d}$  with  $Z_0$ , where  $Z_0$  is a fresh concept variable, and let  $S_1$  be obtained from  $S_0$  by replacing the AUTs

$$\begin{aligned} X_1 : [\mathbb{L}_1, \mathbb{U}_1^{X_1}] \triangleq [\mathbb{L}_2, \mathbb{U}_2^{X_1}] \\ \vdots \\ X_{n_d} : [\mathbb{L}_1, \mathbb{U}_1^{X_{n_d}}] \triangleq [\mathbb{L}_2, \mathbb{U}_2^{X_{n_d}}] \end{aligned}$$

with

$$Z_0 : [\mathbb{L}_1, \prod_{j=1}^{n_d} \mathbb{U}_1^{X_j}] \triangleq [\mathbb{L}_2, \prod_{j=1}^{n_d} \mathbb{U}_2^{X_j}].$$

Clearly,  $(G_0, S_0)^{\{X_1, \dots, X_{n_d}\}} \Rightarrow_{Z_0} (G_1, S_1)$  and, by definition of compatibility,  $S_1$  is compatible with  $G_1, C_1, C_2$  w.r.t.  $\mathfrak{F}$ . By applying the same reasoning to every other node  $e$  of  $G_i$ , for  $i > 0$ , we obtain a sequence  $(G_0, S_0) \Rightarrow (G_1, S_1) \Rightarrow \dots \Rightarrow (G^*, S')$ , for some  $G^*$  and  $S'$  such that every node of  $G^*$  contains exactly one concept variable and  $S'$  is compatible with  $G^*, C_1, C_2$  w.r.t.  $\mathfrak{F}$ . Notice that since every node of  $G^*$  contains exactly one concept variable and, since all variables in  $G$  are singletons and we only introduced fresh variables, all variables in  $G^*$  are pairwise distinct. But then w.l.o.g.  $G^*$  is  $G_{\mathfrak{F}}$ .

As  $S'$  is compatible with  $G^*, C_1, C_2$  w.r.t.  $\mathfrak{F}$ , for every AUT  $X : [L_1^X, U_1^X] \triangleq [L_2^X, U_2^X] \in S'$  we have  $L_i^X \equiv \varphi_i(\mathbf{d})$ , where  $\mathbf{d}$  is the unique node of  $G^*$  containing variable  $X$ , and  $U_i^X \sqsubseteq (C_i \ominus_{\varphi_i}^{\mathbf{d}} G_{\text{sk}})$ ,  $i = 1, 2$ . But then  $(G^*, S') \gg (G^*, S^*)$ , where  $S^*$  is  $S_{\mathfrak{F}}$ . Thus we have  $(G, S) \Rightarrow^* (G^*, S') \gg (G^*, S^*)$  and  $(G_{\mathfrak{F}}, S_{\mathfrak{F}}) \rightsquigarrow^* (G^*, S^*)$ , as required (the latter is by reflexivity of  $\rightsquigarrow^*$ ).

For the induction step, suppose that the lemma holds for all  $G'$  and  $S'$  such that  $G'$  has  $N > 0$  occurrences of concept variables that have multiple occurrences in  $G'$ ,  $S'$  is compatible with  $G'$  w.r.t.  $\mathfrak{F}$ . Let the number of occurrences of concept variables that have multiple occurrences in  $G$  be  $N + 1$ , and let  $S$  be a set of AUTs compatible with  $G$  w.r.t.  $\mathfrak{F}$ . Suppose that a variable  $Z$  has multiple occurrences in  $G$ . We choose  $Z$  in such a way that either  $Z$  has more than 2 occurrences in  $G$  or every variable with multiple occurrences has exactly 2 occurrences in  $G$ .

We choose a node  $\mathbf{d} \in \mathbf{N}_G(Z)$  to be such that either there is a singleton  $Y \in \mathbf{V}_G(\mathbf{d})$  (i.e.,  $Y$  occurs nowhere else in  $G$ ) or for every  $e \in \mathbf{N}_G(Z)$  the set  $\mathbf{V}_G(e)$  does not contain any singletons (in the latter case, we can choose as  $\mathbf{d}$  an arbitrary node of  $\mathbf{N}_G(Z)$ ).

Let  $\mathbf{V}_G(\mathbf{d}) = \{Z = X_1, X_2, \dots, X_l\}$  and let

$$\begin{aligned} X_1 &: [L_1^{X_1}, U_1^{X_1}] \triangleq [L_2^{X_1}, U_2^{X_1}], \\ &\vdots \\ X_l &: [L_1^{X_l}, U_1^{X_l}] \triangleq [L_2^{X_l}, U_2^{X_l}] \end{aligned}$$

be the corresponding AUTs for  $X_1, \dots, X_l$ .

By definition of a set of AUTs compatible with  $G, C_1, C_2$  w.r.t.  $\mathfrak{F}$  for  $i = 1, 2$  and  $j = 1, \dots, l$ , we have

$$\begin{aligned} L_i^{X_j} \sqsubseteq U_i^{X_j} \text{ and } L_i^{X_j} \equiv lcs(\varphi_i(\mathbf{N}_G(X_j))) \quad \text{and} \\ \prod_{j=1}^l U_i^{X_j} \sqsubseteq (C_i \ominus_{\varphi_i}^{\mathbf{d}} G_{\text{sk}}), \text{ for } i = 1, 2. \end{aligned}$$

Notice that, by properties of a least common subsumer, for  $i = 1, 2$  and  $j = 1, \dots, l$ ,

$$\begin{aligned} lcs(\varphi_i(\mathbf{N}_G(X_j))) \equiv \\ lcs(\{\varphi_i(\mathbf{d})\} \cup \{lcs(\varphi_i(\mathbf{N}_G(X_j) \setminus \{\mathbf{d}\})\})) \end{aligned} \quad (1)$$

Consider cases

1. Suppose that for every  $e \in \mathbf{N}_G(Z)$  the set  $\mathbf{V}_G(e)$  does not contain any singletons. In particular,  $\mathbf{V}_G(\mathbf{d})$  contains no singletons.

Let  $G'$  be obtained from  $G$  by replacing  $Z$  in the label of  $\mathbf{d}$  with  $X$  and by replacing  $Z$  in the label of every other node  $e \neq \mathbf{d}$  of  $G$  with  $Y$ , where  $X$  and  $Y$  are fresh variables.

Let  $S'$  be obtained from  $S$  by replacing the AUT

$$Z : [L_1^Z, U_1^Z] \triangleq [L_2^Z, U_2^Z]$$

with the following two AUTs

$$\begin{aligned} X &: [\varphi_1(\mathbf{d}), U_1^Z] \triangleq [\varphi_2(\mathbf{d}), U_2^Z] \\ Y &: [L_1, U_1^Z] \triangleq [L_2, U_2^Z], \end{aligned}$$

where  $L_i = lcs(\varphi_i(\mathbf{N}_G(Z) \setminus \{\mathbf{d}\}))$ , for  $i = 1, 2$ .

We claim that  $S'$  is correctly defined and compatible with  $G'$  w.r.t.  $\mathfrak{F}$ . Condition (c1) of compatibility obviously holds. As for  $i = 1, 2$  and  $j = 1, \dots, l$  we have  $lcs(\varphi_i(\mathbf{N}_G(X_j) \sqsubseteq U_i^{X_j})$  and since  $X$  is a singleton, we have  $lcs(\varphi_i(\mathbf{N}_G(X))) \equiv \varphi_i(\mathbf{d})$ . As  $\mathbf{N}_G(Z) = \mathbf{N}_G(Y) \cup \{\mathbf{d}\}$ , we have  $lcs(\varphi_i(\mathbf{N}_G(Y))) = L_i$ . Thus condition (c2) also holds. The new AUTs are correctly defined as, by (1), both  $\varphi_i(\mathbf{d}) \sqsubseteq U_i^Z$  and  $lcs(\varphi_i(\mathbf{N}_G(Z) \setminus \{\mathbf{d}\})) \sqsubseteq U_i^Z$ .

Finally, for condition (c3) one can see that for every node  $e$  of  $G'$  we have  $\prod_{W \in \mathbf{V}_{G'}(e)} U_i^W = \prod_{W \in \mathbf{V}_G(e)} U_i^W$ . For example, for  $\mathbf{d}$  we have

$$\begin{aligned} \prod_{W \in \mathbf{V}_{G'}(\mathbf{d})} U_i^W &= \left( \prod_{W \in (\mathbf{V}_G(\mathbf{d}) \setminus \{Z\}) \cup \{X\}} U_i^W \right) = \\ &\left( \prod_{W \in \mathbf{V}_G(\mathbf{d}) \setminus \{Z\}} U_i^W \right) \cap U_i^Z = \prod_{W \in \mathbf{V}_G(\mathbf{d})} U_i^W. \end{aligned}$$

So all conditions of compatibility hold. By ind. hypothesis there exist  $(G^*, S^*)$  and  $S''$  such that  $(G_{\mathfrak{F}}, S_{\mathfrak{F}}) \rightsquigarrow^* (G^*, S^*)$  and  $(G', S') \Rightarrow^* (G^*, S'') \gg (G^*, S^*)$ .

Since we assumed that for every  $e \in \mathbf{N}_G(Z)$  the set  $\mathbf{V}_G(e)$  does not contain any singletons, neither  $X$  nor  $Y$  are reduced into any other variable. Thus, both  $X$  and  $Y$  occur in  $G^*$ . Then the set  $S^*$  contains

$$\begin{aligned} X &: [\varphi_1(\mathbf{d}), U_{11}] \triangleq [\varphi_2(\mathbf{d}), U_{21}], \\ Y &: [L_1, U_{12}] \triangleq [L_2, U_{22}] \end{aligned}$$

and, as  $(G^*, S') \gg (G^*, S^*)$ , we have  $U_i^Z \sqsubseteq U_{ij}$ , for  $i = 1, 2$  and  $j = 1, 2$ . But then  $lcs(\varphi_i(\mathbf{d}), L_i) \sqsubseteq U_{i1} \cap U_{i2}$  and so there exist a concept  $G^{**}$  and set of AUTs  $S^{**}$  such that  $(G^*, S^*) \rightsquigarrow_{(\mathbf{m})}^{\sigma} (G^{**}, S^{**})$ , where  $\sigma = \{X \mapsto Z, Y \mapsto Z\}$ . Notice that  $X \cap Y$  is not equivalent to any subconcept of  $G^*$ . It is easy to see that for  $(G''', S''')$  such that  $(G, S) \Rightarrow^* (G''', S''')$  we have  $G''' = G^{**}$  and  $(G^{**}, S''') \gg (G^{**}, S^{**})$ .

2. Suppose that the conditions of Item 1 above do not apply, every variable that has multiple occurrences in  $G$  have exactly two occurrences in  $G$  and  $Z$  are such that  $|\mathbf{N}_G(Z)| = 2$  and both nodes  $\mathbf{d}$  and  $e$  that contain  $Z$  in their labels also contain singletons  $X' \in \mathbf{V}_G(\mathbf{d})$  and  $Y' \in \mathbf{V}_G(e)$ . Let  $(G, S) \Rightarrow^* (G_0, S_0)$ . Let  $Z_0$  be the result of reduction of repeated variables in  $G$  shared between nodes  $\mathbf{d}$  and  $e$ ,  $X_0$  be the result of reduction of singletons in  $\mathbf{d}$  and  $Y_0$  be the result of reduction of singletons in  $e$ . Notice that  $\mathbf{d}$  and  $e$  share no other variables except  $Z_0$ .

Let  $G'$  be obtained from  $G_0$  by replacing  $Z_0 \sqcap X_0$  in the label of  $\mathbf{d}$  with  $X$  and by replacing  $Z_0 \sqcap Y_0$  in the label of  $\mathbf{e}$  with  $Y$ , where  $X$  and  $Y$  are fresh.

Let  $S'$  be obtained from  $S_0$  by replacing the AUTs

$$\begin{aligned} Z_0 &: [\mathbb{L}_1^{Z_0}, \mathbb{U}_1^{Z_0}] \triangleq [\mathbb{L}_2^{Z_0}, \mathbb{U}_2^{Z_0}] \\ X_0 &: [\mathbb{L}_1^{X_0}, \mathbb{U}_1^{X_0}] \triangleq [\mathbb{L}_2^{X_0}, \mathbb{U}_2^{X_0}] \\ Y_0 &: [\mathbb{L}_1^{Y_0}, \mathbb{U}_1^{Y_0}] \triangleq [\mathbb{L}_2^{Y_0}, \mathbb{U}_2^{Y_0}] \end{aligned}$$

with the following two AUTs

$$\begin{aligned} X &: [\varphi_1(\mathbf{d}), (\mathbb{U}_1^{Z_0} \sqcap \mathbb{U}_1^{X_0})] \triangleq [\varphi_2(\mathbf{d}), (\mathbb{U}_2^{Z_0} \sqcap \mathbb{U}_2^{X_0})] \\ Y &: [\varphi_1(\mathbf{e}), (\mathbb{U}_1^{Z_0} \sqcap \mathbb{U}_1^{Y_0})] \triangleq [\varphi_2(\mathbf{e}), (\mathbb{U}_2^{Z_0} \sqcap \mathbb{U}_2^{Y_0})] \end{aligned}$$

We claim that the set of AUTs  $S'$  is correctly defined and compatible with  $G'$ ,  $C_1, C_2$  w.r.t.  $\mathfrak{F}$ .

Notice that since  $Z_0$  only occurs in the labels of  $\mathbf{d}$  and  $\mathbf{e}$ , and  $X_0$  and  $Y_0$  are singletons,  $\mathbb{L}_i^{Z_0} = lcs(\varphi_i(\mathbf{d}), \varphi_i(\mathbf{e})) \equiv lcs(\mathbb{L}_i^{X_0}, \mathbb{L}_i^{Y_0})$ , for  $i = 1, 2$ . Thus, for  $i = 1, 2$  we have

$$\begin{aligned} \varphi_i(\mathbf{d}) &\sqsubseteq lcs(\varphi_i(\mathbf{d}), \varphi_i(\mathbf{e})) = \mathbb{L}_i^{Z_0} \sqsubseteq \mathbb{U}_i^{Z_0}, \\ \varphi_i(\mathbf{d}) &\sqsubseteq lcs(\mathbb{L}_i^{X_0}, \mathbb{L}_i^{Y_0}) \sqsubseteq \mathbb{U}_i^{X_0}, \\ \varphi_i(\mathbf{e}) &\sqsubseteq lcs(\varphi_i(\mathbf{d}), \varphi_i(\mathbf{e})) = \mathbb{L}_i^{Z_0} \sqsubseteq \mathbb{U}_i^{Z_0}, \\ \varphi_i(\mathbf{e}) &\sqsubseteq lcs(\mathbb{L}_i^{X_0}, \mathbb{L}_i^{Y_0}) \sqsubseteq \mathbb{U}_i^{Y_0}. \end{aligned}$$

Therefore,  $\varphi_i(\mathbf{d}) \sqsubseteq \mathbb{U}_i^{Z_0} \sqcap \mathbb{U}_i^{X_0}$ ,  $\varphi_i(\mathbf{e}) \sqsubseteq \mathbb{U}_i^{Z_0} \sqcap \mathbb{U}_i^{Y_0}$ , and the AUTs are defined correctly.

For compatibility, first, note that since  $X$  and  $Y$  are singletons, we have  $\varphi_i(\mathbf{d}) = lcs(\varphi_i(\mathbf{N}_G(X)))$  and  $\varphi_i(\mathbf{e}) = lcs(\varphi_i(\mathbf{N}_G(Y)))$ .

Finally, for  $\mathbf{d}$  we have

$$\begin{aligned} \prod_{W \in \mathbf{V}_{G'}(\mathbf{d})} \mathbb{U}_i^W &= \left( \prod_{W \in (\mathbf{V}_G(\mathbf{d}) \setminus \{Z_0, X_0\}) \cup \{X\}} \mathbb{U}_i^W \right) = \\ & \left( \prod_{W \in \mathbf{V}_G(\mathbf{d}) \setminus \{Z_0, X_0\}} \mathbb{U}_i^W \right) \cap \mathbb{U}_i^X = \prod_{W \in \mathbf{V}_G(\mathbf{d})} \mathbb{U}_i^W. \end{aligned}$$

The same is true for  $\mathbf{e}$ .

Hence, all conditions of compatibility hold. By induction hypothesis there exist  $(G^*, S^*)$  and  $S''$  such that  $(G_{\mathfrak{F}}, S_{\mathfrak{F}}) \rightsquigarrow^* (G^*, S^*)$  and  $(G', S') \Rightarrow^* (G^*, S'') \gg (G^*, S^*)$ .

Let  $\tilde{X}$  be the result of reducing singletons at node  $\mathbf{d}$ ; let  $\tilde{Y}$  be the result of reducing singletons at node  $\mathbf{e}$  [these are not properly defined]

Then the set  $S''$  contains

$$\begin{aligned} \tilde{X} &: [\varphi_1(\mathbf{d}), (\mathbb{U}_1^{Z_0} \sqcap \mathbb{U}_1^{X''}) \sqcap \mathbb{U}''_1^{\tilde{X}}] \triangleq \\ & \quad [\varphi_2(\mathbf{d}), (\mathbb{U}_2^{Z_0} \sqcap \mathbb{U}_2^{X''}) \sqcap \mathbb{U}''_1^{\tilde{X}}], \\ \tilde{Y} &: [\varphi_1(\mathbf{e}), (\mathbb{U}_1^{Z_0} \sqcap \mathbb{U}_1^{Y''}) \sqcap \mathbb{U}''_1^{\tilde{Y}}] \triangleq \\ & \quad [\varphi_2(\mathbf{e}), (\mathbb{U}_2^{Z_0} \sqcap \mathbb{U}_2^{Y''}) \sqcap \mathbb{U}''_2^{\tilde{Y}}], \end{aligned}$$

where  $\mathbb{U}''_1^{\tilde{X}}$ ,  $\mathbb{U}''_1^{\tilde{X}}$ ,  $\mathbb{U}''_1^{\tilde{Y}}$  and  $\mathbb{U}''_2^{\tilde{Y}}$  are some concepts. [originate from reducing singletons] the set  $S^*$  contains

$$\begin{aligned} \tilde{X} &: [\varphi_1(\mathbf{d}), \mathbb{U}_1^{\tilde{X}}] \triangleq [\varphi_2(\mathbf{d}), \mathbb{U}_2^{\tilde{X}}], \\ \tilde{Y} &: [\varphi_1(\mathbf{e}), \mathbb{U}_1^{\tilde{Y}}] \triangleq [\varphi_2(\mathbf{e}), \mathbb{U}_2^{\tilde{Y}}], \quad \text{and} \\ & (\mathbb{U}_i^{Z_0} \sqcap \mathbb{U}_i^{X''}) \sqcap \mathbb{U}''_i^{\tilde{X}} \sqsubseteq \mathbb{U}_i^{\tilde{X}} \\ & (\mathbb{U}_i^{Z_0} \sqcap \mathbb{U}_i^{Y''}) \sqcap \mathbb{U}''_i^{\tilde{Y}} \sqsubseteq \mathbb{U}_i^{\tilde{Y}}, \quad \text{for } i = 1, 2. \end{aligned}$$

Let concepts  $\mathbb{U}_{ij}^{\tilde{X}}$  and  $\mathbb{U}_{ij}^{\tilde{Y}}$ , where  $\{i, j\} \subseteq \{1, 2\}$ , be such that  $\mathbb{U}_i^{\tilde{X}} \equiv \mathbb{U}_{i1}^{\tilde{X}} \sqcap \mathbb{U}_{i2}^{\tilde{X}}$ ,  $\mathbb{U}_i^{\tilde{Y}} \equiv \mathbb{U}_{i1}^{\tilde{Y}} \sqcap \mathbb{U}_{i2}^{\tilde{Y}}$ ,  $lcs(\varphi_i(\mathbf{d}), \varphi_i(\mathbf{e})) \sqsubseteq \mathbb{U}_{i1}^{\tilde{X}} \sqcap \mathbb{U}_{i1}^{\tilde{Y}}$ , for no conjunct  $C \neq \top$  of  $\mathbb{U}_{i2}^{\tilde{X}}$  we have  $lcs(\varphi_i(\mathbf{d}), \varphi_i(\mathbf{e})) \sqsubseteq C$  and for no conjunct  $D \neq \top$  of  $\mathbb{U}_{i2}^{\tilde{Y}}$  we have  $lcs(\varphi_i(\mathbf{d}), \varphi_i(\mathbf{e})) \sqsubseteq \mathbb{U}_{i2}^{\tilde{Y}}$ .

Then there exist a concept  $G^{**}$  and set of AUTs  $S^{**}$  such that  $(G^*, S^*) \rightsquigarrow_{(ssm)}^\sigma (G^{**}, S^{**})$ , where  $\sigma = \{\tilde{X} \mapsto Z_0 \sqcap \tilde{X}', \tilde{Y} \mapsto Z_0 \sqcap \tilde{Y}'\}$ . It is easy to see that for  $(G''', S''')$  such that  $(G, S) \Rightarrow^* (G''', S''')$  we have  $G''' = G^{**}$  and  $(G^{**}, S''') \gg (G^{**}, S^{**})$ .

- Suppose that conditions of Items 1 and 2 above do not apply. Then  $\mathbf{d} \in \mathbf{N}_G(Z)$  is such that there is a singleton  $X' \in \mathbf{V}_G(\mathbf{d})$  and either  $|\mathbf{N}_G(Z)| > 2$  or  $|\mathbf{N}_G(Z)| = 2$  and  $\mathbf{e}$ , the other node where  $X$  occurs, does not contain any singletons.

Let  $G'$  be obtained from  $G$  by replacing  $Z \sqcap X'$  in  $\mathbf{d}$  with  $X$  and by replacing  $Z$  in every other node  $\mathbf{e} \neq \mathbf{d}$  of  $G$  with  $Y$ .

Let  $S'$  be obtained from  $S$  by replacing the AUTs

$$\begin{aligned} Z &: [\mathbb{L}_1^Z, \mathbb{U}_1^Z] \triangleq [\mathbb{L}_2^Z, \mathbb{U}_2^Z] \\ X' &: [\mathbb{L}_1^{X'}, \mathbb{U}_1^{X'}] \triangleq [\mathbb{L}_2^{X'}, \mathbb{U}_2^{X'}] \end{aligned}$$

with the following two AUTs

$$\begin{aligned} X &: [\varphi_1(\mathbf{d}), (\mathbb{U}_1^Z \sqcap \mathbb{U}_1^{X'})] \triangleq [\varphi_2(\mathbf{d}), (\mathbb{U}_2^Z \sqcap \mathbb{U}_2^{X'})] \\ Y &: [\mathbb{L}_1, \mathbb{U}_1^Z] \triangleq [\mathbb{L}_2, \mathbb{U}_2^Z] \end{aligned}$$

where  $\mathbb{L}_i = lcs(\varphi_i(\mathbf{N}_G(Z) \setminus \{\mathbf{d}\}))$ , for  $i = 1, 2$ . By combining the reasons outlined above,  $S'$  is compatible with  $G'$  w.r.t.  $\mathfrak{F}$ .

Notice that  $Y$  either is not a singleton in  $G'$  or  $\mathbf{e}$  does not contain any singletons. Therefore,  $Y$  is not reduced into any other variable. Let  $\tilde{X}$  be the result of reducing singletons at node  $\mathbf{d}$ .

Then the set  $S''$  contains

$$\begin{aligned} \tilde{X} &: [\varphi_1(\mathbf{d}), (\mathbb{U}_1^Z \sqcap \mathbb{U}_1^{X''}) \sqcap \mathbb{U}''_1^{\tilde{X}}] \triangleq \\ & \quad [\varphi_2(\mathbf{d}), (\mathbb{U}_2^Z \sqcap \mathbb{U}_2^{X''}) \sqcap \mathbb{U}''_1^{\tilde{X}}], \\ Y &: [\mathbb{L}_1, \mathbb{U}_1^Z] \triangleq [\mathbb{L}_2, \mathbb{U}_2^Z] \end{aligned}$$

where  $\mathbb{U}''_1^{\tilde{X}}$ ,  $\mathbb{U}''_1^{\tilde{X}}$ ,  $\mathbb{U}''_1^{\tilde{Y}}$  and  $\mathbb{U}''_2^{\tilde{Y}}$  are some concepts. The set  $S^*$  contains

$$\begin{aligned} \tilde{X} &: [\varphi_1(\mathbf{d}), \mathbb{U}_1^{\tilde{X}}] \triangleq [\varphi_2(\mathbf{d}), \mathbb{U}_2^{\tilde{X}}], \\ Y &: [\mathbb{L}_1, \tilde{\mathbb{U}}_1^Z] \triangleq [\mathbb{L}_2, \tilde{\mathbb{U}}_2^Z] \quad \text{and} \\ & (\mathbb{U}_i^Z \sqcap \mathbb{U}_i^{X''}) \sqcap \mathbb{U}''_i^{\tilde{X}} \sqsubseteq \mathbb{U}_i^{\tilde{X}} \\ & \quad \mathbb{U}_i^Z \sqsubseteq \tilde{\mathbb{U}}_i^Z, \quad \text{for } i = 1, 2. \end{aligned}$$

□

## Definitions of Concepts Used in the Experimental Evaluation Section

AdenomaOfColon  $\equiv$

BodyStructure  $\sqcap \exists$ hasUniqueAssociatedProcess.( BenignNeoplasticProcess  $\sqcap$   
 $\exists$ actsSpecificallyOn. (Adenocyte  $\sqcap \exists$ isSpecificStructuralComponentOf. Colon))

AdrenalPhaeochromocytoma  $\equiv$

BodyStructure  $\sqcap \exists$ hasUniqueAssociatedProcess.( BenignNeoplasticProcess  $\sqcap$   
 $\exists$ actsSpecificallyOn. (ChromaffinCell  $\sqcap \exists$ isSpecificStructuralComponentOf. AdrenalMedulla))

Anonymous-257  $\equiv$

GenericBodyStructure  $\sqcap \exists$ IsDivisionOf. (GenericBodyStructure  $\sqcap \exists$ isPairedOrUnpaired. atLeastPaired)

Anonymous-529  $\equiv$

BodyPart  $\sqcap \exists$ IsDivisionOf. (BodyPart  $\sqcap$  isPairedOrUnpaired. atLeastPaired)

Anonymous-158  $\equiv$

absence  $\sqcap \exists$ isExistenceOf. (GeneralisedSubstance  $\sqcap \exists$ isConsequenceOf. (GeneralisedProcess  $\sqcap$   
 $\exists$ LocativeAttribute. OrganicSolidStructure  $\sqcap \exists$ hasIntrinsicAbnormalityStatus. normal)  $\sqcap$   
 $\exists$ hasIntrinsicAbnormalityStatus. normal),

Anonymous-340  $\equiv$

absence  $\sqcap \exists$ isExistenceOf.(GeneralisedProcess  $\sqcap \exists$ isConsequenceOf.(GeneralisedProcess  $\sqcap$   
 $\exists$ LocativeAttribute.OrganicSolidStructure  $\sqcap \exists$ hasAbnormalityStatus. nonNormal)),