

# Statistical Investigation of First-Order Algebraic ODEs and their Rational General Solutions

Georg Grasegger\*      N. Thieu Vo\*      Franz Winkler

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In this report we use a previously published algorithm for finding rational general solutions of algebraic ODEs to do some statistical investigation of AODEs from famous collections. For all first-order AODEs in these lists we give either a strong rational general solution or the decision that such a solution cannot exist.

## 1 Introduction

In [3] we gave an algorithm for deciding the existence of a rational general solution of strongly parametrization first-order AODEs (SP1AODEs). This report shall give, on the one hand a list of strongly parametrizable AODEs and their solutions and on the other hand a statistical investigation on the relative number of such AODEs in well known collections such as Kamke [1] and Polyanin and Sajzew [2]. For detailed explanations on the algorithm in use, we refer to [3]. To get some insight we also give some intermediate computations of the algorithm here. The given first-order AODE is considered as an algebraic curve over  $\mathbb{K}(x)$ . We need a strong rational parametrization of the curve. Then we compute an associated ODE, which, in order to allow a rational solutions, has to be of a specific form. A rational general solution of the associated ODE leads to a rational general solution of the original AODE.

In the following we write `NORATGENSOL`, if an AODE does not have a rational general solution. If it just does not have a strong rational general solution we write `NOSTRONGRATGENSOL`.

## 2 First-Order AODEs from Kamke [1]

In this section we give rational general solutions of strongly parametrizable AODEs from Kamke [1]. We also list those AODEs which do not have a rational general solution. Table 1 shows the absolute and relative numbers of certain classes of ODEs in Kamke's collection. Figure 1 shows these classes on a more illustrative basis.

Type	# of ODEs in Kamke	%	%
first-order	576	100	
first-order AODE	369	64.41	100
SP1AODE	329	57.11	89.61

Table 1: Number of AODEs in Kamke

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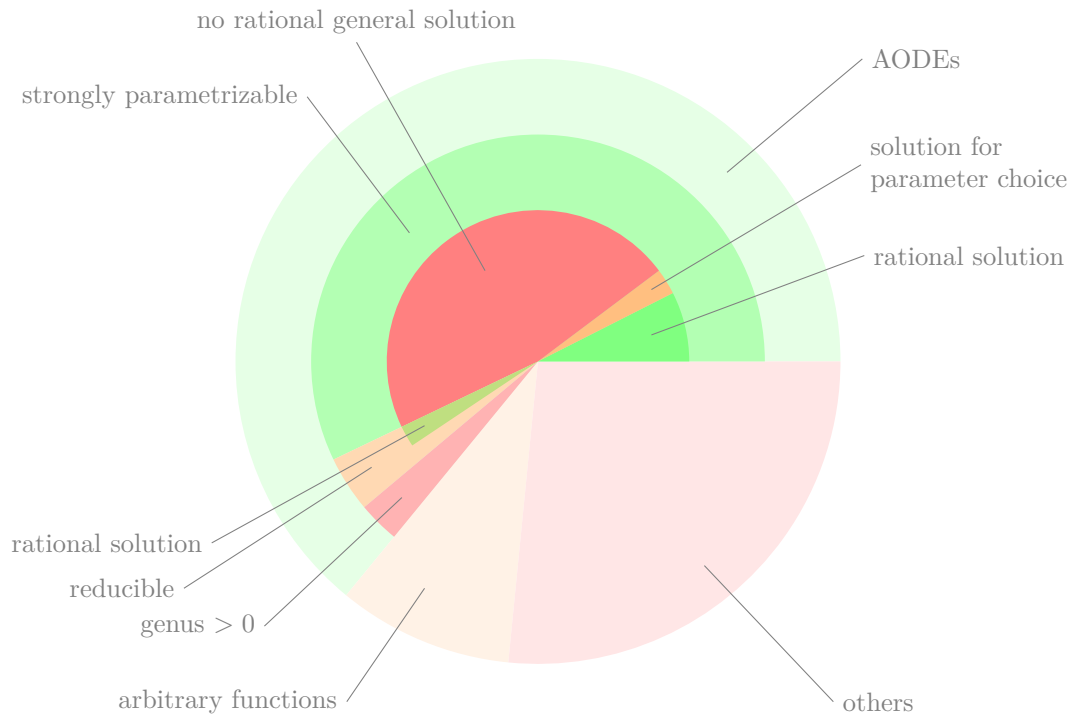


Figure 1: first-order ODEs in Kamke [1]

Note, that around one third of Kamke's collection is not in the class of AODEs. Some of those ODEs depend on arbitrary functions. For certain choices of these functions these ODEs are however algebraic and then of course the algorithm of [3] is applicable.

Kamke is not very precise on the domains of occurring parameters. As long as possible, we chose them to be in a suitable domain to get an AODE. Parameters which appear as coefficients, are usually considered to be in generic case, i. e. there might be exceptional cases where the AODE is reduced to another one or where the given parametrization is not defined. Such exceptional cases are not treated in this paper. Furthermore, when we say there is no rational general solution, then we mean that there is no parametric rational general solution. For certain specific choices of parameters there might be still a rational general solution.

Important for our considerations are those parameters which appear in the exponents. Obviously the exponents of  $y$  and  $y'$  have to be non-negative integers, otherwise the ODE would not be algebraic. We assume furthermore, that these integers are in fact positive, otherwise the ODE might be reduced to a simpler one, or even to an algebraic equation. Let  $F$  define an AODE. In general we require  $F$  to be polynomial in  $x$  as well. We might weaken this requirement by assuming that all exponents of  $x$  are at least integers. In such a case the ODE can be easily transformed into an AODE. Hence, throughout this report, we assume that all parameters appearing in the exponents of  $x$  are integers.

The following tables include all AODEs from Kamke's collection. Each subsection of Kamke is given in a separate table (Table 2 to 5 for AODEs where  $y'$  has degree 1 to 4 (or higher) respectively). Table 6 is a collection of factors of reducible AODEs in Kamke.

Table 2: First-order AODEs of degree one in  $y'$ .

ID	AODE	Parametrization	Solution of AODE
1.12	$y^2 + y' - 1$	$(t, 1 - t^2)$	NoRatGenSol
1.13	$y^2 - b - ax + y'$	$(t, -t^2 + b + ax)$	NoRatGenSol
1.14	$ax^m + y^2 + y'$	$(t, -ax^m - t^2)$	$\begin{cases} -\frac{1}{2} \frac{cx^{-n}(n-1) + n + 1}{x(cx^{-n} - 1)}, & \text{if } m = -2, a = \frac{1-n^2}{4}, n \in \mathbb{Z} \setminus \{0\} \\ \frac{1}{x+c}, & \text{if } a = 0 \\ \text{NoRatGenSol}, & \text{otherwise} \end{cases}$
1.15	$x^4 - 2yx^2 - 2x + y^2 + y' - 1$	$(t, -x^4 + 2tx^2 + 2x - t^2 + 1)$	NoRatGenSol
1.17	$-y^2 - 3y + y' + 4$	$(t, t^2 + 3t - 4)$	NoRatGenSol
1.18	$-y^2 - x(y+1) + y' + 1$	$(t, (t+1)(t+x-1))$	NoRatGenSol
1.19	$y' - (x+y)^2$	$(t, (t+x)^2)$	NoRatGenSol
1.20	$yx^2 - 2x - y^2 + y + y'$	$(t, t^2 - (x^2 + 1)t + 2x)$	NoRatGenSol
1.23	$ay^2 - b + y'$	$(t, b - at^2)$	NoRatGenSol
1.24	$-bx^\alpha + ay^2 + y'$	$(t, bx^\alpha - at^2)$	$\begin{cases} \frac{1}{2} \frac{cx^n(n+1) + n - 1}{ax(cx^n - 1)}, & \text{if } \alpha = -2, ab = -\frac{1-n^2}{4}, n \in \mathbb{Z} \setminus \{0\} \\ \text{NoRatGenSol}, & \text{otherwise} \end{cases}$
1.25	$-cx^{\alpha-1} - bx^{2\alpha} + ay^2 + y'$	$(t, cx^{\alpha-1} + bx^{2\alpha} - at^2)$	$\begin{cases} -\frac{1}{2} \frac{c_1x^n(n+1) + n - 1}{x(c_1x^n - 1)}, & \text{if } \alpha = -1, a(b+c) = \frac{n^2-1}{4}, n \in \mathbb{Z} \setminus \{0\} \\ \text{NoRatGenSol}, & \text{otherwise} \end{cases}$
1.26	$(By - b)(a - Ay) + y'$	$(t, (At - a)(Bt - b))$	$\begin{cases} -\frac{1}{AB(x-c)} + \frac{1}{2} \frac{Ab + aB}{AB}, & \text{if } Ab - aB = 0 \\ \text{NoRatGenSol}, & \text{otherwise} \end{cases}$
1.27	$ay(y-x) + y' - 1$	$(t, at(x-t) + 1)$	NoRatGenSol
1.28	$(y^2 - 2)x - x^3y + y'$	$(t, x(-t^2 + x^2t + 2))$	NoRatGenSol
1.29	$y' - xy(y+3)$	$(t, t(t+3)x)$	NoRatGenSol
1.30	$y^2x^{-a-1} - x^a + y'$	$(t, x^a - t^2x^{-a-1})$	NoRatGenSol
1.31	$y' - ax^n(y^2 + 1)$	$(t, a(t^2 + 1)x^n)$	$\begin{cases} -\frac{1}{2} \frac{cx^k(k+1) + k - 1}{a(cx^k - 1)} + \frac{1}{2a}, & \text{if } n = -1, a^2 = -\frac{k^2}{4}, k \in \mathbb{Z} \setminus \{0\} \\ \text{NoRatGenSol}, & \text{otherwise} \end{cases}$
1.36	$y^3 + axy^2 + y'$	$(t, -t^2(t+ax))$	NoRatGenSol
1.39	$-a_3y^3 - a_2y^2 - a_1y - a_0 + y'$	$(t, a_0 + t(a_1 + t(a_2 + a_3t)))$	NoRatGenSol
1.40	$3a(2x+y)y^2 + y'$	$(t, -3at^2(t+2x))$	NoRatGenSol
1.41	$axy^3 + by^2 + y'$	$(t, -t^2(b+atx))$	NoRatGenSol
1.42	$-x(x+2)y^3 - (x+3)y^2 + y'$	$(t, t^2(tx^2 + 2tx + x + 3))$	NoRatGenSol
1.43	$(b + ax(4a + 3x))y^3 + 3xy^2 + y'$	$(t, -t^2(bt + x(4ta^2 + 3txa + 3)))$	NoRatGenSol
1.44	$2ax^3y^3 + 2xy + y'$	$(t, -2(at^3x^3 + tx))$	NoRatGenSol
1.45	$2(a^2x^3 - b^2x)y^3 + 3by^2 + y'$	$(t, t^2(-2a^2tx^3 + 2b^2tx - 3b))$	NoRatGenSol
1.46	$2x^{-a-1} - x^{-2a} - yx^{-a} - y^3x^a + 3y^2 + y'$	$(t, tx^{-a} + t^3x^a + (x - 2x^a)x^{-2a-1} - 3t^2)$	NoRatGenSol
1.47	$-a(x^n - x)y^3 - y^2 + y'$	$(t, a(x^n - x)t^3 + t^2)$	$\begin{cases} -\frac{1}{x-c}, & \text{if } a = 0 \text{ or } n = 1 \\ \text{NoRatGenSol}, & \text{otherwise} \end{cases}$
1.48	$-x(b+ax)y^3 - cy^2 + y'$	$(t, t^2(c + tx(b+ax)))$	NoRatGenSol

Table 2: First-order AODEs of degree one in  $y'$  (continued).

ID	AODE	Parametrization	Solution of AODE
1.94	$bx^n + y'x + ay$	$\left(t, -\frac{bx^n + at}{x}\right)$	$\begin{cases} cx^{-a} - \frac{bx^n}{a+n}, & \text{if } a \in \mathbb{Z} \setminus \{-n\} \\ cx^n, & \text{if } a = -n \in \mathbb{Z}, b = 0 \\ \text{NoRatGenSol}, & \text{otherwise} \end{cases}$
1.95	$x^2 + y'x + y^2$	$\left(t, -\frac{t^2 + x^2}{x}\right)$	NoRatGenSol
1.96	$-y^2 + xy' + 1$	$\left(t, \frac{t^2 - 1}{x}\right)$	$\frac{x^2 + c}{c - x^2}$
1.97	$bx^2 + y'x + ay^2 - y$	$\left(t, \frac{-at^2 + t - bx^2}{x}\right)$	NoRatGenSol
1.98	$cx^{2b} + y'x + ay^2 - by$	$\left(t, -\frac{cx^{2b} - at^2 + bt}{x}\right)$	$\begin{cases} \frac{n(x^n - c)}{2a(c + x^n)}, & \text{if } b = 0, 4ac + 1 = \frac{1 - n^2}{4}, n \in \mathbb{Z} \setminus \{0\} \\ \text{NoRatGenSol}, & \text{otherwise} \\ \frac{1}{2} \frac{c_1 x^n (n+1) + n - 1}{a(c_1 x^n - 1)} + \frac{b-1}{2a}, & \text{if } b = 0, 4ac + b^2 = n^2, n \in \mathbb{Z} \setminus \{0\} \\ \text{NoRatGenSol}, & \text{otherwise} \end{cases}$
1.99	$-cx^\beta + y'x + ay^2 - by$	$\left(t, \frac{cx^\beta - at^2 + bt}{x}\right)$	$\begin{cases} \frac{1}{2} \frac{c_1 x^n (n+1) + n - 1}{a(c_1 x^n - 1)} + \frac{b-1}{2a}, & \text{if } b = 0, 4ac + b^2 = n^2, n \in \mathbb{Z} \setminus \{0\} \\ \text{NoRatGenSol}, & \text{otherwise} \end{cases}$
1.100	$a + x(y^2 + y')$	$\left(t, -\frac{xt^2 + a}{x}\right)$	NoRatGenSol
1.101	$xy^2 - y + xy'$	$\left(t, t\left(\frac{1}{x} - t\right)\right)$	$\frac{2x}{x^2 - c}$
1.102	$-ax^3 + y^2x + y'x - y$	$\left(t, -t^2 + \frac{t}{x} + ax^2\right)$	NoRatGenSol
1.103	$-x^3 - 2yx^2 + (y' - y^2)x - y$	$\left(t, t^2 + \left(2x + \frac{1}{x}\right)t + x^2\right)$	$\frac{x(x^2 - c + 2)}{c - x^2}$
1.104	$axy^2 + 2y + bx + xy'$	$\left(t, -\frac{axt^2 + 2t + bx}{x}\right)$	NoRatGenSol
1.105	$axy^2 + by + d + cx + xy'$	$\left(t, -\frac{axt^2 + bt + d + cx}{x}\right)$	NoRatGenSol
1.107	$ay^2x^\alpha - cx^\beta + y'x + by$	$\left(t, -\frac{at^2x^\alpha - cx^\beta + bt}{x}\right)$	$\begin{cases} \frac{c_1 x^n (n+1) + n - 1}{2ax^\alpha(c_1 x^n - 1)} + \frac{\alpha - \beta - 1}{2ax^\alpha}, & \text{if } \alpha = -\beta, ac = \frac{n^2 - (\alpha - \beta)^2}{4}, n \in \mathbb{Z} \setminus \{0\} \\ \text{NoRatGenSol}, & \text{otherwise} \end{cases}$
1.111	$y^3 + x(3y^2 + y')$	$\left(t, -\frac{t^2(t+3x)}{x}\right)$	NoRatGenSol
1.129	$y^2 + y' + x(y' - y)$	$\left(t, \frac{t(x-t)}{x+1}\right)$	NoRatGenSol
1.130	$-2x^3 + 2y'x + y$	$\left(t, x^2 - \frac{t}{2x}\right)$	NoRatGenSol
1.133	$y'x^2 - x + y$	$\left(t, \frac{x-t}{x^2}\right)$	NoRatGenSol
1.135	$y'x^2 - yx + y$	$\left(t, \frac{t(x-1)}{x^2}\right)$	NoRatGenSol
1.136	$(y' + 1)x^2 + yx + y^2$	$\left(t, -\frac{t^2 + xt + x^2}{x^2}\right)$	NoRatGenSol
1.137	$y'x^2 - y^2 - xy$	$\left(t, \frac{t(t+x)}{x^2}\right)$	NoRatGenSol
1.138	$(y' - 1)x^2 - y^2 - xy$	$\left(t, \frac{t^2 + xt + x^2}{x^2}\right)$	NoRatGenSol
1.139	$ax^k + (y^2 + y')x^2 - b^2 + b$	$\left(t, -\frac{ax^k + t^2x^2 - b^2 + b}{x^2}\right)$	$\begin{cases} \frac{1}{2} \frac{c_1 x^n (n+1) + n - 1}{ax(c_1 x^n - 1)}, & \text{if } k = 0, a = b^2 - b - \frac{1 - n^2}{4}, n \in \mathbb{Z} \setminus \{0\} \\ \text{NoRatGenSol}, & \text{otherwise} \end{cases}$
1.140	$(y^2 + y')x^2 + 4yx + 2$	$\left(t, -\frac{t^2x^2 + 4tx + 2}{x^2}\right)$	$\frac{x-2c}{(c-x)x}$
1.141	$b + x(y^2 + ay + y')$	$\left(t, -\frac{b+t(a+t)x}{x}\right)$	NoRatGenSol
1.142	$(y' - y^2)x^2 + a(x - x^2y) + 2$	$\left(t, t^2 + a\left(t - \frac{1}{x}\right) - \frac{2}{x^2}\right)$	NoRatGenSol
1.143	$x^2(ay^2 + y') - b$	$\left(t, \frac{b}{x^2} - at^2\right)$	$\begin{cases} \frac{1}{2} \frac{c_1 x^n (n+1) + n - 1}{ax(c_1 x^n - 1)}, & \text{if } ab = \frac{n^2 - 1}{4}, n \in \mathbb{Z} \setminus \{0\} \\ \text{NoRatGenSol}, & \text{otherwise} \\ \frac{1}{2} \frac{c_1 x^n (n+1) + n - 1}{ax(c_1 x^n - 1)}, & \text{if } \alpha = 0, a(b+c) = 1 - n^2, n \in \mathbb{Z} \setminus \{0\} \\ \text{NoRatGenSol}, & \text{otherwise} \end{cases}$
1.144	$bx^\alpha + (ay^2 + y')x^2 + c$	$\left(t, -\frac{bx^\alpha + at^2x^2 + c}{x^2}\right)$	$\begin{cases} \frac{1}{2} \frac{c_1 x^n (n+1) + n - 1}{ax(c_1 x^n - 1)}, & \text{if } \alpha = 0, a(b+c) = 1 - n^2, n \in \mathbb{Z} \setminus \{0\} \\ \text{NoRatGenSol}, & \text{otherwise} \end{cases}$

Table 2: First-order AODEs of degree one in  $y'$  (continued).

ID	AODE	Parametrization	Solution of AODE
1.145	$y'x^2 + ay^2(y - x^2)$	$\left(t, \frac{at^2(x^2 - t)}{x^2}\right)$	NoRatGenSol
1.146	$ay^2 + x(y^3 + xy')$	$\left(t, -\frac{t^2(a+tx)}{x^2}\right)$	NoRatGenSol
1.147	$(ay^3 + y')x^2 + by^2$	$\left(t, -\frac{t^2(atx^2 + b)}{x^2}\right)$	NoRatGenSol
1.148	$y'x^2 + yx + y' - 1$	$\left(t, \frac{1-tx}{x^2+1}\right)$	NoRatGenSol
1.149	$-x^3 + y'x^2 + (y-1)x + y'$	$\left(t, \frac{x^3 - tx + x}{x^2+1}\right)$	NoRatGenSol
1.150	$(y' - 2)x^2 + 2yx + y'$	$\left(t, \frac{2x(x-t)}{x^2+1}\right)$	$\frac{2x^3+3c}{3x^2+3}$
1.151	$(2xy - 1)(y^2 + 1) + (x^2 + 1)y'$	$\left(t, -\frac{(t^2+1)(2tx-1)}{x^2+1}\right)$	NoRatGenSol
1.153	$a - xy + (x^2 - 1)y'$	$\left(t, \frac{tx-a}{x^2-1}\right)$	NoRatGenSol
1.155	$y^2 - 2xy + (x^2 - 1)y' + 1$	$\left(t, \frac{t^2-2xt+1}{1-x^2}\right)$	NoRatGenSol
1.156	$y'x^2 + yx - y^2 - y'$	$\left(t, \frac{t(t-x)}{x^2-1}\right)$	NoRatGenSol
1.157	$a(y^2 - 2xy + 1) + (x^2 - 1)y'$	$\left(t, -\frac{a(t^2-2xt+1)}{x^2-1}\right)$	NoRatGenSol
1.158	$y'x^2 + y(ay + 1)x - y'$	$\left(t, \frac{t(at+1)x}{x^2-1}\right)$	NoRatGenSol
1.160	$(x+2)y^2 - 4y + (x^2 - 4)y'$	$\left(t, \frac{t(t(x+2)-4)}{x^2-4}\right)$	NoRatGenSol
1.161	$x^2 + 3yx - 8y + (x^2 - 5x + 6)y'$	$\left(t, \frac{-x^2-3tx+8t}{x^2-5x+6}\right)$	$\frac{(8-3x)x^3+12c}{12(x-3)(x-2)^2}$
1.162	$y^2 + k(-a+x+y)(-b+x+y) + (x-a)(x-b)y'$	$\left(t, \frac{kt^2+t^2+2kxt+kx^2+ak(b-t-x)-bk(t+x)}{(a-x)(x-b)}\right)$	NoRatGenSol
1.163	$2xa^2 - 2y^2 - xy + 2x^2y'$	$\left(t, \frac{-2xa^2+2t^2+tx}{2x^2}\right)$	NoRatGenSol
1.164	$2xa^2 - 2y^2 - 3xy + 2x^2y'$	$\left(t, \frac{-2xa^2+2t^2+3tx}{2x^2}\right)$	NoRatGenSol
1.165	$2y'x^2 - (4y + y' - 4)x + (y - 1)y$	$\left(t, -\frac{(t-1)(t-4x)}{x(2x-1)}\right)$	$\frac{c-2x^2}{c-x}$
1.166	$(x-1)y^2 - x + 2(x-1)xy'$	$\left(t, \frac{x-t^2(x-1)}{2(x-1)x}\right)$	NoRatGenSol
1.167	$(3y' - 1)x^2 - 3yx - 7y^2$	$\left(t, \frac{7t^2+3xt+x^2}{3x^2}\right)$	NoRatGenSol
1.168	$y^2 - xy + 3(x^2 - 4)y' - 3$	$\left(t, \frac{-t^2+xt+3}{3(x^2-4)}\right)$	NoRatGenSol
1.169	$cy^2 + (b+ax)(y^3 + (b+ax)y')$	$\left(t, -\frac{t^2(c+t(b+ax))}{(b+ax)^2}\right)$	NoRatGenSol
1.170	$-x^4 + y'x^3 - y^2$	$\left(t, \frac{t^2}{x^3} + x\right)$	NoRatGenSol
1.171	$y'x^3 - y^2 - x^2y$	$\left(t, \frac{t(x^2+t)}{x^3}\right)$	$\frac{cx^2}{c-x}$
1.172	$-y^2x^4 + y'x^3 + yx^2 + 20$	$\left(t, xt^2 - \frac{t}{x} - \frac{20}{x^3}\right)$	$\frac{5x^9+4c}{x^2(c-x^9)}$
1.173	$-y^2x^6 + (y' - 2y)x^3 + 3yx^2 + 3$	$\left(t, t^2x^3 + t\left(2 - \frac{3}{x}\right) - \frac{3}{x^3}\right)$	NoRatGenSol
1.174	$y'x^3 + yx^2 + y'x - 2$	$\left(t, \frac{2-tx^2}{x^3+x}\right)$	NoRatGenSol
1.175	$ax^3 - 2yx^2 + (x^2 - 1)y'x + y$	$\left(t, \frac{ax^3-2tx^2+t}{x-x^3}\right)$	NoRatGenSol
1.176	$-x^2 + (x^2 - 1)y'x + (x^2 - 1)y^2$	$\left(t, -\frac{-x^2t^2+t^2+x^2}{x-x^3}\right)$	NoRatGenSol

Table 2: First-order AODEs of degree one in  $y'$  (continued).

ID	AODE	Parametrization	Solution of AODE
1.177	$y'x^3 - (y + y')x^2 + 2yx - y^2$	$\left(t, \frac{t(t+(x-2)x)}{(x-1)x^2}\right)$	$\frac{(c-1)x^2}{c-x}$
1.178	$x^2 + 2(x^2 - 1)y'x + 2(x^2 - 1)y^2 + (5 - 3x^2)y - 3$	$\left(t, -\frac{(t-1)(-x^2+2t(x^2-1)+3)}{2x(x^2-1)}\right)$	NoRatGenSol
1.179	$xy^2 - (x^2 + 1)y - 3x + 3x(x^2 - 1)y'$	$\left(t, -\frac{-xt^2+x^2t+t+3x}{3x-3x^3}\right)$	NoRatGenSol
1.180	$x^2 - y^2 + (c + x(b + ax))(xy' - y)$	$\left(t, \frac{t^2+ct+x(b+ax)t-x^2}{x(c+x(b+ax))}\right)$	NoRatGenSol
1.181	$(y^2 + y')x^4 + a$	$\left(t, -t^2 - \frac{a}{x^4}\right)$	NoRatGenSol
1.182	$y'x^4 + x^2 - (2y^2 + y')x + y$	$\left(t, -\frac{-2xt^2+t+x^2}{x-x^4}\right)$	$\frac{x(cx-1)}{c-x^2}$
1.183	$x(2x^3 - 1)y' - 2(x^3 - 1)y$	$\left(t, -\frac{2t(x^3-1)}{x-2x^4}\right)$	NoRatGenSol
1.184	$(y^2 + y')(c + x(b + ax))^2 + A$	$\left(t, -\frac{t^2(c+x(b+ax))^2+A}{(c+x(b+ax))^2}\right)$	NoRatGenSol
1.185	$y'x^7 + 5y^2x^3 + 2(x^2 + 1)y^3$	$\left(t, -\frac{t^2(5x^3+2t(x^2+1))}{x^7}\right)$	NoRatGenSol
1.186	$-(n-1)yx^{n-1} + y'x^n + x^{2n-2} + y^2$	$\left(t, -x^{-n}(- (n-1)tx^{n-1} + x^{2n-2} + t^2)\right)$	NoRatGenSol
1.187	$y'x^n - bx^{2n-2} - ay^2$	$\left(t, bx^{n-1} + at^2x^{-n}\right)$	NoRatGenSol
1.188	$-bx^9 + y'x^7 - ay^3$	$\left(t, \frac{at^3}{x^7} + bx^2\right)$	NoRatGenSol
1.189	$-ay^n - bx^{(m+1)n} + x^{m(n-1)+n}y'$	$\left(t, x^{-nm+m-n}(at^n + bx^{(m+1)n})\right)$	$\begin{cases} \frac{x^{m+1}(a+b)}{m+1} + c, & \text{if } n=0, m \in \mathbb{Z} \setminus \{-1\} \\ c, & \text{if } n=0, m=-1, a=-b \\ -\frac{x^{m+1}b}{a-1-m} + cx^a, & \text{if } n=1, a \in \mathbb{Z}, m \neq a-1 \\ cx^a, & \text{if } n=1, b=0 \\ \frac{(m+2)x^{m+1}}{2a} - \frac{1}{2} \frac{cx^k(k+1)+k-1}{ax^{-m-1}(cx^k-1)}, & \text{if } n=2, ab = \frac{1-k^2+m^2+2m}{4}, k, m \in \mathbb{Z}, k! = 0 \\ -\frac{(m+1)x^{m+1}}{cmx^{m+1} + cx^{m+1} - a}, & \text{if } n=2, b=0 \\ \text{NoRatGenSol,} & \text{otherwise} \end{cases}$
1.203	$x^3 + y + yy'$	$\left(t, -\frac{x^3+t}{t}\right)$	NoRatGenSol
1.204	$x + y(a + y')$	$\left(t, -\frac{at+x}{t}\right)$	NoRatGenSol
1.207	$y^2 + y'y + 4x(x + 1)$	$\left(t, -\frac{t^2+4x(x+1)}{t}\right)$	NoRatGenSol
1.210	$x(y^2 - 4) + yy'$	$\left(t, \frac{4x}{t} - tx\right)$	NoRatGenSol
1.213	$-x + y(y' - 1) + y'$	$\left(t, \frac{t+x}{t+1}\right)$	NoRatGenSol
1.214	$2x - y + (x + y - 1)y' + 3$	$\left(t, \frac{t-2x-3}{t+x-1}\right)$	NoRatGenSol
1.215	$x - y + (2x + y - 2)y' + 1$	$\left(t, \frac{t-x-1}{t+2x-2}\right)$	NoRatGenSol
1.216	$-2y'x + x + y + yy' + y'$	$\left(t, -\frac{t+x}{t-2x+1}\right)$	NoRatGenSol
1.217	$-y'x^2 - x + yy'$	$\left(t, -\frac{x}{t-x^2}\right)$	NoRatGenSol
1.218	$-y'x^2 + 4yx + yy'$	$\left(t, \frac{4tx}{x^2-t}\right)$	NoRatGenSol
1.220	$-x^3 - y^2x + 2yy'$	$\left(t, \frac{x(t^2+x^2)}{2t}\right)$	NoRatGenSol

Table 2: First-order AODEs of degree one in  $y'$  (continued).

ID	AODE	Parametrization	Solution of AODE
1.221	$x(y' - 1) + 2y(y' - 1) + y' + 1$	$\left(t, \frac{2t+x-1}{2t+x+1}\right)$	NoRatGenSol
1.222	$2x - y + (x + 2y + 7)y' + 4$	$\left(t, \frac{t-2(x+2)}{-2t+x+7}\right)$	NoRatGenSol
1.223	$y(2y' - 1) - x(y' + 2)$	$\left(t, \frac{t+2x}{2t-x}\right)$	NoRatGenSol
1.224	$y(2y' - 1) + x(3 - 6y') + 2$	$\left(t, \frac{t-3x-2}{2(t-3x)}\right)$	NoRatGenSol
1.225	$-x - 2y + (2x + 4y + 3)y' - 1$	$\left(t, \frac{2t+x+1}{4t+2x+3}\right)$	NoRatGenSol
1.226	$-x + 2y + (-2x + 4y - 3)y' - 1$	$\left(t, \frac{-2t+x+1}{4t-2x-3}\right)$	NoRatGenSol
1.227	$7x - 3y + (-3x + 4y - 5)y' + 2$	$\left(t, \frac{-3t+7x+2}{-4t+3x+5}\right)$	NoRatGenSol
1.228	$-8x - 25y + (11x + 4y - 11)y' + 62$	$\left(t, \frac{25t+8x-62}{4t+11(x-1)}\right)$	NoRatGenSol
1.229	$2x - 5y + (-5x + 12y - 8)y' + 3$	$\left(t, \frac{-5t+2x+3}{-12t+5x+8}\right)$	NoRatGenSol
1.231	$(c + bx + ay)y' + y\alpha + x\beta + \gamma$	$\left(t, -\frac{t\alpha+x\beta+\gamma}{c+at+bx}\right)$	NoRatGenSol
1.232	$x^2 + yy'x + y^2$	$\left(t, -\frac{t}{x} - \frac{y}{t}\right)$	NoRatGenSol
1.234	$x^3 - 2x^2 + y(y' + 1)x - y^2$	$\left(t, \frac{t}{x} - 1 - \frac{(x-2)x}{t}\right)$	NoRatGenSol
1.235	$by + (a + xy)y'$	$\left(t, -\frac{bt}{a+tx}\right)$	NoRatGenSol
1.236	$x((y + 4)y' - 2) - y(y + 2)$	$\left(t, \frac{t^2+2t+2x}{tx+4x}\right)$	NoRatGenSol
1.237	$cx + (a + y)y'x + by$	$\left(t, -\frac{bt+cx}{ax+tx}\right)$	NoRatGenSol
1.238	$-b - y(x + y) + (a + x(x + y))y'$	$\left(t, \frac{b+t(t+x)}{a+x(t+x)}\right)$	NoRatGenSol
1.239	$-(y' + 2)x^2 + y(y' - 3)x + y^2$	$\left(t, \frac{t^2-3xt-2x^2}{x(x-t)}\right)$	NoRatGenSol
1.240	$ax - y(y - 2xy')$	$\left(t, \frac{t}{2x} - \frac{a}{2t}\right)$	NoRatGenSol
1.241	$ax^2 - y(y - 2xy')$	$\left(t, \frac{t}{2x} - \frac{ax}{2t}\right)$	NoRatGenSol
1.242	$2y^2 + 2xy'y + 1$	$\left(t, -\frac{2t^2+1}{2tx}\right)$	NoRatGenSol
1.243	$x(x + 2y - 1)y' - y(2x + y + 1)$	$\left(t, \frac{t(t+2x+1)}{x(2t+x-1)}\right)$	NoRatGenSol
1.244	$(2x - y - 1)y - x(x - 2y + 1)y'$	$\left(t, -\frac{t(t-2x+1)}{x(-2t+x+1)}\right)$	NoRatGenSol
1.245	$4y^3x^3 + 12y^2x^2 + 2yy'x + y^2$	$\left(t, -\frac{t^2+12x^2t}{4x^3+2tx}\right)$	NoRatGenSol
1.246	$3(x + y)^2 + x(2x + 3y)y'$	$\left(t, -\frac{3(t+x)^2}{x(3t+2x)}\right)$	NoRatGenSol
1.247	$-7x^2 + yx - 9x - y^2 + (3x + 2)(-2x + y - 1)y' - 3$	$\left(t, \frac{t^2-xt+7x^2+9x+3}{(t-2x-1)(3x+2)}\right)$	NoRatGenSol
1.248	$y^3x^2 + 2(3y'y + y + 1)x + 3(y^2 + y')$	$\left(t, -\frac{3t^2+2xt+2x}{x^2+6tx+3}\right)$	NoRatGenSol
1.249	$by^3x^n + y(axy' + y(y\alpha + \beta))$	$\left(t, -\frac{t^2(t\alpha+\beta)}{bx^n+atx}\right)$	NoRatGenSol
1.250	$-By^2 - Axy - \beta y + (Ax^2 + ax + Byx + c + by)y' - x\alpha - \gamma$	$\left(t, \frac{Bt^2+Ax+bt+\beta t+x\alpha+\gamma}{c+bt+x(a+Bt+Ax)}\right)$	NoRatGenSol
1.251	$yy^2x^2 + y^2x - y' - 1$	$\left(t, \frac{1-t^2x}{tx^2-1}\right)$	NoRatGenSol
1.252	$yy^2x^2 - y^2x - y' + 1$	$\left(t, \frac{t^2x-1}{tx^2-1}\right)$	NoRatGenSol
1.253	$yy^2x^2 + 8y^2x - y' - 8$	$\left(t, \frac{8-8t^2x}{tx^2-1}\right)$	NoRatGenSol
1.254	$x^2y^3 + xy^2 - 2y + x(xy - 2)y'$	$\left(t, -\frac{t(t^2x^2+tx-2)}{x(tx-2)}\right)$	NoRatGenSol
1.255	$xy^2 - y + x(xy - 3)y'$	$\left(t, -\frac{t(tx-1)}{x(tx-3)}\right)$	NoRatGenSol
1.256	$(y - 1)y^2x^2 + (x - 1)y$	$\left(t, -\frac{t(x-1)}{(t-1)x^2}\right)$	NoRatGenSol

Table 2: First-order AODEs of degree one in  $y'$  (continued).

ID	AODE	Parametrization	Solution of AODE
1.257	$y(x^4 - yx + 1) + x(x^4 + yx - 1)y'$	$\left(t, \frac{t(-x^4 + tx - 1)}{x(x^4 + tx - 1)}\right)$	NoRatGenSol
1.258	$-2x^3 + (2yy' - 1)x^2 + y^2$	$\left(t, \frac{-\frac{t^2}{x^2} + 2x + 1}{2t}\right)$	NoRatGenSol
1.260	$-x^2y^3 + 2xy^2 + 2x^2y'y + y + xy'$	$\left(t, \frac{t(t^2x^2 - 2tx - 1)}{x(2tx + 1)}\right)$	NoRatGenSol
1.261	$-2xy^2 + (2x^2y' - 1)y - xy'$	$\left(t, \frac{t(2tx + 1)}{x(2tx - 1)}\right)$	NoRatGenSol
1.262	$-(y' - 2)x^3 + 2yy'x^2 - 4y^2x + y^3$	$\left(t, \frac{t^3 - 4xt^2 + 2x^3}{x^2(x - 2t)}\right)$	NoRatGenSol
1.263	$2yy'x^3 + 3y^2x^2 + 7$	$\left(t, \frac{3t^2x^2 + 7}{2tx^3}\right)$	NoRatGenSol
1.264	$y(3x^3y - 1) + 2x(yx^3 + 1)y'$	$\left(t, \frac{t - 3t^2x^3}{2tx^4 + 2x}\right)$	NoRatGenSol
1.270	$x^2 - y'x + y(y'y' - 1)$	$\left(t, \frac{t - x^2}{t^2 - x}\right)$	NoRatGenSol
1.271	$(y' + 4)x^2 + 2yx + y^2y'$	$\left(t, -\frac{2x(t + 2x)}{t^2 + x^2}\right)$	NoRatGenSol
1.272	$y'x^2 + y^2(y' - 1)$	$\left(t, \frac{t^2}{t^2 + x^2}\right)$	NoRatGenSol
1.273	$2xy + (x^2 + y^2 + a)y'$	$\left(t, -\frac{2tx}{t^2 + x^2 + a}\right)$	NoRatGenSol
1.274	$x^2 + 2yx + b + (x^2 + y^2 + a)y'$	$\left(t, \frac{x^2 + 2tx + b}{t^2 + x^2 + a}\right)$	NoRatGenSol
1.275	$(x^2 + x + y^2)y' - y$	$\left(t, \frac{t}{t^2 + x^2 + x}\right)$	NoRatGenSol
1.276	$-y'x^2 + 2yx + y^2y'$	$\left(t, \frac{2tx}{x^2 - t^2}\right)$	NoRatGenSol
1.277	$(x^4 + y^2)y' - 4x^3y$	$\left(t, \frac{4tx^3}{x^4 + t^2}\right)$	NoRatGenSol
1.279	$y^2(x + y)^2 + y(y + 1) + (x + y(y + 2))y'$	$\left(t, -\frac{t(t^3 + 2xt^2 + x^2t + t + 1)}{t^2 + 2t + x}\right)$	NoRatGenSol
1.280	$(x + y)^2y' - a^2$	$\left(t, \frac{a^2}{(t + x)^2}\right)$	NoRatGenSol
1.281	$2y(y' + 1)x + y^2(y' - 1) - x^2(y' - 1)$	$\left(t, \frac{t^2 - 2xt - x^2}{t^2 + 2xt - x^2}\right)$	NoRatGenSol
1.282	$(3x + y - 1)^2y' - (2y - 1)(6x + 4y - 3)$	$\left(t, \frac{(2t - 1)(4t + 6x - 3)}{(t + 3x - 1)^2}\right)$	NoRatGenSol
1.284	$y'x^2 - xy + 4y^2y'$	$\left(t, \frac{tx}{4t^2 + x^2}\right)$	NoRatGenSol
1.285	$(3y' + 2)x^2 + 2y(y' + 3)x + y^2(4y' + 1)$	$\left(t, -\frac{t^2 + 6xt + 2x^2}{4t^2 + 2xt + 3x^2}\right)$	NoRatGenSol
1.286	$(-3x + 2y + 1)^2y' - (2x - 3y + 4)^2$	$\left(t, \frac{(3t - 2(x + 2))^2}{(2t - 3x + 1)^2}\right)$	NoRatGenSol
1.287	$(-4x + 2y + 1)^2y' - (y - 2x)^2$	$\left(t, \frac{(t - 2x)^2}{(2t - 4x + 1)^2}\right)$	NoRatGenSol
1.288	$-3yy'x^2 - 3y^2x + x + 6y^2y' + y'$	$\left(t, \frac{(3t^2 - 1)x}{6t^2 - 3x^2t + 1}\right)$	NoRatGenSol
1.289	$-6y^2 + 2xy + a + (x - 6y)^2y'$	$\left(t, -\frac{a + 2t(x - 3t)}{(x - 6t)^2}\right)$	NoRatGenSol
1.290	$dx^2 + c(2y + xy')x + y(by + ay'y + 2byy')$	$\left(t, \frac{bt^2 + 2cxt + dx^2}{at^2 + 2bxt + cx^2}\right)$	NoRatGenSol
1.291	$a(x\alpha + y\beta)^2 - (ax + by)\alpha + y'(b(x\alpha + y\beta)^2 - (ax + by)\beta)$	$\left(t, \frac{a(x^2\alpha^2 + x(2t\beta - 1)\alpha + t^2\beta^2) - bt\alpha}{ax\beta - b(x^2\alpha^2 + 2tx\beta\alpha + t\beta(t\beta - 1))}\right)$	NoRatGenSol
1.292	$y'(c + bx + ay)^2 + (y\alpha + x\beta + \gamma)^2$	$\left(t, -\frac{(t\alpha + x\beta + \gamma)^2}{(c + at + bx)^2}\right)$	NoRatGenSol



Table 2: First-order AODEs of degree one in  $y'$  (continued).

ID	AODE	Parametrization	Solution of AODE
1.293	$2y^3 + x(yy' - 5)y - 3x^2y'$	$\left(t, \frac{5tx - 2t^3}{t^2 - 3x}\right)$	NoRatGenSol
1.294	$(x^2 + y^2)(xy' - y) - a(y + xy')$	$\left(t, \frac{t(t^2 + x^2 + a)}{x(t^2 + x^2 - a)}\right)$	NoRatGenSol
1.295	$-y'x^3 + y(y' + 1)x^2 + y^2(y' + 1)x - y^3$	$\left(t, \frac{t(-t^2 + xt + x^2)}{x(-t^2 - xt + x^2)}\right)$	NoRatGenSol
1.296	$x^4 + (y + 1)y'x^3 - 2y^2x^2 + y^2y'x - 2y^3$	$\left(t, \frac{-x^4 + 2t^2x^2 + 2t^3}{tx^3 + x^3 + t^2x}\right)$	NoRatGenSol
1.297	$y^3 - x^2y + 2x(5x^2 + y^2)y'$	$\left(t, -\frac{t^3 - tx^2}{10x^3 + 2t^2x}\right)$	NoRatGenSol
1.298	$y^3 + x(3y^2y' - 2)$	$\left(t, \frac{2}{3t^2} - \frac{t}{3x}\right)$	NoRatGenSol
1.299	$y^3 + x(3yy' - 2)y - x^2y'$	$\left(t, -\frac{t^3 - 2tx}{3t^2x - x^2}\right)$	NoRatGenSol
1.300	$2y^3 + 6xy'y^2 + x$	$\left(t, -\frac{2t^3 + x}{6t^2x}\right)$	NoRatGenSol
1.301	$-3y^3 + x(6yy' + 1)y + x^2y'$	$\left(t, \frac{3t^3 - tx}{6xt^2 + x^2}\right)$	NoRatGenSol
1.302	$x^2y'y^2 + y + xy'$	$\left(t, -\frac{t}{t^2x^2 + x}\right)$	NoRatGenSol
1.303	$x^2y^3 + y + (xy - 1)^2y'$	$\left(t, -\frac{t(t^2x^2 + 1)}{(tx - 1)^2}\right)$	NoRatGenSol
1.304	$5x^2y^3 + (10y'x^3 + x)y^2 + (x^2y' - 3)y + 2xy'$	$\left(t, -\frac{t(5t^2x^2 + tx - 3)}{x(10t^2x^2 + tx + 2)}\right)$	NoRatGenSol
1.305	$y'y^3 - 3y + x^2 - 3xy'$	$\left(t, \frac{3t - x^2}{t^3 - 3x}\right)$	NoRatGenSol
1.306	$-y'x^3 - x^2y + y^3y'$	$\left(t, \frac{tx^2}{t^3 - x^3}\right)$	NoRatGenSol
1.307	$a(yy' - x) + (x^2 + y^2)(x + yy')$	$\left(t, -\frac{x(t^2 + x^2 - a)}{t(t^2 + x^2 + a)}\right)$	NoRatGenSol
1.308	$-x^3 + y^2x + 2y^3y'$	$\left(t, \frac{x(x^2 - t^2)}{2t^3}\right)$	NoRatGenSol
1.309	$-2x^3 - x + (2y^3 + y)y'$	$\left(t, \frac{2x^3 + x}{2t^3 + t}\right)$	NoRatGenSol
1.310	$x^3 + 5yy'x^2 + 5y^2x + 2y^3y'$	$\left(t, -\frac{x^3 + 5t^2x}{2t^3 + 5x^2t}\right)$	NoRatGenSol
1.311	$(3y' + 4)x^3 + 3y(2y' + 3)x^2 - 3y^2(y' - 2)x + y^3(20y' - 1)$	$\left(t, \frac{t^3 - 6xt^2 - 9x^2t - 4x^3}{20t^3 - 3xt^2 + 6x^2t + 3x^3}\right)$	NoRatGenSol
1.312	$\frac{(a-b)(yy' - x)}{a+b} + \left(\frac{x^2}{a} + \frac{y^2}{b}\right)(x + yy')$	$\left(t, -\frac{x\left(\left(t^2 - b\right)a^2 + b\left(t^2 + x^2 + b\right)a + b^2x^2\right)}{t\left(\left(t^2 + b\right)a^2 + b\left(t^2 + x^2 - b\right)a + b^2x^2\right)}\right)$	NoRatGenSol
1.313	$b(3y - x(y' - 2))x^2 + c(y'x^2 + y^2) + ay^2(2y'y - y + 3xy')$	$\left(t, \frac{at^3 - ct^2 - bx^2(3t + 2x)}{a(2t + 3x)t^2 + x^2(c - bx)}\right)$	NoRatGenSol
1.315	$-y'x^4 + 2yx^3 + 2y^3y'x - y^4$	$\left(t, \frac{t^4 - 2tx^3}{2t^3x - x^4}\right)$	NoRatGenSol
1.316	$2xy'y^3 + 2y^2 + y'y - 4$	$\left(t, \frac{4 - 2t^2}{2xt^3 + t}\right)$	NoRatGenSol
1.317	$y^2 - xy + x(2y^3 + y + x)y'$	$\left(t, \frac{t(x-t)}{x(2t^3 + t + x)}\right)$	NoRatGenSol
1.318	$y(y^3 + 3xy'y^2 - 2y - 4xy' + y')$	reducible	see Table 6
1.319	$y^4 + 7xy'y^3 + (y' - 5)y - 5xy'$	$\left(t, -\frac{t(t^3 - 5)}{7xt^3 + t - 5x}\right)$	NoRatGenSol
1.320	$x^2y'y^3 + xy'y - 1$	$\left(t, \frac{1}{x^2t^3 + xt}\right)$	NoRatGenSol

Table 2: First-order AODEs of degree one in  $y'$  (continued).

ID	AODE	Parametrization	Solution of AODE
1.321	$2x^2y'y^3 + x^2y'y^2 - 2y - 2xy' - 1$	$\left(t, \frac{2t+1}{x(2xt^3+xt^2-2)}\right)$	NoRatGenSol
1.322	$5xy^4 + x + (10x^2y^3 - 3y^2 - 2)y'$	$\left(t, \frac{5xt^4+x}{-10x^2t^3+3t^2+2}\right)$	NoRatGenSol
1.323	$y(byx^3+c) + x(axy^3+c)y'$	$\left(t, -\frac{t(bt^3+c)}{x(at^3+c)}\right)$	NoRatGenSol
1.324	$2x^3(y'+1)y^3 - y - xy'$	$\left(t, -\frac{t-2t^3x^3}{x-2t^3x^3}\right)$	NoRatGenSol
1.325	$-x^4 - 2yy'x^3 + 2y^3x + y^4y'$	$\left(t, \frac{x^4-2t^3x}{t^4-2t^3}\right)$	NoRatGenSol
1.326	$x(ay^3 + (bx + ay)^3) + y(bx^3 + (bx + ay)^3)y'$	$\left(t, -\frac{x(a^3t^3+3a^2bxt^2+b^3x^3+a(t^3+3b^2x^2t))}{t(a^3t^3+3a^2bxt^2+3ab^2x^2t+b(b^2+1)x^3)}\right)$	NoRatGenSol
1.327	$y^5 + y + (xy^4 + 2x^2y^3 + 2y + x)y'$	$\left(t, -\frac{t^5+t}{xt^4+2x^2t^3+2t+x}\right)$	NoRatGenSol
1.328	$ax^2y'y^n + y - 2xy'$	$\left(t, \frac{t}{2x-at^nx^2}\right)$	NoRatGenSol
1.329	$x^n(by + axy')y^m + \beta y + xy'\alpha$	$\left(t, -\frac{t(bx^nt^m+\beta)}{x(ax^nt^m+\alpha)}\right)$	$\begin{cases} cx^{-k}, & \text{if } b = ka, \beta = k\alpha, k \in \mathbb{Z} \setminus \{0\} \\ cx \frac{\beta}{\alpha} (ax^n + \alpha) \frac{b}{na} - \frac{\beta}{n\alpha}, & \text{if } m = 0, \alpha \mid \beta, na\alpha \mid (\alpha b - a\beta) \\ cx \frac{b+\beta}{a}, & \text{if } m = n = \alpha = 0, a \mid (b + \beta) \\ cx \frac{b+\beta}{\alpha}, & \text{if } m = 1, a = n = 0, \alpha \mid (b + \beta) \\ \text{NoRatGenSol}, & \text{otherwise} \end{cases}$
1.340	$\frac{e_1((a+x)y'-y)}{(a^2+2xa+x^2+y^2)^3} - \frac{e_2(y+(a-x)y')}{(a^2-2xa+x^2+y^2)^3}$	$\left(t, \frac{t\left(\frac{e_1}{(t^2+(a+x)^2)^3} + \frac{e_2}{(t^2+(a-x)^2)^3}\right)}{\frac{e_2(x-a)}{(t^2+(a-x)^2)^3} + \frac{e_1(a+x)}{(t^2+(a+x)^2)^3}}\right)$	NoRatGenSol

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 Table 3: First-order AODEs of degree two in  $y'$ .

ID	AODE	Parametrization	Associated ODE	Solution of assoc. ODE	Solution of AODE
1.368	$bx^2 + y'^2 + ay$	$\left(-\frac{t^2+bx^2}{a}, t\right)$	$-\frac{a}{2} - \frac{bx}{t}$	NoRatGenSol	NoRatGenSol
1.369	$-a^2 + y'^2 + y'^2$	$\left(\frac{a-at^2}{t^2+1}, \frac{2at}{t^2+1}\right)$	$\frac{1}{2}(-t^2 - 1)$	NoRatGenSol	NoRatGenSol
1.371	$-y^3 + y^2 + y'^2$	$(t^2 + 1, t^3 + t)$	$\frac{1}{2}(t^2 + 1)$	NoRatGenSol	NoRatGenSol
1.372	$-4y^3 + ay + y'^2 + b$	genus 1	—	—	NoStrongRatGenSol
1.374	$(y' - 2)y' - y^2$	$\left(-\frac{(t-2)t}{2(t-1)}, \frac{t^2}{2t-2}\right)$	$-\frac{(t-1)t^2}{t^2-2t+2}$	NoRatGenSol	NoRatGenSol
1.375	$bx + y'(a + y')$	reducible over $\overline{\mathbb{K}(x)}$	—	—	NoStrongRatGenSol
1.376	$by + y'(a + y')$	$\left(-\frac{t(a+t)}{b}, t\right)$	$-\frac{bt}{a+2t}$	NoRatGenSol	NoRatGenSol
1.377	$y'^2 + (x - 2)y' - y + 1$	$(t^2 + (x - 2)t + 1, t)$	0	$c$	$c^2 + (x - 2)c + 1$
1.378	$y'(a + x + y') - y$	$(t(a + t + x), t)$	0	$c$	$c(a + c + x)$
1.379	$y + y'(-x + y' - 1)$	$(t(-t + x + 1), t)$	0	$c$	$c(-c + x + 1)$
1.380	$y'(2x + y') - y$	$(t(t + 2x), t)$	$-\frac{t}{2(t+x)}$	NoRatGenSol	NoRatGenSol
1.381	$y + y'(y' - 2x)$	$(-t(t - 2x), t)$	$\frac{t}{2t-2x}$	NoRatGenSol	NoRatGenSol
1.382	$-bx^2 - c + y'(ax + y')$	reducible over $\overline{\mathbb{K}(x)}$	—	—	NoStrongRatGenSol

Table 3: First-order AODEs of degree two in  $y'$  (continued).

ID	AODE	Parametrization	Associated ODE	Solution of assoc. ODE	Solution of AODE
1.383	$cx^2 + by + y'(ax + y')$	$\left(-\frac{t^2+ax+cx^2}{b}, t\right)$	$-\frac{at+bt+2cx}{2t+ax}$	NoRatGenSol	NoRatGenSol
1.384	$c + y'(b + y') + a(xy' - y)$	$\left(\frac{c+t(b+t+ax)}{a}, t\right)$	0	$c$	$\frac{c(b+c+ax+1)}{a}$
1.385	$-2y'^2x^2 + 2yx + y'^2$	$\left(tx - \frac{t^2}{2x}, t\right)$	$\frac{t^2}{2tx-2x^3}$	NoRatGenSol	NoRatGenSol
1.386	$a(xy' - 2y)x^2 + y'^2$	$\left(\frac{t(ax^3+t)}{2ax^2}, t\right)$	$\frac{t}{x}$	$cx$	$\frac{c(ax^2+c)}{2a}$
1.388	$y'(y' - 2y) - 2x$	$\left(\frac{t}{2} - \frac{x}{t}, t\right)$	$\frac{2t(t^2+1)}{t^2+2x}$	NoRatGenSol	NoRatGenSol
1.389	$4y^2 - 4y'y + y + (y' - 1)y'$	$\left(-\frac{(t-5)t}{(5-2t)^2}, \frac{5t}{(5-2t)^2}\right)$	$t - \frac{2t^2}{5}$	NoRatGenSol	NoRatGenSol
1.390	$-c - bx + y'(ay + y')$	$\left(-\frac{t^2+c+bx}{at}, t\right)$	$\frac{t(b-at^2)}{t^2+c+bx}$	NoRatGenSol	NoRatGenSol
1.391	$(bx + y')(ay + y')$	reducible	—	—	see Table 6
1.396	$-(xy - y')(y^2 + y')$	reducible	—	—	see Table 6
1.397	$-2y^2y'x^3 - 4y^3x^2 + y'^2$	$\left(\frac{t^2}{2x^2(tx+2)}, \frac{t^3}{2x^2(tx+2)}\right)$	$t\left(t + \frac{1}{x}\right)$	$\frac{2x}{c-x^2}$	$\frac{1}{c^2-cx^2}$
1.399	$y'(x + 2y' - 1) - y$	$(t(2t + x - 1), t)$	0	$c$	$c(2c + x - 1)$
1.400	$-2y'^2x^2 + 3yx + 2y'^2$	$\left(\frac{2t(x^2-t)}{3x}, t\right)$	$\frac{t}{2x}$	NoRatGenSol	NoRatGenSol
1.401	$y + y'(3y' - 2x)$	$(t(2x - 3t), t)$	$\frac{t}{6t-2x}$	NoRatGenSol	NoRatGenSol
1.402	$x^2 + 4y'x + 3y'^2 - y$	$(3t^2 + 4xt + x^2, t)$	$-\frac{1}{2}$	$c - \frac{x}{2}$	$3c^2 + xc - \frac{x^2}{4}$
1.403	$y'(b + ay') - y$	$(t(b + at), t)$	$\frac{t}{b+2at}$	NoRatGenSol	NoRatGenSol
1.404	$(y + by')x^2 + ay'^2$	$\left(\frac{t(bx^2+at)}{x^2}, t\right)$	$\frac{t(2at-x^3)}{bx^3+2atx}$	NoRatGenSol	NoRatGenSol
1.405	$y'(y + ay') - x$	$\left(\frac{x}{t} - at, t\right)$	$\frac{t-t^3}{at^2+x}$	NoRatGenSol	NoRatGenSol
1.406	$y'(ay' - y) - x$	$\left(at - \frac{x}{t}, t\right)$	$\frac{t^3+t}{at^2+x}$	NoRatGenSol	NoRatGenSol
1.407	$xy'^2 - y$	$(t^2x, t)$	$-\frac{t-1}{2x}$	NoRatGenSol	NoRatGenSol
1.408	$xy'^2 + x - 2y$	$\left(\frac{1}{2}(t^2 + 1)x, t\right)$	$-\frac{(t-1)^2}{2tx}$	NoRatGenSol	NoRatGenSol
1.409	$y'(xy' - 2) - y$	$(t(tx - 2), t)$	$\frac{(t-1)t}{2-2tx}$	NoRatGenSol	NoRatGenSol
1.410	$y'(xy' + 4) - 2y$	$\left(\frac{1}{2}t(tx + 4), t\right)$	$-\frac{(t-2)t}{2tx+4}$	NoRatGenSol	NoRatGenSol
1.411	$xy'(y' + 1) - y$	$(t(t + 1)x, t)$	$-\frac{t^2}{2tx+x}$	NoRatGenSol	NoRatGenSol
1.412	$a + y'(y + xy')$	$\left(-\frac{xt^2+a}{t}, t\right)$	$\frac{2t^3}{a-t^2x}$	NoRatGenSol	NoRatGenSol
1.413	$-x^2 + y'^2x + yy'$	$\left(\frac{x(x-t^2)}{t}, t\right)$	$-\frac{2t(t^2-x)}{x(t^2+x)}$	NoRatGenSol	NoRatGenSol
1.414	$x^3 + y'^2x + yy'$	$\left(-\frac{x(t^2+x^2)}{t}, t\right)$	$\frac{2t^3+3x^2t}{x^3-t^2x}$	NoRatGenSol	NoRatGenSol
1.415	$-y^4 + y'y + xy'^2$	$\left(\frac{t}{t^2-x}, -\frac{t^3}{(t^2-x)^2x}\right)$	$\frac{t}{x}$	$cx$	$\frac{c}{c^2x-1}$
1.416	$x(y' - 3)y' + y(y' + 1)$	$\left(-\frac{(t-3)tx}{t+1}, t\right)$	$-\frac{2t(t+1)}{(t+3)x}$	NoRatGenSol	NoRatGenSol
1.417	$a + y'(xy' - y)$	$\left(\frac{a}{t} + tx, t\right)$	0	$c$	$\frac{a}{c} + cx$
1.418	$ay + y'(xy' - y)$	$\left(-\frac{t^2x}{a-t}, t\right)$	$\frac{a(t-a)}{(2a-t)x}$	NoRatGenSol	NoRatGenSol

Table 3: First-order AODEs of degree two in  $y'$  (continued).

ID	AODE	Parametrization	Associated ODE	Solution of assoc. ODE	Solution of AODE
1.419	$2yy' + x(y'^2 - 1)$	$\left(-\frac{(t^2-1)x}{2t}, t\right)$	$\frac{t-3t^3}{xt^2+x}$	NoRatGenSol	NoRatGenSol
1.420	$a + y'(xy' - 2y)$	$\left(\frac{xt^2+a}{2t}, t\right)$	$\frac{t^3}{t^2x-a}$	NoRatGenSol	NoRatGenSol
1.421	$x(y'^2 - 1) - 2yy'$	$\left(\frac{(t^2-1)x}{2t}, t\right)$	$\frac{t}{x}$	$cx$	$\frac{c^2x^2-1}{2c}$
1.422	$x(y'^2 + 4) - 2yy'$	$\left(\frac{(t^2+4)x}{2t}, t\right)$	$\frac{t}{x}$	$cx$	$\frac{cx^2}{2} + \frac{2}{c}$
1.423	$xy'^2 - 2yy' + x + 2y$	$\left(\frac{(t^2+1)x}{2(t-1)}, t\right)$	$\frac{t-1}{x}$	$cx + 1$	$\frac{cx^2}{2} + x + \frac{1}{c}$
1.424	$bx + y'(ay + xy')$	$\left(-\frac{(t^2+b)x}{at}, t\right)$	$\frac{t((a+1)t^2+b)}{(b-t^2)x}$	NoRatGenSol	NoRatGenSol
1.425	$(x+1)y'^2 - (x+y)y' + y$	$\left(\frac{t(xt+t-x)}{t-1}, t\right)$	0	$c$	$\frac{c(xc+c-x)}{c-1}$
1.426	$(3x+1)y'^2 - 3(y+2)y' + 9$	$\left(t\left(x + \frac{1}{3}\right) + \frac{3}{2} - 2, t\right)$	0	$c$	$c\left(x + \frac{1}{3}\right) + \frac{3}{c} - 2$
1.427	$(3x+5)y'^2 - (x+3y)y' + y$	$\left(\frac{t(t(3x+5)-x)}{3t-1}, t\right)$	0	$c$	$\frac{c(c(3x+5)-x)}{3c-1}$
1.428	$b(xy' - y) + y'(c - ay + axy')$	$\left(\frac{t(c+(b+at)x)}{b+at}, t\right)$	0	$c$	$\frac{c_1(axc_1+c+bx)}{b+ac_1}$
1.429	$ay'(-y + xy' + 1) + b(y - xy' + y')$	$\left(\frac{t(tx a+a+b-bx)}{at-b}, t\right)$	0	$c$	$\frac{c(cxa+a+b-bx)}{ac-b}$
1.430	$(c_2 + a_2x)y'^2 + (c_1 + a_1x + b_1y)y' + c_0 + a_0x + b_0y$	$\left(-\frac{c_2t^2+a_2xt^2+c_1t+a_1xt+c_0+a_0x}{b_0+b_1t}, t\right)$	A <sub>430</sub>	NoRatGenSol	NoRatGenSol
1.431	$-y^4 + y^2 + x^2y'^2$	$\left(\frac{tx}{2} + \frac{1}{2tx}, \frac{t^4x^4-1}{4t^2x^3}\right)$	$\frac{(tx-1)^2}{2x^2}$	NoRatGenSol	NoRatGenSol
1.432	$x^2 + (a + xy')^2 - 2ay$	$\left(\frac{a^2+2txa+(t^2+1)x^2}{2a}, t\right)$	$-\frac{t^2+1}{a+tx}$	NoRatGenSol	NoRatGenSol
1.433	$(y + x(y' + 2))^2 - 4(a + x(x + y))$	$\left(\frac{-4a+t^2-4x^2}{4x}, \frac{4a-t^2+4tx-4x^2}{4x^2}\right)$	2	$2x + c$	$\frac{c(c+4x)-4a}{4x}$
1.435	$y^2 - 2xy'y + y + x^2y'^2 - x$	$\left(x - t^2, \frac{-t^2-t+x}{x}\right)$	$\frac{t+1}{2x}$	NoRatGenSol	NoRatGenSol
1.436	$-x^4 + (y'^2 - y^2)x^2 - 2yy'x + y^2$	$\left(\frac{x^2-t^2}{2t}, \frac{(x-1)t^2+x^2(x+1)}{2tx}\right)$	$t\left(\frac{1}{x} - 1\right)$	NoRatGenSol	NoRatGenSol
1.437	$y^2 + x^2y'^2 - (a + 2xy)y'$	$\left(\frac{t^2x}{a} + t, \frac{t^2}{a}\right)$	0	$c$	$\frac{c(a+cx)}{a}$
1.438	$2y^2 + 3xy'y + x^2y'^2$	reducible	—	—	see Table 6
1.439	$3y^2 + 3xy'y + x^2y'^2$	reducible	—	—	see Table 6
1.440	$-5y^2 + 4xy'y + x^2y'^2$	reducible	—	—	see Table 6
1.441	$x^2y'^2 - 4x(y+2)y' + 4y(y+2)$	$\left(-\frac{2tx(tx-4)}{(tx-2)^2}, \frac{8t}{(tx-2)^2}\right)$	$-\frac{t^2}{2}$	$-\frac{2}{c-x}$	$\frac{2x(x-2c)}{c^2}$
1.442	$(y + y')x^3 + (y'^2 + y(y' - 1))x^2 - y(y + 2y')x + y^2$	reducible	—	—	see Table 6
1.443	$y'x^2 - xy^2 - y'$	$\left(t, \frac{t^2x}{x^2-1}\right)$	$\frac{t^2x}{x^2-1}$	NoRatGenSol	NoRatGenSol
1.444	$y^2 - (y - 2x)y'y + x^2y'^2$	$\left(\frac{(t+x+1)^2}{t+1}, \frac{(t+x+1)^2}{(t+1)^2}\right)$	-1	$c - x$	$\frac{(c+1)^2}{c-x+1}$
1.445	$(ay^3 + y')(y'^2 + b)$	reducible	—	—	see Table 6
1.446	$y^2 - 2xy'y + (x^2 + 1)y'^2 - 1$	$\left(i\frac{(t^2(x^2+1)-1)}{2t}, \frac{it^2(-i+x)^2-i}{2t(-i+x)}\right)$	$\frac{t}{i-x}$	$\frac{c}{ix+1}$	$\frac{(i+x)c^2-i+x}{2c}$

Table 3: First-order AODEs of degree two in  $y'$  (continued).

ID	AODE	Parametrization	Associated ODE	Solution of assoc. ODE	Solution of AODE
1.447	$(x^2 - 1)y' - 1$	reducible over $\overline{\mathbb{K}(x)}$	—	—	NoSTRONGRATGENSOL
1.448	$-y^2 + (x^2 - 1)y' + 1$	$\left( \frac{t^2x^3 - t^2x - 2tx^2 + 2t + x}{t^2x^2 - t^2 - 1}, \frac{t^2(-x^2) + t^2 + 2tx - 1}{t^2x^2 - t^2 - 1} \right)$	$-\frac{t^2(x^2 - 1) + tx - 1}{x^2 - 1}$	NoRATGENSOL	NoRATGENSOL
1.449	$y^2 + 2xy'y + (x^2 - a^2)y'^2$	reducible	—	—	see Table 6
1.450	$(y'^2 - 1)x^2 - 2yy'x - a^2y'^2$	$\left( \frac{(t^2 - 1)x}{2t} - \frac{a^2t}{2x}, t \right)$	$\frac{t}{x}$	$cx$	$-\frac{a^2c^2 - x^2c^2 + 1}{2c}$
1.451	$y^2 - 2xy'y + (x^2 + a)y'^2 + b$	$\left( -\frac{(x^2 + a)t^2 + b}{2i\sqrt{at}}, \frac{a^2t^2 + i\sqrt{a}x(t^2x^2 + b) + a(t^2x(x + i\sqrt{a}) - b)}{2at(x^2 + a)} \right)$	$-\frac{t(x + i\sqrt{a})}{x^2 + a}$	NoRATGENSOL	NoRATGENSOL
1.452	$(y' + 2)y^2 + 2xy'y + (2x^2 + 1)y'^2 + (x^2 + 2)y' + 1$	$\left( -\frac{t^2 + xt + 1}{t - x}, \frac{-2t^2 - 1}{(t - x)^2} \right)$	0	$c$	$-\frac{c^2 + xc + 1}{c - x}$
1.453	$a^2x^2(y'^2 + 1) - (y - xy')^2$	$\left( \frac{(a^2(t^2 + 1) - t^2)x}{2at}, \frac{(t^2 - 1)a^2 - 2t^2a + t^2}{2(a - 1)at} \right)$	$-\frac{at((t^2 + 1)a^2 - 2t^2a + t^2)}{(a - 1)(a^2(t^2 - 1) - t^2)x}$	NoRATGENSOL	NoRATGENSOL
1.454	$-a^2x^2 + a(xy'^2 - 2yy'x) + y^2$	$\left( \frac{ax(a - t^2 - 1)}{2\sqrt{(a - 1)at}}, \frac{\sqrt{(a - 1)ax(a - t^2 - 1)} + (a - 1)x(a + t^2 - 1)}{2(a - 1)tx} \right)$	$-\frac{\sqrt{(a - 1)at}}{ax}$	NoRATGENSOL	NoRATGENSOL
1.455	$y'(y + xy')x^2 + a$	$\left( -\frac{a}{tx^2} - tx, t \right)$	$-\frac{2t}{x}$	$\frac{c}{x^2}$	$-\frac{a}{c} - \frac{c}{x}$
1.456	$xy^2 - 2(x^2 - 1)y'y + x(x^2 - 1)y'^2 - x$	$\left( \frac{x(t^2x^2 - t^2 - 1)}{t^2x^3 - t^2x - 2tx^2 + 2t + x}, \frac{2t(tx^2 - t - x)}{t^2x^3 - t^2x - 2tx^2 + 2t + x} \right)$	$\frac{t(tx(x^2 - 1) + 1)}{x - x^3}$	NoRATGENSOL	NoRATGENSOL
1.457	$xy'(xy' - 1) - y$	$(tx(tx - 1), t)$	$-\frac{2t(tx - 1)}{x(2tx - 1)}$	NoRATGENSOL	NoRATGENSOL
1.458	$x^2(x^2 - a^2)y' - 1$	reducible over $\overline{\mathbb{K}(x)}$	—	—	NoSTRONGRATGENSOL
1.462	$yy'^2 - 1$	$\left( \frac{1}{t^2}, t \right)$	$-\frac{t^4}{2}$	NoRATGENSOL	NoRATGENSOL
1.464	$2xy' + y(y'^2 - 1)$	$\left( -\frac{2tx}{t^2 - 1}, t \right)$	$\frac{t(t^2 - 1)}{2x}$	NoRATGENSOL	NoRATGENSOL
1.465	$2xy' + y(y'^2 - 9)$	$\left( -\frac{2tx}{t^2 - 9}, t \right)$	$\frac{t(t^2 - 9)(t^2 - 7)}{2(t^2 + 9)x}$	NoRATGENSOL	NoRATGENSOL
1.466	$yy'^2 - 2xy' + y$	$\left( \frac{2tx}{t^2 + 1}, t \right)$	$-\frac{t^3 + t}{2x}$	NoRATGENSOL	NoRATGENSOL
1.467	$yy'^2 - 4xy' + y$	$\left( \frac{4tx}{t^2 + 1}, t \right)$	$-\frac{t(t^2 - 3)(t^2 + 1)}{4(t^2 - 1)x}$	NoRATGENSOL	NoRATGENSOL
1.468	$(y - 4xy')a^2 + yy'^2$	$\left( \frac{4a^2tx}{a^2 + t^2}, t \right)$	$\frac{t(t^2 - 3a^2)(a^2 + t^2)}{4a^2(a^2 - t^2)x}$	NoRATGENSOL	NoRATGENSOL
1.469	$by + y'(ax + yy')$	$\left( -\frac{atx}{t^2 + b}, t \right)$	$\frac{t(t^2 + b)(t^2 + a + b)}{a(t^2 - b)x}$	NoRATGENSOL	NoRATGENSOL
1.470	$y'x^3 + yy'^2 - x^2y$	$\left( \frac{tx^3}{x^2 - t^2}, t \right)$	$\frac{t^3}{x^3}$	NoRATGENSOL	NoRATGENSOL
1.471	$(y' - 1)(x + yy')$	reducible	—	—	see Table 6
1.472	$xy'(y' + 2) + y(y'^2 - 1)$	$\left( -\frac{t(t + 2)x}{t^2 - 1}, t \right)$	$\frac{t(t^2 - 1)}{2x}$	NoRATGENSOL	NoRATGENSOL
1.473	$(y - 2x)y'^2 - 2(x - 1)y' + y - 2$	$\left( \frac{2(xt^2 + (x - 1)t + 1)}{t^2 + 1}, t \right)$	$\frac{t^3 + t}{2 - 2x}$	NoRATGENSOL	NoRATGENSOL

Table 3: First-order AODEs of degree two in  $y'$  (continued).

ID	AODE	Parametrization	Associated ODE	Solution of assoc. ODE	Solution of AODE
1.474	$(5 - 4x)y' + 2y(y'^2 + 1)$	$\left(\frac{t(4x-5)}{2(t^2+1)}, t\right)$	$-\frac{2t(t^2+1)}{4x-5}$	NoRatGenSol	NoRatGenSol
1.475	$2xy' + y(4y'^2 - 1)$	$\left(\frac{2tx}{1-4t^2}, t\right)$	$-\frac{t-4t^3}{2x}$	NoRatGenSol	NoRatGenSol
1.476	$4y'^3x^3 - 4yx^2 + 9yy'^2$	$\left(\frac{4tx^3}{4x^2-9t^2}, t\right)$	$\frac{9t^3}{4x^3}$	NoRatGenSol	NoRatGenSol
1.477	$ayy'^2 - by' + 2xy' - y$	$\left(\frac{t(b-2x)}{at^2-1}, t\right)$	$\frac{t-at^3}{b-2x}$	NoRatGenSol	NoRatGenSol
1.478	$(b + ay)(y'^2 + 1) - c$	$\left(-\frac{bt^2+b-c}{at^2+a}, t\right)$	$-\frac{a(t^2+1)^2}{2c}$	NoRatGenSol	NoRatGenSol
1.479	$(c_2 + a_2x + b_2y)y'^2 + (c_1 + a_1x + b_1y)y' + c_0 + a_0x + b_0y$	$\left(-\frac{c_2t^2+a_2xt^2+c_1t+a_1xt+c_0+a_0x}{b_0+t(b_1+b_2t)}, t\right)$	A479	NoRatGenSol	NoRatGenSol
1.480	$-y^2 + y'(2x + ay')y - x^2y'^2$	$\left(\frac{(t-x+1)^2}{a}, \frac{(t-x+1)^2}{a(t+1)}\right)$	$\frac{3t-x+3}{2t+2}$	NoRatGenSol	NoRatGenSol
1.481	$y'^2x^2 + y(y'^2 + 1)x + y^2y'$	reducible	—	—	see Table 6
1.482	$y'^2x^2 + y(y'^2 - 1)x + (a - y^2)y'$	genus 1	—	—	NoStrongRatGenSol
1.483	$-y^2 + 2x(y'^2 + y' + 1)y - x^2y'^2$	$\left(\frac{(t+x+1)^2}{2(t+1)^2x}, \frac{(2x-1)t^2+2(x^2+x-1)t+x^2-1}{2(t+1)(t(x-1)-1)x}\right)$	$-\frac{(t+1)\left(\left(2x^2-1\right)t^2+2\left(x^2-1\right)t+x^2-1\right)}{2t(t(x-1)-1)x^2}$	NoRatGenSol	NoRatGenSol
1.484	$-y^2 + 2x(y'^2 - 3y' + 1)y - x^2y'^2$	$\left(\frac{(t+x+1)^2}{2(t-1)^2x}, \frac{(2x-1)t^2+2(x^2-x-1)t-3x^2-4x-1}{2(t-1)x(xt+t+1)}\right)$	$-\frac{(t-1)\left(\left(2x^2+1\right)t^2+\left(2-6x^2\right)t+3x^2+1\right)}{2x(x+2)(xt+t+1)}$	NoRatGenSol	NoRatGenSol
1.485	$axy'^2 - (bx^2 + ay^2 + c)y' + bxy$	genus 1	—	—	NoStrongRatGenSol
1.486	$y^2(y'^2 + 1) - a^2$	$\left(-\frac{a(2ta+a)^2-t^2}{t^2+(2ta+a)^2}, \frac{2at(2t+1)}{(2ta+a)^2-t^2}\right)$	$\frac{(t^2+(2ta+a)^2)^2}{2a^2((2ta+a)^2-t^2)}$	NoRatGenSol	NoRatGenSol
1.487	$-6y'^3x^3 + 4yx^2 + y^2y'^2$	genus 1	—	—	NoStrongRatGenSol
1.488	$4a^2 - 4(x + yy')a + y^2(y'^2 + 1)$	$\left(\frac{a(t-i)^2+(t+i)^2x}{t^2+1}, \frac{a((2-i)t^2-2t+(2+i))+i(t+i)^2x}{a(t-i)^2+(t+i)^2x}\right)$	$-\frac{\left(\frac{1}{2}+\frac{i}{2}\right)(t-1)(t-i)(t+i)^2}{a(t-i)^2+(t+i)^2x}$	NoRatGenSol	NoRatGenSol
1.489	$c + bx + y(yy'^2 + 2xy' + ay)$	$\mathcal{P}_{489}$	A489	NoRatGenSol	NoRatGenSol
1.490	$-x^2 - 2yy'x + a + y^2(y'^2 + 2)$	$\left(\frac{(t^2-1)x-i\sqrt{2}\sqrt{at}}{t^2+1}, \frac{\sqrt{2}\left((-i)\sqrt{a}(t^2-1)+t(t-2\sqrt{2})x+x\right)}{\sqrt{2}(t^2-1)x-2i\sqrt{a}t}\right)$	$\frac{\left(\sqrt{2}-t\right)(t-1)(t^2+1)}{\sqrt{2}(t^2-1)x-2i\sqrt{a}t}$	NoRatGenSol	NoRatGenSol
1.491	$(y'^2 + 1)y^2 + (a-1)b + a(-y^2 + 2xy'y + x)$	$\mathcal{P}_{491}$	A491	NoRatGenSol	NoRatGenSol
1.492	$y^2 + (y^2 - a^2)y'^2$	$\left(-\frac{(a-1)a(t+1)(ta+a+t-1)}{t^2+a^2(t+1)^2-2a(t+1)+1}, \frac{(a-1)(t+1)(ta+a+t-1)}{2t(ta+a-1)}\right)$	A492	NoRatGenSol	NoRatGenSol
1.493	$y^2 + 2ay'y + (a^2 - 2xa + y^2)y'^2$	$\left(\frac{a(t+i)^2+(t-i)^2x}{\sqrt{2}(t^2+1)}, -\frac{i\sqrt{2}\left(a(t+i)^2+(t-i)^2x\right)}{a(t+i)\left(\left(\sqrt{2}+2i\right)t+2+i\sqrt{2}\right)-\sqrt{2}(t-i)^2x}\right)$	A493	NoRatGenSol	NoRatGenSol
1.494	$yy'(2a + yy') - x^2(a^2(y'^2 + 1) - 1)$	$\left(\frac{a(t+1)x\left(\sqrt{1-a^2}(t-1)x+ia(t+1)\right)}{a(t^2-1)+2i\sqrt{1-a^2}tx}, \frac{ia(t^2-1)-2tx\sqrt{1-a^2}}{at^2x+ax}\right)$	A494	NoRatGenSol	NoRatGenSol
1.495	$(1 - (a-1)y'^2)x^2 + 2ay'y'x + y^2(y'^2 - a + 1)$	reducible	—	—	see Table 6
1.496	$(x - y)^2(y'^2 + 1) - a^2(y' + 1)^2$	$\left(x - \frac{\left(\frac{1}{2} - \frac{i}{2}\right)a(t^2+i)}{t}, -\frac{i(t^2-1)}{t^2+1}\right)$	$\frac{2it^2}{at^2+a}$	NoRatGenSol	NoRatGenSol
1.497	$-x^2 - 2yy'x + y^2(3y'^2 + 4)$	$\left(\frac{x^9+4tx^5+3t^2x}{2x^8+6tx^4+6t^2}, \frac{2t(x^4+2t)}{x^8+4tx^4+3t^2}\right)$	$-\frac{x^{16}+3tx^{12}+4t^2x^8+15t^3x^4+15t^4}{x^{13}+4tx^9+3t^2x^5}$	NoRatGenSol	NoRatGenSol

Table 3: First-order AODEs of degree two in  $y'$  (continued).

ID	AODE	Parametrization	Associated ODE	Solution of assoc. ODE	Solution of AODE
1.498	$(2 - 3y)^2 y'^2 + 4(y - 1)$	$\left( \frac{22908t^2 + 19168t - 6335}{(152t + 73)^2}, -\frac{4(1064t^2 + 8719t + 3942)}{22516t^2 + 13120t - 29663} \right)$	$-\frac{(152t + 73)^4}{15394(22516t^2 + 13120t - 29663)}$	NoRatGenSol	NoRatGenSol
1.499	$y^2 (y'^2 + 1) - a^2(x + yy')^2$	$\mathcal{P}_{499}$	$A_{499}$	NoRatGenSol	NoRatGenSol
1.500	$a(y^2 (y'^2 + 1) - b) - b(x + yy')^2$	$\mathcal{P}_{500}$	$A_{500}$	NoRatGenSol	NoRatGenSol
1.501	$dy^2 - by'y + (ay^2 + c + bx)y'^2$	$\mathcal{P}_{501}$	$A_{501}$	NoRatGenSol	NoRatGenSol
1.502	$(ay - bx)(b^2 + a^2 y'^2) - c^2(b + ay')^2$	$\left( \frac{xb^3 + c^2 b^2 + at(2c^2 + atx)b + a^2 c^2 t^2}{a(b^2 + a^2 t^2)}, t \right)$	$-\frac{(b^2 + a^2 t^2)^2}{2abc^2(b + at)}$	NoRatGenSol	NoRatGenSol
1.503	$(c_2 + a_2 x + b_2 y)^2 y'^2 + (c_1 + a_1 x + b_1 y)y' + a_0 + c_0 + b_0 y$	genus 1	—	—	NoStrongRatGenSol
1.504	$yx^2 + y^2 y'^2 x - (x^3 + y^3 - a)y'$	genus 2	—	—	NoStrongRatGenSol
1.505	$-x^3 + y^2 (y'^2 + 2)x - 2y^3 y'$	reducible	—	—	see Table 6
1.506	$x^2(2y' - 1)y^3 + ((y' - 2)y'x^3 + 1)y^2 - x^2 y'^2$	genus 1	—	—	NoStrongRatGenSol
1.507	$y^4 (y'^2 + 1) - a^2(y - xy')^2$	genus 1	—	—	NoStrongRatGenSol
1.508	$-y^2 + 2xy'y + (y^4 + x^2(y^2 - 1))y'^2$	$\left( -\frac{2itx}{t^2 + 1}, -\frac{2i(t^3 + t)}{(t^2 + 1)^2 + 2it(t^2 - 1)x} \right)$	$\frac{2i(t^4 + t^2)}{(t^2 + 1)^2 + 2it(t^2 - 1)x}$	NoRatGenSol	NoRatGenSol
1.509	$-6xy'y^5 + 8(x^2 - 1)y'^2 y^4 - 4x^2$	genus 2	—	—	NoStrongRatGenSol
1.510	$2x^2 y' y^5 + x^4 (y'^2 - 1)y^4 - 2x^4 y' y^3 + y^2 - x^2 y'^2$	genus 3	—	—	NoStrongRatGenSol

We used the following definitions. Parametrizations which are too long for Table 3.

$$\begin{aligned}
 \mathcal{P}_{489} &:= \left( \frac{-2it\sqrt{b^2 + 4c - bt^2} + b + 2(t^2 - 1)x}{2\sqrt{a}(t^2 + 1)}, \frac{\sqrt{a}\left(it^2\sqrt{b^2 + 4c} + (-i)\sqrt{b^2 + 4c} - 2bt + 2(t + 1)^2 x\right)}{2it\sqrt{b^2 + 4c} + b(t^2 - 1) - 2(t^2 - 1)x} \right) \\
 \mathcal{P}_{491} &:= \left( \frac{-2t\sqrt{4(a - 1)b + 1} - i(t^2 - 1)(2ax - 1)}{2\sqrt{a - 1}(t^2 + 1)}, \frac{\sqrt{a - 1}\left(t^2\sqrt{4(a - 1)b + 1} - \sqrt{4(a - 1)b + 1} + 2i(t - a(t + 1)^2 x)\right)}{(t^2 - 1)(2ax - 1) - 2it\sqrt{4(a - 1)b + 1}} \right) \\
 \mathcal{P}_{499} &:= \left( -\frac{ax(a^6 x^8 - 2a^3 t x^4 - (a^2 - 1)t^2)}{a^6 x^8 + 2a^3(a^2 - 1)t x^4 - (a^2 - 1)t^2}, \frac{2a^4 x^4(a^3 x^4 + (a^2 - 1)t)}{(a^2 - 1)(a^6 x^8 - 2a^3 t x^4 - (a^2 - 1)t^2)} \right) \\
 \mathcal{P}_{500} &:= \left( \frac{t^2 x \sqrt{\frac{ab}{a-b}} - 2t\sqrt{ab} - x\sqrt{\frac{ab}{a-b}}}{\sqrt{a}(t^2 + 1)}, -\frac{\sqrt{a}\left(t\left(-t\sqrt{a - b\sqrt{ab}} - 2x\sqrt{a - b\sqrt{\frac{ab}{a-b}}} + btx\right) + \sqrt{a - b\sqrt{ab}} + bx\right)}{(a - b)\left(t^2 x\left(-\sqrt{\frac{ab}{a-b}}\right) + 2t\sqrt{ab} + x\sqrt{\frac{ab}{a-b}}\right)} \right) \\
 \mathcal{P}_{501} &:= \left( \frac{\sqrt{\frac{a}{a}}\left(b^2\left(-\left(t^2 - 1\right)\right) + 2bd(t - i)^2 x + 2cd(t - i)^2\right)}{2bd(t^2 + 1)}, -\frac{i\sqrt{\frac{a}{a}}\left(b^2\left(t^2 - 1\right) - 2bd(t - i)^2 x - 2cd(t - i)^2\right)}{ib^2(t + 1)^2 + 2bd(t - i)^2 x + 2cd(t - i)^2} \right)
 \end{aligned}$$

Associated systems which are too long for the table:

$$\begin{aligned}
A_{430} &:= -\frac{(b_0 + b_1 t)(a_0 + t(a_1 + b_0 + (a_2 + b_1)t))}{b_0(c_1 + 2c_2 t + a_1 x + 2a_2 t x) + b_1(c_2 t^2 + a_2 x t^2 - c_0 - a_0 x)} \\
A_{479} &:= -\frac{(b_0 + t(b_1 + b_2 t))(a_0 + t(a_1 + b_0 + t(a_2 + b_1 + b_2 t)))}{-b_2 t(2c_0 + c_1 t + 2a_0 x + a_1 t x) + b_0(c_1 + 2c_2 t + a_1 x + 2a_2 t x) + b_1(c_2 t^2 + a_2 x t^2 - c_0 - a_0 x)} \\
A_{489} &:= \frac{(t^2 + 1) \left( (t + 1) \left( a(t^2 + 1) \left( t\sqrt{b^2 + 4c} - \sqrt{b^2 + 4c} - 2i(t + 1)x \right) - 2(t - 1) \left( t\sqrt{b^2 + 4c} + i(t^2 - 1)x \right) \right) + ib(2a(t^3 + t) + t^4 - 2t^2 + 1) \right)}{\left( -\sqrt{b^2 + 4c} + t \left( t\sqrt{b^2 + 4c} + 2ib - 4ix \right) \right) \left( 2it\sqrt{b^2 + 4c} + b(t^2 - 1) - 2(t^2 - 1)x \right)} \\
A_{491} &:= \frac{(t^2 + 1) \left( \sqrt{4(a - 1)b + 1} (at^4 + 2at^3 - 2at - a - t^4 + 1) - i(at^4 - 2at^3 - 2a(t + 1)^2 x (-2at + t^2 + 1) - 2at^2 - 2at + a + 2t^3 + 2t) \right)}{\left( (t^2 - 1) \sqrt{4(a - 1)b + 1} - 4iatx + 2it \right) \left( (t^2 - 1)(2ax - 1) - 2it\sqrt{4(a - 1)b + 1} \right)} \\
A_{492} &:= \frac{(t + 1)(ta + a + t - 1) \left( t^2 + a^2(t + 1)^2 - 2a(t + 1) + 1 \right)^2}{8at^2(ta + a - 1)^2} \\
A_{493} &:= \frac{(1 - it)(t - i)^2 \left( a(t + i) \left( (\sqrt{2} + 4i)t^2 - 4i + \sqrt{2} \right) - (t - i)^2 \left( (\sqrt{2} - 2i)t + 2 - i\sqrt{2} \right) x \right)}{2 \left( a(t + i)^2 - (t - i)^2 x \right) \left( a(t + i) \left( (\sqrt{2} + 2i)t + 2 + i\sqrt{2} \right) - \sqrt{2}(t - i)^2 x \right)} \\
A_{494} &:= \frac{a(t^2 - 1) \left( a^3(t + 1)^2(t^2 + 1)x - 2(a^2 - 1)a(t^3 + t)x^3 - 12(a^2 - 1)t^2 x^2 - a^2(t^2 - 1)^2 \right) + \sqrt{1 - a^2} \left( -2ix \left( a^3(t^2 - 1)^2(t^2 + 1)x + 4(a^2 - 1)t^3 x^2 + 3a^2 t(t^2 - 1)^2 \right) \right)}{2a^2(t^2 + 1)x^2 \left( a^2(t^2(x^2 + 1) + 2t + x^2 + 1) - ia\sqrt{1 - a^2}(t^2 - 1)x - (t^2 + 1)x^2 \right)} \\
A_{499} &:= \frac{a^{12}(a^2 + 1)x^{16} + 2a^9(5a^4 - 3a^2 - 2)tx^{12} + 6a^6(-2a^4 + a^2 + 1)t^2 x^8 + 2a^3(-3a^6 + 4a^4 + a^2 - 2)t^3 x^4 - (a^2 - 1)^3 t^4}{2a^5(a^2 - 1)x^5(a^6 x^8 - 2a^3 t x^4 - (a^2 - 1)t^2)} \\
A_{500} &:= \frac{(t^2 + 1) \left( 2bt(t^2 - 1) \sqrt{\frac{ab}{a-b}} \sqrt{ab} + a \left( t^4 \sqrt{a - b\sqrt{ab}} - 2t^3 \sqrt{\frac{ab}{a-b}} (\sqrt{ab} - x\sqrt{a-b}) + 2t \sqrt{\frac{ab}{a-b}} (x\sqrt{a-b} + \sqrt{ab}) - \sqrt{a-b}\sqrt{ab} - 4bt^2 x \right) \right)}{2(a-b) \left( t^2 \sqrt{ab} + 2tx \sqrt{\frac{ab}{a-b}} - \sqrt{ab} \right) \left( t^2 x \left( -\sqrt{\frac{ab}{a-b}} \right) + 2t\sqrt{ab} + x\sqrt{\frac{ab}{a-b}} \right)} \\
A_{501} &:= -\frac{bd(t-i)^2(t+i)(t+1) \left( b^2(t^2 - i) - (1+i)bd(t-i)^2 x + (-i)cd(t-i)^2 \right)}{\left( ib^2(t+1)^2 + 2bd(t-i)^2 x + 2cd(t-i)^2 \right) \left( ib^2 t + d(t-i)^2 (bx + c) \right)}
\end{aligned}$$

Table 4: First-order AODEs of degree three in  $y'$ .

ID	AODE	Parametrization	Associated ODE	Solution of assoc. ODE	Solution of AODE
1.518	$y'^3 - (a - y)^2(b - y)^2$	genus 1	—	—	NoSTRONGRATGENSOL
1.520	$y'^3 + y' - y$	$(t^3 + t, t)$	$\frac{t}{3t^2 + 1}$	NoRATGENSOL	NoRATGENSOL
1.521	$y'^3 + xy' - y$	$(t(t^2 + x), t)$	0	$c$	$c(a^2 + x)$
1.522	$y'^3 - (x + 5)y' + y$	$(t(-t^2 + x + 5), t)$	0	$c$	$c(-c^2 + x + 5)$
1.523	$x^3 - ay'x + y'^3$	reducible over $\overline{\mathbb{K}(x)}$	—	—	NoSTRONGRATGENSOL
1.524	$y'^3 - 2yy' + y^2$	$(\frac{2t-1}{t^3}, \frac{2t-1}{t^2})$	$\frac{(1-2t)t^2}{4t-3}$	NoRATGENSOL	NoRATGENSOL
1.525	$y'^3 + ay(2y - xy')$	$(at^2(x - 2t), at(x - 2t))$	$\frac{1}{2}$	$c + \frac{x}{2}$	$-\frac{1}{2}ac(2c + x)^2$
1.526	$(x^2 - y')(xy - y')(y' - y^2)$	reducible	—	—	see Table 6



Table 4: First-order AODEs of degree three in  $y'$  (continued).

ID	AODE	Parametrization	Associated ODE	Solution of assoc. ODE	Solution of AODE
1.527	$-y^5 - xy'y^4 + y'^3$	$\left(\frac{t^3}{t^2x^3 - x^6}, -\frac{t^5}{x^4(t^2 - x^3)^2}\right)$	$\frac{2t}{x}$	$cx^2$	$\frac{c^3}{c^2x - 1}$
1.528	$y'^3 - ay'^2 + abx + by$	$\left(-\frac{t^3 + at^2 - abx}{b}, t\right)$	$\frac{b(a+t)}{(2a-3t)t}$	NoRATGENSOL	NoRATGENSOL
1.529	$y'^2(x + y') - y$	$(t^2(t + x), t)$	$\frac{1-t}{3t+2x}$	NoRATGENSOL	NoRATGENSOL
1.530	$y'^3 - yy'^2 + y^2$	$\left(\frac{1}{t^2 - t^3}, \frac{1}{t - t^2}\right)$	$-\frac{(t-1)t^2}{3t-2}$	NoRATGENSOL	NoRATGENSOL
1.531	$(x^2 - y')(xy^2 - y')(y' - y^4)$	reducible	—	—	see Table 6
1.532	$-d - y + y'(c + y'(b + ay'))$	$(t(c + t(b + at)) - d, t)$	$\frac{t}{3at^2 + 2bt + c}$	NoRATGENSOL	NoRATGENSOL
1.533	$(xy' - y)y'^2 + a$	$\left(\frac{a}{t^2} + tx, t\right)$	0	$c$	$\frac{a}{c^2} + cx$
1.534	$4xy'^3 + y(3 - 6y'^2)$	$\left(\frac{4t^3x}{6t^2 - 3}, t\right)$	$\frac{2t^2 - 1}{4tx}$	NoRATGENSOL	NoRATGENSOL
1.535	$8xy'^3 + y(9 - 12y'^2)$	$\left(\frac{8t^3x}{12t^2 - 9}, t\right)$	$\frac{4t^2 - 3}{8tx}$	NoRATGENSOL	NoRATGENSOL
1.536	$(bx + y')(-a^2y'^2 + x^2y'^2 + 1)$	reducible	—	—	see Table 6
1.537	$y'^6 - 2yx^5 + y'^3x^3 - 3yy'^2x^2 + 3y^2y'x - y^3$	$\left(\frac{(x-t)(t^2(x^5+1)-2tx+x^2)}{t^3x^5}, \frac{(x-t)(t^2(2x^5+1)-2tx+x^2)}{t^3x^6}\right)$	$\frac{t(2t-x)}{x^2}$	$\frac{cx}{c-x^2}$	$-\frac{x(xc^2+1)}{c^3}$
1.538	$2(y + xy')^3 - yy'$	$\left(\frac{t^2}{2(t+x)^3}, \frac{t}{2(t+x)^3}\right)$	$-\frac{4t+x}{t-2x}$	NoRATGENSOL	NoRATGENSOL
1.540	$(2y' - 1)(yy'^2 + x)$	reducible	—	—	see Table 6
1.541	$y^2y'^3 + 2xy' - y$	genus 1	—	—	NoSTRONGRATGENSOL
1.542	$16y^2y'^3 + 2xy' - y$	genus 1	—	—	NoSTRONGRATGENSOL
1.543	$y'x^3 + (y^2y'^3 + y')x - y^3y'^2 - x^2y$	genus 2	—	—	NoSTRONGRATGENSOL
1.544	$y^2y'^2x^7 - 3y^3y'^2x^6 + 3y^4y'x^5 - x^4y^5 + y'^2$	genus 1	—	—	NoSTRONGRATGENSOL

Table 5: First-order AODEs of degree four and higher in  $y'$ .

ID	AODE	Parametrization	Associated ODE	Solution of assoc. ODE	Solution of AODE
1.545	$y'^4 + (a - y)^3(b - y)^2$	genus 1	—	—	NoSTRONGRATGENSOL
1.546	$3x(y'^2 + 1) + y'(y'^3 - 3y' - 6y + 3)$	$\left(\frac{t^4 + 3(x-1)t^2 + 3t + 3x}{6t}, t\right)$	$\frac{t}{t^2 + x}$	NoRATGENSOL	NoRATGENSOL
1.547	$y'^4 - 4y(xy' - 2y)^2$	$\left(\frac{4(tx-2)^2}{t^4}, \frac{4(tx-2)^2}{t^3}\right)$	$-\frac{t^2}{2}$	$-\frac{2}{c-x}$	$c^2(c-x)^2$
1.548	$y'^6 - (a - y)^4(y - b)^3$	genus 1	—	—	NoSTRONGRATGENSOL
1.549	$x^3(y'^2 + 1)^3 - a^2$	reducible over $\mathbb{K}(x)$	—	—	NoSTRONGRATGENSOL
1.553	$ay'^m + by'^n - y$	$(at^m + bt^n, t)$	$\frac{t^2}{amt^m + bnt^n}$	$\begin{cases} \frac{x+c}{2(a+b)}, & \text{if } m=n=2 \\ \frac{x+c}{b}, & \text{if } m=0, n=2 \\ \frac{x+c}{a}, & \text{if } m=2, n=0 \\ \text{NoRATGENSOL,} & \text{otherwise} \end{cases}$	$\begin{cases} \frac{(x+c)^2}{4(a+b)}, & \text{if } m=n=2 \\ a + \frac{(x+c)^2}{b}, & \text{if } m=0, n=2 \\ \frac{(x+c)^2}{a} + b, & \text{if } m=2, n=0 \\ \text{NoRATGENSOL,} & \text{otherwise} \end{cases}$
1.554	$y'^n x^{n-1} - ny'x + y$	$(ntx - t^n x^{n-1}, t)$	$\frac{t-nt}{nx}$	$\begin{cases} c, & \text{if } n=1 \\ \text{NoRATGENSOL,} & \text{otherwise} \end{cases}$	$\begin{cases} c(x-1), & \text{if } n=1 \\ \text{NoRATGENSOL,} & \text{otherwise} \end{cases}$

Table 6 gives results of the algorithm for factors of AODEs from Kamke. The factors are numbered by the original numbering from Kamke with an additional index.

Table 6: Factors of reducible first-order AODEs.

ID	AODE	Parametrization	Associated ODE	Solution of assoc. ODE	Solution of AODE
1.318.1	$y$	$(0, t)$	—	—	—
1.318.2	$-4xy' + y' + y(y^2 + 3xy'y - 2)$	$\left(t, \frac{2t-t^3}{3xt^2-4x+1}\right)$	$-\frac{t(t^2-2)}{3t^2x-4x+1}$	NoRATGENSOL	NoRATGENSOL
1.391.1	$ay + y'$	$(t, -at)$	$-at$	NoRATGENSOL	NoRATGENSOL
1.391.2	$bx + y'$	$(t, -bx)$	$-bx$	$c - \frac{bx^2}{2}$	$c - \frac{bx^2}{2}$
1.396.1	$(xy - y')$	$(t, xt)$	$tx$	NoRATGENSOL	NoRATGENSOL
1.396.2	$(y^2 + y')$	$(t, -t^2)$	$-t^2$	$\frac{1}{x-c}$	$\frac{1}{x-c}$
1.438.1	$y + xy'$	$\left(t, -\frac{t}{x}\right)$	$-\frac{t}{x}$	$\frac{c}{x}$	$\frac{c}{x}$
1.438.2	$2y + xy'$	$\left(t, -\frac{2t}{x}\right)$	$-2\frac{t}{x}$	$\frac{c}{x^2}$	$\frac{c}{x^2}$
1.439.1	$\left(\frac{3}{2} + \frac{i\sqrt{3}}{2}\right)y + xy'$	$\left(t, \frac{-i\sqrt{3}t-3t}{2x}\right)$	$-\frac{1}{2}t\frac{i\sqrt{3}+3}{x}$	NoRATGENSOL	NoRATGENSOL
1.439.2	$\left(\frac{3}{2} - \frac{i\sqrt{3}}{2}\right)y + xy'$	$\left(t, \frac{i\sqrt{3}t-3t}{2x}\right)$	$\frac{1}{2}t\frac{i\sqrt{3}-3}{x}$	NoRATGENSOL	NoRATGENSOL
1.440.1	$y - xy'$	$\left(t, \frac{t}{x}\right)$	$\frac{t}{x}$	$cx$	$cx$
1.440.2	$5y + xy'$	$\left(t, -\frac{5t}{x}\right)$	$-5\frac{t}{x}$	$\frac{c}{x^5}$	$\frac{c}{x^5}$
1.442.1	$(x-1)y + xy'$	$\left(t, \frac{t(1-x)}{x}\right)$	$-t\frac{x-1}{x}$	NoRATGENSOL	NoRATGENSOL
1.442.2	$x(x+y') - y$	$\left(t, \frac{t}{x} - x\right)$	$\frac{t-x^2}{x}$	$(c-x)x$	$(c-x)x$
1.445.1	$ay^3 + y'$	$(t, -at^3)$	$-at^3$	NoRATGENSOL	NoRATGENSOL
1.445.2	$y'x^2 + b$	$\left(t, -\frac{b}{x^2}\right)$	$-\frac{b}{x^2}$	$\frac{b}{x} + c$	$\frac{b}{x} + c$
1.449.1	$y + (x-a)y'$	$\left(t, \frac{t}{a-x}\right)$	$\frac{t}{a-x}$	$\frac{c}{a-x}$	$\frac{c}{a-x}$
1.449.2	$y + (a+x)y'$	$\left(t, -\frac{t}{a+x}\right)$	$-\frac{t}{a+x}$	$\frac{c}{a+x}$	$\frac{c}{a+x}$
1.471.1	$y' - 1$	$(t, 1)$	1	$c+x$	$c+x$
1.471.2	$x + yy'$	$\left(t, -\frac{x}{t}\right)$	$-\frac{x}{t}$	NoRATGENSOL	NoRATGENSOL
1.481.1	$y + xy'$	$\left(t, -\frac{t}{x}\right)$	$-\frac{t}{x}$	$\frac{c}{x}$	$\frac{c}{x}$
1.481.2	$x + yy'$	$\left(t, -\frac{x}{t}\right)$	$-\frac{x}{t}$	NoRATGENSOL	NoRATGENSOL
1.495.1	$\sqrt{a-1}(xy' - y) + x + yy'$	$\left(t, \frac{\sqrt{a-1t-x}}{\sqrt{a-1x+t}}\right)$	$\frac{\sqrt{a-1t-x}}{\sqrt{a-1x+t}}$	NoRATGENSOL	NoRATGENSOL
1.495.2	$-\sqrt{a-1}(xy' - y) + x + yy'$	$\left(t, \frac{\sqrt{a-1t+x}}{\sqrt{a-1x-t}}\right)$	$\frac{\sqrt{a-1t+x}}{\sqrt{a-1x-t}}$	NoRATGENSOL	NoRATGENSOL
1.505.1	$x - yy'$	$\left(t, \frac{x}{t}\right)$	$\frac{x}{t}$	NoRATGENSOL	NoRATGENSOL
1.505.2	$x^2 + yy'x - 2y^2$	$\left(t, \frac{2t^2-x^2}{tx}\right)$	$\frac{2xt^2-x^2}{tx}$	NoRATGENSOL	NoRATGENSOL
1.526.1	$x^2 - y'$	$(t, x^2)$	$x^2$	$\frac{x^3}{3} + c$	$\frac{x^3}{3} + c$
1.526.2	$xy - y'$	$(t, tx)$	$tx$	NoRATGENSOL	NoRATGENSOL
1.526.3	$y' - y^2$	$(t, t^2)$	$t^2$	$\frac{1}{c-x}$	$\frac{1}{c-x}$
1.531.1	$x^2 - y'$	$(t, x^2)$	$x^2$	$\frac{x^3}{3} + c$	$\frac{x^3}{3} + c$
1.531.2	$xy^2 - y'$	$(t, t^2x)$	$t^2x$	$-\frac{2c}{cx^2+1}$	$-\frac{2c}{cx^2+1}$
1.531.3	$y' - y^4$	$(t, t^4)$	$t^4$	NoRATGENSOL	NoRATGENSOL
1.536.1	$bx + y'$	$(t, -bx)$	$-bx$	$c - \frac{bx^2}{2}$	$c - \frac{bx^2}{2}$

Table 6: Factors of reducible first-order AODEs (continued).

ID	AODE	Parametrization	Associated ODE	Solution of assoc. ODE	Solution of AODE
1.536.2	$(x^2 - a^2)y'^2 + 1$	reducible over $\overline{\mathbb{K}(x)}$	—	—	NoSTRONGRATGENSOL
1.540.1	$2y' - 1$	$(t, \frac{1}{2})$	$\frac{1}{2}$	$\frac{1}{2}x + c$	$c + \frac{x}{2}$
1.540.2	$yy'^2 + x$	$(-\frac{x}{t^2}, t)$	$\frac{1}{2} \frac{(t^3+1)*t}{x}$	NoRATGENSOL	NoRATGENSOL

### 3 First-Order AODEs from Polyanin and Sajzew [2]

In this section we give rational general solutions of strongly parametrizable AODEs from Polyanin and Sajzew [2]. We also list those AODEs which do not have a rational general solution. Table 7 shows the absolute and relative numbers of certain classes of ODEs in this collection.

Type	# of ODEs in Polyanin and Sajzew	%	%
first-order	347	100	
first-order AODE	109	31.41	100
SP1AODE	104	29.97	95.41

Table 7: Number of AODEs in Polyanin and Sajzew

Note, that the first Section 1.1 in [2] contains only ODEs depending on arbitrary functions. For certain choices of these functions the ODEs might be algebraic and then of course the algorithm of [3] is applicable. However, we do not treat them here. Each section of [2] is given in a separate table (Table 8-11).

Concerning parameters in the AODEs, we assume the same setting as in the previous section.

Figure 1 shows these classes on a more detailed and illustrative basis what is given in Table 7.

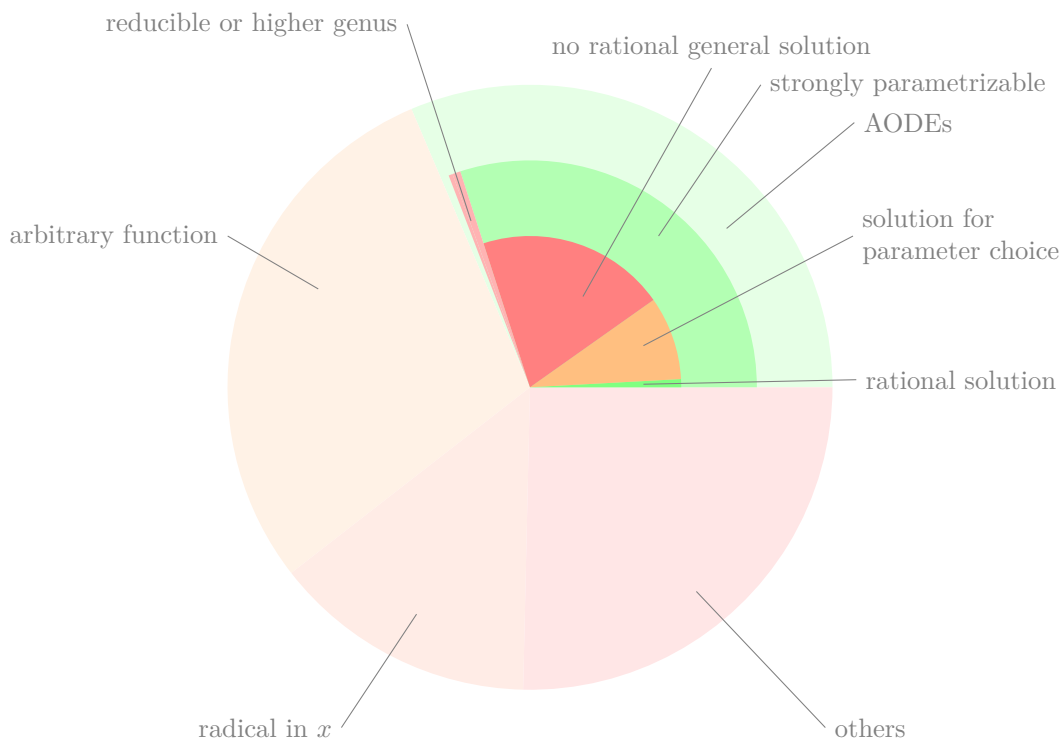


Figure 2: first-order ODEs in Polyanin and Sajzew [2]

Table 8: First-order Riccati AODEs.

ID	AODE	Parametrization	Solution of AODE
1.2.1	$-ay^2 + c - bx + y'$	$(t, at^2 - c + bx)$	NoRatGenSol
1.2.2	$-ay^2x^2 + y'x^2 + b$	$(t, \frac{at^2x^2 - b}{x^2})$	NoRatGenSol
1.2.3	$y^2x^4 + y'x^4 + a^2$	$(t, \frac{-t^2x^4 - a^2}{x^4})$	NoRatGenSol
1.2.4	$-bx^n - ay^2 + y'$	$(t, bx^n + at^2)$	$\begin{cases} -\frac{-ck + c + kx^k + x^k}{2acx + 2ax^{k+1}}, & \text{if } n = -2, ab = \frac{1 - n^2}{4}, n \in \mathbb{Z} \setminus \{0\} \\ \text{NoRatGenSol}, & \text{otherwise} \end{cases}$
1.2.5	$-ax^2 - bx - y^2 - c + y'$	$(t, t^2 + ax^2 + c + bx)$	NoRatGenSol
1.2.6	$-a^2x^4 + y^2x^2 + a(1 - b)x^2 - y'x^2 - b(b + 1)$	$(t, \frac{-a^2x^4 + t^2x^2 + ax^2 - abx^2 - b(b+1)}{x^2})$	NoRatGenSol
1.2.7	$bx^{q-1} + ayx^q + y^2 - y'$	$(t, bx^{q-1} + atx^q + t^2)$	$\begin{cases} \frac{c(-a + n - 1) - (a + n + 1)x^n}{2x(c + x^n)}, & \text{if } q = -1, 4b - a(2 + a) = 1 - n^2, n \in \mathbb{Z} \setminus \{0\} \\ \text{NoRatGenSol}, & \text{otherwise} \end{cases}$
1.2.8	$anx^{n-1} - a^2x^{2n} + y^2 - y'$	$(t, anx^{n-1} - a^2x^{2n} + t^2)$	$\begin{cases} -\frac{-ck + c + (k + 1)x^k}{2x(c + x^k)}, & \text{if } n = -1, -a(1 + a) = \frac{1 - k^2}{4}, k \in \mathbb{Z} \setminus \{0\} \\ \text{NoRatGenSol}, & \text{otherwise} \end{cases}$
1.2.9	$cx^{n-1} + bx^{2n} + ay^2 - y'$	$(t, cx^{n-1} + bx^{2n} + at^2)$	$\begin{cases} -\frac{-c_1k + c_1 + kx^k + x^k}{2ac_1x + 2ax^{k+1}}, & \text{if } n = -1, a(b + c) = \frac{1 - k^2}{4}, k \in \mathbb{Z} \setminus \{0\} \\ \text{NoRatGenSol}, & \text{otherwise} \end{cases}$
1.2.10	$ax^2 + bx + y^2 + c - y' + y(x\alpha + \beta)$	$(t, t^2 + x\alpha t + \beta t + ax^2 + c + bx)$	NoRatGenSol
1.2.11	$-abx^n + ayx^n - b^2 + y^2 - y'$	$(t, -abx^n + atx^n - b^2 + t^2)$	NoRatGenSol
1.2.12	$bx^{n-2} + ay^2x^n - y'$	$(t, bx^{n-2} + at^2x^n)$	$\begin{cases} -\frac{-ck + c + kx^k + x^k}{2acx + 2ax^{k+1}}, & \text{if } n = 0, ab = \frac{1 - k^2}{4}, k \in \mathbb{Z} \setminus \{0\} \\ \text{NoRatGenSol}, & \text{otherwise} \end{cases}$
1.2.13	$-(k + 1)y^2x^k - ax^{m-1} + ayx^{k+m} - y'$	$(t, -kt^2x^k - t^2x^k - ax^{m-1} + atx^{k+m})$	$\begin{cases} \frac{c(a + k - n + 1) + x^n(a + k + n + 1)}{2(k + 1)x^{k+1}(c + x^n)}, & \text{if } -m = k + 1, (a - k)^2 + k - 2a = n^2 - 1, n \in \mathbb{Z} \setminus \{0\} \\ \text{NoRatGenSol}, & \text{otherwise} \end{cases}$
1.2.14	$x\alpha y^2 + \beta y - xy' + x\gamma + \delta$	$(t, \frac{x\alpha t^2 + \beta t + x\gamma + \delta}{x})$	NoRatGenSol
1.2.15	$cx^{2b} - y'x + ay^2 + by$	$(t, \frac{cx^{2b} + at^2 + bt}{x})$	$\begin{cases} \frac{n(c_1 - x^n)}{2a(c_1 + x^n)}, & \text{if } b = 0, 1 - b^2 + 4ac = 1 - n^2, n \in \mathbb{Z} \setminus \{0\} \\ \text{NoRatGenSol}, & \text{otherwise} \end{cases}$
1.2.16	$cx^n - y'x + ay^2 + by$	$(t, \frac{cx^n + at^2 + bt}{x})$	$\begin{cases} -\frac{b(c_1 + x^k) + k(x^k - c_1)}{2a(c_1 + x^k)}, & \text{if } n = 0, 1 - b^2 + 4ac = 1 - k^2, k \in \mathbb{Z} \setminus \{0\} \\ \text{NoRatGenSol}, & \text{otherwise} \end{cases}$
1.2.17	$cx^{2n} - y'x + ay^2 + (bx^n + n)y$	$(t, \frac{btx^n + cx^{2n} + at^2 + nt}{x})$	$\begin{cases} -\frac{b(c_1 + x^k) + k(x^k - c_1)}{2a(c_1 + x^k)}, & \text{if } n = 0, -1 + b^2 - 4ac = k^2 - 1, k \in \mathbb{Z} \setminus \{0\} \\ \text{NoRatGenSol}, & \text{otherwise} \end{cases}$
1.2.18	$bx^n + y^2x - y'x + ay$	$(t, \frac{bx^n + t^2x + at}{x})$	$\begin{cases} \frac{c(-a + k - 1) - (a + k + 1)x^k}{2x(c + x^k)}, & \text{if } n = -1, 2a + a^2 - 4b = k^2 - 1, k \in \mathbb{Z} \setminus \{0\} \\ \text{NoRatGenSol}, & \text{otherwise} \end{cases}$
1.2.19	$byx^n + ay^2x^{2n} - y'x^{n+1} + c$	$(t, x^{-n-1}(btx^n + at^2x^{2n} + c))$	$\begin{cases} \frac{x^{-2n-1}(b(n+1)x^{n+2} + k(\frac{2c_1}{c_1+x^k} - 1) - 2n - 1)}{2a}, & \text{if } (n+b)^2 - 1 - 4ac = k^2 - 1, k \in \mathbb{Z} \setminus \{0\} \\ \text{NoRatGenSol}, & \text{otherwise} \end{cases}$

Table 8: First-order Riccati AODEs (continued).

ID	AODE	Parametrization	Solution of AODE
1.2.20	$y^2 x^{2a} + x^{b+2} - y'x + (b-a)y$	$\left(t, \frac{t^2 x^{2a} + x^{b+2} - at + bt}{x}\right)$	$\begin{cases} \frac{1}{2} x^{-2a} \left(-3a + k \left(\frac{2c}{c+x^k} - 1\right) + 2\right), & \text{if } b+2 = 2a, -1 + a^2 + 2ab + b^2 - 4 = k^2 - 1, k \in \mathbb{Z} \setminus \{0\} \\ \text{NoRatGenSol}, & \text{otherwise} \end{cases}$
1.2.21	$cx^m + ay^2 x^n - y'x + by$	$\left(t, \frac{cx^m + at^2 x^n + bt}{x}\right)$	$\begin{cases} \frac{x^{-n} \left(-b + k \left(\frac{2c_1}{c_1+x^k} - 1\right) - n\right)}{2a}, & \text{if } m = -n, -1 + b^2 + 2bn + n^2 - 4ac = k^2 - 1, k \in \mathbb{Z} \setminus \{0\} \\ \text{NoRatGenSol}, & \text{otherwise} \end{cases}$
1.2.22	$ay^2 x^{2n} - y'x + c + (bx^n - n)y$	$\left(t, \frac{bt x^n + at^2 x^{2n} + c - nt}{x}\right)$	$\begin{cases} -\frac{b - \frac{2ck}{c+x^k} + k}{2a}, & \text{if } n = 0, -1 + b^2 - 4ac = k^2 - 1, k \in \mathbb{Z} \setminus \{0\} \\ \text{NoRatGenSol}, & \text{otherwise} \end{cases}$
1.2.23	$cx^m + ay^2 x^{m+2n} - y'x + (bx^{m+n} - n)y$	$\left(t, \frac{cx^m + bt x^{m+n} + at^2 x^{m+2n} - nt}{x}\right)$	$\begin{cases} \frac{x^{-n} (c(k-b) - (b+k)x^k)}{2a(c+x^k)}, & \text{if } m = -n, -1 + b^2 - 4ac = k^2 - 1, k \in \mathbb{Z} \setminus \{0\} \\ \text{NoRatGenSol}, & \text{otherwise} \end{cases}$
1.2.24	$b_2 + a_2 x + (b_1 + a_1 x)y + (b_0 + a_0 x)(\lambda y^2 + y')$	$\left(t, \frac{-b_0 \lambda t^2 - a_0 x \lambda t^2 - b_1 t - a_1 x t - b_2 - a_2 x}{b_0 + a_0 x}\right)$	NoRatGenSol
1.2.25	$-2xa^2 + 2y^2 + xy - 2x^2 y'$	$\left(t, -\frac{2xa^2 - 2t^2 - tx}{2x^2}\right)$	NoRatGenSol
1.2.26	$-2xa^2 + 2y^2 + 3xy - 2x^2 y'$	$\left(t, -\frac{2xa^2 - 2t^2 - 3tx}{2x^2}\right)$	NoRatGenSol
1.2.27	$ay^2 x^2 - y'x^2 + byx + c$	$\left(t, \frac{at^2 x^2 + bt x + c}{x^2}\right)$	NoRatGenSol
1.2.28	$cy^2 x^2 - y'x^2 + \alpha x^2 + \beta x + (ax^2 + bx)y + \gamma$	$\left(t, \frac{ct^2 x^2 + at x^2 + \alpha x^2 + bt x + \beta x + \gamma}{x^2}\right)$	NoRatGenSol
1.2.29	$bx^n + ay^2 x^2 - y'x^2 + c$	$\left(t, \frac{bx^n + at^2 x^2 + c}{x^2}\right)$	$\begin{cases} -\frac{c_1(1-k) + x^k(k+1)}{2ax(c_1+x^k)}, & \text{if } n = 0, 4a(b+c) = 1 - k^2, k \in \mathbb{Z} \setminus \{0\} \\ \text{NoRatGenSol}, & \text{otherwise} \end{cases}$
1.2.30	$yx^m - y'x^2 + y^2 \alpha x^2 + y\beta x + \delta$	$\left(t, \frac{tx^m + t^2 \alpha x^2 + t\beta x + \delta}{x^2}\right)$	$\begin{cases} \frac{c(-\beta + k - 2) - (\beta + k + 2)x^k}{2\alpha x(c+x^k)}, & \text{if } m = 1, -4\alpha\delta + \beta^2 + 4\beta + 3 = k^2 - 1, k \in \mathbb{Z} \setminus \{0\} \\ \text{NoRatGenSol}, & \text{otherwise} \end{cases}$
1.2.31	$\delta x^m + \gamma x^{2m} + y^2 \alpha x^2 + y\beta x - x^2 y'$	$\left(t, \frac{\delta x^m + \gamma x^{2m} + t^2 \alpha x^2 + t\beta x}{x^2}\right)$	$\begin{cases} \frac{c(-\beta + k - 1) - (\beta + k + 1)x^k}{2\alpha x(c+x^k)}, & \text{if } m = 0, -4\alpha(\gamma + \delta) + \beta^2 + 2\beta = k^2 - 1, k \in \mathbb{Z} \setminus \{0\} \\ \text{NoRatGenSol}, & \text{otherwise} \end{cases}$
1.2.32	$\beta x^n + \alpha x^{2n} + cy^2 x^2 - y'x^2 + (ax^n + b)yx + \gamma$	$\left(t, \frac{\beta x^n + \alpha x^{2n} + at x^{n+1} + ct^2 x^2 + bt x + \gamma}{x^2}\right)$	$\begin{cases} \frac{k(c_1 - x^k) - (a+b+1)(c_1 + x^k)}{2cx(c_1 + x^k)}, & \text{if } n = 0, (a+b)^2 + 2(a+b) - 4c(\alpha + \beta + \gamma) = k^2 - 1, k \in \mathbb{Z} \setminus \{0\} \\ \text{NoRatGenSol}, & \text{otherwise} \end{cases}$
1.2.33	$(x^2 - 1)y' + (y^2 - 2xy + 1)\lambda$	$\left(t, \frac{(-t^2 + 2xt - 1)\lambda}{x^2 - 1}\right)$	NoRatGenSol
1.2.34	$\alpha y^2 + x\beta y + (ax^2 + b)y' + \frac{b(a+\beta)}{\alpha}$	$\left(t, \frac{-t^2 \alpha^2 - tx\beta\alpha - ab - b\beta}{(ax^2 + b)\alpha}\right)$	NoRatGenSol
1.2.35	$\alpha y^2 + x\beta y + (ax^2 + b)y' + \gamma$	$\left(t, \frac{-x\beta t - t^2 \alpha - \gamma}{ax^2 + b}\right)$	NoRatGenSol
1.2.36	$(1-a)x^2 - 2yx + y^2 - b + (ax^2 + b)y'$	$\left(t, \frac{-t^2 + 2xt + ax^2 - x^2 + b}{ax^2 + b}\right)$	NoRatGenSol
1.2.37	$c + (b_1 + b_0 x)y + (a_0 x^2 + a_1 x + a_2)(y^2 + y')$	$\left(t, \frac{-a_0 x^2 t^2 - a_2 t^2 - a_1 x t^2 - b_1 t - b_0 x t - c}{a_0 x^2 + a_1 x + a_2}\right)$	NoRatGenSol
1.2.38	$y^2 + k(x+y-a)(x+y-b) + (x-a)(x-b)y'$	$\left(t, \frac{ak(b-t-x) - bk(t+x) + (k+1)t^2 + 2ktx + kx^2}{(a-x)(x-b)}\right)$	NoRatGenSol
1.2.39	$y^2 \alpha x^3 + \delta x - x^3 y' + y(x^2 \beta - \gamma)$	$\left(t, \frac{t^2 \alpha x^3 + t\beta x^2 + \delta x - t\gamma}{x^3}\right)$	NoRatGenSol

Table 8: First-order Riccati AODEs (continued).

ID	AODE	Parametrization	Solution of AODE
1.2.40	$ay^2x^3 - y'x^3 + (c + bx)yx + \alpha x + \beta$	$\left(t, \frac{at^2x^3 + btx^2 + ctx + \alpha x + \beta}{x^3}\right)$	NoRatGenSol
1.2.41	$x^2 - y^2 + (ax^2 + bx + c)(xy' - y)$	$\left(t, \frac{t^2 + ax^2t + ct + bxt - x^2}{x(ax^2 + bx + c)}\right)$	NoRatGenSol
1.2.42	$(cx^2 + b)y + x(x^2 + a)(\alpha y^2 + y') + x\beta$	$\left(t, \frac{-t^2\alpha x^3 - ctx^2 - at^2\alpha x - \beta x - bt}{x(x^2 + a)}\right)$	NoRatGenSol
1.2.43	$(a + x)(\lambda y^2 + y')x^2 + (c + bx)yx + \alpha x + \beta$	$\left(t, \frac{-t^2\lambda x^3 - btx^2 - at^2\lambda x^2 - ctx - \alpha x - \beta}{x^2(a + x)}\right)$	NoRatGenSol
1.2.44	$(x^2 + a)(\lambda y^2 + y')x^2 + (bx^2 + c)yx + \alpha$	$\left(t, \frac{-t^2\lambda x^4 - btx^3 - at^2\lambda x^2 - ctx - \alpha}{x^2(x^2 + a)}\right)$	NoRatGenSol
1.2.45	$\alpha x^2 + a(x - 1)^2(\lambda y^2 + y')x^2 + \beta x + \gamma$	$\left(t, \frac{-at^2\lambda x^4 + 2at^2\lambda x^3 - at^2\lambda x^2 - \beta x - x^2\alpha - \gamma}{ax^2(x^2 - 2x + 1)}\right)$	NoRatGenSol
1.2.46	$\alpha x^2 + \beta x + (x^2 - l)y\mu x + \gamma$ $+ a(x^2 - 1)^2(\lambda y^2 + y')$	$\left(t, \frac{a\lambda t^2 + \gamma - x(a\lambda t^2(x^2 - 2) + \alpha) + \beta + \mu t(x^2 - l)}{a(x^2 - 1)^2}\right)$	NoRatGenSol
1.2.47	$(y^2 + y')(ax^2 + bx + c)^2 + A$	$\left(t, -\frac{A}{(x(ax+b)+c)^2} - t^2\right)$	NoRatGenSol
1.2.48	$\gamma x^m + (ax^m - 1)(\lambda y^2 + y')x^2$ $+ y(\alpha x^m + \beta)x + \delta$	$\left(t, \frac{-\gamma x^m - t\alpha x^{m+1} - at^2\lambda x^{m+2} + t^2\lambda x^2 - t\beta x - \delta}{x^2(ax^m - 1)}\right)$	$\begin{cases} \frac{\alpha + \beta}{1 - a} - \frac{2ck}{c + x^k} + k + 1, & \text{if } m = 0, \frac{(\alpha + \beta)(\alpha + \beta + 2 - 2a) - 4(a - 1)\lambda(\gamma + \delta)}{4(a - 1)^2} = \frac{k^2 - 1}{4}, k \in \mathbb{Z} \setminus \{0\} \\ ?, & \text{otherwise} \end{cases}$

Table 9: First-order Abel AODEs.

ID	AODE	Parametrization	Associated ODE	Solution of assoc. ODE	Solution of AODE
1.3.1	$-A - y + yy'$	$\left(t, \frac{A+t}{t}\right)$	$\frac{A+t}{t}$	NoRatGenSol	NoRatGenSol
1.3.2	$-B - Ax - y + yy'$	$\left(t, \frac{B+t+Ax}{t}\right)$	$\frac{Ax+B+t}{t}$	NoRatGenSol	NoRatGenSol
1.3.5	$-Ax^4 - yx^3 + yy'x^3 - Bx^2 + B^2$	$\left(t, \frac{Ax^4 + tx^3 + Bx^2 - B^2}{tx^3}\right)$	$\frac{x^3(Ax+t) - B^2 + Bx^2}{tx^3}$	NoRatGenSol	NoRatGenSol
1.3.6	$x^{k-1} - Bkx^k + B^2kx^{2k-1} + A + y - yy'$	$\left(t, \frac{x^{k-1} - Bkx^k + B^2kx^{2k-1} + A + t}{t}\right)$	$\frac{x^{k-1} - Bkx^k + B^2kx^{2k-1} + A + t}{t}$	NoRatGenSol	NoRatGenSol
1.3.7	$yx^3 - yy'x^3 + x^2 - A^2 + A$	$\left(t, \frac{tx^3 + x^2 - A^2 + A}{tx^3}\right)$	$\frac{-A^2 + A + tx^3 + x^2}{tx^3}$	NoRatGenSol	NoRatGenSol
1.3.10	$Ax^m - \frac{2(m+1)x}{(m+3)^2} + y - yy'$	$\left(t, \frac{Ax^m}{t} - \frac{2(m+1)x}{t(m+3)^2} + 1\right)$	$\frac{Ax^m}{t} - \frac{2(m+1)x}{t(m+3)^2} + 1$	NoRatGenSol	NoRatGenSol
1.3.13	$\frac{(2m+1)x^4}{4m^2} + yx^3 - yy'x^3 + Ax^2 - A^2$	$\left(t, -\frac{2mx^4 - x^4 - 4m^2tx^3 - 4Am^2x^2 + 4A^2m^2}{4m^2tx^3}\right)$	$\frac{-4A^2m^2 + 4Am^2x^2 + x^3(4m^2t + 2mx + x)}{4m^2tx^3}$	NoRatGenSol	NoRatGenSol
1.3.14	$2A^2x^3 + 2Ax^2 + \frac{4x}{9} + y - yy'$	$\left(t, \frac{18A^2x^3 + 18Ax^2 + 4x + 9t}{9t}\right)$	$\frac{2x(9A^2x^2 + 9Ax + 2) + 9t}{9t}$	NoRatGenSol	NoRatGenSol
1.3.16	$-A - xy + xyy'$	$\left(t, \frac{A+tx}{tx}\right)$	$\frac{A}{tx} + 1$	NoRatGenSol	NoRatGenSol
1.3.19	$-2x^3 - yx^2 + yy'x^2 - A$	$\left(t, \frac{2x^3 + tx^2 + A}{tx^2}\right)$	$\frac{A}{tx^2} + \frac{2x}{t} + 1$	NoRatGenSol	NoRatGenSol
1.3.33	$-yx^2 + yy'x^2 - A$	$\left(t, \frac{tx^2 + A}{tx^2}\right)$	$\frac{A}{tx^2} + 1$	NoRatGenSol	NoRatGenSol
1.3.44	$-Ax^2 - y + yy' + \frac{9}{625}A$	$\left(t, \frac{625A^2x^2 + 625At - 9}{625At}\right)$	$\frac{Ax^2}{t} - \frac{9}{625At} + 1$	NoRatGenSol	NoRatGenSol
1.3.45	$-Ax^2 + \frac{6x}{25} - y + yy'$	$\left(t, \frac{25Ax^2 - 6x + 25t}{25t}\right)$	$\frac{Ax^2 - \frac{6x}{25}}{t} + 1$	NoRatGenSol	NoRatGenSol
1.3.46	$-Ax^2 - \frac{6x}{25} - y + yy'$	$\left(t, \frac{25Ax^2 + 6x + 25t}{25t}\right)$	$\frac{Ax^2 + \frac{6x}{25}}{t} + 1$	NoRatGenSol	NoRatGenSol
1.3.54	$-6x^5 - yx^4 + yy'x^4 - A$	$\left(t, \frac{6x^5 + tx^4 + A}{tx^4}\right)$	$\frac{A}{tx^4} + \frac{6x}{t} + 1$	NoRatGenSol	NoRatGenSol

Table 9: First-order Abel AODEs (continued).

ID	AODE	Parametrization	Associated ODE	Solution of assoc. ODE	Solution of AODE
1.3.56	$-\frac{15x^8}{4} - yx^7 + yy'x^7 - A$	$\left(t, \frac{15x^8 + 4tx^7 + 4A}{4tx^7}\right)$	$\frac{A}{tx'} + \frac{15x}{4t} + 1$	NO RAT GEN SOL	NO RAT GEN SOL

Table 10: First-order AODEs with other linear derivative  $y'$ .

ID	AODE	Parametrization	Solution of AODE
1.4.1	$4x - 3y(y+1) + (8y-1)y'$	$\left(t, \frac{3t^2 + 3t - 4x}{8t-1}\right)$	NO RAT GEN SOL
1.4.2	$-y^2 - xy - b + (x^2 + yx + a)y'$	$\left(t, \frac{t^2 + xt + b}{x^2 + tx + a}\right)$	NO RAT GEN SOL
1.4.3	$x^2 + 2yx + b + (x^2 + y^2 + a)y'$	$\left(t, \frac{-x^2 - 2tx - b}{t^2 + x^2 + a}\right)$	NO RAT GEN SOL
1.4.4	$-By^2 - b_2y - Axy - c_2 - a_2x + (Ax^2 + a_1x + Byx + c_1 + b_1y)y'$	$\left(t, \frac{Bt^2 + b_2t + Axt + c_2 + a_2x}{Ax^2 + a_1x + Btx + c_1 + b_1t}\right)$	NO RAT GEN SOL
1.4.5	$axy^3 + by^2 - y'$	$\left(t, ax^3 + bt^2\right)$	NO RAT GEN SOL
1.4.6	$(b+ax)y^3 + cy^2 + (b+ax)^2y'$	$\left(t, -\frac{t^2(c+bt+atx)}{b^2+2axb+a^2x^2}\right)$	NO RAT GEN SOL
1.4.7	$x(by^3 + (ax+by)^3) + y(ax^3 + (ax+by)^3)y'$	$\left(t, -\frac{x((ax+bt)^3 + bt^3)}{t((ax+bt)^3 + ax^3)}\right)$	NO RAT GEN SOL
1.4.9	$Ax^k - By^s - y'$	$\left(t, Ax^k - Bt^s\right)$	$\begin{cases} \frac{Ax^{k+1}}{k+1} - Bx + c, & \text{if } s = 0, k \in \mathbb{Z} \setminus \{-1\} \\ \frac{-cn + c + nx^n + x^n}{2Bcx + 2Bx^{n+1}}, & \text{if } s = 2, k = -2, n \in \mathbb{Z} \setminus \{0\} \\ \text{NO RAT GEN SOL,} & \text{otherwise} \end{cases}$
1.4.10	$-y' + ax^{m-n-mn}y^n + bx^m$	$\left(t, ax^{m-n-mn}t^n + bx^m\right)$	$\begin{cases} \frac{(a+b)x^{m+1}}{m+1} + c, & \text{if } n = 0 \\ \frac{bx^{m+1}}{-a+m+1} + cx^a, & \text{if } n = 1, a \in \mathbb{Z} \\ \frac{x^{m+1}(c(k+m+1) + (-k+m+1)x^k)}{2a(c+x^k)}, & \text{if } n = 2, 4ab - m(2+m) = 1 - k^2, k \in \mathbb{Z} \setminus \{0\} \\ \text{NO RAT GEN SOL,} & \text{otherwise} \end{cases}$
1.4.11	$by + x(xy^n + a)y'$	$\left(t, -\frac{bt}{x(xt^n + a)}\right)$	NO RAT GEN SOL
1.4.12	$(a(2n+1)x^2 + cx + b(2n-1))yx^{n-2} - x^{2n-3}(a^2nx^4 + acx^3 + dx^2 + bcx + b^2n) - yy'$	$\mathcal{P}_{4,12}$	NO RAT GEN SOL
1.4.13	$bx^{n-k}y^k + ax^{n-m}y^m + y - xy'$	$\left(t, bx^{n-k-1}t^k + ax^{n-m-1}t^m + \frac{t}{x}\right)$	$\begin{cases} \frac{(a+b)x^n}{n-1} + xc, & \text{if } k = m = 0, n \neq 1 \\ -\frac{kx(x^k - c)}{2a(c+x^k)}, & \text{if } k = 0, m = 2, n = 1, 1 + 4ab = 1 - n^2, n \in \mathbb{Z} \setminus \{0\} \\ -\frac{bx^{b+1}}{ax^b + ac}, & \text{if } k = 1, m = 2, b \in \mathbb{Z} \setminus \{0\} \\ -\frac{x^{2-n}(c(-k+n-1) + (k+n-1)x^k)}{2(a+b)(c+x^k)}, & \text{if } k = m = 2, n = 1 \pm k, k \in \mathbb{Z} \setminus \{0\} \\ cx, & \text{if } k = m > 2, b = -a \\ \text{NO RAT GEN SOL,} & \text{otherwise} \end{cases}$
1.4.14	$\alpha y^3 + \beta y^2 + (bx^n + ayx)y'$	$\left(t, -\frac{t^2(t\alpha + \beta)}{bx^n + atx}\right)$	$\begin{cases} cx^{-k}, & \text{if } n = 1, ka = \beta, kb = \alpha, k \in \mathbb{Z} \setminus \{0\} \\ \text{NO RAT GEN SOL,} & \text{otherwise} \end{cases}$



Table 10: First-order AODEs with other linear derivative  $y'$  (continued).

ID	AODE	Parametrization	Solution of AODE
1.4.15	$x^n(by + axy')y^m + \beta y + xy'\alpha$	$\left( t, -\frac{t(bx^n t^m + \beta)}{x(ax^n t^m + \alpha)} \right)$	$\begin{cases} cx^{-k}, & \text{if } b = ka, \beta = k\alpha, k \in \mathbb{Z} \setminus \{0\} \\ cx \frac{\beta}{\alpha} (ax^n + \alpha) \frac{b}{na} - \frac{\beta}{n\alpha}, & \text{if } m = 0, \alpha \mid \beta, na\alpha \mid (\alpha b - a\beta) \\ cx \frac{b+\beta}{a}, & \text{if } m = n = \alpha = 0, a \mid (b + \beta) \\ cx \frac{b+\beta}{\alpha}, & \text{if } m = 1, a = n = 0, \alpha \mid (b + \beta) \\ \text{NoRatGenSol}, & \text{otherwise} \end{cases}$
1.4.16	$y(bmx^{k+m}y^k + d) + x(any^{k+n}x^k + d)y'$	$\left( t, -\frac{t(bmx^{k+m}t^k + d)}{x(ant^{k+n}x^k + d)} \right)$	$\begin{cases} \frac{d}{bcx + bx^{m+1}}, & \text{if } k = 1, n = 0, m \in \mathbb{Z} \setminus \{0\} \\ \text{NoRatGenSol}, & \text{otherwise} \end{cases}$
1.4.17	$ax^k + bnyx^{n-1} + (ay^m + bx^n + d)y' + \beta$	$\left( t, \frac{-ax^{k+1} - bntx^n - \beta x}{x(at^m + bx^n + d)} \right)$	$\begin{cases} -\frac{ax^{k+1} + (k+1)(\beta x - c)}{(k+1)(a + bx^n + d)}, & \text{if } m = 0 \\ \text{NoRatGenSol}, & \text{otherwise} \end{cases}$

Parametrizations which are too long for the table:

$$\mathcal{P}_{4,12} := \left( t, \frac{x^{n-3} \left( tx(x(a(2n+1)x + c) + b(2n-1)) - x^n (x^2(ax( anx + c) + d) + b^2n + bcx) \right)}{t} \right)$$

Table 11: First-order AODEs with nonlinear derivative  $y'$ .

ID	AODE	Parametrization	Associated ODE	Solution of assoc. ODE	Solution of AODE
1.5.1	$ax^2 + bx - y'^2 + c + y$	$(t^2 - ax^2 - c - bx, t)$	$\frac{2ax+b+t}{2t}$	NoRatGenSol	NoRatGenSol
1.5.3	$-xy'^2 + a + bx + xy\lambda$	$\left( \frac{xt^2 - a - bx}{x\lambda}, t \right)$	$\frac{1}{2} \left( \lambda - \frac{a}{tx^2} \right)$	NoRatGenSol	NoRatGenSol
1.5.4	$y'^2 + ayy' - c - bx$	$\left( \frac{-t^2 + c + bx}{at}, t \right)$	$\frac{t(b - at^2)}{bx + c + t^2}$	NoRatGenSol	NoRatGenSol
1.5.5	$bcyx^2 + ay'x + y'^2$	$\left( -\frac{t(t+ax)}{bcx^2}, t \right)$	$\frac{t(ax - bcx^3 + 2t)}{x(ax + 2t)}$	NoRatGenSol	NoRatGenSol
1.5.6	$ax^2 + y'x + y'^2 + b + d - y + cy'$	$(t^2 + ct + xt + ax^2 + b + d, t)$	$-\frac{2ax}{c+2t+x}$	NoRatGenSol	NoRatGenSol
1.5.7	$ay'x^2 + byx + y'^2$	$\left( -\frac{t(ax^2 + t)}{bx}, t \right)$	$\frac{t(t - x^2(a+b))}{ax^3 + 2tx}$	NoRatGenSol	NoRatGenSol
1.5.8	$xy'^2 - ayy' + b$	$\left( \frac{xt^2 + b}{at}, t \right)$	$\frac{(a-1)t^3}{t^2x - b}$	NoRatGenSol	NoRatGenSol
1.5.9	$xy'^2 + ayy' + bx$	$\left( \frac{(-t^2 - b)x}{at}, t \right)$	$\frac{t((a+1)t^2 + b)}{x(b - t^2)}$	NoRatGenSol	NoRatGenSol
1.5.10	$ay'x^2 + y'^2x + c + by' - yy'$	$\left( \frac{xt^2 + ax^2t + bt + c}{t}, t \right)$	$\frac{2at^2x}{c - t^2x}$	NoRatGenSol	NoRatGenSol
1.5.11	$yy'^2 + axy' + by$	$\left( -\frac{atx}{t^2 + b}, t \right)$	$\frac{t(b+t^2)(a+b+t^2)}{ax(t^2 - b)}$	NoRatGenSol	NoRatGenSol
1.5.12	$xy'^2 + (a - y)y' + b$	$\left( \frac{xt^2 + at + b}{t}, t \right)$	0	c	$a + \frac{b}{c} + cx$
1.5.13	$(c_2 + a_2x + b_2y)y'^2 + (a_1x + b_1y + 1)y' + c_0 + a_0x + b_0y$	$\left( \frac{-c_2t^2 - a_2xt^2 - a_1xt - t - c_0 - a_0x}{b_2t^2 + b_1t + b_0}, t \right)$	$A_{5,13}$	NoRatGenSol	NoRatGenSol
1.5.14	$axy'^2 - (bx^2 + ay^2 + c)y' + bxy$	genus 1	—	—	NoStrongRatGenSol
1.5.15	$-x^2 - 2yy'x + (x^2 - a)y'^2$	$\left( -\frac{-x^2t^2 + at^2 + x^2}{2tx}, t \right)$	$\frac{t}{x}$	cx	$-\frac{ac^2 + c^2(-x^2) + 1}{2c}$

Table 11: First-order AODEs with nonlinear derivative  $y'$  (continued).

ID	AODE	Parametrization	Associated ODE	Solution of assoc. ODE	Solution of AODE
1.5.16	$y^2 - 2xy'y + (x^2 + a)y'^2 + b$	$\left( \frac{i(t^2(a+x^2)+b)}{2\sqrt{at}}, \frac{-b+t^2(\sqrt{a+ix})^2}{2at+2i\sqrt{at}x} \right)$	$-\frac{t(\sqrt{-a+x})}{a+x^2}$	NoRatGenSol	NoRatGenSol
1.5.17	$ax^{m+1} - \frac{(m+1)x^2}{2(m+3)^2} - y'^2 + b + y$	$\left( \frac{(m+1)x^2}{2(m+3)^2} + t^2 - b - ax^{m+1}, t \right)$	$\frac{(m+1)(a(m+3)^2x^m - x)}{2(m+3)^2t} + \frac{1}{2}$	NoRatGenSol	NoRatGenSol
1.5.18	$bx^{m+1} + ax^2 - y'^2 + c + y\lambda$	$\left( \frac{-bx^{m+1} - ax^2 + t^2 - c}{\lambda}, t \right)$	$\frac{2ax + b(m+1)x^m + \lambda t}{2t}$	NoRatGenSol	NoRatGenSol
1.5.19	$bx^ny^m + ay'y + xy'^2$	genus depends on parameters	—	—	—
1.5.23	$x^3 - ay'x + y'^3$	reducible over $\overline{\mathbb{K}(x)}$	—	—	NoRatGenSol
1.5.24	$y'^3 + ay'^2 + d + abx + by$	$\left( \frac{-t^3 - at^2 - d - abx}{b}, t \right)$	$-\frac{b(a+t)}{t(2a+3t)}$	NoRatGenSol	NoRatGenSol
1.5.25	$cy'^{m+1} + by'^2 + xy' + ax^2 + d - y$	$\left( ct^{m+1} + bt^2 + xt + ax^2 + d, t \right)$	$-\frac{2ax}{2bt+c(m+1)t^m+x}$	NoRatGenSol	NoRatGenSol
1.5.26	$by'^m + ay'^n - x$	reducible over $\overline{\mathbb{K}(x)}$	—	—	NoRatGenSol
1.5.27	$by'^m + ay'^n - y$	$(bt^m + at^n, t)$	$\frac{t^2}{ant^n + bmt^m}$	$\begin{cases} \frac{x+c}{2(a+b)}, & \text{if } m=n=2 \\ \frac{x+c}{a}, & \text{if } m=0, n=2 \\ \frac{x+c}{b}, & \text{if } m=2, n=0 \\ \text{NoRatGenSol}, & \text{otherwise} \end{cases}$	$\begin{cases} \frac{(x+c)^2}{4(a+b)}, & \text{if } m=n=2 \\ \frac{(x+c)^2}{a} + b, & \text{if } m=0, n=2 \\ a + \frac{(x+c)^2}{b}, & \text{if } m=2, n=0 \\ \text{NoRatGenSol}, & \text{otherwise} \end{cases}$
1.5.28	$ay'^n + xy' - y$	$(at^n + xt, t)$	0	c	$ac^n + cx$
1.5.29	$ax^ny'^m + xy' - y$	$(ax^nt^m + xt, t)$	$-\frac{ant^m x^{n-1}}{amt^{m-1}x^n + x}$	$\begin{cases} c, & \text{if } m=1, n=0 \\ c, & \text{if } m=n=1, a=0 \\ \frac{c}{x^2}, & \text{if } m=n=1, a=-2 \\ \frac{c}{(ax+1)^2}, & \text{if } m=1, n=2 \\ \text{NoRatGenSol}, & \text{otherwise} \end{cases}$	$\begin{cases} c(ax+x), & \text{if } m=1, n=0 \\ xc, & \text{if } m=n=1, a=0 \\ -\frac{c}{x}, & \text{if } m=n=1, a=-2 \\ \frac{cx}{ax+1}, & \text{if } m=1, n=2 \\ \text{NoRatGenSol}, & \text{otherwise} \end{cases}$
1.5.30	$ay'^{2n}x^n + 2y'x - y$	$(at^{2n}x^n + 2tx, t)$	$-\frac{t}{2x}$	NoRatGenSol	NoRatGenSol
1.5.32	$-y'^k + Bx^r + Ay^s$	genus depends on parameters	—	—	—
1.5.33	$by'^m + axy'^n - y$	$(bt^m + axt^n, t)$	$\frac{t(t-at^n)}{anxt^n + bmt^m}$	$\begin{cases} c(ax+b)\frac{1-a}{a}, & \text{if } m=n=1, a=\pm 1 \\ \text{NoRatGenSol}, & \text{otherwise} \end{cases}$	$\begin{cases} c(ax+b)\frac{1}{a}, & \text{if } m=n=1, a=\pm 1 \\ \text{NoRatGenSol}, & \text{otherwise} \end{cases}$
1.5.34	$ax^n(xy' - y)^m - y'$	$(ax^{n+1}t^m - t, ax^nt^m)$	$\frac{ant^{m+1}x^n}{t-amt^m x^{n+1}}$	$\begin{cases} \frac{anx^{n+1}}{n+1} + c, & \text{if } m=0 \\ \frac{cx^2}{(x-a)^2}, & \text{if } m=1, n=-2 \\ c, & \text{if } m=1, n=0 \\ \text{NoRatGenSol}, & \text{otherwise} \end{cases}$	$\begin{cases} \frac{ax^{n+1} - c(n+1)}{n+1}, & \text{if } m=0 \\ \frac{cx}{a-x}, & \text{if } m=1, n=-2 \\ c(ax-1), & \text{if } m=1, n=0 \\ \text{NoRatGenSol}, & \text{otherwise} \end{cases}$

Associated ODEs which are too long for the table:

$$A_{5,13} := -\frac{(t(b_2t + b_1) + b_0) \left( t(a_2t + a_1 + b_2t^2 + b_1t + b_0) + a_0 \right)}{b_0(2a_2tx + a_1x + 2c_2t + 1) + a_2b_1t^2x - a_1b_2t^2x - a_0x(2b_2t + b_1) + b_1c_2t^2 - 2b_2c_0t - b_1c_0 - b_2t^2}$$

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