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Special Session 7: Symbolic summation and integration

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Refined Parameterized Telescoping Algorithms

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Der Wissenschaftsfonds

Simplify

$$\sum_{k=1}^{m} H_k$$

with $H_k = \sum_{i=1}^{k} \frac{1}{i}$

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Telescoping

Find $g(k)$ s.t,

$$\boxed{H_k = g(k+1) - g(k)}$$

for all $1 \leq k \leq m$ and $m \geq 0$

Simplify

$$\sum_{k=1}^{m} H_k$$

with $H_k = \sum_{i=1}^{k} \frac{1}{i}$ $(H_{k+1} = H_k + \frac{1}{k+1})$

Telescoping

Find $g(k)$ s.t,

$$\boxed{H_k = g(k+1) - g(k)}$$

for all $1 \leq k \leq m$ and $m \geq 0$

We compute

$$g(k) = (H_k - 1)k.$$

Simplify

$$\sum_{k=1}^{m} H_k$$

with $H_k = \sum_{i=1}^{k} \frac{1}{i}$

Telescoping

Find $g(k)$ s.t,

$$\boxed{H_k = g(k+1) - g(k)}$$

for all $1 \leq k \leq m$ and $m \geq 0$

Summing this equation over k from 1 to m gives

$$\boxed{\sum_{k=1}^{m} H_k} = \boxed{g(m+1) - g(1)}$$

Simplify

$$\sum_{k=1}^{m} H_k$$

with $H_k = \sum_{i=1}^{k} \frac{1}{i}$

Telescoping

Find $g(k)$ s.t,

$$\boxed{H_k = g(k+1) - g(k)}$$

for all $1 \le k \le m$ and $m \ge 0$

Summing this equation over k from 1 to m gives

$$\boxed{\sum_{k=1}^{m} H_k} = \boxed{g(m+1) - g(1)}$$

$$= (H_{m+1} - 1)(m+1).$$

Telescoping in the given difference field

FIND a closed form for

$$\sum_{k=1}^{m} H_k.$$

A difference field for the summand

Consider the rational function field

$$\mathbb{F} := \mathbb{Q}$$

with the automorphism $\sigma : \mathbb{F} \to \mathbb{F}$ defined by

$$\sigma(c) = c \quad \forall c \in \mathbb{Q},$$

Telescoping in the given difference field

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A difference field for the summand

Consider the rational function field

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with the automorphism $\sigma : \mathbb{F} \to \mathbb{F}$ defined by

$$\sigma(c) = c \quad \forall c \in \mathbb{Q},$$

$$\sigma(k) = k + 1, \qquad\qquad \mathcal{S}\, k = k + 1,$$

Telescoping in the given difference field

FIND a closed form for

$$\sum_{k=1}^{m} H_k.$$

A difference field for the summand

Consider the rational function field

$$\mathbb{F} := \mathbb{Q}(k)(h)$$

with the automorphism $\sigma : \mathbb{F} \to \mathbb{F}$ defined by

$$\sigma(c) = c \quad \forall c \in \mathbb{Q},$$

$$\sigma(k) = k + 1, \qquad \mathcal{S}\, k = k + 1,$$

$$\sigma(h) = h + \frac{1}{k+1}, \qquad \mathcal{S}\, H_k = H_k + \frac{1}{k+1}.$$

Telescoping in the given difference field

FIND $g \in \mathbb{F}$:

$$\boxed{\sigma(g) - g = h.}$$

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$$g = (h-1)k \in \mathbb{F}.$$

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FIND $g \in \mathbb{F}$:

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This gives

$$\boxed{g(k + 1) - g(k) = H_k}$$

with

$$g(k) = (H_k - 1)k.$$

Telescoping in the given difference field

FIND $g \in \mathbb{F}$:

$$\boxed{\sigma(g) - g = h.}$$

We compute

$$g = (h-1)k \in \mathbb{F}.$$

This gives

$$\boxed{g(k+1) - g(k) = H_k}$$

with

$$g(k) = (H_k - 1)k.$$

Hence,

$$(H_{m+1} - 1)(m+1) = \sum_{k=1}^{m} H_k.$$

CONSTRUCT a difference field $(\mathbb{F}, \sigma)$:

- a rational function field

$$\mathbb{F} := \mathbb{K}$$

- with an automorphism

$$\sigma(c) = c \quad \forall c \in \mathbb{K}$$

CONSTRUCT a difference field $(\mathbb{F}, \sigma)$:

- a rational function field

$$\mathbb{F} := \mathbb{K}(t_1)$$

- with an automorphism

$$\sigma(c) = c \quad \forall c \in \mathbb{K}$$
$$\sigma(t_1) = t_1 + 1,$$

CONSTRUCT a difference field $(\mathbb{F}, \sigma)$:

- a rational function field

$$\mathbb{F} := \mathbb{K}(t_1)$$

- with an automorphism

$$\sigma(c) = c \quad \forall c \in \mathbb{K}$$

$$\sigma(t_1) = q\,t_1, \qquad q \in \mathbb{K}^*,$$

CONSTRUCT a difference field $(\mathbb{F}, \sigma)$:

- a rational function field

$$\mathbb{F} := \mathbb{K}(t_1)$$

- with an automorphism

$$\sigma(c) = c \quad \forall c \in \mathbb{K}$$
$$\sigma(t_1) = a_1\, t_1 + f_1, \quad a_1 \in \mathbb{K}^*, \qquad\qquad f_1 \in \mathbb{K}$$

CONSTRUCT a difference field $(\mathbb{F}, \sigma)$:

- a rational function field

$$\mathbb{F} := \mathbb{K}(t_1)(t_2)$$

- with an automorphism

$$\begin{aligned}
\sigma(c) &= c \quad \forall c \in \mathbb{K} \\
\sigma(t_1) &= a_1\, t_1 + f_1, \qquad a_1 \in \mathbb{K}^*, \qquad\qquad\qquad f_1 \in \mathbb{K} \\
\sigma(t_2) &= t_2 + \frac{1}{t_1 + 1},
\end{aligned}$$

CONSTRUCT a difference field $(\mathbb{F}, \sigma)$:

- a rational function field

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CONSTRUCT a difference field $(\mathbb{F}, \sigma)$:

- a rational function field

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- with an automorphism

$$\begin{aligned} \sigma(c) &= c \quad \forall c \in \mathbb{K} \\ \sigma(t_1) &= a_1\, t_1 + f_1, \quad a_1 \in \mathbb{K}^*, & f_1 \in \mathbb{K} \\ \sigma(t_2) &= a_2\, t_2 + f_2, \quad a_2 \in \mathbb{K}(t_1)^*, & f_2 \in \mathbb{K}(t_1) \end{aligned}$$

CONSTRUCT a difference field $(\mathbb{F}, \sigma)$:

- a rational function field

$$\mathbb{F} := \mathbb{K}(t_1)(t_2)\dots(t_e)$$

- with an automorphism

$$\begin{aligned}
\sigma(c) &= c \quad \forall c \in \mathbb{K} \\
\sigma(t_1) &= a_1\, t_1 + f_1, \quad a_1 \in \mathbb{K}^*, && f_1 \in \mathbb{K} \\
\sigma(t_2) &= a_2\, t_2 + f_2, \quad a_2 \in \mathbb{K}(t_1)^*, && f_2 \in \mathbb{K}(t_1) \\
&\vdots \\
\sigma(t_e) &= a_e\, t_e + f_e, \quad a_e \in \mathbb{K}(t_1, \dots, t_{e-1})^*, && f_e \in \mathbb{K}(t_1, \dots, t_{e-1})
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\sigma(t_1) &= a_1\, t_1 + f_1, \quad a_1 \in \mathbb{K}^*, && f_1 \in \mathbb{K} \\
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\end{aligned}$$

GIVEN $f \in \mathbb{F}$;
FIND $g \in \mathbb{F}$ such that

$$\sigma(g) - g = f.$$

CONSTRUCT a **$\Pi\Sigma^*$-field** $(\mathbb{F},\sigma)$: (Karr, 1981)

- a rational function field

$$\mathbb{F} := \mathbb{K}(t_1)(t_2)\dots(t_e)$$

- with an automorphism

$$\begin{aligned}
\sigma(c) &= c \quad \forall c \in \mathbb{K} \\
\sigma(t_1) &= a_1\, t_1 + f_1, \quad a_1 \in \mathbb{K}^*, \qquad f_1 \in \mathbb{K} \\
\sigma(t_2) &= a_2\, t_2 + f_2, \quad a_2 \in \mathbb{K}(t_1)^*, \qquad f_2 \in \mathbb{K}(t_1) \\
&\vdots \\
\sigma(t_e) &= a_e\, t_e + f_e, \quad a_e \in \mathbb{K}(t_1,\dots,t_{e-1})^*, \quad f_e \in \mathbb{K}(t_1,\dots,t_{e-1})
\end{aligned}$$

such that

$$\text{const}_\sigma\mathbb{F} = \{c \in \mathbb{K}(t_1)(t_2)\dots(t_e) | \sigma(c) = c\} = \mathbb{K}.$$

GIVEN $f \in \mathbb{F}$;
FIND $g \in \mathbb{F}$ such that

$$\sigma(g) - g = f.$$

Simplified telescoping algorithm

FIND a closed form for

$$\sum_{k=1}^{m} H_k.$$

A $\Pi\Sigma^*$-field for the summand

$$\boxed{\text{const}_\sigma \mathbb{F} = \mathbb{Q}}$$

Consider the rational function field

$$\mathbb{F} := \mathbb{Q}(k)(h)$$

with the automorphism $\sigma : \mathbb{F} \to \mathbb{F}$ defined by

$$\sigma(c) = c \quad \forall c \in \mathbb{Q},$$

$$\sigma(k) = k + 1, \qquad \mathcal{S}\, k = k + 1,$$

$$\sigma(h) = h + \frac{1}{k+1}, \qquad \mathcal{S}\, H_k = H_k + \frac{1}{k+1}.$$

FIND $g \in \mathbb{Q}(k)(h)$:

$$\sigma(g) - g = h.$$

FIND $g \in \mathbb{Q}(k)(h)$:

$$\sigma(g) - g = h.$$

Denominator bound: COMPUTE a polynomial $d \in \mathbb{Q}(k)[h]^*$:

$$\forall g \in \mathbb{Q}(k)(h): \quad \sigma(g) - g = h \quad \Rightarrow \quad g\,d \in \mathbb{Q}(k)[h].$$

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FIND $g' \in \mathbb{Q}(k)[h]$ with

$$\sigma(\frac{g'}{d}) - \frac{g'}{d} = h.$$

FIND $g \in \mathbb{Q}(k)(h)$:

$$\sigma(g) - g = h.$$

Denominator bound: COMPUTE a polynomial $d \in \mathbb{Q}(k)[h]^*$:

$$\forall g \in \mathbb{Q}(k)(h): \quad \sigma(g) - g = h \quad \Rightarrow \quad g\,d \in \mathbb{Q}(k)[h].$$

FIND $g' \in \mathbb{Q}(k)[h]$ with

$$\frac{1}{\sigma(d)}\sigma(g') - \frac{1}{d}g = \sigma(\frac{g'}{d}) - \frac{g'}{d} = h.$$

FIND $g \in \mathbb{Q}(k)(h)$:

$$\sigma(g) - g = h.$$

Denominator bound: COMPUTE a polynomial $d \in \mathbb{Q}(k)[h]^*$:

$d = 1$

$$\forall g \in \mathbb{Q}(k)(h): \quad \sigma(g) - g = h \quad \Rightarrow \quad g\,d \in \mathbb{Q}(k)[h].$$

FIND $g' \in \mathbb{Q}(k)[h]$ with

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FIND $g' \in \mathbb{Q}(k)[h]$ with

$$\frac{1}{\sigma(d)}\sigma(g') - \frac{1}{d}g = \sigma(\frac{g'}{d}) - \frac{g'}{d} = h.$$

Degree bound: COMPUTE $m \geq 0$:

$$\forall g \in \mathbb{Q}(k)[h] \quad \sigma(g) - g = h \quad \Rightarrow \quad \deg(g) \leq m.$$

FIND $g \in \mathbb{Q}(k)(h)$:

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FIND $g' \in \mathbb{Q}(k)[h]$ with

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Degree bound: COMPUTE $m \geq 0$:

$m = 2$

$$\forall g \in \mathbb{Q}(k)[h] \quad \sigma(g) - g = h \quad \Rightarrow \quad \deg(g) \leq m.$$

FIND $g \in \mathbb{Q}(k)(h)$:

$$\sigma(g) - g = h.$$

Denominator bound: COMPUTE a polynomial $d \in \mathbb{Q}(k)[h]^*$:

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Polynomial Solution: FIND

$$g = g_2\,h^2 + g_1\,h + g_0 \in \mathbb{Q}(k)[h].$$

FIND $g \in \mathbb{Q}(k)(h)$:

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FIND $g' \in \mathbb{Q}(k)[h]$ with

$$\frac{1}{\sigma(d)}\sigma(g') - \frac{1}{d}g = \sigma(\frac{g'}{d}) - \frac{g'}{d} = h.$$

Degree bound: COMPUTE $m \geq 0$:

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Polynomial Solution: FIND

g=hk-k

$$g = g_2\,h^2 + g_1\,h + g_0 \in \mathbb{Q}(k)[h].$$

ANSATZ $g = g_2\, h^2 + g_1\, h + g_0 \in \mathbb{Q}(k)[h]$

$$\sigma(g) - g = h$$

ANSATZ $g = g_2\, h^2 + g_1\, h + g_0 \in \mathbb{Q}(k)[h]$

$$\Big[\sigma(g_2)\big(h + \tfrac{1}{k+1}\big)^2 + \sigma(g_1 h + g_0)\Big] \\ -\big[g_2\, h^2 + g_1 h + g_0\big] = h$$

ANSATZ $g = g_2\, h^2 + g_1\, h + g_0 \in \mathbb{Q}(k)[h]$

$$\left[\sigma(g_2)\left(h + \tfrac{1}{k+1}\right)^2 + \sigma(g_1 h + g_0)\right] - \left[g_2\, h^2 + g_1 h + g_0\right] = h$$

coeff. comp.

$$\boxed{\sigma(g_2) - g_2 = 0}$$

ANSATZ $g = g_2\, h^2 + g_1\, h + g_0 \in \mathbb{Q}(k)[h]$

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coeff. comp.

$$\boxed{\sigma(g_2) - g_2 = 0}$$

$g_2 = c \in \mathbb{Q}$

ANSATZ $g = g_2\, h^2 + g_1\, h + g_0 \in \mathbb{Q}(k)[h]$

$$\left[\sigma(g_2)\left(h + \tfrac{1}{k+1}\right)^2 + \sigma(g_1 h + g_0)\right] - \left[g_2\, h^2 + g_1 h + g_0\right] = h$$

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$$\boxed{\sigma(g_2) - g_2 = 0}$$

$g_2 = c \in \mathbb{Q}$

$$\left[\sigma(c)\left(h + \tfrac{1}{k+1}\right)^2 + \sigma(g_1 h + g_0)\right] - \left[c\, h^2 + g_1 h + g_0\right] = h$$

ANSATZ $g = g_2\, h^2 + g_1\, h + g_0 \in \mathbb{Q}(k)[h]$

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coeff. comp.

$$\boxed{\sigma(g_2) - g_2 = 0}$$

$g_2 = c \in \mathbb{Q}$

$$\Big[c\big(h+\tfrac{1}{k+1}\big)^2+\sigma(g_1 h+g_0)\Big] - \big[c\, h^2 + g_1 h + g_0\big] = h$$

ANSATZ $g = g_2\, h^2 + g_1\, h + g_0 \in \mathbb{Q}(k)[h]$

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coeff. comp.

$$\sigma(g_2) - g_2 = 0$$

$g_2 = c \in \mathbb{Q}$

$$\sigma(g_1\, h + g_0) - (g_1\, h + g_0) = h - c\Big[\tfrac{2h(k+1)+1}{(k+1)^2}\Big]$$

ANSATZ $g = g_2\, h^2 + g_1\, h + g_0 \in \mathbb{Q}(k)[h]$

$$\left[\sigma(g_2)\left(h + \tfrac{1}{k+1}\right)^2 + \sigma(g_1 h + g_0)\right] - \left[g_2\, h^2 + g_1 h + g_0\right] = h$$

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$$\sigma(g_1\, h + g_0) - (g_1\, h + g_0) = h - c\left[\tfrac{2h(k+1)+1}{(k+1)^2}\right]$$

coeff. comp.

$$\boxed{\sigma(g_1) - g_1 = 1 - c\,\tfrac{2}{k+1}}$$

ANSATZ $g = g_2\, h^2 + g_1\, h + g_0 \in \mathbb{Q}(k)[h]$

$$\Big[\sigma(g_2)\big(h + \tfrac{1}{k+1}\big)^2 + \sigma(g_1 h + g_0)\Big] - \Big[g_2\, h^2 + g_1 h + g_0\Big] = h$$

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coeff. comp.

$$\boxed{\sigma(g_1) - g_1 = 1 - c\,\tfrac{2}{k+1}}$$

$c = 0$, $\quad g_1 = k + d$, $d \in \mathbb{Q}$

ANSATZ $g = g_2\, h^2 + g_1\, h + g_0 \in \mathbb{Q}(k)[h]$

$$\left[\sigma(g_2)\left(h + \tfrac{1}{k+1}\right)^2 + \sigma(g_1 h + g_0)\right] - \left[g_2\, h^2 + g_1 h + g_0\right] = h$$

coeff. comp.

$$\sigma(g_2) - g_2 = 0$$

$g_2 = c \in \mathbb{Q}$

$$\sigma(g_1\, h + g_0) - (g_1\, h + g_0) = h - c\left[\tfrac{2h(k+1)+1}{(k+1)^2}\right]$$

coeff. comp.

$$\sigma(g_1) - g_1 = 1 - c\,\tfrac{2}{k+1}$$

$c = 0$, $g_1 = k + d$, $d \in \mathbb{Q}$

$$\sigma(g_0) - g_0 = -1 - d\frac{1}{k+1}$$

ANSATZ $g = g_2\, h^2 + g_1\, h + g_0 \in \mathbb{Q}(k)[h]$

$$\Big[\sigma(g_2)\big(h + \tfrac{1}{k+1}\big)^2 + \sigma(g_1 h + g_0)\Big] - \big[g_2\, h^2 + g_1 h + g_0\big] = h$$

coeff. comp. → $\boxed{\sigma(g_2) - g_2 = 0}$

g=hk-k

$g_2 = c \in \mathbb{Q}$ →

$$\sigma(g_1\, h + g_0) - (g_1\, h + g_0) = h - c\Big[\tfrac{2h(k+1)+1}{(k+1)^2}\Big]$$

coeff. comp. → $\boxed{\sigma(g_1) - g_1 = 1 - c\,\tfrac{2}{k+1}}$

$c = 0$, $g_1 = k + d$, $d \in \mathbb{Q}$ →

$$\boxed{\sigma(g_0) - g_0 = -1 - d\frac{1}{k+1}}$$

← $g_0 = -k$, $d = 0$

First order difference equations in difference fields

Let $(\mathbb{F}, \sigma)$ be a $\Pi\Sigma$-field with constant field $\mathbb{K}$

Telescoping

- Given $f \in \mathbb{F}$
- Find $g \in \mathbb{F}$:

$$\boxed{\sigma(g) - g = f}$$

First order difference equations in difference fields

Let $(\mathbb{F}, \sigma)$ be a $\Pi\Sigma$-field with constant field $\mathbb{K}$

Telescoping

- Given $f \in \mathbb{F}$
- Find $g \in \mathbb{F}$:

$$\boxed{\sigma(g) - g = f}$$

$$\downarrow \qquad\qquad \uparrow$$

Parameterized Telescoping

- Given $\mathbf{f} = (f_1, \ldots, f_n) \in \mathbb{F}^n$
- Find $\Big\{(c_1, \ldots, c_n, g) \in \mathbb{K}^n \times \mathbb{F} \mid$

$$\boxed{\sigma(g) - g = c_1\, f_1 + \cdots + c_n\, f_n}\Big\}$$

First order difference equations in difference fields

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Telescoping

- Given $f \in \mathbb{F}$
- Find $g \in \mathbb{F}$:

$$\boxed{\sigma(g) - g = f}$$

$\downarrow$ $\uparrow$

Parameterized first order linear difference equation

- Given $\mathbf{f} = (f_1, \dots, f_n) \in \mathbb{F}^n$, $(0,0) \neq (a_0, a_1) \in \mathbb{F}^2$
- Find $\Big\{(c_1, \dots, c_n, g) \in \mathbb{K}^n \times \mathbb{F} \big|$

$$\boxed{a_1\sigma(g) + a_0\, g = c_1\, f_1 + \dots + c_n\, f_n}\Big\}$$

First order difference equations in difference fields

Let $(\mathbb{F}, \sigma)$ be a $\Pi\Sigma$-field with constant field $\mathbb{K}$

Telescoping

- Given $f \in \mathbb{F}$
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Parameterized first order linear difference equation

- Given $\mathbf{f} = (f_1, \ldots, f_n) \in \mathbb{F}^n$, $(0,0) \neq (a_0, a_1) \in \mathbb{F}^2$
- Find

$$\Big\{(c_1, \ldots, c_n, g) \in \mathbb{K}^n \times \mathbb{F} \,\Big|\, \boxed{a_1\sigma(g) + a_0\, g = c_1\, f_1 + \cdots + c_n\, f_n}\Big\} =: V(\mathbf{a}, \mathbf{f}, \mathbb{F})$$

First order difference equations in difference fields

Let $(\mathbb{F}, \sigma)$ be a $\Pi\Sigma$-field with constant field $\mathbb{K}$

Telescoping

- Given $f \in \mathbb{F}$
- Find $g \in \mathbb{F}$:

$$\boxed{\sigma(g) - g = f}$$

$\downarrow$ $\uparrow$

Parameterized first order linear difference equation

- Given $\mathbf{f} = (f_1, \ldots, f_n) \in \mathbb{F}^n$, $(0,0) \neq (a_0, a_1) \in \mathbb{F}^2$
- Find $\Big\{(c_1, \ldots, c_n, g) \in \mathbb{K}^n \times \mathbb{F} |$

$$\boxed{a_1 \sigma(g) + a_0\, g = c_1\, f_1 + \cdots + c_n\, f_n}\Big\} =: V(\mathbf{a}, \mathbf{f}, \mathbb{F})$$

Note:

- $V(\mathbf{a}, \mathbf{f}, \mathbb{F})$ is a subspace of $\mathbb{K}^n \times \mathbb{F}$ over $\mathbb{K}$

First order difference equations in difference fields

Let $(\mathbb{F}, \sigma)$ be a $\Pi\Sigma$-field with constant field $\mathbb{K}$

Telescoping

- Given $f \in \mathbb{F}$
- Find $g \in \mathbb{F}$:

$$\boxed{\sigma(g) - g = f}$$

$\downarrow$ $\uparrow$

Parameterized first order linear difference equation

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- Find

$$\Big\{(c_1, \ldots, c_n, g) \in \mathbb{K}^n \times \mathbb{F} \,\Big|\, \boxed{a_1\sigma(g) + a_0\, g = c_1\, f_1 + \cdots + c_n\, f_n}\Big\} =: V(\mathbf{a}, \mathbf{f}, \mathbb{F})$$

Note:

- $V(\mathbf{a}, \mathbf{f}, \mathbb{F})$ is a subspace of $\mathbb{K}^n \times \mathbb{F}$ over $\mathbb{K}$
- $\dim_{\mathbb{K}} V(\mathbf{a}, \mathbf{f}, \mathbb{F}) \leq n + 1$

First order difference equations in difference fields

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- $\dim_{\mathbb{K}} V(\mathbf{a}, \mathbf{f}, \mathbb{F}) \leq n + 1$
- Task: Compute a basis of $V(\mathbf{a}, \mathbf{f}, \mathbb{F})$

Find a basis of

$\boxed{V(\mathbf{a}, \mathbf{f}, \mathbb{G}(t))}$

$(\mathbb{G}(t), \sigma)$ $\Pi\Sigma^*$-field

Find a basis of

$\boxed{V(\mathbf{a}, \mathbf{f}, \mathbb{G}(t))}$

$\downarrow$

Compute a denominator bound
and search for polynomial solutions

$(\mathbb{G}(t), \sigma)$ $\Pi\Sigma^*$-field

Find a basis of

$$\boxed{V(\mathbf{a}, \mathbf{f}, \mathbb{G}(t))}$$

$\downarrow$

Compute a denominator bound
and search for polynomial solutions

$\downarrow$

Compute a degree bound m

$(\mathbb{G}(t), \sigma)$ $\Pi\Sigma^*$-field

Find a basis of

$V(\mathbf{a}, \mathbf{f}, \mathbb{G}(t))$

$(\mathbb{G}(t), \sigma)$ $\Pi\Sigma^*$-field

↓

Compute a denominator bound
and search for polynomial solutions

↓

Compute a degree bound m

↓

Compute a basis of

$V(\mathbf{a_m}, \mathbf{f_m}, \mathbb{G}[t]_m)$

$\mathbb{G}[t]_m := \{p \in \mathbb{G}[t] \,|\, \deg(p) \leq m\}$

$(\mathbb{G}(t),\sigma)$ $\Pi\Sigma^*$-field

Find a basis of

$V(\mathbf{a},\mathbf{f},\mathbb{G}(t))$

↓

Compute a denominator bound
and search for polynomial solutions

↓

Compute a degree bound m

↓

Compute a basis of

$V(\mathbf{a_m},\mathbf{f_m},\mathbb{G}[t]_m)$

→ coeff. comparison →

Compute a basis of

$V(\tilde{\mathbf{a}}_\mathbf{m},\tilde{\mathbf{f}}_\mathbf{m},\mathbb{G})$

Find a basis of

$V(\mathbf{a}, \mathbf{f}, \mathbb{G}(t))$

$(\mathbb{G}(t), \sigma)$ $\Pi\Sigma^*$-field

↓

Compute a denominator bound
and search for polynomial solutions

↓

Compute a degree bound m

↓

Compute a basis of

$V(\mathbf{a_m}, \mathbf{f_m}, \mathbb{G}[t]_m)$

→ coeff. comparison →

Compute a basis of

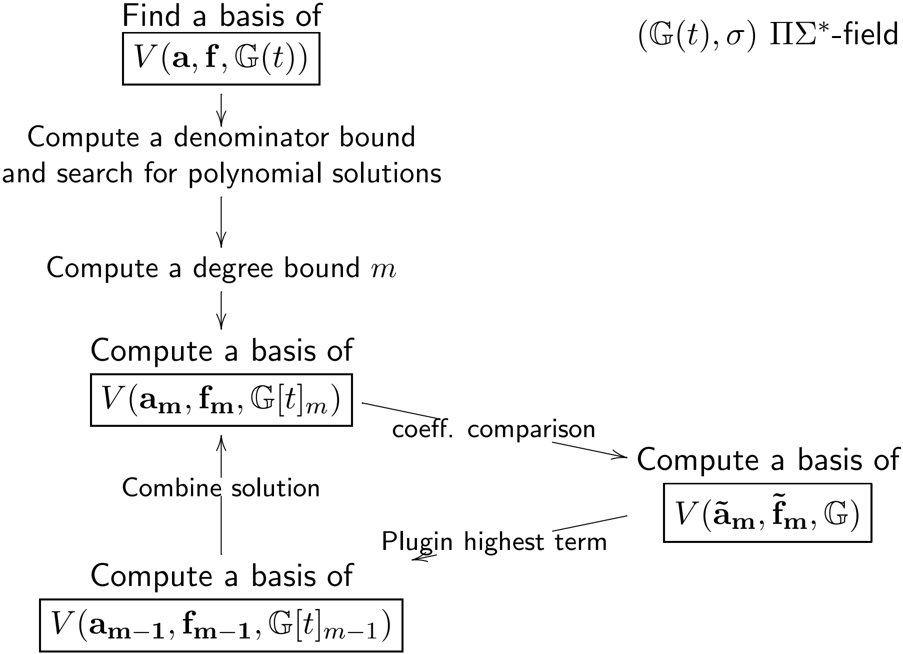
$V(\tilde{\mathbf{a}}_\mathbf{m}, \tilde{\mathbf{f}}_\mathbf{m}, \mathbb{G})$

← Plugin highest term ←

Compute a basis of

$V(\mathbf{a_{m-1}}, \mathbf{f_{m-1}}, \mathbb{G}[t]_{m-1})$

$(\mathbb{G}(t), \sigma)$ $\Pi\Sigma^*$-field



$(\mathbb{G}(t), \sigma)$ $\Pi\Sigma^*$-field

Find a basis of

$V(\mathbf{a}, \mathbf{f}, \mathbb{G}(t))$

↓

Compute a denominator bound
and search for polynomial solutions

↓

Compute a degree bound m

↓

Compute a basis of

$V(\mathbf{a_m}, \mathbf{f_m}, \mathbb{G}[t]_m)$

→ coeff. comparison → Compute a basis of $V(\mathbf{\tilde{a}_m}, \mathbf{\tilde{f}_m}, \mathbb{G})$

← Plugin highest term

↑ Combine solution

Compute a basis of

$V(\mathbf{a_{m-1}}, \mathbf{f_{m-1}}, \mathbb{G}[t]_{m-1})$

→ coeff. comparison → Compute a basis of $V(\mathbf{\tilde{a}_{m-1}}, \mathbf{\tilde{f}_{m-1}}, \mathbb{G})$

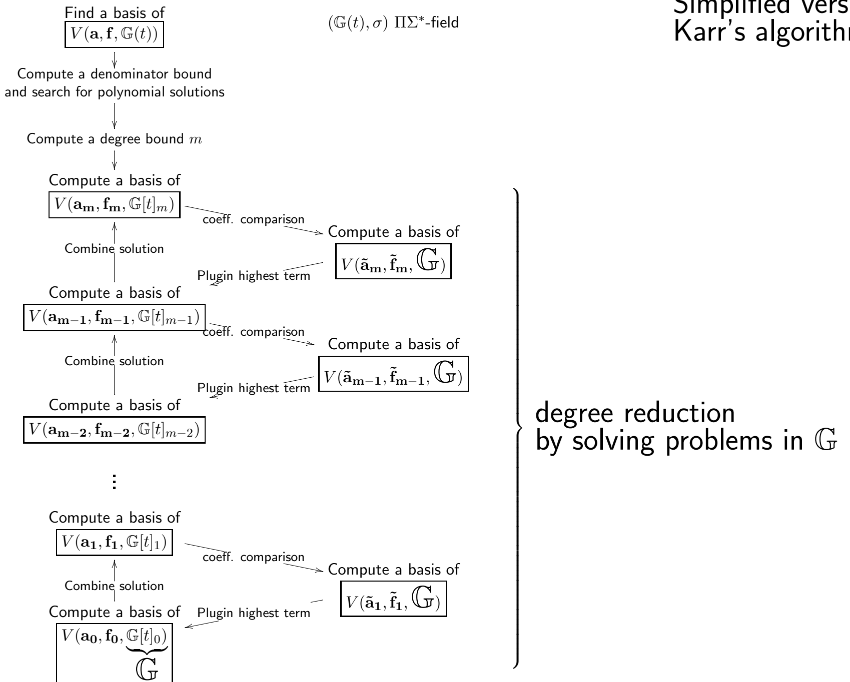
← Plugin highest term

↑ Combine solution

Compute a basis of

$V(\mathbf{a_{m-2}}, \mathbf{f_{m-2}}, \mathbb{G}[t]_{m-2})$

Simplified version of Karr's algorithm



Example 2

FIND $g(k)$:

$$\boxed{g(k+1) - g(k) = \frac{H_k}{k}}$$

for all $1 \leq k \leq m$ and $m \geq 0$.

Find $g \in \mathbb{Q}(k)(h)$ s.t.

$$\sigma(g) - g = \frac{h}{k}$$

Find $g \in \mathbb{Q}(k)[h]$ s.t.

$$\sigma(g) - g = \frac{h}{k}$$

Find $g = g_2 h^2 + g_1 h + g_0 \in \mathbb{Q}(k)[h]_2$ s.t.

$$\sigma(g) - g = \frac{h}{k}$$

Find $g = g_2 h^2 + g_1 h + g_0 \in \mathbb{Q}(k)[h]_2$ s.t.

$$\sigma(g) - g = \frac{h}{k}$$

$$\Big[\sigma(g_2)\big(h + \tfrac{1}{k+1}\big)^2 + \sigma(g_1 h + g_0)\Big] - \big[g_2\, h^2 + g_1 h + g_0\big] = \tfrac{h}{k}$$

Find $g = g_2 h^2 + g_1 h + g_0 \in \mathbb{Q}(k)[h]_2$ s.t.

$$\sigma(g) - g = \frac{h}{k}$$

$$\Big[\sigma(g_2)\big(h + \tfrac{1}{k+1}\big)^2 + \sigma(g_1 h + g_0)\Big] - \Big[g_2\, h^2 + g_1 h + g_0\Big] = \tfrac{h}{k}$$

coeff. comp.

$$\boxed{\sigma(g_2) - g_2 = 0}$$

Find $g = g_2 h^2 + g_1 h + g_0 \in \mathbb{Q}(k)[h]_2$ s.t.

$$\sigma(g) - g = \frac{h}{k}$$

$$\Big[\sigma(g_2)\big(h + \tfrac{1}{k+1}\big)^2 + \sigma(g_1 h + g_0)\Big] - \big[g_2\, h^2 + g_1 h + g_0\big] = \tfrac{h}{k}$$

coeff. comp.

$$\boxed{\sigma(g_2) - g_2 = 0}$$

$g_2 = c \in \mathbb{Q}$

Find $g = g_2 h^2 + g_1 h + g_0 \in \mathbb{Q}(k)[h]_2$ s.t.

$$\sigma(g) - g = \frac{h}{k}$$

$$\Big[\sigma(g_2)\big(h + \tfrac{1}{k+1}\big)^2 + \sigma(g_1 h + g_0)\Big] - \big[g_2\, h^2 + g_1 h + g_0\big] = \tfrac{h}{k}$$

coeff. comp.

$$\boxed{\sigma(g_2) - g_2 = 0}$$

$g_2 = c \in \mathbb{Q}$

$$\sigma(g_1\, h + g_0) - (g_1\, h + g_0) = \tfrac{h}{k} + c\Big[\tfrac{-2h(k+1)-1}{(k+1)^2}\Big]$$

Find $g = g_2 h^2 + g_1 h + g_0 \in \mathbb{Q}(k)[h]_2$ s.t.

$$\sigma(g) - g = \frac{h}{k}$$

$$\left[\sigma(g_2)\left(h + \tfrac{1}{k+1}\right)^2 + \sigma(g_1 h + g_0)\right] - \left[g_2\, h^2 + g_1 h + g_0\right] = \tfrac{h}{k}$$

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coeff. comp.

$$\boxed{\sigma(g_1) - g_1 = \frac{1}{k} + c\,\tfrac{-2}{k+1}}$$

Find $g = g_2 h^2 + g_1 h + g_0 \in \mathbb{Q}(k)[h]_2$ s.t.

$$\sigma(g) - g = \frac{h}{k}$$

$$\left[\sigma(g_2)\left(h + \tfrac{1}{k+1}\right)^2 + \sigma(g_1 h + g_0)\right] - \left[g_2\, h^2 + g_1 h + g_0\right] = \tfrac{h}{k}$$

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coeff. comp.

$$\boxed{\sigma(g_1) - g_1 = \frac{1}{k} + c\,\tfrac{-2}{k+1}}$$

$c = \tfrac{1}{2}$, $g_1 = -\tfrac{1}{k} + d$, $d \in \mathbb{Q}$

Find $g = g_2 h^2 + g_1 h + g_0 \in \mathbb{Q}(k)[h]_2$ s.t.

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$$\boxed{\sigma(g_1) - g_1 = \frac{1}{k} + c\,\tfrac{-2}{k+1}}$$

$c = \frac{1}{2}$, $g_1 = -\frac{1}{k} + d$, $d \in \mathbb{Q}$

$$\boxed{\sigma(g_0) - g_0 = \frac{2}{(k+1)^2} + d\frac{-1}{k+1}}$$

Find $g = g_2 h^2 + g_1 h + g_0 \in \mathbb{Q}(k)[h]_2$ s.t.

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$$\Big[\sigma(g_2)\big(h + \tfrac{1}{k+1}\big)^2 + \sigma(g_1 h + g_0)\Big] - \big[g_2\, h^2 + g_1 h + g_0\big] = \frac{h}{k}$$

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$\leftarrow$ no solution

FIND $g(k)$:

$$\boxed{g(k+1) - g(k) = \frac{H_k}{k}}$$

for all $1 \leq k \leq m$ and $m \geq 0$.

No solution ☹

Refined telescoping

FIND $g(k), \phi(k)$:

$$\boxed{g(k+1) - g(k) + \phi(k) = \frac{H_k}{k}}$$

for all $1 \le k \le m$ and $m \ge 0$.

Find $g = g_2 h^2 + g_1 h + g_0 \in \mathbb{Q}(k)[h]_2$ s.t.

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$$\Big[\sigma(g_2)\big(h + \tfrac{1}{k+1}\big)^2 + \sigma(g_1 h + g_0)\Big] - \big[g_2\, h^2 + g_1 h + g_0\big] = \tfrac{h}{k}$$

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$$\sigma(g_1 h + g_0) - (g_1 h + g_0) = \frac{h}{k} + c\left[\frac{-2h(k+1)-1}{(k+1)^2}\right]$$

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$$\boxed{\sigma(g_0) - g_0 = \frac{2}{(k+1)^2} + d\frac{-1}{k+1}}$$

$$\sigma(0) - 0 + \frac{2}{(k+1)^2} = \frac{2}{(k+1)^2} + 0\,\frac{-1}{k+1}$$

Find $g = g_2 h^2 + g_1 h + g_0 \in \mathbb{Q}(k)[h]_2$ s.t.

$$\sigma(g) - g + \frac{1}{2(k+1)^2} = \frac{h}{k} \qquad g = \frac{1}{2}h + \frac{1}{k}$$

$$\Big[\sigma(g_2)\big(h + \tfrac{1}{k+1}\big)^2 + \sigma(g_1 h + g_0)\Big] - \big[g_2\, h^2 + g_1 h + g_0\big] = \frac{h}{k}$$

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coeff. comp.

$$\sigma(g_1) - g_1 = \frac{1}{k} + c\,\frac{-2}{k+1}$$

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$$\sigma(g_0) - g_0 = \frac{2}{(k+1)^2} + d\frac{-1}{k+1}$$

$$\sigma(0) - 0 + \frac{2}{(k+1)^2} = \frac{2}{(k+1)^2} + 0\,\frac{-1}{k+1}$$

Refined telescoping

FIND $g(k), \phi(k)$:

$$g(k+1) - g(k) + \phi(k) = \frac{H_k}{k}$$

for all $1 \leq k \leq m$ and $m \geq 0$.

We compute

$$g(k) = \frac{1}{2}H_k + \frac{1}{k}$$

$$\phi(k) = \frac{1}{2(k+1)^2}$$

Refined telescoping

FIND $g(k), \phi(k)$:

$$g(k+1) - g(k) + \phi(k) = \frac{H_k}{k}$$

for all $1 \leq k \leq m$ and $m \geq 0$.

Summing this equation over k from 1 to m gives

$$\sum_{k=1}^{m} \frac{H_k}{k} = g(m+1) - g(1) + \sum_{k=1}^{m} \phi(k)$$

.

Refined telescoping

FIND $g(k), \phi(k)$:

$$\boxed{g(k+1) - g(k) + \phi(k) = \frac{H_k}{k}}$$

for all $1 \leq k \leq m$ and $m \geq 0$.

Summing this equation over k from 1 to m gives

$$\boxed{\sum_{k=1}^{m} \frac{H_k}{k}} = \boxed{g(m+1) - g(1) + \sum_{k=1}^{m} \phi(k)}$$

$$= \frac{1}{2}\left(H_m^2 + \sum_{k=1}^{m} \frac{1}{k^2}\right).$$

Parameterized telescoping

Given $\Pi\Sigma^*$-field $(\mathbb{K}(t_1)\dots(t_e),\sigma)$.
Define $\mathbb{F}_i=\mathbb{K}(t_1)\dots(t_i)$, i.e.,

$$\mathbb{F}_0<\mathbb{F}_1<\dots<\mathbb{F}_i<\dots<\mathbb{F}_e$$

Parameterized telescoping

Given $\Pi\Sigma^*$-field $(\mathbb{K}(t_1)\dots(t_e),\sigma)$.
Define $\mathbb{F}_i=\mathbb{K}(t_1)\dots(t_i)$, i.e.,

$$\underbrace{\mathbb{F}_{-1}}_{=\{0\}}\subset\mathbb{F}_0<\mathbb{F}_1<\dots<\mathbb{F}_i<\dots<\mathbb{F}_e$$

Parameterized telescoping

Given $\Pi\Sigma^*$-field $(\mathbb{K}(t_1)\dots(t_e),\sigma)$.
Define $\mathbb{F}_i = \mathbb{K}(t_1)\dots(t_i)$, i.e.,

$$\underbrace{\mathbb{F}_{-1}}_{=\{0\}} \subset \mathbb{F}_0 < \mathbb{F}_1 < \dots < \mathbb{F}_i < \dots < \mathbb{F}_e$$

Given $f_1,\dots,f_n \in \mathbb{F}_e$.
Find **all** $c_1,\dots c_n \in \mathbb{K}$, $g \in \mathbb{F}_e$:

$$\sigma(g) - g = c_1\, f_1 + \dots + c_n\, f_n$$

Special cases are

- telescoping $(n = 1)$
- Zeilberger's creative telescoping paradigm (take $f_i = F(r-1+i,k)$)

First entry telescoping

Given $\Pi\Sigma^*$-field $(\mathbb{K}(t_1)\dots(t_e),\sigma)$.
Define $\mathbb{F}_i=\mathbb{K}(t_1)\dots(t_i)$, i.e.,

$$\underbrace{\mathbb{F}_{-1}}_{=\{0\}}\subset\mathbb{F}_0<\mathbb{F}_1<\dots<\mathbb{F}_i<\dots<\mathbb{F}_e$$

Given $f_1,\dots,f_n\in\mathbb{F}_e$.
Find $c_1,\dots c_n\in\mathbb{K}$, $g\in\mathbb{F}_e$:

$$\sigma(g)-g=\underbrace{c_1}_{\neq 0}\,f_1+\dots+c_n\,f_n$$

Note: Gives rise to faster algorithms

Find $g = g_2 h^2 + g_1 h + g_0 \in \mathbb{Q}(k)[h]_2$ s.t.

$$\sigma(g) - g = \frac{h}{k}$$

$$\left[\sigma(g_2)\left(h + \tfrac{1}{k+1}\right)^2 + \sigma(g_1 h + g_0)\right] - \left[g_2\, h^2 + g_1 h + g_0\right] = \tfrac{h}{k}$$

coeff. comp.

$$\sigma(g_2) - g_2 = 0$$

$g_2 = c \in \mathbb{Q}$

$$\sigma(g_1\, h + g_0) - (g_1\, h + g_0) = \tfrac{h}{k} + c\left[\tfrac{-2h(k+1)-1}{(k+1)^2}\right]$$

coeff. comp.

$$\sigma(g_1) - g_1 = \frac{1}{k} + c\,\tfrac{-2}{k+1}$$

$c = \frac{1}{2}$, $g_1 = -\frac{1}{k} + d$, $d \in \mathbb{Q}$

no solution $\leftarrow$ $$\sigma(g_0) - g_0 = \frac{2}{(k+1)^2} + d\frac{-1}{k+1}$$

First entry telescoping

Given $\Pi\Sigma^*$-field $(\mathbb{K}(t_1)\dots(t_e),\sigma)$.
Define $\mathbb{F}_i=\mathbb{K}(t_1)\dots(t_i)$, i.e.,

$$\underbrace{\mathbb{F}_{-1}}_{=\{0\}}\subset\mathbb{F}_0<\mathbb{F}_1<\dots<\mathbb{F}_i<\dots<\mathbb{F}_e$$

Given $f_1,\dots,f_n\in\mathbb{F}_e$.
Find $c_1,\dots c_n\in\mathbb{K}$, $g\in\mathbb{F}_e$:

$$\sigma(g)-g=\underbrace{c_1}_{\neq 0}f_1+\dots+c_n\,f_n$$

Note: Gives rise to faster algorithms

Refined telescoping

Given $\Pi\Sigma^*$-field $(\mathbb{K}(t_1)\dots(t_e),\sigma)$.
Define $\mathbb{F}_i = \mathbb{K}(t_1)\dots(t_i)$, i.e.,

$$\underbrace{\mathbb{F}_{-1}}_{=\{0\}} \subset \mathbb{F}_0 < \mathbb{F}_1 < \dots < \mathbb{F}_i < \dots < \mathbb{F}_e$$

Given $f_1,\dots,f_n \in \mathbb{F}_e$.
Find $c_1,\dots c_n \in \mathbb{K}$, $g \in \mathbb{F}_e$
$\boxed{\phi \in \mathbb{F}_i \text{ with } i \text{ minimal}}$:

$$\sigma(g) - g + \boxed{\phi} = \underbrace{c_1}_{\neq 0} f_1 + \dots + c_n f_n$$

Find $c_j \in \mathbb{K}$, $g \in \mathbb{F}_e$, $\phi \in \mathbb{F}_e$ s.t.

$$\sigma(g) - g + \phi = \underbrace{c_1}_{\neq 0} f_1 + \cdots + c_n f_n.$$

$$\boxed{V((-1,1),(f_1,\dots,f_n),\mathbb{F}_e)}$$

$$\rightsquigarrow$$

$$\boxed{V((-1,1),(f_1^{(e-1)},\dots,f_{n_{e-1}}^{(e-1)}),\mathbb{F}_{e-1})}$$

Find $g = g_2 h^2 + g_1 h + g_0 \in \mathbb{Q}(k)[h]_2$ s.t.

$$\sigma(g) - g + \frac{1}{2(k+1)^2} = \frac{h}{k} \qquad g = \frac{1}{2}h + \frac{1}{k}$$

$$V((-1,1),(\frac{h}{k}),\mathbb{Q}(k)(h))$$

coeff. comp.

$$\sigma(g_2) - g_2 = 0$$

$g_2 = c \in \mathbb{Q}$

coeff. comp.

$$\sigma(g_1) - g_1 = \frac{1}{k} + c\,\frac{-2}{k+1}$$

$$V((-1,1),(\frac{2}{(k+1)^2},\frac{-1}{k+1}),\mathbb{Q}(k))$$

Find $c_j \in \mathbb{K}$, $g \in \mathbb{F}_e$, $\phi \in \mathbb{F}_e$ s.t.

$$\sigma(g) - g + \phi = \underbrace{c_1}_{\neq 0} f_1 + \cdots + c_n f_n.$$

$$\boxed{V((-1,1),(f_1,\dots,f_n),\mathbb{F}_e)}$$

$$\Downarrow$$

$$\boxed{V((-1,1),(f_1^{(e-1)},\dots,f_{n_{e-1}}^{(e-1)}),\mathbb{F}_{e-1})}$$

Find $c_j \in \mathbb{K}$, $g \in \mathbb{F}_e$, $\phi \in \mathbb{F}_e$ s.t.

$$\sigma(g) - g + \phi = \underbrace{c_1}_{\neq 0} f_1 + \cdots + c_n f_n.$$

$$\boxed{V((-1,1),(f_1,\dots,f_n),\mathbb{F}_e)}$$

$$\boxed{V((-1,1),(f_1^{(e-1)},\dots,f_{n_{e-1}}^{(e-1)}),\mathbb{F}_{e-1})}$$

$$\vdots$$

$$\boxed{V((-1,1),(f_1^{(i)},\dots,f_{n_i}^{(i)}),\mathbb{F}_i)}$$

Find $c_j \in \mathbb{K}$, $g \in \mathbb{F}_e$, $\phi \in \mathbb{F}_e$ s.t.

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Refined telescoping

Given $\Pi\Sigma^*$-field $(\mathbb{K}(t_1)\dots(t_e),\sigma)$.
Define $\mathbb{F}_i=\mathbb{K}(t_1)\dots(t_i)$, i.e.,

$$\underbrace{\mathbb{F}_{-1}}_{=\{0\}}\subset\mathbb{F}_0<\mathbb{F}_1<\dots<\mathbb{F}_i<\dots<\mathbb{F}_e$$

Given $f_1,\dots,f_n\in\mathbb{F}_e$.
Find $c_1,\dots c_n\in\mathbb{K}$, $g\in\mathbb{F}_e$
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$$\sigma(g)-g+\boxed{\phi}=\underbrace{c_1}_{\neq 0}f_1+\dots+c_n f_n$$

Refined telescoping

Given $\Pi\Sigma^*$-extension $(\mathbb{G}(t_1)\dots(t_e),\sigma)$ of a $\boxed{\text{difference field } (\mathbb{G},\sigma)}$.
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Requirements: an algorithm to solve first order linear difference equations in $(\mathbb{G},\sigma)$ plus technical conditions.

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Example 1: $\mathbb{G} = \mathbb{K}$

Examples

$$\sum_{k=1}^{m}(k^2+1)\,k!\,H_k^2 = \sum_{k=1}^{m}\frac{(k+1)!}{k^2} + m\,(m+1)!\,H_m^2 - 2m!\,H_m.$$

$$\sum_{k=1}^{m} k \sum_{j=1}^{k}\frac{H_j}{j^2} = \frac{1}{2}\Bigg(\sum_{k=1}^{m}\frac{1}{k^2} + m(m+1)\sum_{i=1}^{m}\frac{H_i}{i^2} + H_m^2 - (m+1)H_m + m\Bigg)$$

$$\sum_{k=1}^{m}\Bigg(\sum_{j=1}^{k}\binom{r}{j}\Bigg)^2 = -\frac{1}{2}r\sum_{k=1}^{m}\binom{r}{k}^2 + \frac{1}{2}(2m-r+2)\Bigg(\sum_{i=1}^{m}\binom{r}{i}\Bigg)^2 - (m-r)\binom{r}{m}\sum_{i=1}^{m}\binom{r}{i}$$

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Example 2 (free difference field): $\mathbb{G} = \mathbb{K}(\dots,x_{-1},x_0,x_1,\dots)$
with $\sigma(x_i) = x_{i+1}$.

Unspecified sequences (M. Kauers,CS; 2006)

$$\sum_{k=1}^{n} a^k \sum_{j=1}^{k} X_j = \frac{a^{n+1} \sum_{k=1}^{n} X_k - \sum_{k=1}^{n} a^k X_k}{a-1}, \quad a \neq 1$$

$$\sum_{k=0}^{n} k^2 \sum_{i=0}^{k} X_i = \tfrac{1}{6}\Big(n(n+1)(2n+1) \sum_{k=0}^{n} X_k - \sum_{k=0}^{n} k X_k + 3 \sum_{k=0}^{n} k^2 X_k - 2 \sum_{k=0}^{n} k^3 X_k \Big)$$

Refined telescoping

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Example 3 (radicals of order $d>0$): $\mathbb{G}=\mathbb{K}(k)(\dots,x_{-1},x_0,x_1,\dots)$ with $\sigma(k)=k+1$ and $\sigma(x_i)=x_{i+1}$ subject to the relation $x_i^d=k$.

Radical expressions (M. Kauers, CS; 2007)

$$\sum_{k=0}^{n} \frac{1}{\sqrt{k+1}+\sqrt{k}} = \sqrt{n+1}$$

$$\begin{aligned}\sum_{k=2}^{n} H_k\Big(\frac{(\sqrt{k})^3}{k-1} + \sum_{i=1}^{k} \frac{\sqrt{i}}{i+\sqrt{i}}\Big) &= \frac{1}{2}\Big(-3-5n+(5n+3)H_n \\ &\quad -(2n+1)H_n^2 + H_n^{(2)}\Big) + \sum_{k=1}^{n}\sqrt{k} \\ &\quad + \Big((n+1)H_n-(n-1)\Big)\sum_{k=2}^{n}\frac{\sqrt{k}}{k-1}\end{aligned}$$

Conclusion

simplified version of
Karr's algorithm

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refined parameterized
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simpler sum representations/
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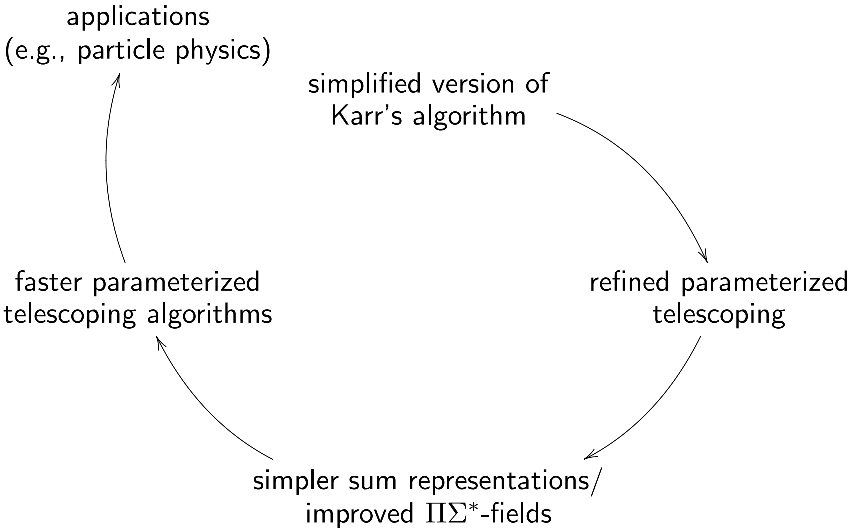
Conclusion



Further details: LNCS Proceedings [arXiv:1307.7887]