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Here I will restrict to the setting of difference rings/fields.
Robin and Herb,

I am willing to bet that Carsten Schneider’s SIGMA package for handling sums with harmonic numbers (among others) can do it in a jiffy. I am Cc-ing this to Carsten.

Carsten: please do it, and Cc- the answer to me.

-Doron
The problem

From: Robin Pemantle [University of Pennsylvania]
To: herb wilf; doron zeilberger

Herb, Doron,

I have a sum that, when I evaluate numerically, looks suspiciously like it comes out to exactly 1.
Is there a way I can automatically decide this?
The sum may be written in many ways, but one is:

$$\sum_{n,k=1}^{\infty} \frac{H_k(H_{n+1} - 1)}{kn(n+1)(k+n)}; \quad H_k := \sum_{i=1}^{k} \frac{1}{i}$$

[Arose in the analysis of the simplex algorithm on the Klee-Minty cube (J. Balogh, R. Pemantle)]
\[ S = \sum_{n=1}^{\infty} \frac{H_{n+1} - 1}{n(n + 1)} \sum_{k=1}^{\infty} \frac{H_k}{k(k + n)} \]

where \[ H_k = \sum_{i=1}^{k} \frac{1}{i}. \]
GIVEN

\[ A'(n) := \sum_{k=1}^{a} \frac{H_k}{k(k + n)} \cdot \]

\[ =: f(n, k) \]
Telescoping

Given

\[ A'(n) := \sum_{k=1}^{a} \frac{H_k}{k(k + n)} \cdot \]

Find \( g(n, k) \):

\[ g(n, k + 1) - g(n, k) = f(n, k) \]

for all \( n, k \geq 1 \).
Telescoping

GIVEN

\[ A'(n) := \sum_{k=1}^{a} \frac{H_k}{k(k + n)} \cdot \]

\[ =: f(n, k) \]

FIND \( g(n, k) \):

\[ g(n, k + 1) - g(n, k) = f(n, k) \]

for all \( n, k \geq 1 \).

\[ g(n, a + 1) - g(n, 1) = \sum_{k=1}^{a} f(n, k) \]
Telescoping

GIVEN

\[ A'(n) := \sum_{k=1}^{a} \frac{H_k}{k(k + n)} \cdot \]

\[ =: f(n, k) \]

FIND \( g(n, k) \):

\[ g(n, k + 1) - g(n, k) = f(n, k) \]

for all \( n, k \geq 1 \).

no solution 😞
Zeilberger’s creative telescoping paradigm

GIVEN

\[
A'(n) := \sum_{k=1}^{a} \frac{H_k}{k(k + n)}.
\]

\[= f(n, k)\]

FIND \(g(n, k)\) and \(c_0(n), c_1(n)\):

\[
g(n, k + 1) - g(n, k) = c_0(n)f(n, k) + c_1(n)f(n + 1, k)
\]

for all \(n, k \geq 1\).

no solution 😞
Zeilberger’s creative telescoping paradigm

GIVEN

\[ A'(n) := \sum_{k=1}^{a} \frac{H_k}{k(k+n)}. \]

\[ =: f(n, k) \]

FIND \( g(n, k) \) and \( c_0(n), c_1(n), c_2(n) \):

\[ g(n, k + 1) - g(n, k) = c_0(n)f(n, k) + c_1(n)f(n + 1, k) + c_2(n)f(n + 2, k) \]

for all \( n, k \geq 1 \).

solution 😊
Zeilberger’s creative telescoping paradigm

**GIVEN**

\[ A'(n) := \sum_{k=1}^{a} \frac{H_k}{k(k+n)} \cdot \]

\[ =: f(n, k) \]

**FIND** \(g(n, k)\) and \(c_0(n), c_1(n), c_2(n)\):

\[ g(n, k + 1) - g(n, k) = c_0(n)f(n, k) + c_1(n)f(n + 1, k) + c_2(n)f(n + 2, k) \]

for all \(n, k \geq 1\).

**Sigma computes:**

\[ c_0(n) = n^2, \quad c_1(n) = -(n + 1)(2n + 1), \quad c_2(n) = (n + 1)(n + 2) \]

and

\[ g(n, k) := -\frac{kH_k + n + k}{(n + k)(n + k + 1)}, \]

\[ g(n, k + 1) := -\frac{(1 + n)H_k + n + k + 2}{(n + k + 1)(n + k + 2)}. \]
Zeilberger’s creative telescoping paradigm

GIVEN

$$A'(n) := \sum_{k=1}^{a} \frac{H_k}{k(k+n)} =: f(n,k)$$

FIND $g(n,k)$ and $c_0(n), c_1(n), c_2(n)$:

$$g(n,k+1) - g(n,k) = c_0(n)f(n,k) + c_1(n)f(n+1,k) + c_2(n)f(n+2,k)$$

for all $n, k \geq 1$.

Summing this equation over $k$ from 1 to $a$ gives:

$$g(n,a+1) - g(n,1) = \sum_{k=1}^{a} \left[ c_0(n)f(n,k) + c_1(n)f(n+1,k) + c_2(n)f(n+2,k) \right]$$
Zeilberger’s creative telescoping paradigm

**GIVEN**

\[ A'(n) := \sum_{k=1}^{a} \frac{H_k}{k(k+n)} \cdot \]

\[ =: f(n, k) \]

**FIND** \( g(n, k) \) and \( c_0(n), c_1(n), c_2(n) \):

\[
g(n, k + 1) - g(n, k) = c_0(n)f(n, k) + c_1(n)f(n + 1, k) + c_2(n)f(n + 2, k)
\]

for all \( n, k \geq 1 \).

Summing this equation over \( k \) from 1 to \( a \) gives:

\[
g(n, a + 1) - g(n, 1) = \sum_{k=1}^{a} c_0(n)f(n, k) + \sum_{k=1}^{a} c_1(n)f(n + 1, k) + \sum_{k=1}^{a} c_2(n)f(n + 2, k)
\]
Zeilberger’s creative telescoping paradigm

GIVEN

\[ A'(n) := \sum_{k=1}^{a} \frac{H_k}{k(k+n)} =: f(n, k) \]

FIND \( g(n, k) \) and \( c_0(n), c_1(n), c_2(n) \):

\[ g(n, k + 1) - g(n, k) = c_0(n)f(n, k) + c_1(n)f(n + 1, k) + c_2(n)f(n + 2, k) \]

for all \( n, k \geq 1 \).

Summing this equation over \( k \) from 1 to \( a \) gives:

\[ g(n, a + 1) - g(n, 1) = c_0(n)\sum_{k=1}^{a} f(n, k) + c_1(n)\sum_{k=1}^{a} f(n + 1, k) + c_2(n)\sum_{k=1}^{a} f(n + 2, k) \]
Zeilberger’s creative telescoping paradigm

GIVEN

\[
A'(n) := \sum_{k=1}^{a} \frac{H_k}{k(k + n)}.
\]

FIND \(g(n, k)\) and \(c_0(n), c_1(n), c_2(n)\):

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g(n, k + 1) - g(n, k) = c_0(n)f(n, k) + c_1(n)f(n + 1, k) + c_2(n)f(n + 2, k)
\]
for all \(n, k \geq 1\).

Summing this equation over \(k\) from 1 to \(a\) gives:

\[
g(n, a + 1) - g(n, 1) = c_0(n)A'(n) + c_1(n)A'(n + 1) + c_2(n)A'(n + 2)
\]
Zeilberger’s creative telescoping paradigm

**GIVEN**

\[
A'(n) := \sum_{k=1}^{a} \frac{H_k}{k(k + n)}.
\]

**FIND** \(g(n, k)\) and \(c_0(n), c_1(n), c_2(n)\):

\[
g(n, k + 1) - g(n, k) = c_0(n)f(n, k) + c_1(n)f(n + 1, k) + c_2(n)f(n + 2, k)
\]

for all \(n, k \geq 1\).

Summing this equation over \(k\) from 1 to \(a\) gives:

\[
g(n, a + 1) - g(n, 1) = c_0(n)A'(n) + c_1(n)A'(n + 1) + c_2(n)A'(n + 2)
\]

\[
\frac{a}{(n+1)(a+n+1)} n^2A'(n) - (n+1)(2n+1)A'(n + 1) + (n+1)(n+2)A'(n+2)
\]

\[- \frac{(a+1)H_a}{(a+n+1)(a+n+2)}\]
Summation principles (in difference field/ring setting)

\[
\begin{align*}
    n^2 A(n) - (n + 1)(2n + 1)A(n + 1) + (n + 1)(n + 2)A(n + 2) &= \frac{1}{n + 1} \\
    A(n) &= \sum_{k=1}^{\infty} \frac{H_k}{k(k + n)}
\end{align*}
\]
A bet (at my cost)

Summation principles (in difference field/ring setting)

\[ n^2 \mathbf{A}(n) - (n + 1)(2n + 1) \mathbf{A}(n + 1) + (n + 1)(n + 2) \mathbf{A}(n + 2) = \frac{1}{n + 1} \]

Recurrence solver

\[ \mathbf{A}(n) = \sum_{k=1}^{\infty} \frac{H_k}{k(k + n)} \]

where

\[ H_{n}^{(2)} = \sum_{i=1}^{n} \frac{1}{i^2} \]

\[ \left\{ \begin{array}{l}
    c_1 \frac{nH_n - 1}{n^2} + c_2 \frac{1}{n} \\
    + \frac{nH_n^2 - 2H_n + nH_n^{(2)}}{2n^2}
\end{array} \right\} \quad \text{where} \quad c_1, c_2 \in \mathbb{R} \]
A bet (at my cost)

Summation principles (in difference field/ring setting)

\[ n^2 A(n) - (n + 1)(2n + 1)A(n + 1) + (n + 1)(n + 2)A(n + 2) = \frac{1}{n + 1} \]

Recurrence solver

\[ A(n) = \sum_{k=1}^{\infty} \frac{H_k}{k(k + n)} \in \left\{ c_1 \frac{nH_n - 1}{n^2} + c_2 \frac{1}{n} \left( \frac{nH_n^2 - 2H_n + nH_n^{(2)}}{2n^2} \right) \right\} \quad \mid c_1, c_2 \in \mathbb{R} \]

where \( H_n^{(2)} = \sum_{i=1}^{n} \frac{1}{i^2} \)
A bet (at my cost)

Summation principles (in difference field/ring setting)

\[ n^2 A(n) - (n + 1)(2n + 1)A(n + 1) + (n + 1)(n + 2)A(n + 2) = \frac{1}{n + 1} \]

Summation package Sigma

(based on difference field algorithms/theory
see, e.g., Karr 1981, Bronstein 2000, Schneider 2001 –)

\[
A(n) = \sum_{k=1}^{\infty} \frac{H_k}{k(k + n)} = 0 \frac{nH_n - 1}{n^2} + \zeta_2 \frac{1}{n} + nH_n^2 - 2H_n + nH_n^{(2)} \frac{1}{2n^2}
\]

where

\[
H_n^{(2)} = \sum_{i=1}^{n} \frac{1}{i^2} \quad \zeta_z = \sum_{i=1}^{\infty} \frac{1}{i^z} (= \zeta(z))
\]

RISC, J. Kepler University Linz
ln[1]:= << Sigma.m

Sigma - A summation package by Carsten Schneider © RISC-Linz

ln[2]:= mySum = \sum_{k=1}^{a} \frac{H_k}{k(k + n)}
A bet (at my cost)

\[ \text{In[1]} := \text{\texttt{<< Sigma.m}} \]

\[ \text{Sigma - A summation package by Carsten Schneider} \odot \text{RISC-Linz} \]

\[ \text{In[2]} := \text{mySum} = \sum_{k=1}^{a} \frac{H_k}{k(k+n)} \]

\[ \text{In[3]} := \text{rec} = \text{GenerateRecurrence[mySum, n][[1]]} \]

\[ \text{Out[3]} = \text{n}^2 \text{SUM}[n] - (n+1)(2n+1)\text{SUM}[n+1] + (n+1)(n+2)\text{SUM}[n+2] == \]

\[ \frac{(-a-1)H_a}{(a+n+1)(a+n+2)} + \frac{a}{(n+1)(a+n+1)} \]
In[1]:= <<Sigma.m

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= mySum = \[\sum_{k=1}^{a} \frac{H_k}{k(k+n)}\]

In[3]:= rec = GenerateRecurrence[mySum, n][[1]]

Out[3]= \[n^2 \text{SUM}[n] - (n + 1)(2n + 1)\text{SUM}[n + 1] + (n + 1)(n + 2)\text{SUM}[n + 2] == \frac{(-a - 1)H_a}{(a + n + 1)(a + n + 2)} + \frac{a}{(n + 1)(a + n + 1)}\]

In[4]:= rec = LimitRec[rec, \text{SUM}[n], \{n\}, a]

Out[4]= \[n^2 \text{SUM}[n] - (n + 1)(2n + 1)\text{SUM}[n + 1] + (n + 1)(n + 2)\text{SUM}[n + 2] = \frac{1}{n + 1}\]
In[1]:= << Sigma.m

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= mySum = \[\sum\limits_{k=1}^{a} \frac{H_k}{k(k+n)}\]

In[3]:= rec = GenerateRecurrence[mySum, n][[1]]

Out[3]= \[n^2\text{SUM}[n] - (n + 1)(2n + 1)\text{SUM}[n + 1] + (n + 1)(n + 2)\text{SUM}[n + 2] ==
\frac{(-a - 1)H_a}{(a + n + 1)(a + n + 2)} + \frac{a}{(n + 1)(a + n + 1)}\]

In[4]:= rec = LimitRec[rec, \text{SUM}[n], \{n\}, a]

Out[4]= \[n^2\text{SUM}[n] - (n + 1)(2n + 1)\text{SUM}[n + 1] + (n + 1)(n + 2)\text{SUM}[n + 2] = \frac{1}{n + 1}\]

In[5]:= recSol = SolveRecurrence[rec, \text{SUM}[n], \text{IndefiniteSummation} \rightarrow \text{True}]

Out[5]= \[\{\{0, \frac{1}{n}\}, \{0, \frac{1}{n^2} - \frac{1}{n}\}, \{1, \frac{\left(\sum_{i=1}^{n} \frac{1}{i}\right)^2}{2n} - \sum_{i=1}^{n} \frac{1}{i} + \frac{\sum_{i=1}^{n} \frac{1}{i^2}}{2n}\}\}\]
In[1]:= << Sigma.m

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= mySum = \[\sum_{k=1}^{a} \frac{H_k}{k(k + n)}\]

In[3]:= rec = GenerateRecurrence[mySum, n][[1]]

Out[3]= \[\begin{align*}
(n + 1)(2n + 1)\text{SUM}[n + 1] + (n + 1)(n + 2)\text{SUM}[n + 2] &= \frac{(-a - 1)H_a}{(a + n + 1)(a + n + 2)} + \frac{a}{(n + 1)(a + n + 1)}
\end{align*}\]

In[4]:= rec = LimitRec[rec, \text{SUM}[n], \{n\}, a]

Out[4]= \[\begin{align*}
(n + 1)(2n + 1)\text{SUM}[n + 1] + (n + 1)(n + 2)\text{SUM}[n + 2] &= \frac{1}{n + 1}
\end{align*}\]

In[5]:= recSol = SolveRecurrence[rec, \text{SUM}[n], \text{IndefiniteSummation} \rightarrow \text{True}]

Out[5]= \{\{0, \frac{1}{n}\}, \{0, \frac{1}{n} - \frac{1}{n^2}\}, \{1, \frac{(\sum_{i=1}^{n} \frac{1}{i})^2}{2n} - \frac{\sum_{i=1}^{n} \frac{1}{i}}{n^2} + \frac{\sum_{i=1}^{n} \frac{1}{i^2}}{2n}\}\}

In[6]:= FindLinearCombination[recSol, \{1, \{\zeta_2, 1/2 + \zeta_2/2\}\}, n, 2]

Out[6]= -\frac{\sum_{i=1}^{n} \frac{1}{i^2}}{n^2} + \frac{(\sum_{i=1}^{n} \frac{1}{i})^2}{2n} + \frac{\sum_{i=1}^{n} \frac{1}{i^2}}{2n} + \frac{\zeta_2}{n}
\[ S = \sum_{n=1}^{\infty} \frac{H_{n+1} - 1}{n(n+1)} \sum_{k=1}^{\infty} \frac{H_k}{k(k+n)} \]

\[ = \frac{\zeta_2}{n} + \frac{n H_n^2 - 2H_n + n H_n^{(2)}}{2n^2} \]
\[ S = \sum_{n=1}^{\infty} \frac{H_{n+1} - 1}{n(n + 1)} \left[ \sum_{k=1}^{\infty} \frac{H_k}{k(k + n)} \right] \]

\[ = \frac{\zeta_2}{n} + \frac{nH_n^2 - 2H_n + nH_n^{(2)}}{2n^2} \]

\[ = -4\zeta_2 + (\zeta_2 - 1) \sum_{i=1}^{\infty} \frac{H_i}{i^2} - \sum_{i=1}^{\infty} \frac{H_i^2}{i^3} \]

\[ + \frac{1}{2} \sum_{i=1}^{\infty} \frac{H_i^3}{i^2} + \frac{1}{2} \sum_{i=1}^{\infty} \frac{H_i H_i^{(2)}}{i^2} \]
\[
S = \sum_{n=1}^{\infty} \frac{H_{n+1} - 1}{n(n+1)} \sum_{k=1}^{\infty} \frac{H_k}{k(k+n)}
\]

\[
= \frac{\zeta_2}{n} + \frac{nH_n^2 - 2H_n + nH_n^{(2)}}{2n^2}
\]

\[
= -4\zeta_2 + (\zeta_2 - 1) \sum_{i=1}^{\infty} \frac{H_i}{i^2} - \sum_{i=1}^{\infty} \frac{H_i^2}{i^3}
\]

\[
+ \frac{1}{2} \sum_{i=1}^{\infty} \frac{H_i^3}{i^2} + \frac{1}{2} \sum_{i=1}^{\infty} \frac{H_i H_i^{(2)}}{i^2}
\]

\[
= -4\zeta_2 - 2\zeta_3 + 4\zeta_2 \zeta_3 + 2\zeta_5 = 0.999222\ldots
\]


\[
S = \sum_{n=1}^{\infty} \frac{H_{n+1} - 1}{n(n+1)} \left( \sum_{k=1}^{\infty} \frac{H_k}{k(k+n)} \right)
\]

\[
= \frac{\zeta_2}{n} + \frac{nH_n^2 - 2H_n + nH_n^{(2)}}{2n^2}
\]

\[
= -4\zeta_2 + (\zeta_2 - 1) \sum_{i=1}^{\infty} \frac{H_i}{i^2} - \sum_{i=1}^{\infty} \frac{H_i^2}{i^3}
\]

\[
+ \frac{1}{2} \sum_{i=1}^{\infty} \frac{H_i^3}{i^2} + \frac{1}{2} \sum_{i=1}^{\infty} \frac{H_i H_i^{(2)}}{i^2}
\]

\[
= -4\zeta_2 - 2\zeta_3 + 4\zeta_2 \zeta_3 + 2\zeta_5 = 0.999222...
\]


Toolbox 1: Indefinite summation

Toolbox 2: Definite summation

Toolbox 3: Special function algorithms
Toolbox 1: Indefinite summation
Telescoping

**GIVEN** \( f(k) = H_k \).

**FIND** \( g(k) \):

\[
f(k) = g(k + 1) - g(k)
\]

for all \( 1 \leq k \leq n \) and \( n \geq 0 \).
Telescoping

**GIVEN** \( f(k) = H_k \).

**FIND** \( g(k) \):

\[
f(k) = g(k + 1) - g(k)
\]

for all \( 1 \leq k \leq n \) and \( n \geq 0 \).

We compute

\[
g(k) = (H_k - 1)k.
\]
Telescoping

GIVEN $f(k) = H_k$.

FIND $g(k)$:

\[ f(k) = g(k + 1) - g(k) \]

for all $1 \leq k \leq n$ and $n \geq 0$.

Summing this equation over $k$ from 1 to $n$ gives

\[ \sum_{k=1}^{n} H_k = g(n + 1) - g(1) \]

\[ = (H_{n+1} - 1)(n + 1). \]
Telescoping in the given difference field

FIND a closed form for

\[ \sum_{k=1}^{n} H_k. \]

A difference field for the summand

Consider the rational function field

\[ \mathbb{F} \]

with the automorphism \( \sigma : \mathbb{F} \to \mathbb{F} \) defined by
Telescoping in the given difference field

FIND a closed form for

\[ \sum_{k=1}^{n} H_k. \]

A difference field for the summand

Consider the rational function field

\[ \mathbb{F} := \mathbb{Q} \]

with the automorphism \( \sigma : \mathbb{F} \rightarrow \mathbb{F} \) defined by

\[ \sigma(c) = c \quad \forall c \in \mathbb{Q}, \]
Telescoping in the given difference field

FIND a closed form for

$$\sum_{k=1}^{n} H_k.$$ 

A difference field for the summand

Consider the rational function field

$$\mathbb{F} := \mathbb{Q}(k)$$

with the automorphism \( \sigma : \mathbb{F} \rightarrow \mathbb{F} \) defined by

\[
\sigma(c) = c \quad \forall c \in \mathbb{Q}, \quad \sigma(k) = k + 1, \quad S_k = k + 1,
\]
Telescoping in the given difference field

FIND a closed form for
\[ \sum_{k=1}^{n} H_k. \]

A difference field for the summand

Consider the rational function field
\[ F := \mathbb{Q}(k)(h) \]

with the automorphism \( \sigma : F \rightarrow F \) defined by
\[
\begin{align*}
\sigma(c) &= c \quad \forall c \in \mathbb{Q}, \\
\sigma(k) &= k + 1, \\
\sigma(h) &= h + \frac{1}{k + 1}, \\
S k &= k + 1, \\
S H_k &= H_k + \frac{1}{k + 1}.
\end{align*}
\]
Telescoping in the given difference field

FIND \( g \in \mathbb{F} \):

\[ \sigma(g) - g = h. \]
Telescoping in the given difference field

FIND $g \in \mathbb{F}$:

\[ \sigma(g) - g = h. \]

We compute

\[ g = (h - 1)k \in \mathbb{F}. \]
Telescoping in the given difference field

**FIND** $g \in \mathbb{F}$:

$$\sigma(g) - g = h.$$ 

We compute

$$g = (h - 1)k \in \mathbb{F}.$$ 

This gives

$$g(k + 1) - g(k) = H_k$$

with

$$g(k) = (H_k - 1)k.$$
Telescoping in the given difference field

**FIND** \( g \in \mathbb{F} \):

\[ \sigma(g) - g = h. \]

We compute

\[ g = (h - 1)k \in \mathbb{F}. \]

This gives

\[ g(k + 1) - g(k) = H_k \]

with

\[ g(k) = (H_k - 1)k. \]

Hence,

\[ (H_{n+1} - 1)(n + 1) = \sum_{k=1}^{n} H_k. \]
Toolbox 1: Indefinite summation
– the basic tactic
(a simplified version of Karr’s algorithm, 1981)
CONSTRUCT a difference field \((F, \sigma)\):

- a rational function field (containing \(\mathbb{Q}\))
  \[ F := \mathbb{K} \]

- with an automorphism
  \[ \sigma(c) = c \quad \forall c \in \mathbb{K} \]
CONSTRUCT a difference field \((F, \sigma)\):

- a rational function field (containing \(\mathbb{Q}\))
  \[ F := \mathbb{K}(t_1) \]
- with an automorphism
  \[
  \begin{align*}
  \sigma(c) &= c \quad \forall c \in \mathbb{K} \\
  \sigma(t_1) &= a_1 t_1 + f_1, \quad a_1 \in \mathbb{K}^*, \quad f_1 \in \mathbb{K}
  \end{align*}
  \]
**CONSTRUCT** a difference field $(\mathbb{F}, \sigma)$:

- a rational function field (containing $\mathbb{Q}$)
  \[ \mathbb{F} := \mathbb{K}(t_1)(t_2) \]

- with an automorphism
  \[
  \begin{align*}
  \sigma(c) &= c \quad \forall c \in \mathbb{K} \\
  \sigma(t_1) &= a_1 t_1 + f_1, \quad a_1 \in \mathbb{K}^*, \quad f_1 \in \mathbb{K} \\
  \sigma(t_2) &= a_2 t_2 + f_2, \quad a_2 \in \mathbb{K}(t_1)^*, \quad f_2 \in \mathbb{K}(t_1)
  \end{align*}
  \]
CONSTRUCT a difference field \((F, \sigma)\):

- a rational function field (containing \(\mathbb{Q}\))

\[
F := \mathbb{K}(t_1)(t_2)\ldots(t_e)
\]

- with an automorphism

\[
\sigma(c) = c \quad \forall c \in \mathbb{K}
\]
\[
\sigma(t_1) = a_1 t_1 + f_1, \quad a_1 \in \mathbb{K}^*, \quad f_1 \in \mathbb{K}
\]
\[
\sigma(t_2) = a_2 t_2 + f_2, \quad a_2 \in \mathbb{K}(t_1)^*, \quad f_2 \in \mathbb{K}(t_1)
\]
\[
\vdots
\]
\[
\sigma(t_e) = a_e t_e + f_e, \quad a_e \in \mathbb{K}(t_1, \ldots, t_{e-1})^*, \quad f_e \in \mathbb{K}(t_1, \ldots, t_{e-1})
\]
CONSTRUCT a difference field $(F, \sigma)$:

- a rational function field (containing $\mathbb{Q}$)

$$F := \mathbb{K}(t_1)(t_2) \ldots (t_e)$$

- with an automorphism

\[ \sigma(c) = c \quad \forall c \in \mathbb{K} \]
\[ \sigma(t_1) = a_1 t_1 + f_1, \quad a_1 \in \mathbb{K}^*, \quad f_1 \in \mathbb{K} \]
\[ \sigma(t_2) = a_2 t_2 + f_2, \quad a_2 \in \mathbb{K}(t_1)^*, \quad f_2 \in \mathbb{K}(t_1) \]
\[ \vdots \]
\[ \sigma(t_e) = a_e t_e + f_e, \quad a_e \in \mathbb{K}(t_1, \ldots, t_{e-1})^*, \quad f_e \in \mathbb{K}(t_1, \ldots, t_{e-1}) \]

such that

\[ \text{const}_\sigma F = \{ c \in \mathbb{K}(t_1)(t_2) \ldots (t_e) | \sigma(c) = c \} = \mathbb{K}. \]
CONSTRUCT a $\Pi\Sigma$-field $(\mathbb{F}, \sigma)$:

- a rational function field (containing $\mathbb{Q}$)
  \[ \mathbb{F} := \mathbb{K}(t_1)(t_2) \ldots (t_e) \]

- with an automorphism
  \[ \sigma(c) = c \quad \forall c \in \mathbb{K} \]
  \[ \sigma(t_1) = a_1 t_1 + f_1, \quad a_1 \in \mathbb{K}^*, \quad f_1 \in \mathbb{K} \]
  \[ \sigma(t_2) = a_2 t_2 + f_2, \quad a_2 \in \mathbb{K}(t_1)^*, \quad f_2 \in \mathbb{K}(t_1) \]
  \[ \vdots \]
  \[ \sigma(t_e) = a_e t_e + f_e, \quad a_e \in \mathbb{K}(t_1, \ldots, t_{e-1})^*, \quad f_e \in \mathbb{K}(t_1, \ldots, t_{e-1}) \]

such that

\[ \text{const}_\sigma \mathbb{F} = \{ c \in \mathbb{K}(t_1)(t_2) \ldots (t_e) | \sigma(c) = c \} = \mathbb{K}. \]

GIVEN $f \in \mathbb{F}$;

FIND, in case of existence, a $g \in \mathbb{F}$ such that

\[ \sigma(g) - g = f. \]
Telescoping in the given difference field

FIND a closed form for

\[ \sum_{k=1}^{n} H_k. \]

**A \( \Pi \Sigma^* \)-field for the summand**

Consider the rational function field

\[ F := \mathbb{Q}(k)(h) \]

with the automorphism \( \sigma : F \rightarrow F \) defined by

\[ \sigma(c) = c \quad \forall c \in \mathbb{Q}, \]
\[ \sigma(k) = k + 1, \quad S \ k = k + 1, \]
\[ \sigma(h) = h + \frac{1}{k+1}, \quad S \ H_k = H_k + \frac{1}{k+1}. \]
FIND $g \in \mathbb{Q}(k)(h)$:

$$\sigma(g) - g = h.$$
FIND $g \in \mathbb{Q}(k)(h)$:

$$\sigma(g) - g = h.$$ 

**Denominator bound:** COMPUTE a polynomial $d \in \mathbb{Q}(k)[h]^*$:

$$\forall g \in \mathbb{Q}(k)(h) : \quad \sigma(g) - g = h \quad \Rightarrow \quad g \cdot d \in \mathbb{Q}(k)[h].$$

FIND $g' \in \mathbb{Q}(k)[h]$ with

$$\sigma\left(\frac{g'}{d}\right) - \frac{g'}{d} = h.$$
FIND \( g \in \mathbb{Q}(k)(h) \):

\[ \sigma(g) - g = h. \]

**Denominator bound:** COMPUTE a polynomial \( d \in \mathbb{Q}(k)[h]^* \):

\[ d = 1 \]

\[ \forall g \in \mathbb{Q}(k)(h) : \quad \sigma(g) - g = h \quad \Rightarrow \quad g d \in \mathbb{Q}(k)[h]. \]

FIND \( g' \in \mathbb{Q}(k)[h] \) with

\[ \sigma\left(\frac{g'}{d}\right) - \frac{g'}{d} = h. \]
FIND \( g \in \mathbb{Q}(k)(h) \):

\[ \sigma(g) - g = h. \]

**Denominator bound:** COMPUTE a polynomial \( d \in \mathbb{Q}(k)[h]^* \):

\[ d = 1 \]

\[ \forall g \in \mathbb{Q}(k)(h) : \quad \sigma(g) - g = h \quad \Rightarrow \quad g \cdot d \in \mathbb{Q}(k)[h]. \]

FIND \( g' \in \mathbb{Q}(k)[h] \) with

\[ \sigma\left(\frac{g'}{d}\right) - \frac{g'}{d} = h. \]

**Degree bound:** COMPUTE \( b \geq 0 \):

\[ \forall g \in \mathbb{Q}(k)[h] \quad \sigma(g) - g = h \quad \Rightarrow \quad \deg(g) \leq b. \]
FIND $g \in \mathbb{Q}(k)(h)$:
\[
\sigma(g) - g = h.
\]

**Denominator bound:** COMPUTE a polynomial $d \in \mathbb{Q}(k)[h]^*$:
\[
d = 1
\]

\[\forall g \in \mathbb{Q}(k)(h) : \quad \sigma(g) - g = h \quad \Rightarrow \quad g d \in \mathbb{Q}(k)[h].\]

FIND $g' \in \mathbb{Q}(k)[h]$ with
\[
\sigma \left( \frac{g'}{d} \right) - \frac{g'}{d} = h.
\]

**Degree bound:** COMPUTE $b \geq 0$:
\[
b = 2
\]

\[\forall g \in \mathbb{Q}(k)[h] \quad \sigma(g) - g = h \quad \Rightarrow \quad \deg(g) \leq b.\]
FIND $g \in \mathbb{Q}(k)(h)$:

$$\sigma(g) - g = h.$$ 

**Denominator bound:** COMPUTE a polynomial $d \in \mathbb{Q}(k)[h]^*$:

$$d = 1$$

$$\forall g \in \mathbb{Q}(k)(h) : \quad \sigma(g) - g = h \quad \Rightarrow \quad gd \in \mathbb{Q}(k)[h].$$

FIND $g' \in \mathbb{Q}(k)[h]$ with

$$\sigma\left(\frac{g'}{d}\right) - \frac{g'}{d} = h.$$ 

**Degree bound:** COMPUTE $b \geq 0$:

$$b = 2$$

$$\forall g \in \mathbb{Q}(k)[h] \quad \sigma(g) - g = h \quad \Rightarrow \quad \deg(g) \leq b.$$ 

**Polynomial Solution:** FIND

$$g = g_2 h^2 + g_1 h + g_0 \in \mathbb{Q}(k)[h].$$
ANSATZ \( g = g_2 h^2 + g_1 h + g_0 \in \mathbb{Q}(k)[h] \)

\[ \sigma(g) - g = h \]
ANSATZ \( g = g_2 h^2 + g_1 h + g_0 \in \mathbb{Q}(k)[h] \)

\[
\left[ \sigma(g_2) \left( h + \frac{1}{k+1} \right)^2 + \sigma(g_1 h + g_0) \right] - \left[ g_2 h^2 + g_1 h + g_0 \right] = h
\]
Toolbox 1: Indefinite summation

ANSATZ $g = g_2 h^2 + g_1 h + g_0 \in \mathbb{Q}(k)[h]$

$$
\left[ \sigma(g_2) \left( h + \frac{1}{k+1} \right)^2 + \sigma(g_1 h + g_0) \right] - [g_2 h^2 + g_1 h + g_0] = h
$$

$\sigma(g_2) - g_2 = 0$
ANSATZ $g = g_2 h^2 + g_1 h + g_0 \in \mathbb{Q}(k)[h]$

\[
\left[\sigma(g_2)\left(h + \frac{1}{k+1}\right)^2 + \sigma(g_1 h + g_0)\right] - \left[g_2 h^2 + g_1 h + g_0\right] = h
\]

$\sigma(g_2) - g_2 = 0$

$g_2 = c \in \mathbb{Q}$
ANSATZ  \( g = g_2 h^2 + g_1 h + g_0 \in \mathbb{Q}(k)[h] \)

\[
\begin{align*}
\left[ \sigma(g_2) \left( h + \frac{1}{k+1} \right)^2 + \sigma(g_1 h + g_0) \right] - \left[ g_2 h^2 + g_1 h + g_0 \right] &= h \\
\sigma(g_2) - g_2 &= 0 \\
g_2 &= c \in \mathbb{Q}
\end{align*}
\]
**ANSATZ** $g = g_2 h^2 + g_1 h + g_0 \in \mathbb{Q}(k)[h]$

\[
\begin{align*}
[\sigma(g_2)(h + \frac{1}{k+1})^2 + \sigma(g_1 h + g_0)] - [g_2 h^2 + g_1 h + g_0] &= h \\
\sigma(g_2) - g_2 &= 0
\end{align*}
\]

\[
\begin{align*}
[c(h + \frac{1}{k+1})^2 + \sigma(g_1 h + g_0)] - [c h^2 + g_1 h + g_0] &= h \\
g_2 &= c \in \mathbb{Q}
\end{align*}
\]
**ANSATZ**

\[ g = g_2 h^2 + g_1 h + g_0 \in \mathbb{Q}(k)[h] \]

\[
\left[ \sigma(g_2) \left( h + \frac{1}{k+1} \right)^2 + \sigma(g_1 h + g_0) \right] - [g_2 h^2 + g_1 h + g_0] = h \\
\]

\[
\Rightarrow \sigma(g_2) - g_2 = 0 \\
\]

\[
\sigma(g_1 h + g_0) - (g_1 h + g_0) = h - c \left[ \frac{2h(k+1)+1}{(k+1)^2} \right] \\
\]

\[ g_2 = c \in \mathbb{Q} \]
**ANSATZ** $g = g_2 h^2 + g_1 h + g_0 \in \mathbb{Q}(k)[h]$ 

\[
\begin{align*}
\sigma(g_2)(h + \frac{1}{k+1})^2 + \sigma(g_1 h + g_0) & - [g_2 h^2 + g_1 h + g_0] = h \\
\sigma(g_1 h + g_0) - (g_1 h + g_0) & = h - c \left[ \frac{2h(k+1)+1}{(k+1)^2} \right]
\end{align*}
\]

\[\sigma(g_2) - g_2 = 0\]

\[g_2 = c \in \mathbb{Q}\]

\[\sigma(g_1) - g_1 = 1 - c \frac{2}{k+1}\]
ANSATZ \( g = g_2 h^2 + g_1 h + g_0 \in \mathbb{Q}(k)[h] \)

\[
\left[ \sigma(g_2) \left( h + \frac{1}{k+1} \right)^2 + \sigma(g_1 h + g_0) \right] - \left[ g_2 h^2 + g_1 h + g_0 \right] = h
\]

\[
\sigma(g_2) - g_2 = 0
\]

\[
g_2 = c \in \mathbb{Q}
\]

\[
\sigma(g_1) - g_1 = 1 - c \frac{2}{k+1}
\]

\[
c = 0, \quad g_1 = k + d, \quad d \in \mathbb{Q}
\]
ANSATZ \( g = g_2 h^2 + g_1 h + g_0 \in \mathbb{Q}(k)[h] \)

\[
\left[ \sigma(g_2) \left( h + \frac{1}{k+1} \right)^2 + \sigma(g_1 h + g_0) \right] - \left[ g_2 h^2 + g_1 h + g_0 \right] = h
\]

\[
\sigma(g_1 h + g_0) - (g_1 h + g_0) = h - c \left[ \frac{2h(k+1)+1}{(k+1)^2} \right]
\]

\[
\sigma(g_0) - g_0 = -1 - d \frac{1}{k+1}
\]

\[
\sigma(g_2) - g_2 = 0
\]

\[
g_2 = c \in \mathbb{Q}
\]

\[
\sigma(g_1) - g_1 = 1 - c \frac{2}{k+1}
\]

\[
c = 0, \quad g_1 = k + d
\]

\[
d \in \mathbb{Q}
\]
Toolbox 1: Indefinite summation

**ANSATZ**

\[ g = g_2 h^2 + g_1 h + g_0 \in \mathbb{Q}(k)[h] \]

\[
\left[ \sigma(g_2) \left( h + \frac{1}{k+1} \right)^2 + \sigma(g_1 h + g_0) \right] - [g_2 h^2 + g_1 h + g_0] = h
\]

\[
\sigma(g_1 h + g_0) - (g_1 h + g_0) = h - c \left[ \frac{2h(k+1)+1}{(k+1)^2} \right]
\]

\[
g_0 = -k, \quad d = 0 \quad \leftarrow \quad \sigma(g_0) - g_0 = -1 - d \frac{1}{k+1}
\]

\[
\sigma(g_2) - g_2 = 0
\]

\[
g_2 = c \in \mathbb{Q}
\]

\[
\sigma(g_1) - g_1 = 1 - c \frac{2}{k+1}
\]

\[
c = 0, \quad g_1 = k + d, \quad d \in \mathbb{Q}
\]
Toolbox 1: Improved indefinite summation
– symbolic simplification

For algorithmic details see:


For special cases see:

A difference field approach (M. Karr, 1981)

GIVEN a \( \Pi \Sigma \)-field \((\mathbb{F}, \sigma)\) with \( f \in \mathbb{F} \).

FIND \( g \in \mathbb{F} \):

\[ \sigma(g) - g = f. \]
A symbolic summation approach

1. FIND an appropriate $\Pi\Sigma$-field $(\mathbb{F}, \sigma)$ with $f \in \mathbb{F}$.

2. FIND $g \in \mathbb{F}$:

\[ \sigma(g) - g = f. \]
A symbolic summation approach

1. FIND an appropriate $\Pi\Sigma$-field $(F, \sigma)$ with $f \in F$.

2. FIND an appropriate extension $E > F$ with $g \in E$:

$$\sigma(g) - g = f.$$
A symbolic summation approach

1. FIND an appropriate $\Pi\Sigma$-field $(F, \sigma)$ with $f \in F$.

2. FIND an appropriate extension $E > F$ with $g \in E$:

$$\sigma(g) - g = f.$$ 

appropriate = degrees in denominators minimal

Example:

$$\sum_{k=1}^{a} \left( \frac{-2 + k}{10(1 + k^2)} + \frac{(1 - 4k - 2k^2)H_k}{10(1 + k^2)(2 + 2k + k^2)} + \frac{(1 - 4k - 2k^2)H_k^{(3)}}{5(1 + k^2)(2 + 2k + k^2)} \right) = ?$$
A symbolic summation approach

1. FIND an appropriate $\Pi\Sigma$-field $(F, \sigma)$ with $f \in F$.

2. FIND an appropriate extension $E > F$ with $g \in E$:

$$\sigma(g) - g = f.$$ 

appropriate $=$ degrees in denominators minimal

Example:

$$\sum_{k=1}^{a} \left( \frac{-2 + k}{10(1 + k^2)} + \frac{(1 - 4k - 2k^2)H_k}{10(1 + k^2)(2 + 2k + k^2)} + \frac{(1 - 4k - 2k^2)H_k^{(3)}}{5(1 + k^2)(2 + 2k + k^2)} \right)$$

$$= \frac{a^2 + 4a + 5}{10(a^2 + 2a + 2)}H_a - \frac{(a - 1)(a + 1)}{5(a^2 + 2a + 2)}H_a^{(3)} - \frac{2}{5} \sum_{k=1}^{a} \frac{1}{k^2}$$
A symbolic summation approach

1. FIND an appropriate \Pi\Sigma-field \((\mathbb{F}, \sigma)\) with \(f \in \mathbb{F}\).

2. FIND an appropriate extension \(\mathbb{E} > \mathbb{F}\) with \(g \in \mathbb{E}\):

\[
\sigma(g) - g = f.
\]

appropriate = sum representations with optimal nesting depth

Example:

\[
\sum_{k=1}^{n} \sum_{j=1}^{k} \frac{1}{i} \sum_{i=1}^{j} \frac{1}{j} \frac{1}{k} = ?
\]
A symbolic summation approach

1. FIND an appropriate $\Pi\Sigma$-field $(\mathbb{F}, \sigma)$ with $f \in \mathbb{F}$.

2. FIND an appropriate extension $\mathbb{E} > \mathbb{F}$ with $g \in \mathbb{E}$:

$$\sigma(g) - g = f.$$ 

appropriate = sum representations with optimal nesting depth

Example:

$$\sum_{k=1}^{n} \sum_{i=1}^{j} \frac{1}{k} = \frac{1}{6} \left( \sum_{i=1}^{n} \frac{1}{i} \right)^3 + \frac{1}{2} \left( \sum_{i=1}^{n} \frac{1}{i^2} \right) \left( \sum_{i=1}^{n} \frac{1}{i} \right) + \frac{1}{3} \sum_{i=1}^{n} \frac{1}{i^3}$$

depth 3 depth 1
A symbolic summation approach

1. FIND an appropriate $\Pi\Sigma$-field $(F, \sigma)$ with $f \in F$.

2. FIND an appropriate extension $E > F$ with $g \in E$:

$$\sigma(g) - g = f.$$  

appropriate = sum representations with minimal number of objects

Example:

$$\sum_{k=0}^{a} (-1)^k H_k^2 \binom{n}{k} = ?$$
A symbolic summation approach

1. FIND an appropriate $\Pi\Sigma$-field $(\mathbb{F}, \sigma)$ with $f \in \mathbb{F}$.

2. FIND an appropriate extension $\mathbb{E} > \mathbb{F}$ with $g \in \mathbb{E}$:

$$\sigma(g) - g = f.$$ 

appropriate = sum representations with minimal number of objects

Example:

$$\sum_{k=0}^{a} (-1)^k H_k^2 \binom{n}{k} = - \frac{1}{n} \sum_{i_1=1}^{a} (-1)^{i_1} \binom{n}{i_1}$$

$$- (a - n) \left( n^2 H_a^2 + 2n H_a + 2 \right) \frac{(-1)^a \binom{n}{a}}{n^3} - \frac{2}{n^2}$$
Simplification of nested product-sum expressions

\[ A(k) : \text{nested product-sum expression (sums/products not in the denominator)} \] \[ \downarrow \quad \text{SigmaReduce}[A,k] \] \[ B(k) : \text{nested product-sum expression (sums/products not in the denominator)} \]

such that

\[ A(k) = B(k) \]
Simplification of nested product-sum expressions

\[ A(k) : \text{nested product-sum expression (sums/products not in the denominator)} \]
\[ \downarrow \quad \text{SigmaReduce}[A,k] \]
\[ B(k) : \text{nested product-sum expression (sums/products not in the denominator)} \]

- such that
  \[ A(k) = B(k) \]
  - such that all the sums in \( B(k) \) are simplified as above
Simplification of nested product-sum expressions

\[ A(k) : \text{nested product-sum expression (sums/products not in the denominator)} \]

\[ \downarrow \Sigma \text{Reduce}[A,k] \]

\[ B(k) : \text{nested product-sum expression (sums/products not in the denominator)} \]

- such that

\[ A(k) = B(k) \]

- such that all the sums in \( B(k) \) are simplified as above

- and such that the arising sums in \( B(k) \) are algebraically independent (i.e., they do not satisfy any polynomial relation)
Toolbox 2: Definite summation
Summation principles (in difference field/ring setting)

\[ n^2 A(n) - (n + 1)(2n + 1)A(n + 1) + (n + 1)(n + 2)A(n + 2) = \frac{1}{n + 1} \]

Summation package Sigma
(based on difference field algorithms/theory
see, e.g., Karr 1981, Bronstein 2000, Schneider 2001 –)

\[ A(n) = \sum_{k=1}^{\infty} \frac{H_k}{k(k + n)} \]

where

\[ H_n^{(2)} = \sum_{i=1}^{n} \frac{1}{i^2} \]
\[ \zeta_z = \sum_{i=1}^{\infty} \frac{1}{i^z} (= \zeta(z)) \]
1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger’s algorithm (1991))

**GIVEN** a **definite sum**

\[ A(n) = \sum_{k=1}^{n} f(n, k); \quad f(n, k): \text{ indefinite nested product-sum in } k; \]

\[ n: \text{ extra parameter} \]

**FIND** a **recurrence** for \( A(n) \)
Back to creative telescoping

Given

\[ f(n, k) = \frac{H_k}{k(k + n)}; \]

Find \( g(n, k) \) and \( c_0(n), c_1(n), c_2(n) \):

\[ g(n, k + 1) - g(n, k) = c_0(n) f(n, k) + c_1(n) f(n + 1, k) + c_2(n) f(n + 2, k) \]
Back to creative telescoping

Given

\[ f(n, k) = \frac{H_k}{k(k + n)}; \]

Find \( g(n, k) \) and \( c_0(n), c_1(n), c_2(n) \):

\[
g(n, k + 1) - g(n, k) = c_0(n) \frac{H_k}{k(k + n)} + c_1(n) f(n + 1, k) + c_2(n) f(n + 2, k)
\]
Back to creative telescoping

Given

\[ f(n, k) = \frac{H_k}{k(k+n)}; \]

Find \( g(n, k) \) and \( c_0(n), c_1(n), c_2(n) \):

\[
g(n, k + 1) - g(n, k) = c_0(n) \frac{H_k}{k(k+n)} + c_1(n) \frac{H_k}{k(k+n+1)} + c_2(n) f(n + 2, k)
\]
Back to creative telescoping

Given

\[ f(n, k) = \frac{H_k}{k(k+n)}; \]

Find \( g(n, k) \) and \( c_0(n), c_1(n), c_2(n) \):

\[
g(n, k + 1) - g(n, k) = c_0(n) \frac{H_k}{k(k+n)} + c_1(n) \frac{H_k}{k(k+n+1)} + c_2(n) \frac{H_k}{k(k+n+2)}
\]
Back to creative telescoping

Given

\[ f(n, k) = \frac{H_k}{k(k + n)}; \]

Find \( g(n, k) \) and \( c_0(n), c_1(n), c_2(n) \):

\[
g(n, k + 1) - g(n, k) = c_0(n) \frac{H_k}{k(k + n)} + c_1(n) \frac{H_k}{k(k + n + 1)} + c_2(n) \frac{H_k}{k(k + n + 2)}
\]

A difference field for the summand:

Construct a rational function field

\[
F
\]

and a field automorphism \( \sigma : F \rightarrow F \) defined by
Back to creative telescoping

Given

\[ f(n, k) = \frac{H_k}{k(k+n)}; \]

Find \( g(n, k) \) and \( c_0(n), c_1(n), c_2(n) \):

\[
g(n, k + 1) - g(n, k) = c_0(n) \frac{H_k}{k(k+n)} + c_1(n) \frac{H_k}{k(k+n+1)} + c_2(n) \frac{H_k}{k(k+n+2)}
\]

A difference field for the **summand**:

Construct a rational function field

\[
\mathbb{F} := \mathbb{Q}(n)
\]

and a field automorphism \( \sigma : \mathbb{F} \rightarrow \mathbb{F} \) defined by

\[
\sigma(c) = c \quad \forall c \in \mathbb{Q}(n),
\]
Back to creative telescoping

Given

\[ f(n, k) = \frac{H_k}{k(k + n)}; \]

Find \( g(n, k) \) and \( c_0(n), c_1(n), c_2(n) \):

\[
g(n, k + 1) - g(n, k) = c_0(n) \frac{H_k}{k(k + n)} + c_1(n) \frac{H_k}{k(k + n + 1)} + c_2(n) \frac{H_k}{k(k + n + 2)}
\]

A difference field for the **summand**:

Construct a rational function field

\[
F := \mathbb{Q}(n)(k)
\]

and a field automorphism \( \sigma : F \rightarrow F \) defined by

\[
\sigma(c) = c \quad \forall c \in \mathbb{Q}(n), \\
\sigma(k) = k + 1, \quad S k = k + 1,
\]
Back to creative telescoping

Given

\[ f(n, k) = \frac{H_k}{k(k + n)} \]

Find \( g(n, k) \) and \( c_0(n), c_1(n), c_2(n) \):

\[
g(n, k + 1) - g(n, k) = c_0(n) \frac{H_k}{k(k + n)} + c_1(n) \frac{H_k}{k(k + n + 1)} + c_2(n) \frac{H_k}{k(k + n + 2)}
\]

A difference field for the summand:

Construct a rational function field

\[ \mathbb{F} := \mathbb{Q}(n)(k)(h) \]

Karr 1981

and a field automorphism \( \sigma : \mathbb{F} \to \mathbb{F} \) defined by

\[
\sigma(c) = c \quad \forall c \in \mathbb{Q}(n),
\]

\[
\sigma(k) = k + 1,
S_k = k + 1,
\]

\[
\sigma(h) = h + \frac{1}{k + 1},
S H_k = H_k + \frac{1}{k + 1},
\]
Back to creative telescoping

Given
\[ f(n, k) = \frac{H_k}{k(k + n)}; \]

Find \( g(n, k) \) and \( c_0(n), c_1(n), c_2(n) \):

\[
g(n, k + 1) - g(n, k) = c_0(n) \frac{H_k}{k(k+n)} + c_1(n) \frac{H_k}{k(k+n+1)} + c_2(n) \frac{H_k}{k(k+n+2)}
\]

FIND \( g \in F \) and \( c_0, c_1, c_2 \in \mathbb{Q}(n) \):

\[
\sigma(g) - g = c_0 \frac{h}{k(k+n)} + c_1 \frac{h}{k(k+n+1)} + c_2 \frac{h}{k(k+n+2)}
\]
Back to creative telescoping

Given
\[ f(n, k) = \frac{H_k}{k(k + n)}; \]

Find \( g(n, k) \) and \( c_0(n), c_1(n), c_2(n) \):

\[ g(n, k + 1) - g(n, k) = c_0(n) \frac{H_k}{k(k+n)} + c_1(n) \frac{H_k}{k(k+n+1)} + c_2(n) \frac{H_k}{k(k+n+2)} \]

FIND \( g \in \mathbb{F} \) and \( c_0, c_1, c_2 \in \mathbb{Q}(n) \):

\[ \sigma(g) - g = c_0 \frac{h}{k(k+n)} + c_1 \frac{h}{k(k+n+1)} + c_2 \frac{h}{k(k+n+2)} \]

\[ \downarrow \]

\[ c_0 = n^2, \ c_1 = -(n + 1)(2n + 1), \ c_2 = (n + 1)(n + 2) \]

\[ g = -\frac{kh + n + k}{(n+k)(n+k+1)} \]
Zeilberger’s creative telescoping paradigm

GIVEN

\[ A'(n) := \sum_{k=1}^{a} \frac{H_k}{k(k+n)}. \]

\[ =: f(n,k) \]

FIND \( g(n,k) \) and \( c_0(n), c_1(n), c_2(n) \):

\[ g(n,k+1) - g(n,k) = c_0(n)f(n,k) + c_1(n)f(n+1,k) + c_2(n)f(n+2,k) \]

for all \( n, k \geq 1 \).

Sigma computes:

\[ c_0(n) = n^2, \quad c_1(n) = -(n+1)(2n+1), \quad c_2(n) = (n+1)(n+2) \]

and

\[ g(n,k) := -\frac{kH_k + n + k}{(n+k)(n+k+1)}, \]

\[ g(n,k+1) := -\frac{(1+n)H_k + n + k + 2}{(n+k+1)(n+k+2)}. \]
Zeilberger’s creative telescoping paradigm

GIVEN

\[ A'(n) := \sum_{k=1}^{a} \frac{H_k}{k(k + n)} \cdot \]

\[ =: f(n, k) \]

FIND \( g(n, k) \) and \( c_0(n), c_1(n), c_2(n) \):

\[
g(n, k + 1) - g(n, k) = c_0(n)f(n, k) + c_1(n)f(n + 1, k) + c_2(n)f(n + 2, k)
\]

for all \( n, k \geq 1 \).

Summing this equation over \( k \) from 1 to \( a \) gives:

\[
g(n, a + 1) - g(n, 1) = \sum_{k=1}^{a} \left[ c_0(n)f(n, k) + c_1(n)f(n + 1, k) + c_2(n)f(n + 2, k) \right]
\]
Zeilberger’s creative telescoping paradigm

**GIVEN**

\[ A'(n) := \sum_{k=1}^{a} \frac{H_k}{k(k + n)}. \]

\[ =: f(n, k) \]

**FIND** \( g(n, k) \) and \( c_0(n), c_1(n), c_2(n) \):

\[
\begin{align*}
g(n, k + 1) - g(n, k) &= c_0(n)f(n, k) + c_1(n)f(n + 1, k) + c_2(n)f(n + 2, k) \\
\end{align*}
\]

for all \( n, k \geq 1 \).

Summing this equation over \( k \) from 1 to \( a \) gives:

\[
\begin{align*}
g(n, a+1) - g(n, 1) &= \sum_{k=1}^{a} c_0(n)f(n, k) + \sum_{k=1}^{a} c_1(n)f(n + 1, k) + \sum_{k=1}^{a} c_2(n)f(n + 2, k) \\
\end{align*}
\]
Zeilberger’s creative telescoping paradigm

**GIVEN**

\[ A'(n) := \sum_{k=1}^{a} \frac{H_k}{k(k+n)}. \]

\[ =: f(n,k) \]

**FIND** \( g(n,k) \) and \( c_0(n), c_1(n), c_2(n) \):

\[
\begin{align*}
  g(n,k + 1) - g(n,k) &= c_0(n) f(n,k) + c_1(n) f(n+1,k) + c_2(n) f(n+2,k)
\end{align*}
\]

for all \( n, k \geq 1 \).

Summing this equation over \( k \) from 1 to \( a \) gives:

\[
\begin{align*}
  g(n,a+1) - g(n,1) &= c_0(n) \sum_{k=1}^{a} f(n,k) + c_1(n) \sum_{k=1}^{a} f(n+1,k) + c_2(n) \sum_{k=1}^{a} f(n+2,k)
\end{align*}
\]
Zeilberger’s creative telescoping paradigm

GIVEN

\[ A'(n) := \sum_{k=1}^{a} \frac{H_k}{k(k + n)}. \]

\[ =: f(n, k) \]

FIND \( g(n, k) \) and \( c_0(n), c_1(n), c_2(n) \):

\[ g(n, k + 1) - g(n, k) = c_0(n) f(n, k) + c_1(n) f(n + 1, k) + c_2(n) f(n + 2, k) \]

for all \( n, k \geq 1 \).

Summing this equation over \( k \) from 1 to \( a \) gives:

\[ g(n, a+1) - g(n, 1) = c_0(n) A'(n) + c_1(n) A'(n + 1) + c_2(n) A'(n + 2) \]
Zeilberger's creative telescoping paradigm

**GIVEN**

\[ A'(n) := \sum_{k=1}^{a} \frac{H_k}{k(k+n)}. \]

\[ =: f(n, k) \]

**FIND** \( g(n, k) \) and \( c_0(n), c_1(n), c_2(n) \):

\[
\begin{align*}
g(n, k + 1) - g(n, k) &= c_0(n) f(n, k) + c_1(n) f(n + 1, k) + c_2(n) f(n + 2, k)
\end{align*}
\]

for all \( n, k \geq 1 \).

Summing this equation over \( k \) from 1 to \( a \) gives:

\[
\begin{align*}
g(n, a + 1) - g(n, 1) &= c_0(n) A'(n) + c_1(n) A'(n + 1) + c_2(n) A'(n + 2) \\
\left. \left| \begin{array}{c} a \\ (n+1)(a+n+1) \\
\end{array} \right| n^2 A'(n) - (n+1)(2n+1) A'(n + 1) + (n+1)(n+2) A'(n + 2) \\
\right. \\
- \left. \left| \begin{array}{c} (a+1)H_a \\ (a+n+1)(a+n+2) \\
\end{array} \right| \right.
\end{align*}
\]
1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

**GIVEN** a **definite sum**

\[ A(n) = \sum_{k=1}^{n} f(n, k); \quad f(n, k): \text{indefinite nested product-sum in } k; \]
\[ n: \text{extra parameter} \]

**FIND** a **recurrence** for \( A(n) \)
1. Creative telescoping

(for the special case of hypergeometric terms see Zeilberger’s algorithm (1991))

GIVEN a definite sum

\[ A(n) = \sum_{k=1}^{n} f(n, k); \]

\( f(n, k) \): indefinite nested product-sum in \( k \);
\( n \): extra parameter

FIND a recurrence for \( A(n) \)

2. Recurrence solving

GIVEN a recurrence

\[ a_0(n), \ldots, a_d(n), h(n): \]

indefinite nested product-sum expressions in \( n \).

\[ a_0(n)A(n) + \cdots + a_d(n)A(n + d) = h(n); \]

FIND all solutions expressible by indefinite nested products/sums in \( n \).

(d’Alembertian solutions)

(Abramov/Bronstein/Petkovšek/CS, in preparation)
1. **Creative telescoping** (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

**Given** a definite sum

\[ A(n) = \sum_{k=1}^{n} f(n, k); \]

\( f(n, k) \): indefinite nested product-sum in \( k \);
\( n \): extra parameter

**Find** a recurrence for \( A(n) \)

2. **Recurrence solving**

**Given** a recurrence

\[ a_0(n), \ldots, a_d(n), h(n): \]

indefinite nested product-sum expressions in \( n \).

\[ a_0(n)A(n) + \cdots + a_d(n)A(n + d) = h(n); \]

**Find** all solutions expressible by indefinite nested products/sums in \( n \).
(d’Alembertian solutions)

(Abramov/Bronstein/Petkovšek/CS, in preparation)

**Note:** the sum solutions are highly nested
(possibly with denominators of high degrees)
1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger’s algorithm (1991))

GIVEN a definite sum

\[ A(n) = \sum_{k=1}^{n} f(n, k); \]
\[ f(n, k): \text{indefinite nested product-sum in } k; \]
\[ n: \text{extra parameter} \]

FIND a recurrence for \( A(n) \)

2. Recurrence solving

GIVEN a recurrence \( a_0(n), \ldots, a_d(n), h(n): \)
\[ \text{indefinite nested product-sum expressions in } n. \]

\[ a_0(n)A(n) + \cdots + a_d(n)A(n + d) = h(n); \]

FIND all solutions expressible by indefinite nested products/sums in \( n \).
(d’Alembertian solutions)
(Abramov/Bronstein/Petkovšek/CS, in preparation)

3. Simplify the solutions (using difference field theory) s.t.
   - the sums are algebraically independent;
   - the sums are flattened;
   - the sums can be given in terms of special functions.
1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger’s algorithm (1991))

GIVEN a definite sum

\[ A(n) = \sum_{k=1}^{n} f(n, k); \]

\( f(n, k) \): indefinite nested product-sum in \( k \);
\( n \): extra parameter

FIND a recurrence for \( A(n) \)

2. Recurrence solving

GIVEN a recurrence

\[ a_0(n), \ldots, a_d(n), h(n): \]

indefinite nested product-sum expressions in \( n \).

\[ a_0(n)A(n) + \cdots + a_d(n)A(n + d) = h(n); \]

FIND all solutions expressible by indefinite nested products/sums in \( n \).
(d’Alembertian solutions)
(Abramov/Bronstein/Petkovšek/CS, in preparation)

4. Find a “closed form”

\[ A(n) = \text{combined solutions in terms of indefinite nested sums in } n. \]
In[1]:= << Sigma.m

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= mySum = \[\sum_{k=1}^{a} \frac{H_k}{k(k + n)}\]
In[1]:= \texttt{<< Sigma.m}

\textbf{Sigma} - A summation package by Carsten Schneider © RISC-Linz

In[2]:= mySum = \sum_{k=1}^{a} \frac{H_k}{k(k+n)}

In[3]:= rec = \texttt{GenerateRecurrence[mySum, n][[1]]}

Out[3]= n^2 \text{SUM}[n] - (n+1)(2n+1)\text{SUM}[n+1] + (n+1)(n+2)\text{SUM}[n+2] ==
\frac{(-a-1)H_a}{(a+n+1)(a+n+2)} + \frac{a}{(n+1)(a+n+1)}

In[4]:= rec = \texttt{LimitRec[rec, SUM[n], \{n\}, a]}

Out[4]= n^2 \text{SUM}[n] - (n+1)(2n+1)\text{SUM}[n+1] + (n+1)(n+2)\text{SUM}[n+2] = \frac{1}{n+1}
In[1]:= << Sigma.m

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= mySum = \[\sum\limits_{k=1}^{a} \frac{H_k}{k(k + n)}\]

In[3]:= rec = GenerateRecurrence[mySum, n][[1]]

Out[3]= n^2 \text{SUM}[n] - (n + 1)(2n + 1)\text{SUM}[n + 1] + (n + 1)(n + 2)\text{SUM}[n + 2] ==
\(-a - 1)H_a\]
\((a + n + 1)(a + n + 2)\) + \(a\)
\(n + 1)(a + n + 1)\)

In[4]:= rec = LimitRec[rec, \text{SUM}[n], \{n\}, a]

Out[4]= n^2 \text{SUM}[n] - (n + 1)(2n + 1)\text{SUM}[n + 1] + (n + 1)(n + 2)\text{SUM}[n + 2] = \frac{1}{n + 1}

In[5]:= recSol = SolveRecurrence[rec, \text{SUM}[n], \text{IndefiniteSummation} \to \text{False}]

RISC, J. Kepler University Linz
In[1]:= \texttt{\textlangle\textrangle Sigma.m}

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= mySum = \sum_{k=1}^{a} \frac{H_k}{k(k+n)}

In[3]:= rec = \texttt{GenerateRecurrence}[mySum, n][[1]]

Out[3]= \begin{align*}
&n^2\text{SUM}[n] - (n + 1)(2n + 1)\text{SUM}[n + 1] + (n + 1)(n + 2)\text{SUM}[n + 2] = \frac{(-a - 1)H_a}{(a + n + 1)(a + n + 2)} + \frac{a}{(n + 1)(a + n + 1)}
\end{align*}

In[4]:= rec = \texttt{LimitRec}[rec, \text{SUM}[n], \{n\}, a]

Out[4]= \begin{align*}
&n^2\text{SUM}[n] - (n + 1)(2n + 1)\text{SUM}[n + 1] + (n + 1)(n + 2)\text{SUM}[n + 2] = \frac{1}{n + 1}
\end{align*}

In[5]:= recSol = \texttt{SolveRecurrence}[rec, \text{SUM}[n], \text{IndefiniteSummation} \rightarrow \text{False}]

Out[5]= \begin{align*}
&\{\{0, \frac{1}{n}\}, \{0, -\frac{1}{n^2} \text{\sum}_{i=1}^{n} \frac{1}{i}\} \} \}, \{1, -\frac{1}{n^2} \text{\sum}_{i=1}^{n} \frac{1}{i} \} \}, \{\sum_{k=1}^{n} \frac{1}{k}\} \}
\end{align*}
In[1]:= << Sigma.m

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= mySum = \sum_{k=1}^{a} \frac{H_k}{k(k+n)}

In[3]:= rec = GenerateRecurrence[mySum, n][[1]]

Out[3]= n^2\text{SUM}[n] - (n + 1)(2n + 1)\text{SUM}[n + 1] + (n + 1)(n + 2)\text{SUM}[n + 2] == 
\frac{(-a - 1)H_a}{(a + n + 1)(a + n + 2)} + \frac{a}{(n + 1)(a + n + 1)}

In[4]:= rec = LimitRec[rec, \text{SUM}[n], \{n\}, a]

Out[4]= n^2\text{SUM}[n] - (n + 1)(2n + 1)\text{SUM}[n + 1] + (n + 1)(n + 2)\text{SUM}[n + 2] = \frac{1}{n + 1}

In[5]:= recSol = SolveRecurrence[rec, \text{SUM}[n], \text{IndefiniteSummation} \rightarrow \text{True}]

Out[5]= \{\{0, \frac{1}{n}\}, \{0, \frac{1}{n^2} - \frac{1}{n}\}, \{1, \frac{(\sum_{i=1}^{n} \frac{1}{i})^2}{2n} - \sum_{i=1}^{n} \frac{1}{i^2} + \sum_{i=1}^{n} \frac{1}{2i}\}\}
In[1]:= << Sigma.m

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= mySum = \[\sum_{k=1}^{a} \frac{H_k}{k(k + n)}\]

In[3]:= rec = GenerateRecurrence[mySum, n][[1]]

Out[3]= n^2 \text{SUM}[n] - (n + 1)(2n + 1)\text{SUM}[n + 1] + (n + 1)(n + 2)\text{SUM}[n + 2] ==
\frac{(-a - 1)H_a}{(a + n + 1)(a + n + 2)} + \frac{a}{(n + 1)(a + n + 1)}

In[4]:= rec = LimitRec[rec, \text{SUM}[n], \{n\}, a]

Out[4]= n^2 \text{SUM}[n] - (n + 1)(2n + 1)\text{SUM}[n + 1] + (n + 1)(n + 2)\text{SUM}[n + 2] = \frac{1}{n + 1}

In[5]:= recSol = SolveRecurrence[rec, \text{SUM}[n], \text{IndefiniteSummation} \rightarrow \text{True}]

Out[5]= \{\{0, \frac{1}{n}\}, \{0, \frac{\sum_{i=1}^{n} \frac{1}{i}}{n} - \frac{1}{n^2}\}, \{1, \frac{\left(\sum_{i=1}^{n} \frac{1}{i}\right)^2}{2n} - \frac{\sum_{i=1}^{n} \frac{1}{i}}{n^2} + \frac{\sum_{i=1}^{n} \frac{1}{i^2}}{2n}\}\}

In[6]:= FindLinearCombination[recSol, \{1, \{\zeta_2, 1/2 + \zeta_2/2\}\}, n, 2]

Out[6]= -\frac{\sum_{i=1}^{n} \frac{1}{i}}{n^2} + \frac{\left(\sum_{i=1}^{n} \frac{1}{i}\right)^2}{2n} + \frac{\sum_{i=1}^{n} \frac{1}{i^2}}{2n} + \frac{\zeta_2}{n}
Sigma’s summation spiral

1. Combination of solutions
2. Simplified solutions
3. Indefinite summation
4. d’Alembertian solutions
5. Creative telescoping
6. Recurrence
7. Solving
8. Definite sum

Flow from combination of solutions to indefinite summation, then to d’Alembertian solutions, to creative telescoping, to recurrence, to solving, to definite sum, back to combination of solutions.
Toolbox 3: Special function algorithms
Computer algebra and special functions:

**Harmonic sums** (Borwein, Hoffman, Broadhurst, Vermaseren, Remmiddi, Blümlein, ...)
Harmonic sums (Borwein, Hoffman, Broadhurst, Vermaseren, Remmiddi, Blümlein,…) 

\[
\sum_{i=1}^{n} \frac{1}{i^2} \sum_{j=1}^{i} \frac{1}{j}
\]

Integral representation:

\[
\int_{0}^{1} \frac{x^n - 1}{1 - x} \left( \int_{0}^{x} \frac{1}{1-z} \frac{1}{y} dy - \zeta_2 \right) dx, \quad \zeta_z := \sum_{i=1}^{\infty} \frac{1}{i^z}
\]
Computer algebra and special functions:

**Harmonic sums** (Borwein, Hoffman, Broadhurst, Vermaseren, Remiddi, Blümlein, . . .)

\[
\sum_{i=1}^{n} \frac{1}{i^2} \sum_{j=1}^{i} \frac{1}{j}
\]

**Integral representation:**

\[
= \int_{0}^{1} \frac{x^n - 1}{1 - x} \left( \int_{0}^{x} \int_{0}^{y} \frac{1}{1 - z} \frac{dz}{y} - \zeta_2 \right) dx,
\]

\[
\zeta_z := \sum_{i=1}^{\infty} \frac{1}{i^z}
\]

**Asymptotic expansion:**

\[
= \left( \frac{1}{30n^5} - \frac{1}{6n^3} + \frac{1}{2n^2} - \frac{1}{n} \right) \ln(n)
\]

\[
- \frac{1}{100n^5} - \frac{1}{6n^4} + \frac{13}{36n^3} - \frac{1}{4n^2} - \frac{1}{n} + 2\zeta_3 + O\left( \frac{\ln(n)}{n^6} \right).
\]

**Limit computations**

**Numerical evaluation**
Generalized algorithms for generalized harmonic sums

\[ \sum_{k=1}^{N} 2^{k} \sum_{i=1}^{k} \frac{2^{-i} \sum_{j=1}^{i} H_{j}}{i} \]

\[ = -\frac{21 \zeta_{2}^{2}}{20} \frac{1}{N} + \frac{1}{8N^{2}} + \frac{295}{216N^{3}} - \frac{1115}{96N^{4}} + O(N^{-5}) \]

\[ + \left( \frac{1}{2N} - \frac{3}{4N^{2}} + \frac{19}{12N^{3}} - \frac{5}{N^{4}} + O(N^{-5}) \right) \zeta_{2} \]

\[ + 2^{N} \left( \frac{3}{2N} + \frac{3}{2N^{2}} + \frac{9}{2N^{3}} + \frac{39}{2N^{4}} + O(N^{-5}) \right) \zeta_{3} \]

\[ + \left( \frac{1}{N} + \frac{3}{4N^{2}} - \frac{157}{36N^{3}} + \frac{19}{N^{4}} + O(N^{-5}) \right) (\log(N) + \gamma) \]

\[ + \left( \frac{1}{2N} - \frac{3}{4N^{2}} + \frac{19}{12N^{3}} - \frac{5}{N^{4}} + O(N^{-5}) \right) (\log(N) + \gamma)^{2} \]

Toolbox 3: Special function algorithms

▶ Generalized algorithms for cyclotomic harmonic sums

\[
\sum_{k=1}^{N} \frac{\sum_{j=1}^{j} \frac{1}{1+2i}}{(1+2k)^2} = \left( -3 + \frac{35\zeta_3}{16} \right) \zeta_2 - \frac{31\zeta_5}{8} + O(N^{-5})
\]

\[
+ \frac{1}{N} - \frac{33}{32N^2} + \frac{17}{16N^3} - \frac{4795}{4608N^4} + O(N^{-5})
\]

\[
+ \log(2)\left(6\zeta_2 - \frac{1}{N} + \frac{9}{8N^2} - \frac{7}{6N^3} + \frac{209}{192N^4} + O(N^{-5})\right)
\]

\[
+ ( -\frac{7}{4} - \frac{7}{16N} + \frac{7}{16N^2} - \frac{77}{192N^3} + \frac{21}{64N^4} + O(N^{-5})) \zeta_3
\]

\[
+ \left( \frac{1}{16N^2} - \frac{1}{8N^3} + \frac{65}{384N^4} + O(N^{-5}) \right) (\log(N) + \gamma)
\]

Generalized algorithms for nested binomial sums

\[
\sum_{j=1}^{N} \frac{4^j H_{j-1}}{\binom{2j}{j} j^2} = 7\zeta_3 + \sqrt{\pi} \sqrt{N} \left\{ -\frac{2}{N} + \frac{5}{12N^2} - \frac{21}{320N^3} - \frac{223}{10752N^4} + \frac{671}{49152N^5} \\
+ \frac{11635}{1441792N^6} - \frac{1196757}{136314880N^7} - \frac{376193}{50331648N^8} + \frac{201980317}{18253611008N^9} \\
+ O(N^{-10}) \right\} \ln(N) - \frac{4}{N} + \frac{5}{18N^2} - \frac{263}{2400N^3} + \frac{579}{12544N^4} + \frac{10123}{1105920N^5} \\
- \frac{1705445}{71368704N^6} - \frac{27135463}{11164188672N^7} + \frac{197432563}{7927234560N^8} + \frac{405757489}{775778467840N^9} \\
+ O(N^{-10}) \right\}
\]

Discovery of algebraic relations

multiple Zeta-values

\[ \sum_{i=1}^{\infty} \frac{1}{i^3} \sum_{j=1}^{i} \frac{(-1)^j}{j^2} \sum_{k=1}^{j} \frac{1}{k} \]


combining known relations of the sum and integral representations
Discovery of algebraic relations (J. Ablinger, J. Blümlein, CS)

\[
\sum_{i=1}^{\infty} \frac{1}{i^3} \sum_{j=1}^{i} \frac{(-1)^j}{j^2} \sum_{k=1}^{j} \frac{1}{k}
\]


combining known relations of the sum and integral representations

cyclotomic Zeta-values

\[
\sum_{i=1}^{\infty} \frac{1}{i^3} \sum_{j=1}^{i} \frac{(-1)^j}{(2j + 1)^2} \sum_{k=1}^{j} \frac{1}{k}
\]
Discovery of algebraic relations (J. Ablinger, J. Blümlein, CS)

multiple Zeta-values

\[\sum_{i=1}^{\infty} \frac{1}{i^3} \sum_{j=1}^{i} \frac{(-1)^j}{j^2} \sum_{k=1}^{j} \frac{1}{k}\]


combining known relations of the sum and integral representations

cyclotomic Zeta-values

\[\sum_{i=1}^{\infty} \frac{1}{i^3} \sum_{j=1}^{i} \frac{(-1)^j}{(2j + 1)^2} \sum_{k=1}^{j} \frac{1}{k}\]

generalized multiple Zeta-values

\[\sum_{i=1}^{\infty} \frac{1}{i^3} \sum_{j=1}^{i} \frac{1}{2j} \sum_{k=1}^{j} \frac{1}{k}\]
The full machinery:

Toolbox 1 + Toolbox 2 + Toolbox 3
The problem

From: Robin Pemantle [University of Pennsylvania]
To: herb wilf; doron zeilberger

Herb, Doron,

I have a sum that, when I evaluate numerically, looks suspiciously like it comes out to exactly 1.
Is there a way I can automatically decide this?
The sum may be written in many ways, but one is:

\[ \sum_{n,k=1}^{\infty} \frac{H_k(H_{n+1} - 1)}{kn(n+1)(k+n)}; \quad H_k := \sum_{i=1}^{k} \frac{1}{i} \]

[Arose in the analysis of the simplex algorithm on the Klee-Minty cube (J. Balogh, R. Pemantle)]
The full machinery:

\[
\text{In[1]}:= \text{<< Sigma.m}
\]
\[
\text{Sigma by Carsten Schneider © RISC-Linz}
\]

\[
\text{In[2]}:= \text{<< EvaluateMultiSums.m}
\]
\[
\text{EvaluateMultiSums by Carsten Schneider © RISC-Linz}
\]

\[
\text{In[4]}:= \text{EvaluateMultiSum} \left[ \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{H_k (H_{n+1} - 1)}{kn(n + 1)(k + n)} \right]
\]
The full machinery:

\[ \text{Out[4]} = 3 \sum_{i=1}^{\infty} \frac{\sum_{j=1}^{i} \frac{\sum_{k=1}^{j} \frac{1}{k}}{k}}{i^2} - 2 \sum_{j=1}^{\infty} \frac{\sum_{k=1}^{j} \frac{1}{k}}{j^3} + \frac{1}{3} \left( 3 \sum_{j=1}^{\infty} \frac{\sum_{k=1}^{j} \frac{1}{k}}{j^2} - 3 \sum_{k=1}^{\infty} \frac{\sum_{l=1}^{k} \frac{1}{l}}{k^4} \right) - \]

\[ 2 \sum_{k=1}^{\infty} \frac{\sum_{l=1}^{k} \frac{1}{l^3}}{k^2} \left( \sum_{l=1}^{\infty} \frac{1}{l^2} \right) - \sum_{k=1}^{\infty} \frac{\sum_{l=1}^{k} \frac{1}{l^2}}{k^2} + \sum_{l=1}^{\infty} \frac{1}{l^3} - 1 \right) + z_2 \left( \sum_{k=1}^{\infty} \frac{\sum_{l=1}^{k} \frac{1}{l^2}}{k^2} - 1 \right) + \sum_{l=1}^{\infty} \frac{1}{l^5} \]
The full machinery:

\[ \text{In[1]:= } << \text{Sigma.m} \]
\[ \text{Sigma by Carsten Schneider © RISC-Linz} \]

\[ \text{In[2]:= } << \text{EvaluateMultiSums.m} \]
\[ \text{EvaluateMultiSums by Carsten Schneider © RISC-Linz} \]

\[ \text{In[3]:= } << \text{HarmonicSums.m} \]
\[ \text{HarmonicSums by Jakob Ablinger © RISC-Linz} \]

\[ \text{In[4]:= } \text{EvaluateMultiSum}\left[ \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{H_k(H_{n+1} - 1)}{kn(n+1)(k+n)} \right] \]
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EvaluateMultiSums by Carsten Schneider © RISC-Linz

\[ \text{In[3]} := \text{\texttt{\textless \textless HarmonicSums.m}} \]

HarmonicSums by Jakob Ablinger © RISC-Linz

\[ \text{In[4]} := \text{EvaluateMultiSum}\left[ \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{H_k(H_{n+1} - 1)}{kn(n+1)(k+n)} \right] \]

\[ \text{Out[4]} = -4\zeta_2 - 2\zeta_3 + 4\zeta_2\zeta_3 + 2\zeta_5 \]
The full machinery:

\[ \text{In}[1]:= \text{\textless\textless Sigma.m} \]

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\[ \text{In}[3]:= \text{\textless\textless HarmonicSums.m} \]

HarmonicSums by Jakob Ablinger © RISC-Linz

\[ \text{In}[4]:= \text{EvaluateMultiSum}\left[ \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{H_k(H_{n+1} - 1)}{kn(n + 1)(k + n)} \right] \]

\[ \text{Out}[4]= -4\zeta_2 - 2\zeta_3 + 4\zeta_2\zeta_3 + 2\zeta_5 \]

\[ \text{In}[5]:= \text{EvaluateMultiSum}\left[ \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{H_k^2(H_{n+1} - 1)^2}{k(k + n)n} \right] \]
The full machinery:

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HarmonicSums by Jakob Ablinger © RISC-Linz

In[4]:= EvaluateMultiSum[\(\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{H_k(H_{n+1} - 1)}{kn(n + 1)(k + n)}\)]

Out[4]= \(-4\zeta_2 - 2\zeta_3 + 4\zeta_2\zeta_3 + 2\zeta_5\)

In[5]:= EvaluateMultiSum[\(\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{H_k^2(H_{n+1} - 1)^2}{k(k + n)n}\)]

Out[5]= \(-10\zeta_3 + \zeta_2^2 \left(\frac{58\zeta_3}{5} - \frac{29}{5}\right) - 10\zeta_5 + \zeta_2(-\zeta_3 + 13\zeta_5 - 4) + \frac{457\zeta_7}{8}\)
The full machinery:

\[
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\text{Sigma by Carsten Schneider } \copyright \text{ RISC-Linz}
\]

\[
\text{In[2]:= } << \text{EvaluateMultiSums.m} \\
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\]

\[
\text{In[3]:= } << \text{HarmonicSums.m} \\
\text{HarmonicSums by Jakob Ablinger } \copyright \text{ RISC-Linz}
\]

\[
\text{In[4]:= } \text{EvaluateMultiSum}\left[ \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{H_k (H_{n+1} - 1)}{kn(n+1)(k+n)} \right] \\
\text{Out[4]= } -4\zeta_2 - 2\zeta_3 + 4\zeta_2 \zeta_3 + 2\zeta_5
\]

\[
\text{In[5]:= } \text{EvaluateMultiSum}\left[ \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \frac{H_k (H_{n+1} - 1)}{k (k+n)^2 n^2} \right]
\]
The full machinery:

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\[ \text{In[3]} := \text{<< HarmonicSums.m} \]
\[ \text{HarmonicSums by Jakob Ablinger © RISC-Linz} \]

\[ \text{In[4]} := \text{EvaluateMultiSum}\left[ \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{H_k(H_{n+1} - 1)}{kn(n + 1)(k + n)} \right] \]
\[ \text{Out[4]} = -4\zeta_2 - 2\zeta_3 + 4\zeta_2\zeta_3 + 2\zeta_5 \]

\[ \text{In[5]} := \text{EvaluateMultiSum}\left[ \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \frac{H_k(H_{n+1} - 1)}{k(k + n)^2n^2} \right] \]
\[ \text{Out[5]} = 2\zeta_3 + \frac{\zeta_2^2}{2} \left( \frac{17\zeta_3}{10} + \frac{17}{10} \right) + \zeta_2(2\zeta_3 - 3\zeta_5 - 4) - \frac{9\zeta_5}{2} + \frac{3\zeta_7}{16} \]
The full machinery:

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In[2]:= << EvaluateMultiSums.m

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In[3]:= << HarmonicSums.m

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In[4]:= EvaluateMultiSum[∑_{n=1}^{∞} ∑_{k=1}^{∞} \frac{H_k(H_{n+1} - 1)}{kn(n + 1)(k + n)}]

Out[4]= -4\zeta_2 - 2\zeta_3 + 4\zeta_2\zeta_3 + 2\zeta_5

In[5]:= EvaluateMultiSum[∑_{n=1}^{∞} ∑_{k=1}^{∞} ∑_{l=1}^{∞} \frac{H_kH_nH_{n+l+k}}{k(k + n)(k + n + l + 1 + 1)^2}]

RISC, J. Kepler University Linz
The full machinery:

\[ \text{In[1]} := \text{\textasciitilde\textasciitilde\texttt{Sigma.m}} \]
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\[ \text{In[2]} := \text{\textasciitilde\textasciitilde\texttt{EvaluateMultiSums.m}} \]
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\[ \text{In[3]} := \text{\textasciitilde\textasciitilde\texttt{HarmonicSums.m}} \]
HarmonicSums by Jakob Ablinger © RISC-Linz

\[ \text{In[4]} := \text{\texttt{EvaluateMultiSum[}} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{H_k (H_{n+1} - 1)}{kn(n + 1)(k + n)} \text{]} \]

\[ \text{Out[4]} = -4\zeta_2 - 2\zeta_3 + 4\zeta_2 \zeta_3 + 2\zeta_5 \]

\[ \text{In[5]} := \text{\texttt{EvaluateMultiSum[}} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{H_k H_n H_{n+l+k}}{k (k + n) (k + n + l + 1)^2} \text{]} \]

\[ \text{Out[5]} = 3\zeta_3^2 - \frac{15\zeta_5}{2} + \zeta_2 (9\zeta_5 - 6\zeta_3) + \frac{149\zeta_7}{16} + \frac{114}{35} \zeta_2^3 \]
Example 1: Unfair permutations

joint work with H. Prodinger, S. Wagner
Example 1: Unfair permutations

- We are given $n$ players.
Example 1: Unfair permutations

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- Player $i$: chooses randomly a number (all different)
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The player with the highest number gets $n$ dices
Example 1: Unfair permutations

- We are given $n$ players.
- Player $i$: chooses randomly a number (all different)
- The player with the highest number gets $n$ dices
  The player with the second highest number gets $n - 1$ dices.
We are given $n$ players.

Player $i$: chooses randomly a number (all different)

The player with the highest number gets $n$ dices
The player with the second highest number gets $n - 1$ dices.

The player with the lowest number (looser) gets 1 dice.
Example 1: Unfair permutations

- We are given $n$ players.
- Player $i$: chooses randomly a number (all different)
- The player with the highest number gets $n$ dices
  The player with the second highest number gets $n - 1$ dices.
  
  
  
  The player with the lowest number (looser) gets 1 dice.
- We get a random permutation

\[
\begin{pmatrix}
1 & 2 & 3 & \ldots & n \\
\alpha_1 & \alpha_2 & \alpha_3 & \ldots & \alpha_n
\end{pmatrix} \in S_n
\]
Example 1: Unfair permutations

- We are given $n$ players.
- Player $i$: chooses randomly $i$ numbers and takes the largest (best) one
- The player with the highest number gets $n$ dices
  The player with the second highest number gets $n - 1$ dices.
  
  
  The player with the lowest number (looser) gets 1 dice.
- We get an unfair permutation

\[
\begin{pmatrix}
1 & 2 & 3 & \ldots & n \\
a_1 & a_2 & a_3 & \ldots & a_n
\end{pmatrix} \in S_n
\]
Example 1: Unfair permutations

- We are given $n$ players.
- Player $i$: chooses randomly $i$ numbers and takes the largest (best) one.
- The player with the highest number gets $n$ dices.
  The player with the second highest number gets $n - 1$ dices.
  
  \[
  \vdots
  \]
  
  The player with the lowest number (looser) gets 1 dice.

- We get an unfair permutation

\[
\begin{pmatrix}
1 & 2 & 3 & \ldots & n \\
\ a_1 & a_2 & a_3 & \ldots & a_n \\
\end{pmatrix} \in S_n
\]

anti-inversion:

\[
i < j \quad \text{and} \quad a_i < a_j
\]

\[
\uparrow
\]

\[
i < j \quad \text{and} \quad j \ \text{beats} \ i
\]
We are given \( n \) players.

Player \( i \): chooses randomly \( i \) numbers and takes the largest (best) one.

The player with the highest number gets \( n \) dices.

The player with the second highest number gets \( n - 1 \) dices.

... 

The player with the lowest number (looser) gets 1 dice.

We get an unfair permutation

\[
\begin{pmatrix}
1 & 2 & 3 & \ldots & n \\
a_1 & a_2 & a_3 & \ldots & a_n
\end{pmatrix}
\in S_n
\]

anti-inversion:  
\( i < j \) and \( a_i < a_j \)  

\[\uparrow\]

probability:  
\[\frac{j}{i + j}\]

\( i < j \) and \( j \) beats \( i \)
Example 1: Unfair permutations

- We are given \( n \) players.
- Player \( i \): chooses randomly \( i \) numbers and takes the largest (best) one
- The player with the highest number gets \( n \) dices
- The player with the second highest number gets \( n - 1 \) dices.
- The player with the lowest number (looser) gets 1 dice.
- We get an unfair permutation

\[
\begin{pmatrix}
1 & 2 & 3 & \ldots & n \\
a_1 & a_2 & a_3 & \ldots & a_n
\end{pmatrix} \in S_n
\]

- anti-inversion: \( i < j \) and \( a_i < a_j \)
- \( j \) out of \( i + j \)
- \( i < j \) and \( j \) beats \( i \)

- probability:

\[
\frac{j}{i + j}
\]

- expected number of anti-inversions:

\[
\sum_{1 \leq i < j \leq n} \frac{j}{i + j}
\]
Theorem (Prodinger, Wagner).

\[ A_n = \text{no. of anti-inversions of a random unfair permutation of length } n. \]

Then the mean of \( A_n \) is

\[
\sum_{1 \leq i < j \leq n} \frac{j}{i + j}
\]
Theorem (Prodinger, Wagner).

\( A_n = \text{no. of anti-inversions of a random unfair permutation of length } n. \)

Then the mean of \( A_n \) is

\[
\sum_{1 \leq i < j \leq n} \frac{j}{i + j} = \frac{1}{16} (-8n^2 - 8n - 1) H_n + \frac{1}{8} (2n + 1)^2 H_{2n} - \frac{5n}{8}
\]
Theorem (Prodinger, Wagner).

$A_n =$ no. of anti-inversions of a random unfair permutation of length $n$.

Then the mean of $A_n$ is

$$
\sum_{1 \leq i < j \leq n} \frac{j}{i + j} = \frac{1}{16} \left( -8n^2 - 8n - 1 \right) H_n + \frac{1}{8} (2n + 1)^2 H_{2n} - \frac{5n}{8}
$$

$$
= 0.3465735903n^2 - 0.4034264097n + O(\log n)
$$

fair case $= 0.25n^2 - 0.25n$
Theorem (Prodinger, Wagner).

\( A_n = \) no. of anti-inversions of a random unfair permutation of length \( n \).

Then the mean of \( A_n \) is

\[
\sum_{1 \leq i < j \leq n} \frac{j}{i + j} = \frac{1}{16} \left( -8n^2 - 8n - 1 \right) H_n + \frac{1}{8} \left( 2n + 1 \right)^2 H_{2n} - \frac{5n}{8}
\]

The variance of \( A_n \) is

\[
2 \sum_{1 \leq i < j < k \leq n} \frac{kj}{(i + j)(i + j + k)} + 2 \sum_{1 \leq i < j < k \leq n} \frac{k}{i + j + k}
\]

\[
+ 2 \sum_{1 \leq i < j < k \leq n} \frac{kj}{(i + j)(i + j + k)} + 2 \sum_{1 \leq i < j < k \leq n} \frac{kj}{(i + k)(i + j + k)}
\]

\[
- 2 \sum_{1 \leq i < j < k \leq n} \frac{j}{i + j} \cdot \frac{k}{j + k} - 2 \sum_{1 \leq i < j < k \leq n} \frac{k}{i + k} \cdot \frac{k}{j + k}
\]

\[
- 2 \sum_{1 \leq i < j < k \leq n} \frac{j}{i + j} \cdot \frac{k}{i + k} - \sum_{1 \leq i < j \leq n} \frac{j^2}{(i + j)^2} + \sum_{1 \leq i < j \leq n} \frac{j}{i + j}
\]
Theorem (Prodinger, Wagner).

\[ A_n = \text{no. of anti-inversions of a random unfair permutation of length } n. \]

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2 \sum_{1 \leq i < j < k \leq n} \frac{kj}{(i+j)(i+j+k)} + 2 \sum_{1 \leq i < j < k \leq n} \frac{k}{i+j+k}
\]
\[
+ 2 \sum_{1 \leq i < j < k \leq n} \frac{kj}{(i+j)(i+j+k)} + 2 \sum_{1 \leq i < j < k \leq n} \frac{kj}{(i+k)(i+j+k)}
\]
\[
- 2 \sum_{1 \leq i < j < k \leq n} \frac{j}{i+j} \cdot \frac{k}{j+k} - 2 \sum_{1 \leq i < j < k \leq n} \frac{k}{i+k} \cdot \frac{k}{j+k}
\]
\[
- 2 \sum_{1 < i < j < k \leq n} \frac{j}{i+j} \cdot \frac{k}{i+k} - \sum_{1 < i < j < n} \frac{j^2}{(i+j)^2} - \sum_{1 < i < j < n} \frac{j}{i+j}
\]
Example 1: Unfair permutations

$$\sum_{k=3}^{n} \sum_{j=2}^{k-1} \sum_{i=1}^{j-1} \frac{jk}{(i + j)(j + k)}$$
Example 1: Unfair permutations

\[ \sum_{k=3}^{n} \sum_{j=2}^{k-1} \sum_{i=1}^{j-1} \frac{jk}{(i+j)(j+k)} \]
Example 1: Unfair permutations

\[
\sum_{k=3}^{n} \sum_{j=2}^{k-1} \sum_{i=1}^{j-1} \frac{jk}{(i + j)(j + k)}
\]

\[
\frac{1}{j + k} \sum_{r=1}^{j} \frac{1}{-1 + 2r} - \frac{j H_j}{2(j + k)} - \frac{k}{2(j + k)}
\]
Example 1: Unfair permutations

\[ \sum_{k=3}^{n} \sum_{j=2}^{k-1} \left[ \frac{1}{j+k} \sum_{r=1}^{j} \frac{1}{1+2r} - \frac{jkH_j}{2(j+k)} - \frac{k}{2(j+k)} \right] \]
Example 1: Unfair permutations

$$\sum_{k=3}^{n} \sum_{j=2}^{k-1} \left[ \frac{1}{j+k} \sum_{r=1}^{j} \frac{1}{-1+2r} - \frac{jkH_j}{2(j+k)} - \frac{k}{2(j+k)} \right]$$
Example 1: Unfair permutations

\[
\sum_{k=3}^{n} \sum_{j=2}^{k-1} \left[ \frac{1}{j + k} \sum_{r=1}^{j} \frac{1}{-1 + 2r} - \frac{jkH_j}{2(j + k)} - \frac{k}{2(j + k)} \right]
\]

\[
- k^2 \sum_{s=1}^{\infty} \frac{1}{s} \sum_{r=1}^{k} \frac{1}{-1 + 2r} + \left( (k - 1)k + k^2H_k \right) \sum_{r=1}^{k} \frac{1}{-1 + 2r}
\]

\[
- \frac{1}{4} k^2H_k^2 - \frac{1}{4} k^2H_k^{(2)} - \frac{1}{4} k(2k - 3)H_k + \frac{1}{4}
\]

\[
\text{summation spiral}
\]
$$\sum_{k=3}^{n} \left[ -k^2 \sum_{r=1}^{k} \frac{1}{-1 + 2r} \sum_{s=1}^{1} \frac{1}{s} + \left( (k - 1)k + k^2 H_k \right) \sum_{r=1}^{k} \frac{1}{-1 + 2r} \right.$$

$$\left. - \frac{1}{4}k^2 H_k^2 - \frac{1}{4}k^2 H_k^{(2)} - \frac{1}{4}k(2k - 3)H_k + \frac{1}{4} \right]$$
Example 1: Unfair permutations

\[
\sum_{k=3}^{n} \left[ -k^2 \sum_{s=1}^{k} \frac{1}{-1 + 2r} \sum_{r=1}^{s} \frac{1}{s} + \left( (k - 1)k + k^2 H_k \right) \sum_{r=1}^{k} \frac{1}{-1 + 2r} \right. \\
- \frac{1}{4} k^2 H_k^2 - \frac{1}{4} k^2 H_k^{(2)} - \frac{1}{4} k(2k - 3) H_k + \frac{1}{4} \right]
\]

||

\[
n(n+1)(2n+1) \left[ -\frac{1}{6} \sum_{s=1}^{n} \frac{1}{-1 + 2r} \sum_{r=1}^{s} \frac{1}{s} - \frac{1}{12} \right] H_n - \frac{1}{24} H_n^2
\]

\[
+ \left( \frac{1}{6} \sum_{r=1}^{n} \frac{1}{-1 + 2r} - \frac{1}{24} H_n^{(2)} + \frac{1}{6} \sum_{r=1}^{n} \frac{1}{-1 + 2r} \right]
\]

\[-\frac{1}{8} (2n+1)^2 \sum_{r=1}^{n} \frac{1}{-1 + 2r} + \frac{1}{12} (n+1)(4n+1) H_n + \frac{7n}{24} \]
Example 1: Unfair permutations

\[
\sum_{k=3}^{n} \sum_{j=2}^{k-1} \sum_{i=1}^{j-1} \frac{jk}{(i + j)(j + k)}
\]

\[
n(n + 1)(2n + 1) \left[ - \frac{1}{6} \sum_{s=1}^{n} \frac{1}{s - 1 + 2r} - \frac{1}{12} \right] H_n - \frac{1}{24} H_n^2
\]

\[
+ \left( \frac{1}{6} \sum_{r=1}^{n} \frac{1}{-1 + 2r} - \frac{1}{24} H_n^{(2)} + \frac{1}{6} \sum_{r=1}^{n} \frac{1}{-1 + 2r} \right)
\]

\[
- \frac{1}{8} (2n + 1)^2 \sum_{r=1}^{n} \frac{1}{-1 + 2r} + \frac{1}{12} (n + 1)(4n + 1) H_n + \frac{7n}{24}
\]
Theorem (Prodinger, Wagner).

\( A_n = \text{no. of anti-inversions of a random unfair permutation of length } n. \)

Then the mean of \( A_n \) is

\[
\sum_{1 \leq i < j \leq n} \frac{j}{i + j} = \frac{1}{16} \left( -8n^2 - 8n - 1 \right) H_n + \frac{1}{8} \left( 2n + 1 \right)^2 H_{2n} - \frac{5n}{8}
\]

The variance of \( A_n \) is

\[
2 \sum_{1 \leq i < j < k \leq n} \frac{k \cdot j}{(i + j)(i + j + k)} + 2 \sum_{1 \leq i < j < k \leq n} \frac{k}{i + j + k} \\
+ 2 \sum_{1 \leq i < j < k \leq n} \frac{k \cdot j}{(i + j)(i + j + k)} + 2 \sum_{1 \leq i < j < k \leq n} \frac{k}{(i + k)(i + j + k)} \\
- 2 \sum_{1 \leq i < j < k \leq n} \frac{j}{i + j} \cdot \frac{k}{j + k} - 2 \sum_{1 \leq i < j < k \leq n} \frac{k}{i + k} \cdot \frac{k}{j + k} \\
- 2 \sum_{1 < i < j < k \leq n} \frac{j}{i + j} \cdot \frac{k}{i + k} - \sum_{1 < i < j \leq n} \frac{j^2}{(i + j)^2} + \sum_{1 < i < j \leq n} \frac{j}{i + j}
\]
Theorem (Prodinger, Wagner, CS).

\[ A_n = \text{no. of anti-inversions of a random unfair permutation of length } n. \]

Then the mean of \( A_n \) is

\[
\sum_{1 \leq i < j \leq n} \frac{j}{i + j} = \frac{1}{16} \left( -8n^2 - 8n - 1 \right) H_n + \frac{1}{8} (2n + 1)^2 H_{2n} - \frac{5n}{8}
\]

The variance of \( A_n \) is

\[
\frac{n(29 + 126n + 72n^2)}{216} + \frac{35 + 108n + 81n^2 - 27n^3}{162} H_n
\]

\[
+ \frac{-3 - 16n - 10n^2 + 8n^3}{12} H_{2n} + \frac{-16 + 27n - 54n^3}{108} H_{3n}
\]

\[
+ \frac{n(1 + 3n + 2n^2)}{6} \left( 3H_{2n}^{(2)} - 2H_n^{(2)} + 4 \sum_{1 \leq i \leq 2n} \frac{(-1)^i H_i}{i} \right)
\]

\[
+ \frac{8}{27} \sum_{i=1}^{n} \frac{1}{3i - 2} + \frac{(-1)^n n}{4} \left( \sum_{i=1}^{n} \frac{(-1)^i}{i} - \sum_{i=1}^{3n} \frac{(-1)^i}{i} \right),
\]
Example 2: Super-congruences

(S. Ahlgren, E. Mortenson, R. Osburn, Sigma)
Example 2: Super-congruences

Sigma’s contribution to harmonic number congruences

S. Ahlgren (2001):

\[ \frac{p-1}{2} \sum_{j=0}^{\frac{p-1}{2}} \left( \binom{\frac{p-1}{2}}{j} \right)^2 \left( j + \frac{p-1}{2} \right) \left( H_{j + \frac{p-1}{2}} - H_{\frac{p-1}{2}} \right) \equiv 0 \mod p \]
Sigma’s contribution to harmonic number congruences

- S. Ahlgren (2001):
  \[ \sum_{j=0}^{\frac{p-1}{2}} \left( \binom{\frac{p-1}{2}}{j} \right)^2 \left( j + \binom{\frac{p-1}{2}}{j} \right) (H_{j+\frac{p-1}{2}} - H_{\frac{p-1}{2}}) \equiv 0 \pmod{p} \]

- E. Mortenson (2003):
  \[ \sum_{j=0}^{\frac{p-1}{2}} \left( \binom{\frac{p-1}{2}}{j} \right)^2 \left( j + \binom{\frac{p-1}{2}}{j} \right) (1 + 3jH_{j+\frac{p-1}{2}} - 3jH_j) \equiv 0 \pmod{p} \]
  \[ \sum_{j=0}^{\frac{p-1}{2}} \left( \binom{\frac{p-1}{2}}{j} \right) \left( j + \binom{\frac{p-1}{2}}{j} \right) (1 + 2jH_{j+\frac{p-1}{2}} - 2jH_j) \equiv 0 \pmod{p} \]
Sigma's contribution to harmonic number congruences

S. Ahlgren (2001):

\[
\sum_{j=0}^{\frac{p-1}{2}} \left( \binom{\frac{p-1}{2}}{j} \right)^2 \left( j + \binom{\frac{p-1}{2}}{j} \right) \left( H_j + \frac{p-1}{2} - H_{p-1} \right) \equiv 0 \pmod{p}
\]

E. Mortenson (2003):

\[
\sum_{j=0}^{\frac{p-1}{2}} \left( \binom{\frac{p-1}{2}}{j} \right)^2 \left( j + \binom{\frac{p-1}{2}}{j} \right) \left( 1 + 3jH_j + \frac{p-1}{2} - 3jH_j \right) \equiv 0 \pmod{p}
\]
\[
\sum_{j=0}^{\frac{p-1}{2}} \left( \binom{\frac{p-1}{2}}{j} \right)^2 \left( j + \binom{\frac{p-1}{2}}{j} \right) \left( 1 + 2jH_j + \frac{p-1}{2} - 2jH_j \right) \equiv 0 \pmod{p}
\]

R. Osburn:

\[
p^2 E_2(p) + p E_1(p) + p^0 E_0(p) \equiv (-1)^{\frac{p-1}{2}} \pmod{p^3}
\]
For a prime $p > 2$, 

$$p^2 E_2(p) + p E_1(p) + p^0 E_0(p) \equiv (-1)^{\frac{p-1}{2}} \mod p^3$$
Example 2: Super-congruences

For a prime $p > 2$,

$$p^2 E_2(p)$$

$$+ p E_1(p)$$

$$+ p^0 \sum_{j=0}^{\frac{p-1}{2}} \left( \binom{2j}{j} \right)^2 16^{-j} \equiv (-1)^{\frac{p-1}{2}} \mod p^3$$
For a prime \( p > 2 \),

\[
p^2 E_2(p)
\]

\[
+ p \left[ \sum_{j=0}^{\frac{p-1}{2}} (-1)^j \left( \binom{p-1}{j} \left( j + \frac{p-1}{2} \right) \right) (1 + j \left( + H_{j + \frac{p-1}{2}} + H_{-j + \frac{p-1}{2}} - 2H_j \right) \right) \right]
\]

\[
+ p^0 \sum_{j=0}^{\frac{p-1}{2}} \left( \binom{2j}{j} \right)^2 16^{-j} \equiv (-1)^{\frac{p-1}{2}} \mod p^3
\]
For a prime $p > 2$,

$$p^2 \left\{ \sum_{j=1}^{p-3} \frac{(-1)^j}{\binom{p-1}{2j} \binom{j+p-1}{j}} \right\}$$

$$+ \sum_{j=0}^{p-1} (-1)^j \left( \binom{p-1}{2j} \right) \left( j + \frac{p-1}{2} \right) \left( 1 + 4j \left( H_{j+p-1} - H_j \right) \right)$$

$$+ j^2 \left( 2 \left( H_{j+p-1} - H_j \right)^2 + H_{j+p-1}^{(2)} - H_j^{(2)} \right)$$

$$+ \frac{p^2}{p^2} \left\{ \sum_{j=0}^{p-1} (-1)^j \left( \binom{p-1}{2j} \right) \left( j + \frac{p-1}{2} \right) \left( 1 + j \left( + H_{j+p-1} + H_{j+p-1} - 2H_j \right) \right) \right\}$$

$$+ p \left\{ \sum_{j=0}^{p-1} (-1)^j \left( \binom{p-1}{2j} \right) \left( j + \frac{p-1}{2} \right) \left( 1 + j \left( + H_{j+p-1} + H_{j+p-1} - 2H_j \right) \right) \right\}$$

$$+ p^0 \left\{ \sum_{j=0}^{p-1} \binom{2j}{j}^2 16^{-j} \right\} \equiv (-1)^{\frac{p-1}{2}} \mod p^3$$
For a prime $p > 2$,

\[
p^2 \left[ \sum_{j=1}^{\frac{p-3}{2}} \left( \frac{(-1)^j}{\binom{p-1}{j} \binom{j+p-1}{j}} \right) \right. \\
+ \sum_{j=0}^{\frac{p-1}{2}} (-1)^j \binom{p-1}{j} \binom{j+p-1}{j} (1 + 4j(H_{j+p-1} - H_j)) \\
\left. + j^2 \left( 2(H_{j+p-1} - H_j)^2 + H_j^{(2)} - H_{j+p-1}^{(2)} \right) \right] \\
\left. \right. \\
+ p \left[ \sum_{j=0}^{\frac{p-1}{2}} (-1)^j \binom{p-1}{j} \binom{j+p-1}{j} (1 + j( + H_{j+p-1} + H_{-j+p-1} - 2H_j)) \right] \\
\left. \right. \\
\left. \right. \\
+ p^0 \sum_{j=0}^{\frac{p-1}{2}} \binom{2j}{j}^2 16^{-j} \equiv (-1) \binom{p-1}{2} \mod p^3
\]
\[
\sum_{j=1}^{p-3 \over 2} \left( \frac{(-1)^j}{\binom{p-1}{2} j} \left( \frac{p-1}{2} \right) \left( \frac{p-1}{2} j \right) \right) + \sum_{j=0}^{p-1 \over 2} (-1)^j \left( \frac{p-1}{2} \right) \left( \frac{p-1}{2} \right) \left( 1 + 4j \left( H_j + \frac{p-1}{2} - H_j \right) \right) + j^2 \left( 2 \left( H_j + \frac{p-1}{2} - H_j \right)^2 + H_j^{(2)} - H_j^{(2)} \right)
\]
\[
\sum_{j=1}^{n-1} \left( \frac{(-1)^j}{\binom{n}{j} \binom{j+n}{j}} \right) + \sum_{j=0}^{n} (-1)^j \binom{n}{j} \binom{j+n}{j} \left( 1 + 4j \left( H_{j+n} - H_j \right) + j^2 \left( 2 \left( H_{j+n} - H_j \right)^2 + H_j^{(2)} - H_{j+n}^{(2)} \right) \right)
\]
Example 2: Super-congruences

\[
\sum_{j=1}^{n-1} \left( \frac{(-1)^j}{\binom{n}{j} \binom{j+n}{j}} \right) + \sum_{j=0}^{n} (-1)^j \binom{n}{j} \binom{j+n}{j} (1 + 4j(H_{j+n} - H_j)) + j^2 \left( 2(H_{j+n} - H_j)^2 + H_j^{(2)} - H_{j+n}^{(2)} \right)
\]

\[
(-1)^n ((n + 1)(2n + 1) - \binom{2n}{n})
\]
Example 2: Super-congruences

\[
\begin{align*}
\sum_{j=1}^{\frac{p-3}{2}} & \left( \frac{(-1)^j}{\binom{p-1}{j} \binom{j+p-1}{j}} \right) \\
+ \sum_{j=0}^{\frac{p-1}{2}} & (-1)^j \binom{p-1}{2j} \binom{j+p-1}{j} \left( 1 + 4j \left( H_{j+\frac{p-1}{2}} - H_j \right) \\
& \quad + j^2 \left( 2(H_{j+\frac{p-1}{2}} - H_j)^2 + H_j^{(2)} - H_{j+\frac{p-1}{2}}^{(2)} \right) \right)
\end{align*}
\]

||

\[
(-1)^{\frac{p-1}{2}} \left( \left( \frac{p-1}{2} + 1 \right)p - \left( \frac{p-1}{2} \right) \right)
\]
For a prime $p > 2$,

\[
p^2 \left[ \sum_{j=1}^{\frac{p-3}{2}} \left( \frac{-1}{j} \right) \left( \frac{p-1}{\frac{p-1}{2}j} \right) \left( \frac{p-1}{j} \right) \right]

+ \sum_{j=0}^{\frac{p-1}{2}} (-1)^j \left( \frac{p-1}{2}j \right) \left( \frac{p-1}{j} \right) \left( 1 + 4j \left( H_{j+\frac{p-1}{2}} - H_j \right) \right)

+ j^2 \left( 2 \left( H_{j+\frac{p-1}{2}} - H_j \right)^2 + H_j^{(2)} - H_{j+\frac{p-1}{2}}^{(2)} \right) \right]

+ p \left[ \sum_{j=0}^{\frac{p-1}{2}} (-1)^j \left( \frac{p-1}{2}j \right) \left( \frac{p-1}{j} \right) \left( 1 + j \left( + H_{j+\frac{p-1}{2}} + H_{-j+\frac{p-1}{2}} - 2H_j \right) \right) \right]

+ p^0 \sum_{j=0}^{\frac{p-1}{2}} \left( \frac{2j}{j} \right)^2 16^{-j} \equiv (-1)^{\frac{p-1}{2}} \mod p^3

\]
For a prime $p > 2$,

\[
p^2 \left[ (-1)^{\frac{p-1}{2}} \left( \left( \frac{p-1}{2} + 1 \right)p - \left( \frac{p-1}{2} \right) \right) \right. \\
\left. + p \left[ \sum_{j=0}^{\frac{p-1}{2}} (-1)^j \left( \frac{p-1}{2} \right)_j \left( \left( j + \frac{p-1}{2} \right)_j \right) \left( 1 + j \left( + H_{\frac{p-1}{2}} + H_{-\frac{p-1}{2}} - 2H_j \right) \right) \right. \\
\left. + p^0 \sum_{j=0}^{\frac{p-1}{2}} \binom{2j}{j}^2 16^{-j} \equiv (-1)^{\frac{p-1}{2}} \mod p^3 \right]
\]

\[\text{Example 2: Super-congruences}\]
For a prime $p > 2$,

\[
p^2 \left[ - (-1)^{\frac{p-1}{2}} \left( \frac{p-1}{2} \right) \right] + p \left[ \sum_{j=0}^{\frac{p-1}{2}} (-1)^j \left( \frac{p-1}{j} \right) \left( j + \frac{p-1}{2} \right) \left( 1 + j \left( + H_{j+\frac{p-1}{2}} + H_{-j+\frac{p-1}{2}} - 2H_j \right) \right) \right] + p^0 \sum_{j=0}^{\frac{p-1}{2}} \binom{2j}{j}^2 16^{-j} \equiv (-1)^{\frac{p-1}{2}} \mod p^3
\]

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For a prime $p > 2$,

$$
\left[ p^2 \left[ - (-1)^{\frac{p-1}{2}} \binom{p-1}{\frac{p-1}{2}} \right] + p \left[ \sum_{j=0}^{\frac{p-1}{2}} (-1)^j \binom{\frac{p-1}{2}}{j} \binom{j + \frac{p-1}{2}}{j} (1 + j (H_{j + \frac{p-1}{2}} + H_{-j + \frac{p-1}{2}} - 2H_j)) \right] + p^0 \sum_{j=0}^{\frac{p-1}{2}} \binom{2j}{j}^2 16^{-j} \equiv (-1)^{\frac{p-1}{2}} \mod p^3 \right]
$$

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Example 2: Super-congruences

\[
\sum_{j=0}^{\frac{p-1}{2}} (-1)^j \left( \binom{\frac{p-1}{2}}{j} \right) \left( j + \binom{\frac{p-1}{2}}{j} \right) \left( 1 + j \left( -2H_j + H_{j+\frac{p-1}{2}} + H_{-j+\frac{p-1}{2}} \right) \right)
\]
Example 2: Super-congruences

\[
\sum_{j=0}^{n} (-1)^j \binom{n}{j} \binom{j+n}{j} \left( 1 + j \left( -2H_j + H_{j+n} + H_{-j+n} \right) \right)
\]
Example 2: Super-congruences

\[
\sum_{j=0}^{n} (-1)^j \binom{n}{j} \binom{j+n}{j} \left(1 + j \left(-2H_j + H_{j+n} + H_{-j+n}\right)\right)
\]

\[
-\frac{3}{2} (-1)^n n(n+1) \sum_{j=1}^{n} \frac{2j}{j} + (-1)^n (2n+1) \binom{2n}{n}
\]

|| summation spiral
Example 2: Super-congruences

\[
\sum_{j=0}^{p-1} (-1)^j \left( \binom{p-1}{j} \right) \left( j + \frac{p-1}{2} \right) (1 + j \left( -2H_j + H_{j+p-1} + H_{-j+p-1} \right))
\]

||

\[-\frac{3}{2} (-1)^{p-1} \left( \frac{p^2}{4} - \frac{1}{4} \right) \sum_{j=1}^{\frac{p-1}{2}} \binom{2j}{j} + (-1)^{p-1} \frac{p-1}{2} p \left( \frac{p-1}{2} \right)\]
For a prime $p > 2$,

\[
p^2 \left[ \right.
\]

\[
- (-1)^{\frac{p-1}{2}} \binom{p-1}{\frac{p-1}{2}}
\]

\[
+ p \left[ \sum_{j=0}^{\frac{p-1}{2}} (-1)^j \binom{\frac{p-1}{2}}{j} \binom{j + \frac{p-1}{2}}{j} (1 + j \left( + H_{j + \frac{p-1}{2}} + H_{-j + \frac{p-1}{2}} - 2H_j \right)) \right]
\]

\[
+ p^0 \sum_{j=0}^{\frac{p-1}{2}} \binom{2j}{j}^2 16^{-j} \equiv (-1)^{\frac{p-1}{2}} \mod p^3
\]

\[
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\]
For a prime \( p > 2 \),

\[
\begin{align*}
  p^2 & \left[ -\left(-1\right)^{\frac{p-1}{2}} \binom{p-1}{\frac{p-1}{2}} \right] \\
  + p & \left[ -\frac{3}{2} \left(-1\right)^{\frac{p-1}{2}} \left(\frac{p^2}{4} - \frac{1}{4}\right) \sum_{j=1}^{\frac{p-1}{2}} \frac{2j}{j} + \left(-1\right)^{\frac{p-1}{2}} p \binom{p-1}{\frac{p-1}{2}} \right] \\
  + p^0 & \sum_{j=0}^{\frac{p-1}{2}} \left(\binom{2j}{j}\right)^2 16^{-j} \equiv \left(-1\right)^{\frac{p-1}{2}} \mod p^3
\end{align*}
\]
For a prime $p > 2$,

\[
p^2 \left[ 0 \right]
\]

\[
+ p \left[ - \frac{3}{2} (-1)^{\frac{p-1}{2}} \left( \frac{p^2}{4} - \frac{1}{4} \right) \sum_{j=1}^{\frac{p-1}{2}} \frac{(2j)}{j} \right]
\]

\[
+ p^0 \sum_{j=0}^{\frac{p-1}{2}} \left( \frac{2j}{j} \right)^2 16^{-j} \equiv (-1)^{\frac{p-1}{2}} \mod p^3
\]
For a prime $p > 2$,

$$p^2 \left[ \sum_{j=0}^{p-1} \left( \begin{array}{c} 2j \\ j \end{array} \right)^2 16^{-j} \right]$$

$$+ p \left[ \frac{3}{8} (-1)^{\frac{p-1}{2}} \sum_{j=1}^{\frac{p-1}{2}} \left( \begin{array}{c} 2j \\ j \end{array} \right) \right]$$

$$\equiv (-1)^{\frac{p-1}{2}} \mod p^3$$
For a prime $p > 2$,

\[
p^2 \left[ \sum_{j=1}^{\frac{p-3}{2}} \frac{(-1)^j}{\left( \frac{p-1}{2} \right) \left( j + \frac{p-1}{2} \right)} \right] \\
+ \sum_{j=0}^{\frac{p-1}{2}} (-1)^j \left( \frac{p-1}{2} \right) \left( j + \frac{p-1}{2} \right) \left( 1 + 4j \left( H_{j + \frac{p-1}{2}} - H_j \right) \right) \\
+ j^2 \left( 2 \left( H_{j + \frac{p-1}{2}} - H_j \right)^2 + H_j^{(2)} - H_j^{(2)} \left( j + \frac{p-1}{2} \right) \right) \left( 1 + j \left( + H_{j + \frac{p-1}{2}} + H_{-j + \frac{p-1}{2}} - 2H_j \right) \right) \right] \\
+p \left[ \sum_{j=0}^{\frac{p-1}{2}} (-1)^j \left( \frac{p-1}{2} \right) \left( j + \frac{p-1}{2} \right) \left( 1 + j \left( + H_{j + \frac{p-1}{2}} + H_{-j + \frac{p-1}{2}} - 2H_j \right) \right) \right] \\
+p^0 \sum_{j=0}^{\frac{p-1}{2}} \left( \frac{2j}{j} \right)^2 16^{-j} \equiv \left( -1 \right)^{\frac{p-1}{2}} \mod p^3
\]
Sigma’s contribution to harmonic number congruences

- S. Ahlgren (2001):
\[
\sum_{j=0}^{\frac{p-1}{2}} \left( \binom{\frac{p-1}{2}}{j} \right)^2 \left( j + \frac{p-1}{2} \right) \left( H_j + \frac{p-1}{2} - H_{\frac{p-1}{2}} \right) \equiv 0 \mod p
\]

- E. Mortenson (2003):
\[
\sum_{j=0}^{\frac{p-1}{2}} \left( \binom{\frac{p-1}{2}}{j} \right)^2 \left( j + \frac{p-1}{2} \right) \left( 1 + 3j H_j + \frac{p-1}{2} - 3j H_j \right) \equiv 0 \mod p
\]

\[
p \frac{3}{8} (-1)^{\frac{p-1}{2}} \sum_{j=1}^{\frac{p-1}{2}} \frac{(2j)^2}{j} + \sum_{j=0}^{\frac{p-1}{2}} \binom{2j}{j}^2 16^{-j} \equiv (-1)^{\frac{p-1}{2}} \mod p^3
\]
Example 3: Feynman integrals

joint work with J. Ablinger, A. Behring, J. Blümlein, A. Hasselhuhn, A. de Freitas, C. Raab, M. Round, F. Wissbrock (RISC–DESY)
Evaluation of Feynman diagrams
(long term project with J. Blümlein, Deutsches Elektronen–Synchrotron)

Behavior of particles
Evaluation of Feynman diagrams
(long term project with J. Blümlein, Deutsches Elektronen–Synchrotron)

Behavior of particles

\[ \int \Phi(n, \epsilon, x) \, dx \]

Feynman integrals
Evaluation of Feynman diagrams
(long term project with J. Blümlein, Deutsches Elektronen-Synchrotron)

Behavior of particles

\[ \int \Phi(n, \epsilon, x) dx \]

Feynman integrals

\[ \sum f(n, \epsilon, k) \]

multi sums

DESY
Example 3: Feynman diagrams

**Evaluation of Feynman diagrams**

(long term project with J. Blümlein, Deutsches Elektronen–Synchrotron)

![Feynman diagram]

Behavior of particles → \[ \int \Phi(n, \epsilon, x)dx \]

Feynman integrals

\[ \sum f(n, \epsilon, k) \]

multi sums

symbolic summation

simple sum expressions

DESY

RISC, J. Kepler University Linz
Example 3: Feynman diagrams

Evaluation of Feynman diagrams
(long term project with J. Blümlein, Deutsches Elektronen–Synchrotron)

Behavior of particles

\[ \int \Phi(n, \epsilon, x) dx \]

Feynman integrals

Evaluations required for the LHC experiment at CERN

processable by physicists

simple sum expressions

symbolic summation

multi sums

DESY

\[ \sum f(n, \epsilon, k) \]
Example 3: Feynman diagrams

\[ F_{-3}(n) \varepsilon^{-3} + F_{-2}(n) \varepsilon^{-2} + F_{-1}(n) \varepsilon^{-1} + F_0(n) + \ldots \]
Simplify

\[
\sum_{j=0}^{n-3} \sum_{k=0}^{j+n-3} \sum_{l=0}^{-j+n-3} \sum_{q=0}^{-l+n-q-3} \sum_{s=1}^{-l+n-q-s-3} \sum_{r=0}^{-l+n-q-r-s-3} (-1)^{-j+k-l+n-q-3} \times
\]
\[
\frac{(j+1) \binom{k}{j+2} (n-1) \binom{-j+n-3}{q} (-l+n-q-3) \binom{-l+n-q-s-3}{s} r! (-l+n-q-r-s-3)! (s-1)!}{(-l+n-q-2)! (-j+n-1) (n-q-r-s-2) (q+s+1)}
\]
\[
4H_{-j+n-1} - 4H_{-j+n-2} - 2H_k
\]
\[
- (H_{-l+n-q-2} + H_{-l+n-q-r-s-3} - 2H_{r+s})
\]
\[
+ 2H_{s-1} - 2H_{r+s} \]
\]
\[
F_0(N) = \frac{7}{12} H_N^4 + \frac{(17N + 5)H_N^3}{3N(N + 1)} + \left( \frac{35N^2 - 2N - 5}{2N^2(N + 1)^2} + \frac{13H_N^{(2)}}{2} + \frac{5(-1)^N}{2N^2} \right) H_N^2 \\
+ \left( -\frac{4(13N + 5)}{N^2(N + 1)^2} + \left( \frac{4(-1)^N(2N + 1)}{N(N + 1)} - \frac{13}{N} \right) H_N^{(2)} + \left( \frac{29}{3} - (-1)^N \right) H_N^{(3)} \right) \\
+ (2 + 2(-1)^N) S_{2,1}(N) - 28 S_{-2,1}(N) + \frac{20(-1)^N}{N^2(N + 1)} H_N + \left( \frac{3}{4} + (-1)^N \right) H_N^{(2)^2} \\
- 2(-1)^N S_{-2}(N)^2 + S_{-3}(N) \left( \frac{2(3N - 5)}{N(N + 1)} + (26 + 4(-1)^N) H_N + \frac{4(-1)^N}{N + 1} \right) \\
+ \left( \frac{(-1)^N(5 - 3N)}{2N^2(N + 1)} - \frac{5}{2N^2} \right) H_N^{(2)} + S_{-2}(N) \left( 10 H_N^2 + \frac{8(-1)^N(2N + 1)}{N(N + 1)} \right) \\
+ \frac{4(3N - 1)}{N(N + 1)} H_N + \frac{8(-1)^N(3N + 1)}{N(N + 1)^2} + \left( -22 + 6(-1)^N \right) H_N^{(2)} - \frac{16}{N(N + 1)} \\
+ \left( \frac{(-1)^N(9N + 5)}{N(N + 1)} - \frac{29}{3N} \right) H_N^{(3)} + \left( \frac{19}{2} - 2(-1)^N \right) H_N^{(4)} + \left( -6 + 5(-1)^N \right) S_{-4}(N) \\
+ \left( -\frac{2(-1)^N(9N + 5)}{N(N + 1)} - \frac{2}{N} \right) S_{2,1}(N) + (20 + 2(-1)^N) S_{2,-2}(N) + \left( -17 + 13(-1)^N \right) S_{3,1}(N) \\
- \frac{8(-1)^N(2N + 1) + 4(9N + 1)}{N(N + 1)} S_{-2,1}(N) - \frac{2(-1)^N(2N + 1) + 4(9N + 1)}{N(N + 1)} S_{-3,1}(N) + \left( 3 - 5(-1)^N \right) S_{2,1,1}(N) \\
+ 32 S_{-2,1,1}(N) + \left( 3 \frac{H_N^2}{2} - \frac{3H_N}{N} + \frac{3}{2} (-1)^N S_{-2}(N) \right) \zeta(2)
\]
Example 3: Feynman diagrams

\[ F_0(N) = \]

\[
\frac{7}{12} H_N^4 + \frac{(17N + 5)H_N^3}{3N(N+1)} + \left( \frac{35N^2 - 2N - 5}{2N^2(N+1)^2} + \frac{13H_N^2(2)}{2} + \frac{5(-1)^N}{2N^2} \right) H_N^2
\]

\[
+ \left( \frac{-4}{N(N+1)} \right) \frac{(2N+1)^N}{1} \frac{1}{N(N+1)} - \frac{13}{N} \right) H_N^2 + \left( \frac{29}{3} - (-1)^N \right) H_N^3
\]

\[
+ (2 + 1) \left( \sum_{i=1}^{N} \frac{1}{i} \right) - 28S_{-2,1}(N) + \left( \frac{20(-1)^N}{N^2(N+1)} \right) H_N + \left( \frac{3}{4} + (-1)^N \right) H_N^2
\]

\[
- 2(-1)^N S_{-2}(N)^2 + S_{-3}(N) \left( \frac{2(3N - 5)}{N(N+1)} + (26 + 4(-1)^N) H_N + \frac{4(-1)^N}{N+1} \right)
\]

\[
+ \left( \frac{(-1)^N(5 - 3N)}{2N^2(N+1)} - \frac{5}{2N^2} \right) H_N^2 + S_{-2}(N) \left( 10H_N^2 + \left( \frac{8(-1)^N(2N+1)}{N(N+1)} \right) H_N^2 - \frac{16}{N(N+1)} \right)
\]

\[
+ \left( \frac{4(3N - 1)}{N(N+1)} \right) H_N + \left( \frac{8(-1)^N(3N+1)}{N(N+1)^2} + (-22 + 6(-1)^N) H_N^2 \right)
\]

\[
+ \left( \frac{(-1)^N(9N + 5)}{N(N+1)} - \frac{29}{3N} \right) H_N^3 + \left( \frac{19}{2} - 2(-1)^N \right) H_N^4 + \left( -6 + 5(-1)^N \right) S_{-4}(N)
\]

\[
+ \left( - \frac{2(-1)^N(9N + 5)}{N(N+1)} - \frac{2}{N} \right) S_{2,1}(N) + \left( 20 + 2(-1)^N \right) S_{2,-2}(N) + \left( -17 + 13(-1)^N \right) S_{3,1}(N)
\]

\[
- \frac{8(-1)^N(2N + 1) + 4(9N + 1)}{N(N+1)} S_{-2,1}(N) - \left( 24 + 4(-1)^N \right) S_{-3,1}(N) + \left( 3 - 5(-1)^N \right) S_{2,1,1}(N)
\]

\[
+ 32S_{-2,1,1}(N) + \left( \frac{3}{2} H_N^2 - \frac{3H_N}{N} + \frac{3}{2} (-1)^N S_{-2}(N) \right) \zeta(2)
\]

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\[
F_0(N) = \frac{7}{12} H_N^4 + \frac{(17N + 5)H_N^3}{3N(N + 1)} + \left( \frac{35N^2 - 2N - 5}{2N^2(N + 1)^2} + \frac{13H_N^{(2)}}{2} + \frac{5(-1)^N}{2N^2} \right) H_N^2
\]

\[
H_N = \sum_{i=1}^{N} \frac{1}{i}
\]

\[
H_N^{(2)} = \sum_{i=1}^{N} \frac{1}{i^2}
\]

\[
\sum_{i=1}^{N} \frac{1}{i^2} = \frac{N(N+1)(2N+1)}{6}
\]

\[
\sum_{i=1}^{N} \frac{1}{i^3} = \frac{N^2(N+1)^2}{4}
\]

\[
\sum_{i=1}^{N} \frac{1}{i^4} = \frac{N(N+1)(2N+1)(3N^2 + 3N - 1)}{12}
\]

\[
\sum_{i=1}^{N} \frac{1}{i^5} = \frac{N^2(N+1)^2(2N^2 + 2N - 1)}{12}
\]

\[
\sum_{i=1}^{N} \frac{1}{i^6} = \frac{N(N+1)(2N+1)(3N^4 + 3N^3 - N^2 - N + 1)}{120}
\]

\[
\sum_{i=1}^{N} \frac{1}{i^7} = \frac{N^2(N+1)^2(2N^3 + 2N^2 - N - 1)}{12}
\]

\[
\sum_{i=1}^{N} \frac{1}{i^8} = \frac{N(N+1)(2N+1)(3N^6 + 3N^5 - N^4 - N^3 + N^2)}{120}
\]

\[
\sum_{i=1}^{N} \frac{1}{i^9} = \frac{N^2(N+1)^2(2N^4 + 2N^3 - N^2 - N + 1)}{12}
\]

\[
\sum_{i=1}^{N} \frac{1}{i^{10}} = \frac{N(N+1)(2N+1)(3N^8 + 3N^7 - N^6 - N^5 + N^4 - N^3 + N^2)}{120}
\]

\[
\sum_{i=1}^{N} \frac{1}{i^{11}} = \frac{N^2(N+1)^2(2N^5 + 2N^4 - N^3 - N^2 + N)}{12}
\]

\[
\sum_{i=1}^{N} \frac{1}{i^{12}} = \frac{N(N+1)(2N+1)(3N^{10} + 3N^9 - N^8 - N^7 + N^6 - N^5 + N^4 - N^3 + N^2)}{120}
\]

\[
\sum_{i=1}^{N} \frac{1}{i^{13}} = \frac{N^2(N+1)^2(2N^7 + 2N^6 - N^5 - N^4 + N^3 - N^2 + N)}{12}
\]

\[
\sum_{i=1}^{N} \frac{1}{i^{14}} = \frac{N(N+1)(2N+1)(3N^{12} + 3N^{11} - N^{10} - N^9 + N^8 - N^7 + N^6 - N^5 + N^4 - N^3 + N^2)}{120}
\]

\[
\sum_{i=1}^{N} \frac{1}{i^{15}} = \frac{N^2(N+1)^2(2N^9 + 2N^8 - N^7 - N^6 + N^5 - N^4 + N^3 - N^2 + N)}{12}
\]

\[
\sum_{i=1}^{N} \frac{1}{i^{16}} = \frac{N(N+1)(2N+1)(3N^{14} + 3N^{13} - N^{12} - N^{11} + N^{10} - N^9 + N^8 - N^7 + N^6 - N^5 + N^4 - N^3 + N^2)}{120}
\]
Example 3: Feynman diagrams

\[ F_0(N) = \]

\[
\frac{7}{12} H_N^4 + \frac{(17N + 5)H_N^3}{3N(N + 1)} + \left( \frac{35N^2 - 2N - 5}{2N^2(N + 1)^2} + \frac{13H_N^{(2)}}{2} + \frac{5(-1)^N}{2N^2} \right) H_N^2
\]

\[
+ \left( -\frac{4}{N} \right) \left( \frac{(-1)^N (2N + 1)}{N(N + 1)} - \frac{13}{N} \right) H_N^{(2)} + \left( \frac{29}{3} - (-1)^N \right) H_N^{(3)}
\]

\[
+ (2 + 4(-1)^N) S_{-2,1}(N) + 28S_{-2,1}(N) + \frac{20(-1)^N}{N^2(N + 1)} H_N^{(2)^2}
\]

\[
- 2(-1)^N S_{-2}(N)^2 + S_{-3}(N) \left( \frac{2(3N - 5)}{N(N + 1)} \right) + (26 + 4(3N - 5)) S_{-2}(N) + \frac{8(-1)^N}{N(N + 1)} S_{-2}(N)^2
\]

\[
+ (\frac{(-1)^N (5 - 3N)}{2N^2(N + 1)} - \frac{5}{2N^2}) H_N^{(2)} + S_{-2}(N) \left( 10H_N^2 + \left( \frac{9(-1)}{2N + 1} \right) H_N^{(2)} - \frac{16}{N(N + 1)} \right)
\]

\[
+ \left( \frac{-4(3N - 5)}{N(N + 1)} \right) \left( \sum_{i=1}^{N} \frac{1}{i} \right) + \left( \frac{-2(-1)^N}{N(N + 1)} \right) \left( \sum_{i=1}^{N} \frac{1}{i} \right)
\]

\[
+ \left( \frac{-2(-1)^N}{N(N + 1)} \right) \left( \sum_{i=1}^{N} \frac{1}{i^2} \right)
\]

\[
S_{-2,1,1}(N) = \sum_{i=1}^{N} \left( -\frac{1}{i} \right) \sum_{j=1}^{N} \frac{1}{j} \sum_{k=1}^{j} \frac{1}{k} + \left( -6 + 5(-1)^N \right) S_{-4}(N)
\]

\[
+ \left( -17 + 13(-1)^N \right) S_{3,1}(N)
\]

\[
+ \left( -17 + 13(-1)^N \right) S_{2,-2}(N) + \left( 3 - 5(-1)^N \right) S_{2,1,1}(N)
\]

\[
\left( 3 + \frac{3}{2} H_N^2 - \frac{3H_N}{N} + \frac{3}{2} (-1)^N S_{-2}(N) \right) \zeta(2)
\]

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Summarizing:

If you have
Summarizing:

If you have unfair permutations/monster sums/...
Summarizing:

If you have

unfair permutations/monster sums/...

super congruences/identities/...,
Summarizing:

If you have unfair permutations/monster sums/ . . .

super congruences/identities/ . . . ,

give the presented machinery a try!