

Nominal Anti-Unification

Alexander Baumgartner¹, Temur Kutsia¹, Jordi Levy², and Mateu Villaret³

¹ Research Institute for Symbolic Computation
Johannes Kepler University Linz, Austria
{[abaumgar](mailto:abaumgar@risc.jku.at),[kutsia](mailto:kutsia@risc.jku.at)}@risc.jku.at

² Artificial Intelligence Research Institute (IIIA)
Spanish Council for Scientific Research (CSIC), Barcelona, Spain
levy@iiia.csic.es

³ Departament d'Informàtica i Matemàtica Aplicada (IMA)
Universitat de Girona (UdG), Girona, Spain.
villaret@ima.udg.edu

Abstract. We study nominal anti-unification, which is concerned with computing least general generalizations for given terms-in-context. In general, the problem does not have a least general solution, but if the set of atoms permitted in generalizations is finite, then there exists a least general generalization which is unique modulo variable renaming and α -equivalence. We present an algorithm that computes it. The algorithm relies on a subalgorithm that constructively decides equivariance between two terms-in-context. We prove soundness and completeness properties of both algorithms and analyze their complexity. Nominal anti-unification can be applied to problems where generalization of first-order terms is needed (inductive learning, clone detection, etc.), but bindings are involved.

1 Introduction

Binders are very common in computer science, logic, mathematics, linguistics. Functional abstraction λ , universal quantifier \forall , limit \lim , integral \int are some well-known examples of binders. To formally represent and study systems with binding, Pitts and Gabbay [12–14] introduced nominal techniques, based on the idea to give explicit names to bound entities. It makes a syntactic distinction between *atoms*, which can be bound, and *variables*, which can be substituted. This approach led to the development of the theory of nominal sets, nominal logic, nominal algebra, nominal rewriting, nominal logic programming, etc.

Equation solving between nominal terms (maybe together with freshness constraints) has been investigated by several authors, who designed and analyzed algorithms for nominal unification [4, 5, 17–19, 29], nominal matching [6], equivariant unification [8], and permissive nominal unification [9, 10]. However, in contrast to unification, its dual problem, anti-unification, has not been studied for nominal terms previously.

The anti-unification problem for two terms t_1 and t_2 is concerned with finding a term t that is more general than the original ones, i.e., t_1 and t_2 should be substitutive instances of t . The interesting generalizations are the least general ones,

which retain the common structure of t_1 and t_2 as much as possible. Plotkin [22] and Reynolds [24] initiated research on anti-unification in the 1970s, developing generalization algorithms for first-order terms. Since then, anti-unification has been studied in various theories, including some of those with binding constructs: calculus of constructions [21], $M\lambda$ [11], second-order lambda calculus with type variables [20], simply-typed lambda calculus where generalizations are higher-order patterns [3], just to name a few.

The problem we address in this paper is to compute generalizations for nominal terms. More precisely, we consider this problem for nominal *terms-in-context*, which are pairs of a freshness context and a nominal term, aiming at computing their least general generalizations (lgg). However, it turned out that without a restriction, there is no lgg for terms-in-context, in general. Even more, a *minimal* complete set of generalizations does not exist. This is in sharp contrast with the related problem of anti-unification for higher-order patterns, which always have a single lgg [3]. The reason is one can make terms-in-context less and less general by adding freshness constraints for the available (infinitely many) atoms, see Example 2. Therefore, we restrict the set of atoms which are permitted in generalizations to be fixed and finite. In this case, there exists a single lgg (modulo α -equivalence and variable renaming) for terms-in-context and we design an algorithm to compute it in $O(n^5)$ time.

There is a close relation between nominal and higher-order pattern unification: One can be translated into the other by the solution-preserving translation defined in [7, 17, 19] or the translation defined for permissive terms in [9, 10]. We show that for anti-unification, this method, in general, is not applicable. Even if one finds conditions under which such a translation-based approach to anti-unification works, due to complexity reasons it is still better to use the direct nominal anti-unification algorithm developed in this paper.

Computation of nominal lgg's requires to solve the equivariance problem: Given two terms s_1 and s_2 , find a permutation of atoms which, when applied to s_1 , makes it α -equivalent to s_2 (under the given freshness context). This is necessary to guarantee that the computed generalization is *least* general. For instance, if the given terms are $s_1 = f(a, b)$ and $s_2 = f(b, a)$, where a, b are atoms, the freshness context is empty, and the atoms permitted in the generalization are a, b , and c , then the term-in-context $\langle \{c\#X, c\#Y\}, f(X, Y) \rangle$ generalizes $\langle \emptyset, s_1 \rangle$ and $\langle \emptyset, s_2 \rangle$, but it is not their lgg. To compute the latter, we need to reflect the fact that generalizations of the atoms are related to each other: One can be obtained from the other by swapping a and b . This leads to an lgg $\langle \{c\#X\}, f(X, (a\ b)\cdot X) \rangle$. To compute the permutation $(a\ b)$, an equivariance problem should be solved. Equivariance is already present in α -Prolog [27] and Isabelle [26]. We develop a rule-based algorithm for equivariance problems, which computes in quadratic time the justifying permutation if the input terms are equivariant, and fails otherwise.

Both anti-unification and equivariance algorithms are implemented in the anti-unification algorithm library [2] and can be accessed from <http://www.risc.jku.at/projects/stout/software/>.

Various variants of anti-unification, such as first-order, higher-order, or equational anti-unification have been used in inductive logic programming, logical and relational learning [23], reasoning by analogy [15], program synthesis [25], program verification [20], etc. Nominal anti-unification can, hopefully, contribute in solving similar problems in nominal setting or in first-order settings where bindings play an important role.

In this paper, we mainly follow the notation from [19].

2 Nominal Terms

In *nominal signatures* we have *sorts of atoms* (typically ν) and *sorts of data* (typically δ) as disjoint sets. *Atoms* (typically a, b, \dots) have one of the sorts of atoms. *Variables* (typically X, Y, \dots) have a sort of atom or a sort of data, i.e. of the form $\nu \mid \delta$. In nominal terms, variables can be instantiated and atoms can be bound. Nominal function symbols (typically f, g, \dots) have an arity of the form $\tau_1 \times \dots \times \tau_n \rightarrow \delta$, where δ is a sort of data and τ_i are sorts given by the grammar $\tau ::= \nu \mid \delta \mid \langle \nu \rangle \tau$. Abstractions have sorts of the form $\langle \nu \rangle \tau$.

A *swapping* (ab) is a pair of atoms of the same sort. A *permutation* is a (possibly empty) sequence of swappings. We use upright Greek letters (e.g., π, ρ) to denote permutations. *Nominal terms* (typically t, s, u, r, q, \dots) are given by the grammar:

$$t ::= f(t_1, \dots, t_n) \mid a \mid a.t \mid \pi.X$$

where f is an n -ary function symbol, a is an atom, π is a permutation, and X is a variable. They are called respectively *application*, *atom*, *abstraction*, and *suspension*. The sorts of application and atomic terms are defined as usual, the sort of $a.t$ is $\langle \nu \rangle \tau$ where ν is the sort of a and τ is the sort of t , and the sort of $\pi.X$ is the one of X .

The *inverse* of a permutation $\pi = (a_1 b_1) \dots (a_n b_n)$ is the permutation $(a_n b_n) \dots (a_1 b_1)$, denoted by π^{-1} . The empty permutation is denoted by Id . The *effect of a swapping* over an atom is defined by $(ab) \bullet a = b$, $(ab) \bullet b = a$ and $(ab) \bullet c = c$, when $c \notin \{a, b\}$. It is extended to the rest of terms: $(ab) \bullet f(t_1, \dots, t_n) = f((ab) \bullet t_1, \dots, (ab) \bullet t_n)$, $(ab) \bullet (c.t) = ((ab) \bullet c).((ab) \bullet t)$, and $(ab) \bullet \pi.X = (ab)\pi.X$, where $(ab)\pi$ is the permutation obtained by concatenating (ab) and π . The *effect of a permutation* is defined by $(a_1 b_1) \dots (a_n b_n) \bullet t = (a_1 b_1) \bullet ((a_2 b_2) \dots (a_n b_n) \bullet t)$. The effect of the empty permutation is $Id \bullet t = t$. We extend it to suspensions and write X as the shortcut of $Id.X$.

The set of variables of a term t is denoted by $\text{Vars}(t)$. A term t is called *ground* if $\text{Vars}(t) = \emptyset$. The set of *atoms* of a term t or a permutation π is the set of all atoms which appear in it and is denoted by $\text{Atoms}(t)$, $\text{Atoms}(\pi)$ respectively. For instance, $\text{Atoms}(f(a.g(a), (bc).X), d) = \{a, b, c, d\}$. We write $\text{Atoms}(t_1, \dots, t_n)$ for the set $\text{Atoms}(t_1) \cup \dots \cup \text{Atoms}(t_n)$.

The *length* of a term t is the number of all appearances of atoms, variables, and function symbols in it and we denote it by $\|t\|$, e.g., $\|f(a.(ab).X, X, Y)\| = 7$. The number of variable occurrences in a term t is denoted by $\|t\|_{\text{vars}}$, e.g., $\|f(a.(ab).X, X, Y)\|_{\text{vars}} = 3$.

Positions in terms are defined with respect to their tree representation in the usual way, as strings of integers. However, suspensions are put in a single leaf node. For instance, the tree form of the term $f(a.b.g((ab) \cdot X, a), h(c))$, and the corresponding positions are shown in Fig. 1. The symbol f stands in the position ϵ (the empty sequence). The suspension is put in one node of the tree, at the position 1.1.1.1. The abstraction operator and the corresponding bound atom together occupy one node as well. For any term t , $t|_p$ denotes the *subterm of t at position p* . For instance, $f(a.b.g((ab) \cdot X, a), h(c))|_{1.1} = b.g((ab) \cdot X, a)$.

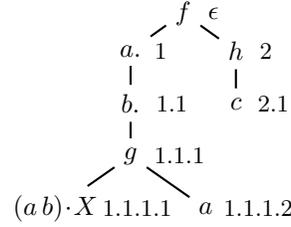


Fig. 1. The tree form and positions of the term $f(a.b.g((ab) \cdot X, a), h(c))$.

The *path to a position* in a term is defined as the sequence of expressions from the root to the node at that position (not including) in the tree form of the term, e.g., the path to the position 1.1.1.2 in $f(a.b.g((ab) \cdot X, a), h(c))$ is f, a, b, g . Since suspensions are always in leaves, they never appear in a path.

Every permutation π naturally defines a bijective function from the set of atoms to the sets of atoms, that we will also represent as π . Suspensions are uses of variables with a permutation of atoms waiting to be applied once the variable is instantiated. Occurrences of an atom a are said to be bound if they are in the scope of an abstraction of a , otherwise are said to be free. We denote by $\text{FA}(t)$ the set of all atoms which occur freely in t : $\text{FA}(f(t_1, \dots, t_n)) = \bigcup_{i=1}^n \text{FA}(t_i)$, $\text{FA}(a) = \{a\}$, $\text{FA}(a.t) = \text{FA}(t) \setminus \{a\}$, and $\text{FA}(\pi \cdot X) = \text{Atoms}(\pi)$. $\text{FA}^s(t)$ is the set of all atoms which occur freely in t ignoring suspensions: $\text{FA}^s(f(t_1, \dots, t_n)) = \bigcup_{i=1}^n \text{FA}^s(t_i)$, $\text{FA}^s(a) = \{a\}$, $\text{FA}^s(a.t) = \text{FA}^s(t) \setminus \{a\}$, and $\text{FA}^s(\pi \cdot X) = \emptyset$.

The head of a term t , denoted $\text{Head}(t)$, is defined as: $\text{Head}(f(t_1, \dots, t_n)) = f$, $\text{Head}(a) = a$, $\text{Head}(a.t) = .$, and $\text{Head}(\pi \cdot X) = X$.

Substitutions are defined in the standard way, as a mapping from variables to terms of the same sort. We use Greek letters $\sigma, \vartheta, \varphi$ to denote substitutions. The identity substitution is denoted by ϵ . Furthermore, we use the postfix notation for substitution applications, i.e. $t\sigma$ denotes the application of a substitution σ to a term t , and similarly, the composition of two substitutions σ and ϑ is written as $\sigma\vartheta$. Composition of two substitutions is performed as usual. Their application allows atom capture, for instance, $a.X\{X \mapsto a\} = a.a$, and forces the permutation effect: $\pi \cdot X\{X \mapsto t\} = \pi \bullet t$, for instance, $(ab) \cdot X\{X \mapsto f(a, (ab) \cdot Y)\} = f(b, (ab)(ab) \cdot Y)$. The notions of substitution *domain* and *range* are also standard and are denoted, respectively, by Dom and Ran .

A *freshness constraint* is a pair of the form $a\#X$ stating that the instantiation of X cannot contain free occurrences of a . A *freshness context* is a finite set of freshness constraints. We will use ∇ and Γ to denote freshness contexts. $\text{Vars}(\nabla)$ and $\text{Atoms}(\nabla)$ denote respectively the set of variables and atoms of ∇ .

We say that a substitution σ *respects* a freshness context ∇ , if for all X , $\text{FA}^s(X\sigma) \cap \{a \mid a\#X \in \nabla\} = \emptyset$.

The predicate \approx , which stands for α -equivalence between terms, and the freshness predicate $\#$ were defined in [28, 29] by the following theory:

$$\begin{array}{c} \frac{}{\nabla \vdash a \approx a} (\approx\text{-atom}) \quad \frac{\nabla \vdash t \approx t'}{\nabla \vdash a.t \approx a.t'} (\approx\text{-abs-1}) \\ \\ \frac{a \neq a' \quad \nabla \vdash t \approx (a a') \bullet t' \quad \nabla \vdash a \# t'}{\nabla \vdash a.t \approx a'.t'} (\approx\text{-abs-2}) \\ \\ \frac{a \# X \in \nabla \text{ for all } a \text{ such that } \pi \bullet a \neq \pi' \bullet a}{\nabla \vdash \pi.X \approx \pi'.X} (\approx\text{-susp.}) \\ \\ \frac{\nabla \vdash t_1 \approx t'_1 \quad \dots \quad \nabla \vdash t_n \approx t'_n}{\nabla \vdash f(t_1, \dots, t_n) \approx f(t'_1, \dots, t'_n)} (\approx\text{-application}) \end{array}$$

where the freshness predicate $\#$ is defined by

$$\begin{array}{c} \frac{a \neq a'}{\nabla \vdash a \# a'} (\#\text{-atom}) \quad \frac{(\pi^{-1} \bullet a \# X) \in \nabla}{\nabla \vdash a \# \pi.X} (\#\text{-susp.}) \\ \\ \frac{\nabla \vdash a \# t_1 \quad \dots \quad \nabla \vdash a \# t_n}{\nabla \vdash a \# f(t_1, \dots, t_n)} (\#\text{-application}) \\ \\ \frac{}{\nabla \vdash a \# a.t} (\#\text{-abst-1}) \quad \frac{a \neq a' \quad \nabla \vdash a \# t}{\nabla \vdash a \# a'.t} (\#\text{-abst-2}) \end{array}$$

Their intended meanings are:

1. $\nabla \vdash a \# t$ holds, if for every substitution σ such that $t\sigma$ is a ground term and σ respects the freshness context ∇ , we have a is not free in $t\sigma$;
2. $\nabla \vdash t \approx u$ holds, if for every substitution σ such that $t\sigma$ and $u\sigma$ are ground terms and σ respects the freshness context ∇ , $t\sigma$ and $u\sigma$ are α -equivalent.

Based on the definition of the freshness predicate, we can design an algorithm which solves the following problem:

Given: A set of freshness formulas $\{a_1 \# t_1, \dots, a_n \# t_n\}$.

Compute: A *minimal* (with respect to \subseteq) freshness context ∇ such that $\nabla \vdash a_1 \# t_1, \dots, \nabla \vdash a_n \# t_n$.

Such a ∇ may or may not exist, and the algorithm should detect it.

We give a rule-based description of the algorithm, which we call **FC** for it is supposed to compute a freshness context. The rules operate on pairs $F; \nabla$, where F is a set of atomic freshness formulas of the form $a \# t$, and ∇ is a freshness context. \cup stands for disjoint union. The rules are the following:

Del-FC: Delete in FC

$$\{a \# b\} \cup F; \nabla \Longrightarrow F; \nabla, \quad \text{if } a \neq b.$$

Abs-FC1: Abstraction in FC 1

$$\{a\#a.t\} \cup F; \nabla \Longrightarrow F; \nabla.$$

Abs-FC2: Abstraction in FC 2

$$\{a\#b.t\} \cup F; \nabla \Longrightarrow \{a\#t\} \cup F; \nabla, \quad \text{if } a \neq b.$$

Dec-FC: Decomposition in FC

$$\begin{aligned} \{a\#f(t_1, \dots, t_n)\} \cup F; \nabla \Longrightarrow \\ \{a\#t_1, \dots, a\#t_n\} \cup F; \nabla. \end{aligned}$$

Sus-FC: Suspension in FC

$$\{a\#\pi.X\} \cup F; \nabla \Longrightarrow F; \{\pi^{-1} \bullet a\#X\} \cup \nabla.$$

To compute a minimal freshness context which justifies the atomic freshness formulas $a_1\#t_1, \dots, a_n\#t_n$, we start with $\{a_1\#t_1, \dots, a_n\#t_n\}; \emptyset$ and apply the rules of FC as long as possible. It is easy to see that the algorithm terminates. The state to which no rule applies has either the form $\emptyset; \nabla$ or $\{a\#a\} \cup F; \nabla$, where ∇ is a freshness context. In the former case we say that the algorithm succeeds and computes ∇ , writing this fact as $\text{FC}(\{a_1\#t_1, \dots, a_n\#t_n\}) = \nabla$. In the latter case we say that FC fails and write $\text{FC}(\{a_1\#t_1, \dots, a_n\#t_n\}) = \perp$.

The following theorem is easy to verify:

Theorem 1. *Let F be a set of freshness formulas and ∇ be a freshness context. Then $\text{FC}(F) \subseteq \nabla$ iff $\nabla \vdash a\#t$ for all $a\#t \in F$.*

Corollary 1. *$\text{FC}(F) = \perp$ iff there is no freshness context that would justify all formulas in F .*

Given a freshness context ∇ and a substitution σ , we define $\nabla\sigma = \text{FC}(\{a\#X\sigma \mid a\#X \in \nabla\})$. The following lemma is straightforward:

Lemma 1. *σ respects ∇ iff $\nabla\sigma \neq \perp$.*

When $\nabla\sigma \neq \perp$, we call $\nabla\sigma$ the *instance* of ∇ under σ .

It is not hard to see that (a) if σ respects ∇ , then σ respects any $\nabla' \subseteq \nabla$, and (b) if σ respects ∇ and ϑ respects $\nabla\sigma$, then $\sigma\vartheta$ respects ∇ and $(\nabla\sigma)\vartheta = \nabla(\sigma\vartheta)$.

Definition 1. *A term-in-context is a pair $\langle \nabla, t \rangle$ of a freshness context and a term. A term-in-context $\langle \nabla_1, t_1 \rangle$ is more general than a term-in-context $\langle \nabla_2, t_2 \rangle$, written $\langle \nabla_1, t_1 \rangle \preceq \langle \nabla_2, t_2 \rangle$, if there exists a substitution σ , which respects ∇_1 , such that $\nabla_1\sigma \subseteq \nabla_2$ and $\nabla_2 \vdash t_1\sigma \approx t_2$.*

We write $\nabla \vdash t_1 \preceq t_2$ if there exists a substitution σ such that $\nabla \vdash t_1\sigma \approx t_2$.

Two terms-in-context p_1 and p_2 are equivalent (or equi-general), written $p_1 \simeq p_2$, iff $p_1 \preceq p_2$ and $p_2 \preceq p_1$. The strict part of \preceq is denoted by \prec , i.e., $p_1 \prec p_2$ iff $p_1 \preceq p_2$ and not $p_2 \preceq p_1$. We also write $\nabla \vdash t_1 \simeq t_2$ iff $\nabla \vdash t_1 \preceq t_2$ and $\nabla \vdash t_2 \preceq t_1$.

Example 1. We give some examples to demonstrate the relations we have just defined:

- $\langle \{a\#X\}, f(a) \rangle \simeq \langle \emptyset, f(a) \rangle$. We can use $\{X \mapsto b\}$ for the substitution applied to the first pair.
- $\langle \emptyset, f(X) \rangle \preceq \langle \{a\#X\}, f(X) \rangle$ (with $\sigma = \varepsilon$), but not $\langle \{a\#X\}, f(X) \rangle \preceq \langle \emptyset, f(X) \rangle$.
- $\langle \emptyset, f(X) \rangle \preceq \langle \{a\#Y\}, f(Y) \rangle$ with $\sigma = \{X \mapsto Y\}$.
- $\langle \{a\#X\}, f(X) \rangle \not\preceq \langle \emptyset, f(Y) \rangle$, because in order to satisfy $\{a\#X\}\sigma \subseteq \emptyset$, the substitution σ should map X to a term t which contains neither a (freely) nor variables. But then $\emptyset \vdash f(t) \approx f(Y)$ does not hold. Hence, together with the previous example, we get $\langle \emptyset, f(Y) \rangle \prec \langle \{a\#X\}, f(X) \rangle$.
- $\langle \{a\#X\}, f(X) \rangle \not\preceq \langle \{a\#X\}, f(a) \rangle$. Notice that $\sigma = \{X \mapsto a\}$ does not respect $\{a\#X\}$.
- $\langle \{b\#X\}, (ab) \cdot X \rangle \preceq \langle \{c\#X\}, (ac) \cdot X \rangle$ with the substitution $\sigma = \{X \mapsto (ab)(ac) \cdot X\}$. Hence, we get $\langle \{b\#X\}, (ab) \cdot X \rangle \simeq \langle \{c\#X\}, (ac) \cdot X \rangle$, because the \succeq part can be shown with the help of the substitution $\{X \mapsto (ac)(ab) \cdot X\}$.

Definition 2. A term-in-context $\langle \Gamma, r \rangle$ is called a generalization of two terms-in-context $\langle \nabla_1, t \rangle$ and $\langle \nabla_2, s \rangle$ if $\langle \Gamma, r \rangle \preceq \langle \nabla_1, t \rangle$ and $\langle \Gamma, r \rangle \preceq \langle \nabla_2, s \rangle$. It is the least general generalization, (lgg in short) of $\langle \nabla_1, t \rangle$ and $\langle \nabla_2, s \rangle$ if there is no generalization $\langle \Gamma', r' \rangle$ of $\langle \nabla_1, t \rangle$ and $\langle \nabla_2, s \rangle$ which satisfies $\langle \Gamma, r \rangle \prec \langle \Gamma', r' \rangle$.

Note that if we have infinite number of atoms in the language, the relation \prec is not well-founded: $\langle \emptyset, X \rangle \prec \langle \{a\#X\}, X \rangle \prec \langle \{a\#X, b\#X\}, X \rangle \prec \dots$. As a consequence, two terms-in-context may not have an lgg and not even a minimal complete set of generalizations:⁴

Example 2. Let $p_1 = \langle \emptyset, a_1 \rangle$ and $p_2 = \langle \emptyset, a_2 \rangle$ be two terms-in-context. Then in any complete set of generalizations of p_1 and p_2 there is an infinite chain $\langle \emptyset, X \rangle \prec \langle \{a_3\#X\}, X \rangle \prec \langle \{a_3\#X, a_4\#X\}, X \rangle \prec \dots$, where $\{a_1, a_2, a_3, \dots\}$ is the set of all atoms of the language. Hence, p_1 and p_2 do not have a minimal complete set of generalizations.

This example is a proof of the theorem, which characterizes the generalization type of nominal anti-unification:⁵

Theorem 2. The problem of anti-unification for terms-in-context is of nullary type.

However, if we restrict the set of atoms which can be used in the generalizations to be fixed and finite, then the anti-unification problem becomes unitary. (We do not prove this property here, it will follow from the Theorems 4 and 5 in Sect. 6.)

⁴ Minimal complete sets of generalizations are defined in the standard way. For a precise definition, see, e.g., [1, 16].

⁵ Generalization types are defined analogously to unification types: unary, finitary, infinitary, and nullary, see [16].

Definition 3. We say that a term t (resp., a freshness context ∇) is based on a set of atoms A iff $\text{Atoms}(t) \subseteq A$ (resp., $\text{Atoms}(\nabla) \subseteq A$). A term-in-context $\langle \nabla, t \rangle$ is based on A if both t and ∇ are based on it. We extend the notion of A -basedness to permutations, calling π A -based if it contains only atoms from A . Such a permutation defines a bijection, in particular, from A to A . If p_1 and p_2 are A -based terms-in-context, then their A -based generalizations are terms-in-context which are generalizations of p_1 and p_2 and are based on A . An A -based lgg of A -based terms-in-context p_1 and p_2 is a term-in-context p , which is an A -based generalization of p_1 and p_2 and there is no A -based generalization p' of p_1 and p_2 which satisfies $p \prec p'$.

The problem we would like to solve is the following:

- Given:** Two nominal terms t and s of the same sort, a freshness context ∇ , and a finite set of atoms A such that t , s , and ∇ are based on A .
- Find:** A term r and a freshness context Γ , such that the term-in-context $\langle \Gamma, r \rangle$ is an A -based least general generalization of the terms-in-context $\langle \nabla, t \rangle$ and $\langle \nabla, s \rangle$.

Our anti-unification problem is parametric on the set of atoms we consider as the base, and finiteness of this set is essential to ensure the existence of an lgg.

3 Motivation of Using a Direct Nominal Anti-Unification Algorithm

In [19], relation between nominal unification (NU) and higher-order pattern unification (HOPU) has been studied. In particular, it was shown how to translate NU problems into HOPU problems and how to obtain nominal unifiers back from higher-order pattern unifiers. It is tempting to use the same translation for nominal anti-unification (NAU), using the algorithm from [3] to solve higher-order anti-unification problems over patterns (HOPAU), but it turns out that the generalization computed in this way is not always based on the same set of the atoms as the input:

Example 3. We consider the following problem: Let the set of atoms be $A_1 = \{a, b\}$. The terms to be generalized are $a.b$ and $b.a$, and the freshness context is $\nabla = \emptyset$. According to [19], translation to higher-order patterns gives the anti-unification problem $\lambda a, b, a. b \triangleq \lambda a, b, b. a$, whose lgg is $\lambda a, b, c. X(a, b)$. However, we can not translate this lgg back to an A_1 -based term-in-context, because it contains more bound variables than there are atoms in A_1 .

On the other hand, the translation would work for the set of atoms $A_2 = \{a, b, c\}$: Back-translating $\lambda a, b, c. X(a, b)$ gives the A_2 -based lgg $\langle \{c\#X\}, c.X \rangle$.

The reason why the translation-based approach does not work for A -based NAU is that A is finite, while in higher-order anti-unification there is an infinite

supply of fresh (bound) variables. If we assumed A to be infinite, there would still be a mismatch between NAU and the corresponding HOPAU: NAU, as we saw, is nullary in this case, while HOPAU is unitary. The reason of this contrast is that from infinitely many nominal generalizations, there is only one which is a well-typed higher-order generalization.

One might think that the translation-based approach would still work, if one considers only nominal anti-unification problems where the set of atoms is large enough for the input terms-in-context. However, there is a reason that speaks against NAU-to-HOPAU translation: complexity. The translation approach leads to a quadratic increase of the input size (Lemma 5.6 in [19]). The HOPAU algorithm in [3] runs in cubic time with respect to the size of its input. Hence, the translation-based approach leads to an algorithm with runtime complexity $O(n^6)$. In contrast, the algorithm developed in this paper has runtime complexity $O(n^5)$, and requires no back and forth translations.

4 The Lattice of More General Terms-In-Context

The notion of *more general term* defines an order relation between classes of terms (modulo some notion of *variable renaming*). In most cases, we have actually a meet-semilattice, since, given two terms, there always exists a greatest lower bound (meet) that corresponds to their anti-unifier. On the contrary, the least upper bound (join) of two terms only exists if they are unifiable. For instance, the two first-order terms $f(a, X_1)$ and $f(X_2, b)$ have a meet $f(Y_1, Y_2)$, and, since they are unifiable, also a join $f(a, b)$. Notice that unifiability and existence of a join are equivalent if both terms do not share variables (for instance $f(a, X)$ and $f(X, b)$ are both smaller than $f(a, b)$, hence joinable, but they are not unifiable). With this restriction one does not lose generality: The unification problem $t_1 \approx^? t_2$ (sharing variables), can be reduced to $f(t_1, t_2) \approx^? f(X, X)$ (not sharing variables), where f is some binary symbol and X a fresh variable. Therefore, in the first-order case, the problem of searching a most general unifier is equivalent to the search of the join of two terms, and the search of a least general generalization to the search of the meet. Notice that meet and join are unique up to some notion of *variable renaming*. For instance, the join of $f(a, X, X')$ and $f(Y, b, Y')$ is $f(a, b, Z)$ for any renaming of Z by any variable.

In the nominal case, we consider the set of terms-in-context (modulo variable renaming) with the more general relation. The following lemma establishes a correspondence between joinability and unifiability.

Lemma 2. *Given two terms-in-context $\langle \nabla_1, t_1 \rangle$ and $\langle \nabla_2, t_2 \rangle$ with disjoint sets of variables, $\langle \nabla_1, t_1 \rangle$ and $\langle \nabla_2, t_2 \rangle$ are joinable if, and only if, $\{t_1 \approx^? t_2\} \cup \nabla_1 \cup \nabla_2$ has a solution (is unifiable).*

Proof. If they are joinable, there exist $\langle \Gamma, s \rangle$ such that $\langle \nabla_i, t_i \rangle \preceq \langle \Gamma, s \rangle$. Hence, there exist σ_1 and σ_2 such that σ_i respects ∇_i and $\nabla_i \sigma_i \subseteq \Gamma$ and $\Gamma \vdash t_i \sigma_i \approx s$. Now, since $\text{Vars}(t_1) \cap \text{Vars}(t_2) = \emptyset$, define $\sigma(X) = \sigma_1(X)$, if $X \in \text{Vars}(t_1)$, and $\sigma(X) = \sigma_2(X)$, if $X \in \text{Vars}(t_2)$. We have $\Gamma \vdash t_1 \sigma \approx s \approx t_2 \sigma$ and $(\nabla_1 \cup \nabla_2) \sigma \subseteq$

Γ . Hence, according to the properties of FC as stated in Theorem 1, we have $\Gamma \vdash a\#X\sigma$, for any $a\#X \in \nabla_1 \cup \nabla_2$. Therefore, the pair $\langle \Gamma, \sigma \rangle$ is a nominal unifier of $\{t_1 \approx^? t_2\} \cup \nabla_1 \cup \nabla_2$.

Conversely, we can prove easily that if $\langle \Gamma, \sigma \rangle$ solves $\{t_1 \approx^? t_2\} \cup \nabla_1 \cup \nabla_2$, then $\langle \nabla_i, t_i \rangle \preceq \langle \Gamma, s \rangle$, where $s = t_1\sigma$. Let's prove now that when $\langle \Gamma, \sigma \rangle$ is the most general nominal unifier, then $\langle \Gamma, s \rangle$ is the join of $\langle \nabla_1, t_1 \rangle$ and $\langle \nabla_2, t_2 \rangle$, i.e. whenever $\langle \nabla_i, t_i \rangle \preceq \langle \Gamma', s' \rangle$, we have $\langle \Gamma, s \rangle \preceq \langle \Gamma', s' \rangle$:

From $\langle \nabla_i, t_i \rangle \preceq \langle \Gamma', s' \rangle$ we have that there exists σ' such that $\langle \Gamma', \sigma' \rangle$ is a nominal unifier of $\{t_1 \approx^? t_2\} \cup \nabla_1 \cup \nabla_2$ and $\Gamma' \vdash t_1\sigma' \approx s' \approx t_2\sigma'$. Since $\langle \Gamma, \sigma \rangle$ is most general, there exists a substitution φ , which respects Γ , such that $\Gamma\varphi \subseteq \Gamma'$ and $\Gamma' \vdash t_i\sigma' \approx t_j\sigma\varphi$, for all $i, j \in \{1, 2\}$. The existence of this substitution φ proves $\langle \Gamma, s \rangle \preceq \langle \Gamma', s' \rangle$. \square

Like in first-order unification, the previous lemma allows us to reduce any nominal unification problem $P = \{a_1\#u_1, \dots, a_m\#u_m, t_1 \approx s_1, \dots, t_n \approx s_n\}$ into the joinability of the two terms-in-context $\langle \emptyset, f(X, X) \rangle$ and $\langle \text{FC}(\{a_1\#u_1, \dots, a_m\#u_m\}), f(g(t_1, \dots, t_n), g(s_1, \dots, s_n)) \rangle$ where f and g are any appropriate function symbols, and X is a fresh variable.

The nominal anti-unification problem is already stated in terms of finding the meet of two terms-in-context, with the only proviso that all terms and contexts must be based on some finite set of atoms.

5 Nominal Anti-Unification Algorithm

The triple $X : t \triangleq s$, where X, t, s have the same sort, is called the *anti-unification triple*, shortly AUT, and the variable X is called a *generalization variable*. We say that a set of AUTs P is based on a finite set of atoms A , if for all $X : t \triangleq s \in P$, the terms t and s are based on A .

Definition 4. *The nominal anti-unification algorithm is formulated in a rule-based way working on tuples $P; S; \Gamma; \sigma$ and two global parameters A and ∇ , where*

- P and S are sets of AUTs such that if $X : t \triangleq s \in P \cup S$, then this is the sole occurrence of X in $P \cup S$;
- P is the set of AUTs to be solved;
- A is a finite set of atoms;
- The freshness context ∇ does not constrain generalization variables;
- S is a set of already solved AUTs (the store);
- Γ is a freshness context (computed so far) which constrains generalization variables;
- σ is a substitution (computed so far) mapping generalization variables to nominal terms;
- P, S, ∇ , and Γ are A -based.

We call such a tuple a state. The rules below operate on states.

Dec: Decomposition

$$\begin{aligned} & \{X : h(t_1, \dots, t_m) \triangleq h(s_1, \dots, s_m)\} \cup P; S; \Gamma; \sigma \\ & \implies \{Y_1 : t_1 \triangleq s_1, \dots, Y_m : t_m \triangleq s_m\} \cup P; S; \Gamma; \sigma \{X \mapsto h(Y_1, \dots, Y_m)\}, \end{aligned}$$

where h is a function symbol or an atom, Y_1, \dots, Y_m are fresh variables of the corresponding sorts, $m \geq 0$.

Abs: Abstraction

$$\{X : a.t \triangleq b.s\} \cup P; S; \Gamma; \sigma \implies \{Y : (ca) \bullet t \triangleq (cb) \bullet s\} \cup P; S; \Gamma; \sigma \{X \mapsto c.Y\},$$

where Y is fresh, $c \in A$, $\nabla \vdash c \# a.t$ and $\nabla \vdash c \# b.s$.

Sol: Solving

$$\{X : t \triangleq s\} \cup P; S; \Gamma; \sigma \implies P; S \cup \{X : t \triangleq s\}; \Gamma \cup \Gamma'; \sigma,$$

if none of the previous rules is applicable, i.e. one of the following conditions hold:

- (a) both terms have distinct heads: $\text{Head}(t) \neq \text{Head}(s)$, or
- (b) both terms are suspensions: $t = \pi_1 \cdot Y_1$ and $s = \pi_2 \cdot Y_2$, where π_1, π_2 and Y_1, Y_2 are not necessarily distinct, or
- (c) both are abstractions and rule **Abs** is not applicable: $t = a.t'$, $s = b.s'$ and there is no atom $c \in A$ satisfying $\nabla \vdash c \# a.t'$ and $\nabla \vdash c \# b.s'$.

The set Γ' is defined as $\Gamma' := \{a \# X \mid a \in A \wedge \nabla \vdash a \# t \wedge \nabla \vdash a \# s\}$.

Mer: Merging

$$\begin{aligned} & P; \{X : t_1 \triangleq s_1, Y : t_2 \triangleq s_2\} \cup S; \Gamma; \sigma \implies \\ & P; \{X : t_1 \triangleq s_1\} \cup S; \Gamma \{Y \mapsto \pi \cdot X\}; \sigma \{Y \mapsto \pi \cdot X\}, \end{aligned}$$

where π is an $\text{Atoms}(t_1, s_1, t_2, s_2)$ -based permutation such that $\nabla \vdash \pi \bullet t_1 \approx t_2$, and $\nabla \vdash \pi \bullet s_1 \approx s_2$.

The rules transform states to states. One can easily observe this by inspecting the rules.

Given a finite set of atoms A , two nominal A -based terms t and s , and an A -based freshness context ∇ , to compute A -based generalizations for $\langle \nabla, t \rangle$ and $\langle \nabla, s \rangle$, we start with $\{X : t \triangleq s\}; \emptyset; \emptyset; \varepsilon$, where X is a fresh variable, and apply the rules as long as possible. We denote this procedure by \mathfrak{N} . A *Derivation* is a sequence of state transformations by the rules. The state to which no rule applies has the form $\emptyset; S; \Gamma; \varphi$, where **Mer** does not apply to S . We call it the *final state*. When \mathfrak{N} transforms $\{X : t \triangleq s\}; \emptyset; \emptyset; \varepsilon$ into a final state $\emptyset; S; \Gamma; \varphi$, we say that the *result computed* by \mathfrak{N} is $\langle \Gamma, X\varphi \rangle$.

Note that the **Dec** rule works also for the AUTs of the form $X : a \triangleq a$. In the **Abs** rule, it is important to have the corresponding c in A . If we take $A = A_2$ in Example 3, then **Abs** can transform the AUT between t and s there, but if $A = A_1$ in the same example, then **Abs** is not applicable. In this case the **Sol** rule takes over, because the condition (c) of this rule is satisfied.

The condition (b) of **Sol** helps to compute, e.g, $\langle \emptyset, X \rangle$ for identical terms-in-context $\langle \emptyset, (ab) \cdot Y \rangle$ and $\langle \emptyset, (a b) \cdot Y \rangle$. Although one might expect that computing $\langle \emptyset, (a b) \cdot Y \rangle$ would be more natural, from the generalization point of view it does not matter, because $\langle \emptyset, X \rangle$ is as general as $\langle \emptyset, (a b) \cdot Y \rangle$.

Example 4. We illustrate \mathfrak{N} with the help of some examples:

- Let $t = f(a, b)$, $s = f(b, c)$, $\nabla = \emptyset$, and $A = \{a, b, c, d\}$. Then \mathfrak{N} performs the following transformations:

$$\begin{aligned} & \{X : f(a, b) \triangleq f(b, c)\}; \emptyset; \emptyset; \varepsilon \Longrightarrow_{\text{Dec}} \\ & \{Y : a \triangleq b, Z : b \triangleq c\}; \emptyset; \emptyset; \{X \mapsto f(Y, Z)\} \Longrightarrow_{\text{Sol}}^2 \\ & \emptyset; \{Y : a \triangleq b, Z : b \triangleq c\}; \{c\#Y, d\#Y, a\#Z, d\#Z\}; \{X \mapsto f(Y, Z)\} \Longrightarrow_{\text{Mer}} \\ & \emptyset; \{Y : a \triangleq b\}; \{c\#Y, d\#Y\}; \{X \mapsto f(Y, (ab)(bc) \cdot Y)\} \end{aligned}$$

Hence, $p = \langle \{c\#Y, d\#Y\}, f(Y, (ab)(bc) \cdot Y) \rangle$ is the computed result. It generalizes the input pairs: $p\{Y \mapsto a\} \preceq \langle \nabla, t \rangle$ and $p\{Y \mapsto b\} \preceq \langle \nabla, s \rangle$. The substitutions $\{Y \mapsto a\}$ and $\{Y \mapsto b\}$ can be read from the final store. Note that $\langle \{c\#Y\}, f(Y, (ab)(bc) \cdot Y) \rangle$ would be also an A -based generalization of $\langle \nabla, t \rangle$ and $\langle \nabla, s \rangle$, but it is strictly more general than p .

- Let $t = f(b, a)$, $s = f(Y, (ab) \cdot Y)$, $\nabla = \{b\#Y\}$, and $A = \{a, b\}$. Then \mathfrak{N} computes the term-in-context $\langle \emptyset, f(Z, (ab) \cdot Z) \rangle$. It generalizes the input pairs.
- Let $t = f(g(X), X)$, $s = f(g(Y), Y)$, $\nabla = \emptyset$, and $A = \emptyset$. It is a first-order anti-unification problem. \mathfrak{N} computes $\langle \emptyset, f(g(Z), Z) \rangle$. It generalizes the input pairs.
- Let $t = f(a.b, X)$, $s = f(b.a, Y)$, $\nabla = \{c\#X\}$, $A = \{a, b, c, d\}$. Then \mathfrak{N} computes the term-in-context $p = \langle \{c\#Z_1, d\#Z_1\}, f(c.Z_1, Z_2) \rangle$. It generalizes the input pairs: $p\{Z_1 \mapsto b, Z_2 \mapsto X\} = \langle \emptyset, f(c.b, X) \rangle \preceq \langle \nabla, t \rangle$ and $p\{Z_1 \mapsto a, Z_2 \mapsto Y\} = \langle \emptyset, f(c.a, Y) \rangle \preceq \langle \nabla, s \rangle$.

6 Properties of the Nominal Anti-Unification Algorithm

The Soundness Theorem states that the result computed by \mathfrak{N} is indeed an A -based generalization of the input terms-in-context:

Theorem 3 (Soundness of \mathfrak{N}). *Given terms t and s and a freshness context ∇ , all based on a finite set of atoms A , if $\{X : t \triangleq s\}; \emptyset; \emptyset; \varepsilon \Longrightarrow^+ \emptyset; S; \Gamma; \sigma$ is a derivation obtained by an execution of \mathfrak{N} , then $\langle \Gamma, X\sigma \rangle$ is an A -based generalization of $\langle \nabla, t \rangle$ and $\langle \nabla, s \rangle$.*

Proof. Since all atoms introduced by the rules of \mathfrak{N} in Γ and σ are from A , $\langle \Gamma, X\sigma \rangle$ is A -based. To prove that $\langle \Gamma, X\sigma \rangle$ generalizes both $\langle \nabla, t \rangle$ and $\langle \nabla, s \rangle$, we use well-founded induction on the length of derivations. In fact, we will prove a more general statement:

Assume $P_0; S_0; \Gamma_0; \vartheta_0 \Longrightarrow^+ \emptyset; S_n; \Gamma_n; \vartheta_0 \vartheta_1 \cdots \vartheta_n$ is a derivation in \mathfrak{N} (with ∇ and A) with the following property: If $Z_0 : t_0 \triangleq s_0 \in S_0$, then $a\#Z_0 \in \Gamma_0$ for an $a \in A$ iff $\nabla \vdash a\#t_0$ and $\nabla \vdash a\#s_0$. Notice that requiring this property does not imply a lose of generality: our algorithm starts with no equation in the store, and each time an equation is moved to the store the **Sol** rule adds the required freshness constraints (by inspection of **Sol**). Moreover, freshness constraints are only removed from the freshness context when **Mer** removes the corresponding equation from the store (by inspection of **Mer**). Then for any $Z_0 : t_0 \triangleq s_0 \in P_0 \cup S_0$ we have $\langle \Gamma_n \setminus \Gamma_0, Z_0 \vartheta_1 \cdots \vartheta_n \rangle \preceq \langle \nabla, t_0 \rangle$ and $\langle \Gamma_n \setminus \Gamma_0, Z_0 \vartheta_1 \cdots \vartheta_n \rangle \preceq \langle \nabla, s_0 \rangle$.

Assume the statement is true for any derivation of the length $l < n$ and prove it for a derivation $P_0; S_0; \Gamma_0; \vartheta_0 \Longrightarrow^+ \emptyset; S_n; \Gamma_n; \vartheta_0 \vartheta_1 \cdots \vartheta_n$ of the length n .

Below the composition $\vartheta_i \vartheta_{i+1} \cdots \vartheta_k$ is abbreviated as ϑ_i^k with $k \geq i$.

Let $Z_0 : t_0 \triangleq s_0$ be an AUT selected for transformation from $P_0 \cup S_0$. We consider each rule:

Dec: $Z_0 = X$, $t_0 = \mathbf{h}(t_1, \dots, t_m)$, $s_0 = \mathbf{h}(s_1, \dots, s_m)$, $\Gamma_1 = \Gamma_0$, and $\vartheta_1 = \{X \mapsto \mathbf{h}(Y_1, \dots, Y_m)\}$. By the induction hypothesis (IH), $\langle \Gamma_n \setminus \Gamma_1, Y_i \vartheta_2^n \rangle \preceq \langle \nabla, t_i \rangle$ and $\langle \Gamma_n \setminus \Gamma_1, Y_i \vartheta_2^n \rangle \preceq \langle \nabla, s_i \rangle$ for all $1 \leq i \leq m$. Hence by definition of \preceq for terms-in-context, there exist substitutions σ and φ such that:

- $\vartheta_2^n \sigma$ and $\vartheta_2^n \varphi$ respect $\Gamma_n \setminus \Gamma_1$,
- $(\Gamma_n \setminus \Gamma_1) \sigma \subseteq \nabla$ and $(\Gamma_n \setminus \Gamma_1) \varphi \subseteq \nabla$, and
- $\nabla \vdash Y_i \vartheta_2^n \sigma \approx t_i$ and $\nabla \vdash Y_i \vartheta_2^n \varphi \approx s_i$ for all $1 \leq i \leq m$.

Finally, since $\vartheta_1 = \{X \mapsto \mathbf{h}(Y_1, \dots, Y_m)\}$ and $\Gamma_n \setminus \Gamma_0 = \Gamma_n \setminus \Gamma_1$, we obtain $\langle \Gamma_n \setminus \Gamma_0, Z_0 \vartheta_1^n \rangle \preceq \langle \nabla, t_0 \rangle$ and $\langle \Gamma_n \setminus \Gamma_0, Z_0 \vartheta_1^n \rangle \preceq \langle \nabla, s_0 \rangle$.

Abs: $Z_0 = X$, $t_0 = a.t$, $s_0 = b.s$, $\Gamma_1 = \Gamma_0$, and $\vartheta_1 = \{X \mapsto c.Y\}$, where $\nabla \vdash c\#a.t$ and $\nabla \vdash c\#b.s$. P_1 contains the AUT $Y : (ca) \bullet t \triangleq (cb) \bullet s$. By the IH, $\langle \Gamma_n \setminus \Gamma_1, Y_i \vartheta_2^n \rangle \preceq \langle \nabla, (ca) \bullet t \rangle$ and $\langle \Gamma_n \setminus \Gamma_1, Y_i \vartheta_2^n \rangle \preceq \langle \nabla, (cb) \bullet s \rangle$ hence $\nabla \vdash Y \vartheta_2^n \sigma \approx (ca) \bullet t$ and $\nabla \vdash Y \vartheta_2^n \varphi \approx (cb) \bullet s$ for some σ and φ that in addition satisfy the other properties (as above) for \preceq . Then, since we also have that $\nabla \vdash c\#a.t$ and $\nabla \vdash c\#b.s$ we can prove, with the \approx -abs rules, that $\nabla \vdash c.Y \vartheta_2^n \sigma \approx a.t$ and $\nabla \vdash c.Y \vartheta_2^n \varphi \approx b.s$. Finally, since $\vartheta_1 = \{X \mapsto c.Y\}$ and $\Gamma_n \setminus \Gamma_0 = \Gamma_n \setminus \Gamma_1$, we obtain $\langle \Gamma_n \setminus \Gamma_0, Z_0 \vartheta_1^n \rangle \preceq \langle \nabla, t_0 \rangle$ and $\langle \Gamma_n \setminus \Gamma_0, Z_0 \vartheta_1^n \rangle \preceq \langle \nabla, s_0 \rangle$.

Sol: $Z_0 = X$, $t_0 = t$, $s_0 = s$, $\Gamma_1 \setminus \Gamma_0 = \{a\#X \mid a \in A, \nabla \vdash a\#t\}$, and $\nabla \vdash a\#s$ and $\vartheta_1 = \varepsilon$. By the IH we have that $(\Gamma_n \setminus \Gamma_1) \sigma \subseteq \nabla$, $(\Gamma_n \setminus \Gamma_1) \varphi \subseteq \nabla$, $\nabla \vdash X \vartheta_2^n \sigma \approx t$, and $\nabla \vdash X \vartheta_2^n \varphi \approx s$ for some σ and φ respecting $\Gamma_n \setminus \Gamma_1$.

Since $\vartheta_1 = \varepsilon$, from the IH we get $\nabla \vdash X \vartheta_1^n \sigma \approx t$. To show that $(\Gamma_n \setminus \Gamma_0) \sigma \subseteq \nabla$ take $a\#Y \in \Gamma_n \setminus \Gamma_0$ for some a .

- If $a\#Y \in \Gamma_n \setminus \Gamma_1$, then $\{a\#Y\} \sigma \subseteq \nabla$ by the IH,
- otherwise, if $a\#Y \notin \Gamma_n \setminus \Gamma_1$, then $a\#Y \in \Gamma_1 \setminus \Gamma_0$ with $X = Y$ and $X \vartheta_2^n = X$. By the IH, $\nabla \vdash X \sigma \approx t$, besides, we know $\nabla \vdash a\#t$. Therefore, we know

$\nabla \vdash a\#X\sigma$, which by Theorem 1 implies $\{a\#X\}\sigma = \{a\#Y\}\sigma \subseteq \nabla$. Thus, $(\Gamma_n \setminus \Gamma_0)\sigma \subseteq \nabla$.

Hence, we proved $\langle \Gamma_n \setminus \Gamma_0, X\vartheta_1^n \rangle \preceq \langle \nabla, t \rangle$, which is the same as $\langle \Gamma_n \setminus \Gamma_0, Z_0\vartheta_1^n \rangle \preceq \langle \nabla, t_0 \rangle$. $\langle \Gamma_n \setminus \Gamma_0, Z_0\vartheta_1^n \rangle \preceq \langle \nabla, s_0 \rangle$ can be proved analogously.

Mer: First, we show that the following holds : For all $k \geq 0$, If $Z_k : t_k \triangleq s_k \in S_k$ and $c\#Z_k \in \Gamma_k$ for a $c \in A$, then $\nabla \vdash c\#t_k$ and $\nabla \vdash c\#s_k$.

Proceed by induction on k . If $k = 0$, then it follows from the assumption on $P_0; S_0, \Gamma_0; \vartheta_0$. Assume it is true for k and show it for $k + 1$. Take $Z_{k+1} : t_{k+1} \triangleq s_{k+1} \in S_{k+1}$. We have two alternatives: Either $Z_{k+1} : t_{k+1} \triangleq s_{k+1}$ has been a subject of the Mer rule at this step, or not. If not, then either it was already in S_k or was introduced at this step. In either case, by IH or because it has been introduced with Sol rule, if $c\#Z_k \in \Gamma_k$ for a $c \in A$, then $\nabla \vdash c\#t_k$ and $\nabla \vdash c\#s_k$. If $Z_{k+1} : t_{k+1} \triangleq s_{k+1}$ was a subject of the Mer rule, then there exists some $U_k : r_k \triangleq q_k \in S_k$, such that $\nabla \vdash \pi_k \bullet t_{k+1} \approx r_k$, $\nabla \vdash \pi_k \bullet s_{k+1} \approx q_k$. Moreover, for all $d\#U_k \in S_k$ we now have $\pi_k^{-1} \bullet d\#Z_{k+1} \in S_{k+1}$, and all $c\#Z_{k+1} \in S_k$ are retained in S_{k+1} . For these c 's, since $Z_{k+1} : t_{k+1} \triangleq s_{k+1} \in S_k$, by the induction hypothesis we have $\nabla \vdash c\#t_{k+1}$ and $\nabla \vdash c\#s_{k+1}$. As for $\pi_k^{-1} \bullet d\#Z_{k+1} \in S_{k+1}$, here we need to show $\nabla \vdash \pi_k^{-1} \bullet d\#t_{k+1}$ and $\nabla \vdash \pi_k^{-1} \bullet d\#s_{k+1}$. By the induction hypothesis we know $\nabla \vdash d\#r_k$. Then $\nabla \vdash \pi_k^{-1} \bullet d\#\pi_k^{-1} \bullet r_k$ and since $\nabla \vdash \pi_k \bullet t_{k+1} \approx r_k$, we get $\nabla \vdash \pi_k^{-1} \bullet d\#t_{k+1}$. $\nabla \vdash \pi_k^{-1} \bullet d\#s_{k+1}$ can be shown similarly, using $\nabla \vdash d\#q_k$.

Now we turn to proving the Mer case itself. In this case, there exist $X : t_1 \triangleq s_1 \in S_0$, $Y : t_2 \triangleq s_2 \in S_0$, and π such that $\nabla \vdash \pi \bullet t_1 \approx t_2$ and $\nabla \vdash \pi \bullet s_1 \approx s_2$. Moreover, by the construction of the derivation, $X : t_1 \triangleq s_1$ is either retained in S_n , or is removed from there because there exist an AUT $Z : t_n \triangleq s_n \in S_n$ and a permutation ρ such that $\nabla \vdash \rho \bullet t_n \approx t_1$, $\nabla \vdash \rho \bullet s_n \approx s_1$, and $X\vartheta_1^n = \rho \cdot Z$. We can turn these two cases into one, permitting $Z = X$, $t_n = t_1$, $s_n = s_1$, and $\rho = Id$ to cover also the first case.

Therefore, we can say that there exists a AUT $Z : t_n \triangleq s_n \in S_n$ such that for some permutation ρ , $X\vartheta_1^n = X\vartheta_2^n = \rho \cdot Z$, $Y\vartheta_1^n = \pi \bullet X\vartheta_2^n = \pi\rho \cdot Z$, $\Gamma_n \setminus \Gamma_0 = \{(\pi\rho)^{-1}a\#Z \mid a\#Y \in \Gamma_0\}$, $\nabla \vdash \pi\rho \bullet t_n \approx t_2$, $\nabla \vdash \pi\rho \bullet s_n \approx s_2$, $\nabla \vdash \rho \bullet t_n \approx t_1$, and $\nabla \vdash \rho \bullet s_n \approx s_1$.

We want to prove $\langle \Gamma_n \setminus \Gamma_0, Z_0\vartheta_1^n \rangle \preceq \langle \nabla, t_0 \rangle$. First, we take σ such that $Z\sigma = t_n$ and show $(\Gamma_n \setminus \Gamma_0)\sigma \subseteq \nabla$. For this, we try to prove $\{b\#U\}\sigma \subseteq \nabla$ for all $b\#U \in \Gamma_n \setminus \Gamma_0$. By the IH, we have $\{b\#U\}\sigma \subseteq \nabla$ for all $b\#U \in \Gamma_n \setminus \Gamma_1$. Note that $Z\sigma = t_n$ does not restrict generality, because if $U = Z$, then by the proposition we proved at the beginning of the Mer case we have that $b\#U \in \Gamma_n \setminus \Gamma_1$ implies $\nabla \vdash b\#t_n$.

Therefore, $\{b\#U\}\sigma = FC(\{b\#Z\sigma\}) = FC(\{b\#t_n\})$ and by Theorem 1 we indeed have $\{b\#U\}\sigma \subseteq \nabla$. Now assume $b\#U \in (\Gamma_n \setminus \Gamma_0) \setminus (\Gamma_n \setminus \Gamma_1)$. Then $b\#U \in \Gamma_n \cap (\Gamma_1 \setminus \Gamma_0)$. That means, $b\#U = \pi^{-1} \bullet a\#X$, where $a\#Y \in \Gamma_0$. Moreover, the AUT $X : t_1 \triangleq s_1$ has been retained in S_n . From the latter we have, in fact, $Z = X$, $t_n = t_1$, and $s_n = s_1$. Then $\{b\#U\}\sigma = FC(\{\pi^{-1} \bullet a\#X\sigma \mid a\#Y \in \Gamma_0\}) = FC(\{a\#\pi \bullet t_1 \mid a\#Y \in \Gamma_0\})$. On the other hand, from the assumption on $P_0; S_0; \Gamma_0; \vartheta_0$ we know that for all $a\#Y \in \Gamma_0$ we have $\nabla \vdash a\#t_2$,

from which by $\nabla \vdash \pi \bullet t_1 \approx t_2$ we get $\nabla \vdash a\#\pi \bullet t_1$. Hence, we can apply Theorem 1 to $\text{FC}(\{a\#\pi \bullet t_1 \mid a\#Y \in I_0\})$, obtaining $\{b\#U\}\sigma \subseteq \nabla$ also in this case. Hence, $(\Gamma_n \setminus I_0)\sigma \subseteq \nabla$.

It remains to prove $\nabla \vdash Z_0\vartheta_1^n\sigma \approx t_0$. First, assume $Z_0 = X$, $t_0 = t_1$, $s_0 = s_1$. Then we have $\nabla \vdash Z_0\vartheta_2^n\sigma \approx t_0$, because $Z_0\vartheta_2^n\sigma = \rho \bullet Z\sigma = \rho \bullet t_n$ and we know that $\nabla \vdash \rho \bullet t_n \approx t_1$. Since $Z_0\vartheta_2^n = Z_0\vartheta_1^n$, we get $\nabla \vdash Z_0\vartheta_1^n\sigma \approx t_0$. Hence, we proved $\langle \Gamma_n \setminus I_0, Z_0\vartheta_1^n \rangle \preceq \langle \nabla, t_0 \rangle$ for this case. $\langle \Gamma_n \setminus I_0, Z_0\vartheta_1^n \rangle \preceq \langle \nabla, s_0 \rangle$ can be proved similarly.

Now let $Z_0 = Y$, $t_0 = t_2$, $s_0 = s_2$ and prove again $\nabla \vdash Z_0\vartheta_1^n\sigma \approx t_0$. Then $Z_0\vartheta_1^n\sigma = \pi\rho \bullet Z\sigma = \pi\rho \bullet t_n$. But we have already seen that $\nabla \vdash \pi\rho \bullet t_n \approx t_2$. Hence, $\nabla \vdash Z_0\vartheta_1^n\sigma \approx t_0$ is proved. It implies $\langle \Gamma_n \setminus I_0, Z_0\vartheta_1^n \rangle \preceq \langle \nabla, t_0 \rangle$ for this case. $\langle \Gamma_n \setminus I_0, Z_0\vartheta_1^n \rangle \preceq \langle \nabla, s_0 \rangle$ can be proved similarly. \square

The Completeness Theorem states that for any given A -based generalization of two input terms-in-context, \mathfrak{N} can compute one which is at most as general than the given one.

Theorem 4 (Completeness of \mathfrak{N}). *Given terms t and s and freshness contexts ∇ and Γ , all based on a finite set of atoms A . If $\langle \Gamma, r \rangle$ is an A -based generalization of $\langle \nabla, t \rangle$ and $\langle \nabla, s \rangle$, then there exists a derivation $\{X : t \triangleq s\}; \emptyset; \emptyset; \varepsilon \Longrightarrow^+ \emptyset; S; \Gamma'; \sigma$ obtained by an execution of \mathfrak{N} , such that $\langle \Gamma, r \rangle \preceq \langle \Gamma', X\sigma \rangle$.*

Proof. By structural induction on r . We can assume without loss of generality that $\langle \Gamma, r \rangle$ is an lgg of $\langle \nabla, t \rangle$ and $\langle \nabla, s \rangle$.

Let r be an atom a . Then $t = s = a$. Therefore, the Dec rule gives $\langle \emptyset, a \rangle$ as the computed answer. To show that $\langle \Gamma, a \rangle \preceq \langle \emptyset, a \rangle$, it is enough to take a substitution σ such that $X\sigma \neq b$ for each $b\#X \in \Gamma$. Note that it is not necessary $b \in A$.

Let r be an abstraction $c.r'$. Then $t = a.t'$, $s = b.s'$, $c \in A$, $\nabla \vdash c\#t$, $\nabla \vdash c\#s$, and $\langle \Gamma, r' \rangle$ is an A -based generalization of $\langle \nabla, t' \rangle$ and $\langle \nabla, s' \rangle$. In this case, the Abs rule can be applied, which gives $\{Y : (ca) \bullet t' \triangleq (cb) \bullet s'\}; \emptyset; \emptyset; \sigma_1$, where $\sigma_1 = \{X \mapsto c.Y\}$. By the induction hypothesis, we can compute Γ' and σ_2 such that $\langle \Gamma, r' \rangle \preceq \langle \Gamma', Y\sigma_2 \rangle$. Let $\sigma = \sigma_1\sigma_2$. We get $\langle \Gamma, r \rangle = \langle \Gamma, c.r' \rangle \preceq \langle \Gamma', c.Y\sigma_2 \rangle = \langle \Gamma', X\sigma \rangle$.

Let r be a suspension $\pi.Z$. Since $\langle \Gamma, r \rangle$ is an lgg of $\langle \nabla, t \rangle$ and $\langle \nabla, s \rangle$, the context Γ contains all constraints $\pi^{-1} \bullet a\#Z$ such that $\nabla \vdash a\#t$ and $\nabla \vdash a\#s$, and the following alternatives are possible:

- (a) t and s have distinct heads: $\text{Head}(t) \neq \text{Head}(s)$, or
- (b) t and s are both suspensions: $t = \pi_1.Y_1$ and $s = \pi_2.Y_2$, where π_1, π_2 and Y_1, Y_2 are not necessarily distinct, or
- (c) t and s are abstractions, but A does not contain an appropriate fresh atom to uniformly rename the bound atoms in t and s .

These alternatives give exactly the conditions of the Sol rule. Hence, we can apply it, getting $\emptyset; \{X : t \triangleq s\}; \Gamma'; \sigma$, where $\Gamma' = \{a\#X \mid a \in A \wedge \nabla \vdash a\#t \wedge \nabla \vdash a\#s\}$ and $\sigma = \varepsilon$. Then $\langle \Gamma, r \rangle \preceq \langle \Gamma', X\sigma \rangle$, which can be confirmed by the substitution $\{Z \mapsto \pi^{-1}.X\}$.

Let r be a term $f(r_1, \dots, r_n)$. Then $t = f(t_1, \dots, t_n)$, $s = f(s_1, \dots, s_n)$, and $\langle \Gamma, r_i \rangle$ is a generalization of $\langle \nabla, t_i \rangle$ and $\langle \nabla, s_i \rangle$. We proceed by the Dec rule, obtaining $\{Y_i : t_i \triangleq s_i \mid 1 \leq i \leq n\}; \emptyset; \emptyset; \{X \mapsto f(Y_1, \dots, Y_n)\}$. By the induction hypothesis, we can construct derivations D_1, \dots, D_n computing the substitutions $\sigma_1, \dots, \sigma_n$, respectively, such that $\langle \Gamma, r_i \rangle \preceq \langle \Gamma'_i, Y_i \sigma_i \rangle$ for $1 \leq i \leq n$. We combine these derivations, together with the initial Dec step, into one derivation of the form $D = \{X : t \triangleq s\}; S_0; \Gamma'_0; \sigma_0 \Longrightarrow \{Y_i : t_i \triangleq s_i \mid 1 \leq i \leq n\}; S_1; \Gamma'_1; \sigma_0 \sigma_1 \Longrightarrow^* \emptyset; S_n; \Gamma'_n; \sigma_0 \sigma_1 \cdots \sigma_n$, where $\Gamma'_0 = \Gamma'_1 = \emptyset$, $\sigma_0 = \varepsilon$, and $\sigma_1 = \{X \mapsto f(Y_1, \dots, Y_n)\}$. If r does not contain the same variable more than once, $\langle \Gamma, r_i \rangle \preceq \langle \Gamma'_i, Y_i \sigma_i \rangle$ for all $1 \leq i \leq n$ imply $\langle \Gamma, r \rangle = \langle \Gamma, f(r_1, \dots, r_n) \rangle \preceq \langle \Gamma', f(Y_1, \dots, Y_n) \rangle = \langle \Gamma', X \sigma \rangle$. If r contains the same variable at positions \mathfrak{p}_1 and \mathfrak{p}_2 (in subterms of the form $\pi_1 \cdot Z$ and $\pi_2 \cdot Z$), it indicates that

- (a) the path to \mathfrak{p}_1 is the same (modulo bound atom renaming) in t and s . It equals (modulo bound atom renaming) the path to \mathfrak{p}_1 in r , and
- (b) the path to \mathfrak{p}_2 is the same (modulo bound atom renaming) in t and s . It equals (modulo bound atom renaming) the path to \mathfrak{p}_2 in r .
- (c) there exists a substitution ϑ_1 , which respects Γ , such that $\Gamma \vdash \pi_1 \cdot Z \vartheta_1 \approx \tau_1 \bullet t|_{\mathfrak{p}_1}$ and $\Gamma \vdash \pi_2 \cdot Z \vartheta_1 \approx \tau_2 \bullet t|_{\mathfrak{p}_2}$, where τ_1 and τ_2 are permutations which rename atoms bound in t by fresh ones,
- (d) there exists a substitution ϑ_2 , which respects Γ , such that $\Gamma \vdash \pi_1 \cdot Z \vartheta_2 \approx \rho_1 \bullet s|_{\mathfrak{p}_1}$ and $\Gamma \vdash \pi_2 \cdot Z \vartheta_2 \approx \rho_2 \bullet s|_{\mathfrak{p}_2}$, where ρ_1 and ρ_2 are permutations which rename atoms bound in s by fresh ones,

Then, because of (a) and (b), we should have two AUTs in S_n : One, between (renamed variants of) $t|_{\mathfrak{p}_1}$ and $s|_{\mathfrak{p}_1}$, and the other one between (renamed variants of) $t|_{\mathfrak{p}_2}$ and $s|_{\mathfrak{p}_2}$. The possible renaming of bound atoms is caused by the fact that Abs might have been applied to obtain the AUTs. From (c) and (d) we know that $\tau_1, \tau_2, \rho_1, \rho_2$ are the names of those renaming permutations. Let those AUTs be $Z_1 : \tau_1 \bullet t|_{\mathfrak{p}_1} \triangleq \rho_1 \bullet s|_{\mathfrak{p}_1}$ and $Z_2 : \tau_2 \bullet t|_{\mathfrak{p}_2} \triangleq \rho_2 \bullet s|_{\mathfrak{p}_2}$.

From (c) we get $\Gamma \vdash Z \vartheta_1 \approx \pi_1^{-1} \tau_1 \bullet t|_{\mathfrak{p}_1}$ and $\Gamma \vdash Z \vartheta_1 \approx \pi_2^{-1} \tau_2 \bullet t|_{\mathfrak{p}_2}$, which imply $\Gamma \vdash \pi_1^{-1} \tau_1 \bullet t|_{\mathfrak{p}_1} \approx \pi_2^{-1} \tau_2 \bullet t|_{\mathfrak{p}_2}$ and, finally, $\Gamma \vdash \pi_2 \pi_1^{-1} \bullet \tau_1 \bullet t|_{\mathfrak{p}_1} \approx \tau_2 \bullet t|_{\mathfrak{p}_2}$. Similarly, from (d) we get $\Gamma \vdash \pi_2 \pi_1^{-1} \rho_1 \bullet s|_{\mathfrak{p}_1} \approx \rho_2 \bullet s|_{\mathfrak{p}_2}$.

That means, we can make the step with the Mer rule for $Z_1 : \tau_1 \bullet t|_{\mathfrak{p}_1} \triangleq \rho_1 \bullet s|_{\mathfrak{p}_1}$ and $Z_2 : \tau_2 \bullet t|_{\mathfrak{p}_2} \triangleq \rho_2 \bullet s|_{\mathfrak{p}_2}$ with the substitution $\sigma'_1 = \{Z_2 \mapsto \pi_2 \pi_1^{-1} \cdot Z_1\}$. We can repeat this process for all duplicated variables in r , extending D to the derivation $\{X : t \triangleq s\}; S_0; \Gamma'_0; \sigma_0 \Longrightarrow \{Y_i : t_i \triangleq s_i \mid 1 \leq i \leq n\}; S_1; \Gamma'_1; \sigma_0 \Longrightarrow^* \emptyset; S_n; \Gamma'_n; \sigma_0 \sigma_1 \cdots \sigma_n \Longrightarrow^+ \emptyset; S_{n+m}; \Gamma'_{n+m}; \sigma_0 \sigma_1 \cdots \sigma_n \sigma'_1 \cdots \sigma'_m$, where $\sigma'_1, \dots, \sigma'_m$ are substitutions introduced by the applications of the Mer rule. Let $\sigma = \sigma_0 \sigma_1 \cdots \sigma_n \sigma'_1 \cdots \sigma'_m$ and $\Gamma' = \Gamma'_{n+m}$. By this construction, we have $\langle \Gamma, r \rangle \preceq \langle \Gamma', X \sigma \rangle$, which finishes the proof. \square

Depending on the selection of AUTs to perform a step, there can be different derivations in \mathfrak{N} starting from the same AUT, leading to different generalizations. The next theorem states that all those generalizations are the same modulo variable renaming and α -equivalence.

Theorem 5 (Uniqueness Modulo \simeq). *Let t and s be terms and ∇ be a freshness context that are based on the same finite set of atoms. Let $\{X : t \triangleq s\}; \emptyset; \emptyset; \varepsilon \Longrightarrow^+ \emptyset; S_1; \Gamma_1; \sigma_1$ and $\{X : t \triangleq s\}; \emptyset; \emptyset; \varepsilon \Longrightarrow^+ \emptyset; S_2; \Gamma_2; \sigma_2$ be two maximal derivations in \mathfrak{N} . Then $\langle \Gamma_1, X\sigma_1 \rangle \simeq \langle \Gamma_2, X\sigma_2 \rangle$.*

Proof. It is not hard to notice that if it is possible to change the order of applications of rules (but sticking to the same selected AUTs for each rule) then the result remains the same (modulo fresh variable and atom names): Let D_1 and D_2 be two two-step derivations $D_1 = P_1; S_1; \Gamma_1; \sigma_1 \Longrightarrow_{R_1} P_2; S_2; \Gamma_2; \sigma_1\vartheta_1 \Longrightarrow_{R_2} P_3; S_3; \Gamma_3; \sigma_1\vartheta_1\vartheta_2$ and $D_2 = P_1; S_1; \Gamma_1; \sigma_1 \Longrightarrow_{R_2} P'_2; S'_2; \Gamma'_2; \sigma_1\vartheta_2 \Longrightarrow_{R_1} P'_3; S'_3; \Gamma'_3; \sigma_1\vartheta_2\vartheta_1$, where R_1 and R_2 are (not necessarily different) rules and each of them transforms *exactly the same AUT(s)* in both D_1 and D_2 . Then these AUT(s) are already present in $P_1 \cup S_1$: They are introduced neither by R_1 nor by R_2 . Therefore, $\text{Dom}(\vartheta_2) \cap \text{Ran}(\vartheta_1) = \text{Dom}(\vartheta_1) \cap \text{Ran}(\vartheta_2) = \emptyset$. Moreover, if we assume that the fresh variables and atoms introduced by the rules are the same in both derivations, then $P_3 = P'_3$, $S_3 = S'_3$, $\Gamma_3 = \Gamma'_3$, and $\sigma_1\vartheta_1\vartheta_2 = \sigma_1\vartheta_2\vartheta_1$.

Decomposition, Abstraction, and Solving rules transform the selected AUT in a unique way. We show that it is irrelevant in which order we decide equivariance in the Merging rule.

Let $P; \{Z : t_1 \triangleq s_1, Y : t_2 \triangleq s_2\} \cup S; \Gamma; \sigma \Longrightarrow P; \{Z : t_1 \triangleq s_1\} \cup S; \Gamma\{Y \mapsto \pi \cdot Z\}; \sigma\{Y \mapsto \pi \cdot Z\}$ be the merging step with $\nabla \vdash \pi \bullet t_1 \approx t_2$ and $\nabla \vdash \pi \bullet s_1 \approx s_2$. If we do it in the other way around, we would get the step $P; \{Z : t_1 \triangleq s_1, Y : t_2 \triangleq s_2\} \cup S; \Gamma; \sigma \Longrightarrow P; \{Y : t_2 \triangleq s_2\} \cup S; \Gamma\{Z \mapsto \pi^{-1} \cdot Y\}; \sigma\{Z \mapsto \pi^{-1} \cdot Y\}$.

Let $\vartheta_1 = \sigma\varphi_1$ with $\varphi_1 = \{Y \mapsto \pi \cdot Z\}$ and $\vartheta_2 = \sigma\varphi_2$ with $\varphi_2 = \{Z \mapsto \pi^{-1} \cdot Y\}$. Our goal is to prove that $\langle \Gamma\varphi_1, X\vartheta_1 \rangle \simeq \langle \Gamma\varphi_2, X\vartheta_2 \rangle$. For this, we need to prove both $\langle \Gamma\varphi_1, X\vartheta_1 \rangle \preceq \langle \Gamma\varphi_2, X\vartheta_2 \rangle$ and $\langle \Gamma\varphi_2, X\vartheta_2 \rangle \preceq \langle \Gamma\varphi_1, X\vartheta_1 \rangle$.

First, prove $\langle \Gamma\varphi_1, X\vartheta_1 \rangle \preceq \langle \Gamma\varphi_2, X\vartheta_2 \rangle$. We should find such a φ that $\Gamma\varphi_1\varphi \subseteq \Gamma\varphi_2$ and $\Gamma\varphi_2 \vdash X\vartheta_1\varphi \approx X\vartheta_2$.

Take $\varphi = \varphi_2$. Note that for any term t , we have $\Gamma\varphi_2 \vdash t\varphi_1\varphi_2 \approx t\varphi_2$, because $\varphi_1\varphi_2 = \{Z \mapsto \pi\pi^{-1} \cdot Y\}$ and we have $\Gamma\varphi_2 \vdash \pi\pi^{-1} \cdot Y \approx Y$. Therefore, $\Gamma\varphi_2 \vdash X\vartheta_1\varphi \approx X\sigma\varphi_1\varphi_2 \approx X\sigma\varphi_2 \approx X\vartheta_2$ holds.

As for $\Gamma\varphi_1\varphi \subseteq \Gamma\varphi_2$, note that φ_2 respects $\Gamma\varphi_1$, because it replaces a variable with a suspension and the FC algorithm will have to apply only **Sus-E** rule. We introduce notations Γ_U and $\overline{\Gamma}_U$ for any freshness context Γ and a variable U , denoting $\Gamma_U := \{a\#U \mid a\#U \in \Gamma\}$ and $\overline{\Gamma}_U := \Gamma \setminus \Gamma_U$. Then $\Gamma\varphi_1 = \overline{\Gamma}_Y \cup \Gamma_Y\varphi_1$ and $\Gamma\varphi_2 = \overline{\Gamma}_Z \cup \Gamma_Z\varphi_2$.

Under this notation, $\Gamma\varphi_1\varphi_2 = \overline{\Gamma}_Y\varphi_2 \cup \Gamma_Y\varphi_1\varphi_2$. Take $\overline{\Gamma}_Y\varphi_2$. We have $\overline{\Gamma}_Y\varphi_2 = (\overline{\Gamma}_Y \setminus (\overline{\Gamma}_Y)_Z) \cup ((\overline{\Gamma}_Y)_Z)\varphi_2 = (\overline{\Gamma}_Y \setminus \Gamma_Z) \cup \Gamma_Z\varphi_2$. Since $\Gamma_Z \cap \Gamma_Z\varphi_2 = \emptyset$, then we obtain $(\overline{\Gamma}_Y \setminus \Gamma_Z) \cup \Gamma_Z\varphi_2 = (\overline{\Gamma}_Y \cup \Gamma_Z\varphi_2) \setminus \Gamma_Z$. As for $\Gamma_Y\varphi_1\varphi_2$, it is easy to see that $\Gamma_Y\varphi_1\varphi_2 = \Gamma_Y$.

Hence, we get $\Gamma\varphi_1\varphi_2 = ((\overline{\Gamma}_Y \cup \Gamma_Z\varphi_2) \setminus \Gamma_Z) \cup \Gamma_Y$. Since $\Gamma_Z \cap \Gamma_Y = \emptyset$, we get $((\overline{\Gamma}_Y \cup \Gamma_Z\varphi_2) \setminus \Gamma_Z) \cup \Gamma_Y = ((\overline{\Gamma}_Y \cup \Gamma_Z\varphi_2) \cup \Gamma_Y) \setminus \Gamma_Z = (\Gamma \cup \Gamma_Z\varphi_2) \setminus \Gamma_Z$. Since $\Gamma_Z\varphi_2 \cap \Gamma_Z = \emptyset$, we get $(\Gamma \cup \Gamma_Z\varphi_2) \setminus \Gamma_Z = (\Gamma \setminus \Gamma_Z) \cup \Gamma_Z\varphi_2$. Hence, $\Gamma\varphi_1\varphi_2 = \overline{\Gamma}_Z \cup \Gamma_Z\varphi_2 = \Gamma\varphi_2$.

We proved $\langle \Gamma\varphi_1, X\vartheta_1 \rangle \preceq \langle \Gamma\varphi_2, X\vartheta_2 \rangle$. With a similar reasoning we can show $\langle \Gamma\varphi_2, X\vartheta_2 \rangle \preceq \langle \Gamma\varphi_1, X\vartheta_1 \rangle$. \square

Theorems 3, 4, and 5 imply that nominal anti-unification is unitary: For any A -based ∇ , t , and s , there exists an A -based lgg of $\langle \nabla, t \rangle$ and $\langle \nabla, s \rangle$, which is unique modulo \simeq and can be computed by the algorithm \mathfrak{N} .

Now we study how lgg's of terms-in-context depend on the set of atoms the terms-in-context are based on. The following lemma states the precise dependence.

Lemma 3. *Let A_1 and A_2 be two finite sets of atoms with $A_1 \subseteq A_2$ such that the A_1 -based terms-in-context $\langle \nabla, t \rangle$ and $\langle \nabla, s \rangle$ have an A_1 -based lgg $\langle \Gamma_1, r_1 \rangle$ and an A_2 -based lgg $\langle \Gamma_2, r_2 \rangle$. Then $\Gamma_2 \vdash r_1 \preceq r_2$.*

Proof. $\langle \Gamma_1, r_1 \rangle$ and $\langle \Gamma_2, r_2 \rangle$ are unique modulo \simeq . Let D_i be the derivation in \mathfrak{N} that computes $\langle \Gamma_i, r_i \rangle$, $i = 1, 2$. The number of atoms in A_1 and A_2 makes a difference in the rule **Abs**: If there are not enough atoms in A_1 , an **Abs** step in D_2 is replaced by a **Sol** step in D_1 . It means that for all positions \mathfrak{p} of r_1 , $r_2|_{\mathfrak{p}}$ is also defined. Moreover, there might exist a subterm $r_1|_{\mathfrak{p}}$, which has a form of suspension, while $r_2|_{\mathfrak{p}}$ is an abstraction. For such positions, $r_1|_{\mathfrak{p}} \preceq r_2|_{\mathfrak{p}}$. For the other positions \mathfrak{p}' of r_1 , $r_1|_{\mathfrak{p}'}$ and $r_2|_{\mathfrak{p}'}$ may differ only by names of generalization variables or by names of bound atoms.

Another difference might be in the application of **Sol** in both derivations: It can happen that this rule produces a larger Γ' in D_2 than in D_1 , when transforming the same AUT.

Hence, if there are positions $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ in r_1 such that $r_1|_{\mathfrak{p}_i} = \pi_i \cdot X$, then there exists a substitution φ_X such that $\Gamma_2 \vdash \pi_i \cdot X \varphi \approx r_2|_{\mathfrak{p}_i}$, $1 \leq i \leq n$. Taking the union of all φ_X 's where $X \in \text{Vars}(r_1)$, we get φ with the property $\Gamma_2 \vdash r_1 \varphi \approx r_2$. \square

Note that, in general, we can not replace $\Gamma_2 \vdash r_1 \preceq r_2$ with $\Gamma_2 \vdash r_1 \simeq r_2$ in Lemma 3. The following example illustrates this:

Example 5. Let $t = a.b$, $s = b.a$, $\nabla = \emptyset$, $A_1 = \{a, b\}$, and $A_2 = \{a, b, c\}$. Then for $\langle \nabla, t \rangle$ and $\langle \nabla, s \rangle$, $\langle \emptyset, X \rangle$ is an A_1 -based lgg and $\langle \{c\#X\}, c.X \rangle$ is an A_2 -based lgg. Obviously, $\{c\#X\} \vdash X \preceq c.X$ but not $\{c\#X\} \vdash c.X \preceq X$.

This example naturally leads to a question: Under which additional conditions can we have $\Gamma_2 \vdash r_1 \simeq r_2$ instead of $\Gamma_2 \vdash r_1 \preceq r_2$ in Lemma 3? To formalize a possible answer to it, we need some notation.

Let the terms t, s and the freshness context ∇ be based on the same set of atoms A . The maximal subset of A , *fresh* for t, s , and ∇ , denoted $\text{fresh}(A, t, s, \nabla)$, is defined as $A \setminus (\text{Atoms}(t, s) \cup \text{Atoms}(\nabla))$.

If $A_1 \subseteq A_2$ are two sets of atoms such that t, s, ∇ are at the same time based on both A_1 and A_2 , then $\text{fresh}(A_1, t, s, \nabla) \subseteq \text{fresh}(A_2, t, s, \nabla)$.

Let $\|t\|_{\text{Abs}}$ stand for the number of abstraction occurrences in t . $|A|$ stands for the cardinality of the set of atoms A . We say that a set of atoms A is *saturated* for A -based t, s and ∇ , if $|\text{fresh}(A, t, s, \nabla)| \geq \min\{\|t\|_{\text{Abs}}, \|s\|_{\text{Abs}}\}$.

The following lemma answers the question posed above:

Lemma 4. *Under the conditions of Lemma 3, if A_1 is saturated for t, s, ∇ , then $\Gamma_2 \vdash r_1 \simeq r_2$.*

Proof. Let D_i be the derivation in \mathfrak{N} that computes $\langle \Gamma_i, r_i \rangle$, $i = 1, 2$. Note that in each of these derivations, the number of **Abs** steps does not exceed $\min\{\|t\|_{\text{Abs}}, \|s\|_{\text{Abs}}\}$. Since A_1 is saturated for t, s, ∇ and $A_1 \subseteq A_2$, A_2 is also saturated for t, s, ∇ . Hence, whenever an AUT between two abstractions is encountered in the derivation D_i , there is always $c \in A_1$ available which satisfies the condition of the **Abs** rule. Therefore, such AU-E's are never transformed by **Sol**. We can assume without loss of generality that the sequence of steps in D_1 and D_2 are the same. We may also assume that we take the same fresh variables, and the same atoms from $\text{fresh}(A_1, t, s, \nabla)$ in the corresponding steps in D_1 and D_2 . Then the only difference between these derivations is in the Γ 's, caused by the **Sol** rule which might eventually make Γ_2 larger than Γ_1 . The σ 's computed by the derivations are the same and, therefore, r_1 and r_2 are the same (modulo the assumptions on the variable and fresh atom names). Hence, $\Gamma_2 \vdash r_1 \simeq r_2$. \square

In other words, this lemma answers the following pragmatic question: Given t, s and ∇ , how to choose a set of atoms A so that (a) t, s, ∇ are A -based and (b) in the A -based lgg $\langle \Gamma, r \rangle$ of $\langle \nabla, t \rangle$ and $\langle \nabla, s \rangle$, the term r generalizes s and t in the “best way”, maximally preserving similarities and uniformly abstracting differences between s and t . The answer is: Besides all the atoms occurring in t, s , or ∇ , A should contain at least m more atoms, where $m = \min\{\|t\|_{\text{Abs}}, \|s\|_{\text{Abs}}\}$.

Besides that, the lemma also gives the condition when the NAU-to-HOPAU translation can be used for solving NAU problems: The set of permitted atoms should be saturated.

7 Deciding Equivariance

Computation of π in the condition of the rule **Mer** above requires an algorithm that solves the following problem: Given nominal terms t, s and a freshness context ∇ , find an $\text{Atoms}(t, s)$ -based permutation π such that $\nabla \vdash \pi \bullet t \approx s$. This is the problem of deciding whether t and s are equivariant with respect to ∇ . In this Section we describe a rule-based algorithm for this problem, called \mathfrak{E} .

Note that our problem differs from the problem of equivariant unification considered in [8]: We do not solve unification problems, since we do not allow variable substitution. We only look for permutations to *decide equivariance constructively* and provide a dedicated algorithm for that.

The algorithm \mathfrak{E} works on tuples of the form $E; \nabla; A; \pi$ (also called states). E is a set of equivariance equations of the form $t \approx s$ where t, s are nominal terms, ∇ is a freshness context, and A is a finite set of atoms which are available for computing π . The latter holds the permutation to be returned in case of success.

The algorithm is split into two phases. The first one is a simplification phase where function applications, abstractions and suspensions are decomposed as

long as possible. The second phase is the permutation computation, where given a set of equivariance equations between atoms of the form $a \approx b$ we compute the permutation which will be returned in case of success. The rules of the first phase are the following:

Dec-E: Decomposition

$$\{f(t_1, \dots, t_m) \approx f(s_1, \dots, s_m)\} \cup E; \nabla; A; Id \Longrightarrow \{t_1 \approx s_1, \dots, t_m \approx s_m\} \cup E; \nabla; A; Id.$$

Alp-E: Alpha Equivalence

$$\{a.t \approx b.s\} \cup E; \nabla; A; Id \Longrightarrow \{(\acute{c} a) \bullet t \approx (\acute{c} b) \bullet s\} \cup E; \nabla; A; Id,$$

where \acute{c} is a fresh atom of the same sort as a and b .

Sus-E: Suspension

$$\{\pi_1.X \approx \pi_2.X\} \cup E; \nabla; A; Id \Longrightarrow \{\pi_1 \bullet a \approx \pi_2 \bullet a \mid a \in A \wedge a \# X \notin \nabla\} \cup E; \nabla; A; Id.$$

The rules of the second phase are the following:

Rem-E: Remove

$$\{a \approx b\} \cup E; \nabla; A; \pi \Longrightarrow E; \nabla; A \setminus \{b\}; \pi, \quad \text{if } \pi \bullet a = b.$$

Sol-E: Solve

$$\{a \approx b\} \cup E; \nabla; A; \pi \Longrightarrow E; \nabla; A \setminus \{b\}; (\pi \bullet a b)\pi,$$

if $\pi \bullet a, b \in A$ and $\pi \bullet a \neq b$.

Note that in Alp-E, \acute{c} is fresh means that $\acute{c} \notin A$ and, therefore, \acute{c} will not appear in π . These atoms are an auxiliary means which play a role during the computation but do not appear in the final result.

Given nominal terms t, s , freshness context ∇ , we construct a state $\{t \approx s\}; \nabla; \text{Atoms}(t, s); Id$. We will prove that when the rules transform this state into $\emptyset; \nabla; A; \pi$, then π is an $\text{Atoms}(t, s)$ -based permutation such that $\nabla \vdash \pi \bullet t \approx s$. When no rule is applicable, and the set of equations is not empty, we will also prove that there is no solution, hence we fail and return \perp .

Example 6. We illustrate the algorithm \mathfrak{E} on examples:

- Consider the equivariance problem $E = \{a \approx a, a.(ab)(cd).X \approx b.X\}$ and $\nabla = \{a \# X\}$:

$$\begin{aligned} &\{a \approx a, a.(ab)(cd).X \approx b.X\}; \{a \# X\}; \{a, b, c, d\}; Id \Longrightarrow_{\text{Alp-E}} \\ &\{a \approx a, (\acute{c} a)(ab)(cd).X \approx (\acute{c} b).X\}; \{a \# X\}; \{a, b, c, d\}; Id \Longrightarrow_{\text{Sus-E}} \\ &\{a \approx a, \acute{e} \approx \acute{e}, c \approx d, d \approx c\}; \{a \# X\}; \{a, b, c, d\}; Id \Longrightarrow_{\text{Rem-E}} \\ &\{\acute{e} \approx \acute{e}, c \approx d, d \approx c\}; \{a \# X\}; \{b, c, d\}; Id \Longrightarrow_{\text{Rem-E}} \\ &\{c \approx d, d \approx c\}; \{a \# X\}; \{b, c, d\}; Id \Longrightarrow_{\text{Sol-E}} \end{aligned}$$

$$\{d \approx c\}; \{a\#X\}; \{b, c\}; (cd) \Longrightarrow_{\text{Rem-E}} \\ \emptyset; \{a\#X\}; \{b\}; (cd).$$

- For $E = \{a.f(b, X) \approx b.f(a, X)\}$ and $\nabla = \{a\#X\}$, \mathfrak{E} returns \perp .
- For $E = \{a.f(b, (ab)\cdot X) \approx b.f(a, X)\}$ and $\nabla = \{a\#X\}$, \mathfrak{E} returns (ba) .
- For $E = \{a.b.(ab)(ac)\cdot X = b.a.(ac)\cdot X\}$ and $\nabla = \emptyset$, \mathfrak{E} returns Id .
- For $E = \{a.b.(ab)(ac)\cdot X = a.b.(bc)\cdot X\}$ and $\nabla = \emptyset$, \mathfrak{E} returns \perp .

The Soundness Theorem for \mathfrak{E} states that, indeed, the permutation the algorithm computes shows that the input terms are equivariant:

Theorem 6 (Soundness of \mathfrak{E}). *Let $\{t \approx s\}; \nabla; A; Id \Longrightarrow^* \emptyset; \nabla; B; \pi$ be a derivation in \mathfrak{E} , then π is an A -based permutation such that $\nabla \vdash \pi \bullet t \approx s$.*

Proof. We assume the success state with π being the computed permutation. Since Sol-E is the only rule which adds a new swapping to π and the swapped atoms are required to be from A , π is A -based.

The proof is by induction on the length of the derivation, and then, by case analysis on the applied rule. Let Γ be the freshness environment containing all statements $\acute{c}\#X$ form by a fresh atom \acute{c} introduced along all the derivation and a variable X of the initial equation.

For any transformation step $E; \nabla; A; \pi \Longrightarrow E'; \nabla; A'; \pi'$ we will prove that if $\nabla \cup \Gamma \vdash \pi' \bullet t'_i \approx s'_i$, for any $t'_i \approx s'_i \in E'$, then $\nabla \cup \Gamma \vdash \pi \bullet t_i \approx s_i$ for any $t_i \approx s_i \in E$, for any possible applied rule. By induction, we will have $\nabla \cup \Gamma \vdash \pi \bullet t \approx s$ for the initial equivariance equation $t \approx s$. Since Γ is not relevant to prove $t \approx s$, we have also $\nabla \vdash \pi \bullet t \approx s$.

Soundness of Dec-E: From $\nabla \vdash \pi \bullet t_1 \approx s_1, \dots, \nabla \vdash \pi \bullet t_n \approx s_n$, follows directly, by the theory of alpha-equivalence $\nabla \vdash f(\pi \bullet t_1, \dots, \pi \bullet t_n) \approx f(s_1, \dots, s_n)$ and by the rule of swapping application $\pi \bullet f(t_1, \dots, t_n) = f(\pi \bullet t_1, \dots, \pi \bullet t_n)$, that $\nabla \vdash \pi \bullet f(t_1, \dots, t_n) \approx f(s_1, \dots, s_n)$. In this case the permutation, the set of atoms and the freshness context are not transformed by the rule.

Soundness of Alp-E: Let ∇ be a freshness context containing Γ , in particular $\acute{c}\#X$ for any variable $X \in \text{Vars}(t, s)$. Assume $\nabla \vdash \pi(a\acute{c}) \bullet t \approx (b\acute{c}) \bullet s$ by induction hypothesis. From this, using \approx -abs-1 and the fact that π does not affect to \acute{c} , we can deduce $\nabla \vdash \pi \bullet c.(a\acute{c}) \bullet t \approx c.(b\acute{c}) \bullet s$. We can also construct a proof for $\nabla \vdash \acute{c}\#t$ and $\nabla \vdash \acute{c}\#s$. Therefore, using \approx -abs-2, we can deduce $\nabla \vdash \acute{c}.(a\acute{c}) \bullet t \approx a.t$ and $\nabla \vdash \acute{c}.(b\acute{c}) \bullet s \approx b.s$. Now using the lemmas about the transitivity of \approx and additivity of permutation application:

$$\text{If } \nabla \vdash t \approx s \text{ and } \nabla \vdash s \approx u \text{ then } \nabla \vdash t \approx u \\ \text{If } \nabla \vdash t \approx s \text{ then } \nabla \vdash \pi \bullet t \approx \pi \bullet s$$

we can deduce $\nabla \vdash \pi \bullet (a.t) \approx b.s$. This proof proves the soundness of Alp-E. Notice that π does not change in this rule.

Soundness of Sus-E: By induction hypothesis, assume $\nabla \vdash \pi \pi_1 \bullet a \approx \pi_2 \bullet a$, for any atom a such that $a \in A$ and $a\#X \notin \nabla$. Assume also $\Gamma \subset \nabla$, hence, for all fresh atoms, we have $\acute{c}\#X \in \nabla$. The rest of atoms b are not fresh and satisfy

$b \notin A$ and $b\#X \notin \nabla$. Since π_1 and π_2 only affect to atoms from A or fresh⁶, and π is A -based, we have $\pi \pi_1 \bullet b = \pi_2 \bullet b = b$. Therefore, $\nabla \vdash \pi \pi_1 \bullet a \approx \pi_2 \bullet a$, for any atom $a\#X \notin \nabla$, and by \approx -susp we deduce $\nabla \vdash \pi \pi_1 \cdot X \approx \pi_2 \cdot X$.

In the second phase we have to take into account that, in all derivations of the form $E; \nabla; A; \pi \Longrightarrow^* \emptyset; \nabla; B; \pi' \pi$, permutation π' only affects to atoms from A . This can be proved by inspection of the rules.

Soundness of Rem-E: Let be the complete derivation as follows

$$\begin{array}{l} \{a \approx b\} \cup E; \nabla; A; \pi \Longrightarrow \\ E; \nabla; A \setminus \{b\}; \pi \Longrightarrow^* \\ \emptyset; \nabla; B; \pi' \pi \end{array}$$

By induction hypothesis, $\pi' \pi$ solves E . Since the rule has been applied we also have $\pi \bullet a = b$. Now, the property above proves $\pi' \bullet b = b$, since $b \notin A \setminus \{b\}$. Therefore $\pi' \pi \bullet a = b$.

Soundness of Sol-E: let the derivation be:

$$\begin{array}{l} \{a \approx b\} \cup E; \nabla; A; \pi \Longrightarrow \\ E; \nabla; A \setminus \{b\}; (\pi \bullet a b) \pi \Longrightarrow^* \\ \emptyset; \nabla; B; \pi' (\pi \bullet a b) \pi \end{array}$$

By induction hypothesis, $\pi' (\pi \bullet a b) \pi$ solves E . Since $b \notin A \setminus \{b\}$, we have $\pi' \bullet b = b$. Hence $\pi' (\pi \bullet a b) \pi \bullet a = \pi' \bullet b = b$, and the computed permutation also solves the equivariance equation $a \approx b$. \square

We now prove an invariant lemma that is used in the proof of completeness Theorem 7.

Lemma 5 (Invariant Lemma). *Let A be a finite set of atoms, E_1 be a set of equivariance equations for terms based on A , π_1 be an A -based permutation and $A_1 \subseteq A$. Let $E_1; \nabla; A_1; \pi_1 \Longrightarrow E_2; \nabla; A_2; \pi_2$ be any step performed by a rule in \mathfrak{E} . Let $\Gamma = \{\acute{c}\#X \mid X \in \text{Vars}(E_1), \acute{c} \text{ is a fresh variable}\}$. Let μ be an A -based permutation such that $\nabla \cup \Gamma \vdash \mu \bullet t \approx s$, for all $t \approx s \in E_1$. Then*

1. $\nabla \cup \Gamma \vdash \mu \bullet t' \approx s'$, for all $t' \approx s' \in E_2$.
2. If $\mu^{-1} \bullet b = \pi_1^{-1} \bullet b$, for all $b \in A \setminus A_1$, then $\mu^{-1} \bullet b = \pi_2^{-1} \bullet b$, for all $b \in A \setminus A_2$.

Proof. By case distinction on the applied rule.

Dec-E: The proposition is obvious.

Alp-E: In this case it follows from the definitions of \approx and permutation application.

Sus-E: In this case $t = \tau_1 \cdot X$, $s = \tau_2 \cdot X$, and by the assumption we have $\nabla \vdash \mu \tau_1 \cdot X \approx \tau_2 \cdot X$. By the definition of \approx , it means that we have $a\#X \in \nabla$, for all atoms a such that $\mu \tau_1 \bullet a \neq \tau_2 \bullet a$. Hence, for all $a \in A$ with $a\#X \notin \nabla$ we have $\nabla \vdash \mu \tau_1 \bullet a \approx \tau_2 \bullet a$. This implies that μ also solves the equations in E_2 , hence item 1 of the lemma.

⁶ Notice that π_1 and π_2 can only contain swappings of the original equation (i.e. A -based) and swappings introduced by Alp-E.

Item 2 of the lemma is trivial for these three rules, since $A_1 = A_2$ and $\pi_1 = \pi_2 = Id$.

Rem-E: The item 1 is trivial. To prove the item 2, note that $t = a$, $s = b$, $\pi_1 = \pi_2$ and we only need to show $\mu^{-1} \bullet b = \pi_2^{-1} \bullet b$. By the assumption we have $\nabla \vdash \mu \bullet a \approx b$. Since a and b are atoms, the latter simply means that $\mu \bullet a = b$. From the rule condition we also know that $\pi_1 \bullet a = b$. From these two equalities we get $\mu^{-1} \bullet b = a = \pi_2^{-1} b$.

Sol-E: The item 1 is trivial also in this case. To prove the item 2, note that $t = a$, $s = b$, $\pi_2 = (\pi_1 \bullet a b)\pi_1$ and we only need to show $\mu^{-1} \bullet b = \pi_2^{-1} \bullet b$. By the assumption we have $\nabla \vdash \mu \bullet a \approx b$, which means that $\mu \bullet a = b$ and, hence, $a = \mu^{-1} \bullet b$. As for $\pi_2^{-1} \bullet b$, we have $\pi_2^{-1} \bullet b = \pi_1^{-1}(\pi_1 \bullet a b) \bullet b = \pi_1^{-1} \bullet (\pi_1 \bullet a) = a$. Hence, we get $\mu^{-1} \bullet b = a = \pi_2^{-1} \bullet b$. \square

Theorem 7 (Completeness of \mathfrak{E}). *Let A be a finite set of atoms, t, s be A -based terms, and ∇ be a freshness context. If $\nabla \vdash \mu \bullet t \approx s$ holds for some A -based permutation μ , then there exists a derivation $\{t \approx s\}; \nabla; A; Id \Longrightarrow^* \emptyset; \Gamma; B; \pi$, obtained by an execution of \mathfrak{E} , such that $\pi \bullet a = \mu \bullet a$ for any atom $a \in FA(t)$.*

Proof. First show that under the conditions of the theorem, if $\{t \approx s\}; \nabla; A; Id \Longrightarrow^* \emptyset; \Gamma; B; \pi$ is a derivation obtained by an execution of \mathfrak{E} , then $\pi \bullet a = \mu \bullet a$, for any atom $a \in FA(t)$. Afterwards we prove that (under the conditions of the theorem) there is no failing derivation with the rules of \mathfrak{E} starting from $\{t \approx s\}; \nabla; A; Id$. Since all derivations are finite, it will imply the existence of $\{t \approx s\}; \nabla; A; Id \Longrightarrow^* \emptyset; \Gamma; B; \pi$.

Let $\{t \approx s\}; \nabla; A; Id \Longrightarrow^* E'; \Gamma'; B'; \pi' \Longrightarrow^* \emptyset; \Gamma; B; \pi$ be a derivation, where $E'; \Gamma'; B'; \pi'$ is the first state in the second phase of the algorithm. It means that E' contains equations between atoms only, and the atoms of t (except, maybe, some bound ones which disappear after the application of the Alp-E rule) appear in the left hand sides of equations in E . By Lemma 5, $\Gamma' \vdash \mu \bullet a_1 \approx a_2$, for all $a_1 \approx a_2 \in E'$. By Theorem 6 and Lemma 5 the same is true for π . Therefore, $\Gamma' \vdash \mu \bullet a_1 \approx \pi \bullet a_1$, for all $a_1 \approx a_2 \in E'$. For atoms, $\nabla \vdash a \approx b$ iff $a = b$. Hence, we get $\mu \bullet a_1 = \pi \bullet a_1$, for all $a_1 \in S$, where $FA(t) \subseteq S \subseteq \text{Atoms}(t)$. It proves $\pi \bullet a = \mu \bullet a$, for all $a \in FA(t)$, when the desired successful derivation exists.

Now we show that no derivation with the rules of \mathfrak{E} starting from $\{t \approx s\}; \nabla; A; Id$ fails. Assume by contradiction that there exists such a failing derivation. Let $E'; \nabla'; A'; \pi'$ be the final state in it, to which no rule applies. Analyzing the rules in \mathfrak{E} , one can easily conclude that it can be caused by one of the following two cases:

1. E' contains an equivariance equation of the form $f(t_1, \dots, t_n) \approx g(s_1, \dots, s_m)$, where $f \neq g$.
2. E' contains an equivariance equation of the form $a \approx b$, where $\pi' \bullet a \neq b$, such that $\pi' \bullet a \notin A'$ or $b \notin A'$.

In the first case, by Lemma 5 $\nabla \vdash \mu \bullet f(t_1, \dots, t_n) \approx g(s_1, \dots, s_m)$ should hold, but $f \neq g$ forbids it. Hence, this case is impossible.

Now we analyze the second case. Consider each condition.

Condition 1: $\pi' \bullet a \notin A'$. Then either $\pi' \bullet a$ is a fresh atom, or $\pi' \bullet a \in A \setminus A'$.

- $\pi' \bullet a$ is a fresh atom: Since π' does not affect fresh atoms, we get $a \neq b$. On the other hand, we have $\Gamma' \vdash \mu \bullet a \approx b$ and, hence, $\mu \bullet a = b$, because $\mu \bullet a$ and b are atoms. Since μ is A -based, $b \notin A$ implies $a = b$. A contradiction.
- $\pi' \bullet a \in A \setminus A'$: By Lemma 5 we get $\mu^{-1}\pi' \bullet a = \pi'^{-1}\pi' \bullet a = a$. Therefore, $\pi' \bullet a = \mu \bullet a$ and we get $\mu \bullet a \neq b$, which contradicts $\Gamma' \vdash \mu \bullet a \approx b$, because $\mu \bullet a$ and b are atoms.

Condition 2: $b \notin A'$. Then either b is a fresh atom, or $b \in A \setminus A'$.

- b is a fresh atom: We obtain a contradiction by a reasoning similar to the case when $\pi' \bullet a$ is a fresh atom.
- $b \in A \setminus A'$: The atom b has been removed from the set of atoms in the derivation earlier either at Sus-E, Rem-E, or Sol-E step, which indicates that there is $c \in A \setminus A'$ such that $c = \pi'^{-1} \bullet b$. Moreover, $c \neq a$. From Lemma 5 we get $c = \mu^{-1} \bullet b$ which, together with $c \neq a$, implies $\mu \bullet a \neq b$. But it contradicts $\Gamma' \vdash \mu \bullet a \approx b$.

The obtained contradiction proves that no derivation with the rules of \mathfrak{E} starting from $\{t \approx s\}; \nabla; A; Id$ fails. \square

8 Complexity Analysis

We represent a permutations π as two hash tables. One for the permutation itself, we call it T_π , and one for the inverse of the permutation, called $T_{\pi^{-1}}$. The key of a hash tables is an atom and we associate another atom, the mapping, with it. For instance the permutation $\pi = (ab)(ac)$ is represented as $T_\pi = \{a \mapsto c, b \mapsto a, c \mapsto b\}$ and $T_{\pi^{-1}} = \{a \mapsto b, b \mapsto c, c \mapsto a\}$. We write $T_\pi(a)$ to obtain from the hash table T_π the atom which is associated with the key a . If no atom is associated with the key a then $T_\pi(a)$ returns a . We write $T_\pi(a \mapsto b)$, to set the mapping such that $T_\pi(a) = b$. As the set of atoms is small, we can assume a perfect hash function. It follows, that both defined operations are done in constant time, leading to constant time application of a permutation. Swapping application to a permutation $(ab)\pi$ is also done in constant time in the following way: Obtain $c = T_{\pi^{-1}}(a)$ and $d = T_{\pi^{-1}}(b)$ and perform the following updates:

- (a) $T_\pi(c \mapsto b)$ and $T_\pi(d \mapsto a)$,
- (b) $T_{\pi^{-1}}(b \mapsto c)$ and $T_{\pi^{-1}}(a \mapsto d)$.

We also represent set membership of atoms to a set of atoms A with a hash table \in_A from atoms to Booleans such that $\in_A(a) = true$ iff $a \in A$. We also have a list L_A of the atoms representing the entries of the table such that $\in_A(a) = true$ to easily know all atoms in A .

Finally we also represent set membership of freshness constraints to a freshness environment ∇ with a hash table \in_∇ .

Theorem 8. *Given a set of equivariance equations E , and a freshness context ∇ . Let m be the size of ∇ , and let n be the size of E . The algorithm \mathfrak{E} has $O(n^2 + m)$ time complexity.*

Proof. Collecting the atoms from E in a separate set A does not affect the space complexity and can be done in time $O(n)$. The freshness environment ∇ will not be modified by rule applications and membership test in the rule **Sus-E** can be done in constant time. We only have to construct the corresponding hash tables in time $O(m)$. We analyze complexity of both phases.

For the first phase, notice that all rules can be applied only $O(n)$ many times, since **Dec-E** removes two function symbols and **Alp-E** two abstraction, and **Sus-E** two suspensions. The resulting equations after this phase only contain atoms. However, notice that the size of these equations is not necessarily linear. Every time we apply **Alp-E** a new swapping is applied to both subterms. This swappings may increase the size of suspensions occurring below the abstraction. Since there are $O(n)$ many suspensions and $O(n)$ many abstractions, the final size of suspensions is $O(n^2)$. This is the size of the atom equations at the beginning of the second phase. We can see that the application of **Dec-E** rule has $O(1)$ time complexity (with the appropriate representation of equations).

The application of **Alp-E** rule requires to find a fresh atom not in A , this can be done in constant time. Later, a swapping has to be applied twice. Swapping application requires traversing the term hence has $O(n)$ time complexity. The application of **Sus-E** requires to traverse L_A ($O(n)$) and check for freshness membership in \in_{∇} ($O(1)$). Finally it has to add equations like $(\pi_1 \bullet a \approx \pi_2)$, this requires to build T_{π_1} and T_{π_2} that can be done in $O(n)$ time complexity and allow us to build each equation in $O(1)$ time. Summing up, this phase has $O(n^2)$ time complexity.

For the second phase, notice that both rules **Rem-E** and **Sol-E** remove an equation and do not introduce any other one. Hence, potentially having $O(n^2)$ many equations in this phase, these equations can be applied $O(n^2)$ many times. We construct a hash table T_{π} for π that will be maintained and used by both rules. Each application has time complexity $O(1)$. **Rem-E** uses T_{π} to check for applicability and if it is applied, it only removes b from A , hence updating \in_A (notice that we do not care about L_A in this second phase of the algorithm). **Sol-E** uses \in_A and T_{π} to check for applicability and if it is applied, it only removes b from A (hence updating \in_A), and updates T_{π} . Summing up, this phase maintains the overall $O(n^2)$ time complexity. \square

Theorem 9. *The nominal anti-unification algorithm \mathfrak{N} has $O(n^5)$ time complexity and $O(n^4)$ space complexity, where n is the input size.*

Proof. By design of the rules and theorem 5 we can arrange a maximal derivation like $\{X_0 : t_0 \triangleq s_0\}; \emptyset; \emptyset; \varepsilon \Longrightarrow_{\text{Dec, Abs, Sol}}^* \emptyset; S_l; \Gamma_l; \sigma_l \Longrightarrow_{\text{Mer}}^* \emptyset; S_m; \Gamma_m; \sigma_m$, postponing the application of **Mer** until the end. Rules **Dec**, **Abs** and **Sol** can be applied $O(n)$ many times. However, notice that every application of **Abs** may increase the size of every suspension below. Hence, the size of the store S_l is $O(n^2)$, although it only contain $O(n)$ equations, after an exhaustive derivation $\{X_0 : t_0 \triangleq s_0\}; \emptyset; \emptyset; \varepsilon \Longrightarrow_{\text{Dec, Abs, Sol}}^* \emptyset; S_l; \Gamma_l; \sigma_l$.

Now we turn to analyzing the transformation phase $\emptyset; S_l; \Gamma_l; \sigma_l \Longrightarrow_{\text{Mer}}^* \emptyset; S_m; \Gamma_m; \sigma_m$. Let $S_l = \{X_1 : t_1 \triangleq s_1, \dots, X_k : t_k \triangleq s_k\}$ and n_i be the size of

$X_i : t_i \triangleq s_i$, $1 \leq i \leq k$, then $\sum_{i=1}^k n_i = O(n^2)$ and $k = O(n)$. From theorem 8 we know that solving the equivariance problem for two AUPs $X_i : t_i \triangleq s_i$ and $X_j : t_j \triangleq s_j$ and an arbitrary freshness context ∇ requires $O((n_i + n_j)^2 + m)$ time and space, where m is the size of ∇ with $m = O(n)$.

Merging requires to solve this problem for each pair of AUPs. This leads to the time complexity $\sum_{i=1}^k \sum_{j=i+1}^k O((n_i + n_j)^2 + m) \leq O(\sum_{i=1}^k \sum_{j=1}^k (n_i + n_j)^2) + O(\sum_{i=1}^k \sum_{j=1}^k m)$. The second sum is $\sum_{i=1}^k \sum_{j=1}^k m = k^2 m = O(n^3)$. Now we estimate an upper bound for the sum $\sum_{i=1}^k \sum_{j=1}^k (n_i + n_j)^2 = \sum_{i=1}^k \sum_{j=1}^k n_i^2 + \sum_{i=1}^k \sum_{j=1}^k 2n_i n_j + \sum_{i=1}^k \sum_{j=1}^k n_j^2 \leq \sum_{i=1}^k k n_i^2 + 2 \left(\sum_{i=1}^k n_i \right) \left(\sum_{j=1}^k n_j \right) + \sum_{i=1}^k \left(\sum_{j=1}^k n_j \right)^2 \leq k \left(\sum_{i=1}^k n_i \right)^2 + 2O(n^2)O(n^2) + \sum_{i=1}^k O(n^2) = kO(n^2)^2 + 2O(n^2)^2 + kO(n^2)^2 = O(n^5)$, resulting into the stated bounds.

The space is bounded by the space required by a single call to the equivariance algorithm with an input of size $O(n^2)$, hence $O(n^4)$. \square

9 Conclusion

The problem of anti-unification for nominal terms-in-context is sensitive to the set of atoms permitted in generalizations: If this set is infinite, there is no least general generalization. Otherwise there exists a unique lgg. If this set is finite and satisfies the notion of being saturated, defined in the paper, then the lgg retains the common structure of the input nominal terms maximally.

We illustrated that, similar to some other theories where unification, generalization, and the subsumption relation are defined, the nominal terms-in-contexts form a join-meet lattice with respect to the subsumption relation, where the existence of join is unifiability, and the meet corresponds to least general generalization.

We designed an anti-unification algorithm for nominal terms-in-context. It contains a subalgorithm that constructively decides whether two terms are equivariant with respect to the given freshness context. We proved termination, soundness, and completeness of these algorithms, investigated their complexities, and implemented them. Given a fixed set of atoms A , the nominal anti-unification algorithm computes a least general A -based term-in-context generalization of the given A -based terms-in-context, and requires $O(n^5)$ time and $O(n^4)$ space for that, where n is the size of the input. The computed lgg is unique modulo α -equivalence and variable renaming.

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