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Sophisticated computer algebra technologies for Feynman integrals

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joint with J. Blümlein, J. Ablinger, A. Behring, A. De Freitas,
A. Hasselhuhn, C. Raab, M. Round, F. Wißbrock (DESY and/or RISC)

The general tactic

Feynman integrals

The general tactic

Feynman integrals

$\downarrow$ non-trivial transformations (DESY)

multiple sums

The general tactic

Feynman integrals

$\downarrow$ non-trivial transformations (DESY)

multiple sums

$\downarrow$ symbolic summation

compact expression in terms of special functions

$$F(\varepsilon,N)=\int\frac{d^{4+\varepsilon}k_1}{(2\pi)^{4+\varepsilon}}\frac{d^{4+\varepsilon}k_2}{(2\pi)^{4+\varepsilon}}\frac{d^{4+\varepsilon}k_3}{(2\pi)^{4+\varepsilon}}\frac{(\Delta.k_3)^N}{k_2^4((k_1-k_3)^2-m^2)(k_1-k_2)^2((k_3-p)^2-m^2)}$$

$$||?$$

$$F_{-3}(N)\varepsilon^{-3}+F_{-2}(N)\varepsilon^{-2}+F_{-1}(N)\varepsilon^{-1}+F_0(N)\varepsilon^0+\dots$$

$$F(\varepsilon,N)=\int\frac{d^{4+\varepsilon}k_1}{(2\pi)^{4+\varepsilon}}\frac{d^{4+\varepsilon}k_2}{(2\pi)^{4+\varepsilon}}\frac{d^{4+\varepsilon}k_3}{(2\pi)^{4+\varepsilon}}\frac{(\Delta.k_3)^N}{k_2^4((k_1-k_3)^2-m^2)(k_1-k_2)^2((k_3-p)^2-m^2)}$$

$$||$$

$$\sum_{k=1}^{N}(-1)^k e^{-\frac{3\varepsilon\gamma}{2}}\Gamma\Big(-1-\frac{3\varepsilon}{2}\Big)\times$$
$$\times B\Big(2+k,\frac{\varepsilon}{2}\Big)B(-\varepsilon+k,-\varepsilon)B\Big(1-\frac{\varepsilon}{2}+k,1+\frac{\varepsilon}{2}\Big)\binom{N}{k}$$

where

$$B(a,b)=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

$$F(\varepsilon,N)=\int\frac{d^{4+\varepsilon}k_1}{(2\pi)^{4+\varepsilon}}\frac{d^{4+\varepsilon}k_2}{(2\pi)^{4+\varepsilon}}\frac{d^{4+\varepsilon}k_3}{(2\pi)^{4+\varepsilon}}\frac{(\Delta.k_3)^N}{k_2^4((k_1-k_3)^2-m^2)(k_1-k_2)^2((k_3-p)^2-m^2)}$$

$$||$$

$$\underbrace{\begin{aligned}\sum_{k=1}^{N}(-1)^k e^{-\frac{3\varepsilon\gamma}{2}}\Gamma\Big(-1-\frac{3\varepsilon}{2}\Big)\times\\ \times B\Big(2+k,\frac{\varepsilon}{2}\Big)B(-\varepsilon+k,-\varepsilon)B\Big(1-\frac{\varepsilon}{2}+k,1+\frac{\varepsilon}{2}\Big)\binom{N}{k}\end{aligned}}_{}$$

$$=f_{-3}(N,k)\varepsilon^{-3}+f_{-2}(N,k)\varepsilon^{-2}+f_{-1}(N,k)\varepsilon^{-1}+\dots$$

$$F(\varepsilon,N)=\int\frac{d^{4+\varepsilon}k_1}{(2\pi)^{4+\varepsilon}}\frac{d^{4+\varepsilon}k_2}{(2\pi)^{4+\varepsilon}}\frac{d^{4+\varepsilon}k_3}{(2\pi)^{4+\varepsilon}}\frac{(\Delta.k_3)^N}{k_2^4((k_1-k_3)^2-m^2)(k_1-k_2)^2((k_3-p)^2-m^2)}$$

$$||$$

$$\underbrace{\sum_{k=1}^{N}(-1)^k e^{-\frac{3\varepsilon\gamma}{2}}\Gamma\Big(-1-\frac{3\varepsilon}{2}\Big)\times \times B\Big(2+k,\frac{\varepsilon}{2}\Big)B(-\varepsilon+k,-\varepsilon)B\Big(1-\frac{\varepsilon}{2}+k,1+\frac{\varepsilon}{2}\Big)\binom{N}{k}}_{=f_{-3}(N,k)\varepsilon^{-3}+f_{-2}(N,k)\varepsilon^{-2}+f_{-1}(N,k)\varepsilon^{-1}+\dots}$$

$$||$$

$$\Big(\sum_{k=1}^{N}f_{-3}(N,k)\Big)\varepsilon^{-3}+\Big(\sum_{k=1}^{N}f_{-2}(N,k)\Big)\varepsilon^{-2}+\Big(\sum_{k=1}^{N}f_{-1}(N,k)\Big)\varepsilon^{-1}+\dots$$

$$F(\varepsilon,N)=\int\frac{d^{4+\varepsilon}k_1}{(2\pi)^{4+\varepsilon}}\frac{d^{4+\varepsilon}k_2}{(2\pi)^{4+\varepsilon}}\frac{d^{4+\varepsilon}k_3}{(2\pi)^{4+\varepsilon}}\frac{(\Delta.k_3)^N}{k_2^4((k_1-k_3)^2-m^2)(k_1-k_2)^2((k_3-p)^2-m^2)}$$

$$||$$

$$\underbrace{\sum_{k=1}^{N}(-1)^k e^{-\frac{3\varepsilon\gamma}{2}}\Gamma\Big(-1-\frac{3\varepsilon}{2}\Big)\times \times B\Big(2+k,\frac{\varepsilon}{2}\Big)B(-\varepsilon+k,-\varepsilon)B\Big(1-\frac{\varepsilon}{2}+k,1+\frac{\varepsilon}{2}\Big)\binom{N}{k}}_{=f_{-3}(N,k)\varepsilon^{-3}+f_{-2}(N,k)\varepsilon^{-2}+f_{-1}(N,k)\varepsilon^{-1}+\dots}$$

$$||$$

$$\Big(\sum_{k=1}^{N}f_{-3}(N,k)\Big)\varepsilon^{-3}+\Big(\sum_{k=1}^{N}f_{-2}(N,k)\Big)\varepsilon^{-2}+\Big(\boxed{\sum_{k=1}^{N}f_{-1}(N,k)}\Big)\varepsilon^{-1}+\dots$$

Simplify

$$F_{-1}(N)=\sum_{k=1}^{N}(-1)^{k+1}\binom{N}{k}\Big(\frac{(2+3k)\big(-2+3k+7k^2+3k^3\big)}{3k^2(1+k)^3}+\frac{2S_2(k)}{1+k}+\frac{\zeta_2}{2(1+k)}\Big)$$

where

$$S_a(N)=\sum_{i=1}^{N}\frac{\text{sign}(a)^i}{i^a}\text{ and }\zeta_a=\sum_{i=1}^{\infty}\frac{1}{i^a}$$

Simplify

$$F_{-1}(N)=\sum_{k=1}^{N}(-1)^{k+1}\binom{N}{k}\Big(\frac{(2+3k)\big(-2+3k+7k^2+3k^3\big)}{3k^2(1+k)^3}+\frac{2S_2(k)}{1+k}+\frac{\zeta_2}{2(1+k)}\Big)$$

$\downarrow$ (summation package `Sigma.m`)

$$\begin{aligned}&\big(16N^3+144N^2+413N+384\big)(N+1)^2F_{-1}(N)\\&\quad-(N+2)(2N+5)\big(16N^3+112N^2+221N+113\big)F_{-1}(N+1)\\&\quad+(N+3)^2\big(16N^3+96N^2+173N+99\big)F_{-1}(N+2)\\&\quad=\frac{1}{2}\big(4N^2+21N+29\big)\zeta_2+\frac{-64N^5-500N^4-1133N^3+203N^2+3516N+3090}{3(N+2)(N+3)}\end{aligned}$$

Simplify

$$F_{-1}(N)=\sum_{k=1}^{N}(-1)^{k+1}\binom{N}{k}\Big(\frac{(2+3k)\big(-2+3k+7k^2+3k^3\big)}{3k^2(1+k)^3}+\frac{2S_2(k)}{1+k}+\frac{\zeta_2}{2(1+k)}\Big)$$

$\downarrow$ (summation package `Sigma.m`)

$$\begin{aligned}&\big(16N^3+144N^2+413N+384\big)(N+1)^2F_{-1}(N)\\&\quad-(N+2)(2N+5)\big(16N^3+112N^2+221N+113\big)F_{-1}(N+1)\\&\quad+(N+3)^2\big(16N^3+96N^2+173N+99\big)F_{-1}(N+2)\\&\quad=\frac{1}{2}\big(4N^2+21N+29\big)\zeta_2+\frac{-64N^5-500N^4-1133N^3+203N^2+3516N+3090}{3(N+2)(N+3)}\end{aligned}$$

$\downarrow$ (summation package `Sigma.m`)

$$\begin{aligned}&\Big\{c_1\frac{1-4N}{N+1}+c_2\frac{-14N-13}{(N+1)^2}\\&+\frac{(4N-1)S_1(N)}{N+1}+\frac{(1-4N)S_1(N)^2}{6(N+1)}+\frac{(14N+13)S_1(N)}{3(N+1)^2}\\&+\frac{175N^2+334N+155}{12(N+1)^3}+\frac{(1-4N)S_2(N)}{6(N+1)}+\frac{\zeta_2}{8(N+1)}\Big|c_1,c_2\in\mathbb{Q}\Big\}\end{aligned}$$

Simplify

$$F_{-1}(N)=\sum_{k=1}^{N}(-1)^{k+1}\binom{N}{k}\Big(\frac{(2+3k)\big(-2+3k+7k^2+3k^3\big)}{3k^2(1+k)^3}+\frac{2S_2(k)}{1+k}+\frac{\zeta_2}{2(1+k)}\Big)$$

$$\bigcap$$

$$\Big\{c_1\,\frac{1-4N}{N+1}+c_2\,\frac{-14N-13}{(N+1)^2}
+\frac{(4N-1)S_1(N)}{N+1}+\frac{(1-4N)S_1(N)^2}{6(N+1)}+\frac{(14N+13)S_1(N)}{3(N+1)^2}
+\frac{175N^2+334N+155}{12(N+1)^3}+\frac{(1-4N)S_2(N)}{6(N+1)}+\frac{\zeta_2}{8(N+1)}\Big|c_1,c_2\in\mathbb{Q}\Big\}$$

Simplify

$$F_{-1}(N)=\sum_{k=1}^{N}(-1)^{k+1}\binom{N}{k}\Big(\frac{(2+3k)\big(-2+3k+7k^2+3k^3\big)}{3k^2(1+k)^3}+\frac{2S_2(k)}{1+k}+\frac{\zeta_2}{2(1+k)}\Big)$$

|| (recurrence finding and solving)

$$\Big(\frac{1}{12}-\frac{1}{8}\zeta_2\Big)\frac{1-4N}{N+1}+1\,\frac{-14N-13}{(N+1)^2}$$
$$+\frac{(4N-1)S_1(N)}{N+1}+\frac{(1-4N)S_1(N)^2}{6(N+1)}+\frac{(14N+13)S_1(N)}{3(N+1)^2}$$
$$+\frac{175N^2+334N+155}{12(N+1)^3}+\frac{(1-4N)S_2(N)}{6(N+1)}+\frac{\zeta_2}{8(N+1)}$$

Iterative application from inside to outside transforms

definite multi-sums

$$\downarrow$$

indefinite nested sums

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+s-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+s-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\boxed{\sum_{s=0}^{n-j+s-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}}$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+s-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\boxed{\sum_{s=0}^{n-j+s-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}}$$

$$||$$

$$\boxed{\binom{j+1}{r}\Big(\frac{(-1)^r(-j+n-2)!r!}{(n+1)(-j+n+r-1)(-j+n+r)!}+\frac{(-1)^{n+r}(j+1)!(-j+n-2)!(-j+n-1)_r r!}{(n-1)n(n+1)(-j+n+r)!(-j-1)_r(2-n)_j}\Big)}$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+s-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\boxed{\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\binom{j+1}{r}\Big(\frac{(-1)^r(-j+n-2)!r!}{(n+1)(-j+n+r-1)(-j+n+r)!}+\frac{(-1)^{n+r}(j+1)!(-j+n-2)!(-j+n-1)_r r!}{(n-1)n(n+1)(-j+n+r)!(-j-1)_r(2-n)_j}\Big)}$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+s-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\boxed{\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\binom{j+1}{r}\Big(\frac{(-1)^r(-j+n-2)!r!}{(n+1)(-j+n+r-1)(-j+n+r)!}+\frac{(-1)^{n+r}(j+1)!(-j+n-2)!(-j+n-1)_r r!}{(n-1)n(n+1)(-j+n+r)!(-j-1)_r(2-n)_j}\Big)}$$

$$||$$

$$\boxed{\Big(\frac{n^2-n+1}{(n-1)^2n^2(n+1)(2-n)_j}+\frac{\sum_{i=1}^{j}\frac{(2-n)_i}{(-i+n-1)^2(i+1)!}}{(n+1)(2-n)_j}+\frac{(-1)^{j+n}(-j-2)(-j+n-2)!}{(j-n+1)(n+1)^2n!}\Big)(j+1)!-\frac{1}{(n+1)^2(-j+n-1)}}$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+s-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\sum_{j=0}^{n-2}\Bigg(\Big(\frac{n^2-n+1}{(n-1)^2n^2(n+1)(2-n)_j}+\frac{\displaystyle\sum_{i=1}^{j}\frac{(2-n)_i}{(-i+n-1)^2(i+1)!}}{(n+1)(2-n)_j}+$$
$$\frac{(-1)^{j+n}(-j-2)(-j+n-2)!}{(j-n+1)(n+1)^2n!}\Big)(j+1)!-\frac{1}{(n+1)^2(-j+n-1)}\Bigg)$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+s-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$\|$$

$$\sum_{j=0}^{n-2}\left(\left(\frac{n^2-n+1}{(n-1)^2n^2(n+1)(2-n)_j}+\frac{\sum_{i=1}^{j}\frac{(2-n)_i}{(-i+n-1)^2(i+1)!}}{(n+1)(2-n)_j}+\frac{(-1)^{j+n}(-j-2)(-j+n-2)!}{(j-n+1)(n+1)^2n!}\right)(j+1)!-\frac{1}{(n+1)^2(-j+n-1)}\right)$$

$$\|$$

$$\frac{-n^2-n-1}{n^2(n+1)^3}+\frac{(-1)^n\left(n^2+n+1\right)}{n^2(n+1)^3}-\frac{2S_{-2}(n)}{n+1}+\frac{S_1(n)}{(n+1)^2}+\frac{S_2(n)}{-n-1}$$

Note: $S_a(n)=\sum_{i=1}^{n}\frac{\mathsf{sign}(a)^i}{i^{|a|}}$, $a\in\mathbb{Z}\setminus\{0\}$.

Automatic toolbox:

In[1]:= $<<$ **Sigma.m**
Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= $<<$ **HarmonicSums.m**
HarmonicSums by Jakob Ablinger © RISC-Linz

In[3]:= $<<$ **EvaluateMultiSums.m**
EvaluateMultiSums by Carsten Schneider © RISC-Linz

In[4]:= $\mathbf{EvaluateMultiSum}[\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!},$
$\{\{s,0,n\text{-}j+r\text{-}2\},\{r,0,j+1\},\{j,0,n\text{-}2\}\}]$

Out[4]= $\frac{-n^2-n-1}{n^2(n+1)^3}+\frac{(-1)^n\left(n^2+n+1\right)}{n^2(n+1)^3}-\frac{2S_{-2}(n)}{n+1}+\frac{S_1(n)}{(n+1)^2}+\frac{S_2(n)}{-n-1}$

Sigma.m is based on difference ring/field theory

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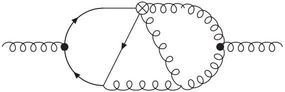
Consider a massive 3–loop ladder graph

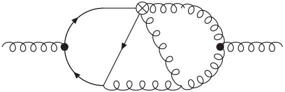
[Ablinger, Blümlein,Hasselhuhn,Klein, CS,Wissbrock, Nucl. Phys. B, 2013, arXiv:1206.2252 [hep-ph]]



$$= F_{-3}(N)\varepsilon^{-3} + F_{-2}(N)\varepsilon^{-2} + F_{-1}(N)\varepsilon^{-1} + \boxed{F_0(N)}$$

All diagrams are produced with axodraw (J. Vermaseren)

Consider a massive 3–loop ladder graph [Ablinger, Blümlein, Hasselhuhn, Klein, CS, Wissbrock, Nucl. Phys. B, 2013, arXiv:1206.2252 [hep-ph]]



$$= F_{-3}(N)\varepsilon^{-3} + F_{-2}(N)\varepsilon^{-2} + F_{-1}(N)\varepsilon^{-1} + \boxed{F_0(N)}$$

Simplify

$$||$$

$$\sum_{j=0}^{N-3}\sum_{k=0}^{j}\sum_{l=0}^{k}\sum_{q=0}^{-j+N-3}\sum_{s=1}^{-l+N-q-3}\sum_{r=0}^{-l+N-q-s-3}(-1)^{-j+k-l+N-q-3}\times$$
$$\times\frac{\binom{j+1}{k+1}\binom{k}{l}\binom{N-1}{j+2}\binom{-j+N-3}{q}\binom{-l+N-q-3}{s}\binom{-l+N-q-s-3}{r}r!(-l+N-q-r-s-3)!(s-1)!}{(-l+N-q-2)!(-j+N-1)(N-q-r-s-2)(q+s+1)}$$
$$\Big[4S_1(-j+N-1)-4S_1(-j+N-2)-2S_1(k)$$
$$-\left(S_1(-l+N-q-2)+S_1(-l+N-q-r-s-3)-2S_1(r+s)\right)$$
$$+2S_1(s-1)-2S_1(r+s)\Big] + \textbf{3 further 6–fold sums}$$

$\boxed{F_0(N)} =$ (using Sigma.m, EvaluateMultiSums.m and J. Ablinger's HarmonicSums.m package)

$$\frac{7}{12}S_1(N)^4+\frac{(17N+5)S_1(N)^3}{3N(N+1)}+\Big(\frac{35N^2-2N-5}{2N^2(N+1)^2}+\frac{13S_2(N)}{2}+\frac{5(-1)^N}{2N^2}\Big)S_1(N)^2$$
$$+\Big(-\frac{4(13N+5)}{N^2(N+1)^2}+\Big(\frac{4(-1)^N(2N+1)}{N(N+1)}-\frac{13}{N}\Big)S_2(N)+\Big(\frac{29}{3}-(-1)^N\Big)S_3(N)$$
$$+\big(2+2(-1)^N\big)S_{2,1}(N)-28S_{-2,1}(N)+\frac{20(-1)^N}{N^2(N+1)}\Big)S_1(N)+\Big(\frac{3}{4}+(-1)^N\Big)S_2(N)^2$$
$$-2(-1)^N S_{-2}(N)^2+S_{-3}(N)\Big(\frac{2(3N-5)}{N(N+1)}+\big(26+4(-1)^N\big)S_1(N)+\frac{4(-1)^N}{N+1}\Big)$$
$$+\Big(\frac{(-1)^N(5-3N)}{2N^2(N+1)}-\frac{5}{2N^2}\Big)S_2(N)+S_{-2}(N)\Big(10S_1(N)^2+\Big(\frac{8(-1)^N(2N+1)}{N(N+1)}$$
$$+\frac{4(3N-1)}{N(N+1)}\Big)S_1(N)+\frac{8(-1)^N(3N+1)}{N(N+1)^2}+\big(-22+6(-1)^N\big)S_2(N)-\frac{16}{N(N+1)}\Big)$$
$$+\Big(\frac{(-1)^N(9N+5)}{N(N+1)}-\frac{29}{3N}\Big)S_3(N)+\Big(\frac{19}{2}-2(-1)^N\Big)S_4(N)+\big(-6+5(-1)^N\big)S_{-4}(N)$$
$$+\Big(-\frac{2(-1)^N(9N+5)}{N(N+1)}-\frac{2}{N}\Big)S_{2,1}(N)+\big(20+2(-1)^N\big)S_{2,-2}(N)+\big(-17+13(-1)^N\big)S_{3,1}(N)$$
$$-\frac{8(-1)^N(2N+1)+4(9N+1)}{N(N+1)}S_{-2,1}(N)-\big(24+4(-1)^N\big)S_{-3,1}(N)+\big(3-5(-1)^N\big)S_{2,1,1}(N)$$
$$+32S_{-2,1,1}(N)+\left(\frac{3}{2}S_1(N)^2-\frac{3S_1(N)}{N}+\frac{3}{2}(-1)^N S_{-2}(N)\right)\zeta_2$$

The general tactic

Feynman integrals

↓ non-trivial transformations (DESY)

multiple sums

↓ symbolic summation

compact expression in terms of special functions

Tactic 1: Expand the summand and simplify

Ablinger, Blümlein, Klein, CS, LL2010, arXiv:1006.4797 [math-ph]
Blümlein, Hasselhuhn, CS, RADCOR'11, arXiv:1202.4303 [math-ph]
CS, ACAT 2013, arXiv:1310.0160 [cs.SC]

The general tactic

Feynman integrals

$\downarrow$ non-trivial transformations (DESY)

multiple sums

$\downarrow$ symbolic summation

compact expression in terms of special functions

Tactic 2: Expand a recurrence in ε

Blümlein, Klein, CS, Stan, J. Symbol. Comput. 2012; arXiv:1011.2656 [cs.SC]
Ablinger, Blümlein, Round, CS, LL2012, arXiv:1210.1685 [cs.SC]

$$F(N)=\sum_{k=1}^{N}(-1)^k e^{-\frac{3\varepsilon\gamma}{2}}\Gamma\big(-1-\frac{3\varepsilon}{2}\big)B\big(2+k,\frac{\varepsilon}{2}\big)B(-\varepsilon+k,-\varepsilon)B\big(1-\frac{\varepsilon}{2}+k,1+\frac{\varepsilon}{2}\big)\binom{N}{k}$$

$\downarrow$ (summation package `Sigma.m`)

$$2(N+1)^2F(N)+\big(3\varepsilon^2+3\varepsilon N+9\varepsilon-4N^2-12N-8\big)F(N+1)$$
$$-(2\varepsilon-N-1)(\varepsilon+2N+6)F(N+2)=0\varepsilon^{-3}-\frac{16}{3}\varepsilon^{-2}+\frac{40}{3}\varepsilon^{-1}-\big(2\zeta_2-\frac{68}{3}\big)\varepsilon^0+\dots$$

$$F(N)=\sum_{k=1}^{N}(-1)^k e^{-\frac{3\varepsilon\gamma}{2}}\Gamma\Big(-1-\frac{3\varepsilon}{2}\Big)B\Big(2+k,\frac{\varepsilon}{2}\Big)B(-\varepsilon+k,-\varepsilon)B\Big(1-\frac{\varepsilon}{2}+k,1+\frac{\varepsilon}{2}\Big)\binom{N}{k}$$

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$$F(1)=\frac{2}{3}\varepsilon^{-3}-\frac{11}{6}\varepsilon^{-2}+\Big(\frac{\zeta_2}{4}+\frac{79}{24}\Big)\varepsilon^{-1}+\dots$$
$$F(2)=\frac{8}{9}\varepsilon^{-3}-\frac{73}{27}\varepsilon^{-2}+\Big(\frac{\zeta_2}{3}+\frac{1415}{324}\Big)\varepsilon^{-1}+\dots$$

↓

$$F(N)=F_{-3}(N)\,\varepsilon^{-3}+F_{-2}(N)\,\varepsilon^{-2}+F_{-1}(N)\,\varepsilon^{-1}+\dots$$

Ansatz (for power series)

$$
\begin{aligned}
&a_0(\varepsilon, N)\Big[F(N)\Big] \\
&+a_1(\varepsilon, N)\Big[F(N+1)\Big] \\
&+ \\
&\vdots \\
&+a_d(\varepsilon, N)\Big[F(N+d)\Big]
\end{aligned}
$$

$$= h_0(N) + h_1(N)\varepsilon + h_1(N)\varepsilon^2 + \dots$$

given (in terms of indefinite nested sums and products)

Ansatz (for power series)

$$a_0(\varepsilon, N)\Big[F_0(N) + F_1(N)\varepsilon + F_2(N)\varepsilon^2 + \dots\Big]$$

$$+a_1(\varepsilon, N)\Big[F(N+1)\Big]$$

$$+$$

$$\vdots$$

$$+a_d(\varepsilon, N)\Big[F(N+d)\Big]$$

$$= h_0(N) + h_1(N)\varepsilon + h_1(N)\varepsilon^2 + \dots$$

given (in terms of indefinite nested sums and products)

Ansatz (for power series)

$$
\begin{aligned}
&a_0(\varepsilon, N)\Big[F_0(N) + F_1(N)\varepsilon + F_2(N)\varepsilon^2 + \dots\Big] \\
+&a_1(\varepsilon, N)\Big[F_0(N+1) + F_1(N+1)\varepsilon + F_2(N+1)\varepsilon^2 + \dots\Big] \\
+& \\
&\vdots \\
+&a_d(\varepsilon, N)\Big[F(N+d)\Big]
\end{aligned}
$$

$$= h_0(N) + h_1(N)\varepsilon + h_1(N)\varepsilon^2 + \dots$$

given (in terms of indefinite nested sums and products)

Ansatz (for power series)

$$a_0(\varepsilon,N)\Big[F_0(N)+F_1(N)\varepsilon+F_2(N)\varepsilon^2+\dots\Big]$$
$$+a_1(\varepsilon,N)\Big[F_0(N+1)+F_1(N+1)\varepsilon+F_2(N+1)\varepsilon^2+\dots\Big]$$
$$+$$
$$\vdots$$
$$+a_d(\varepsilon,N)\Big[F_0(N+d)+F_1(N+d)\varepsilon+F_2(N+d)\varepsilon^2+\dots\Big]$$
$$=h_0(N)+h_1(N)\varepsilon+h_1(N)\varepsilon^2+\dots$$

given (in terms of indefinite nested sums and products)

Ansatz (for power series)

$$
\begin{aligned}
&a_0(\varepsilon,N)\Big[F_0(N)+F_1(N)\varepsilon+F_2(N)\varepsilon^2+\dots\Big]\\
+&a_1(\varepsilon,N)\Big[F_0(N+1)+F_1(N+1)\varepsilon+F_2(N+1)\varepsilon^2+\dots\Big]\\
+&\\
&\vdots\\
+&a_d(\varepsilon,N)\Big[F_0(N+d)+F_1(N+d)\varepsilon+F_2(N+d)\varepsilon^2+\dots\Big]\\
&\qquad\qquad\qquad\qquad = h_0(N)+h_1(N)\varepsilon+h_1(N)\varepsilon^2+\dots
\end{aligned}
$$

$\Downarrow$ constant terms must agree

$$
\boxed{a_0(0,N)F_0(N)+a_1(0,N)F_0(N+1)+\dots+a_d(0,N)F_0(N+d)=h_0(N)}
$$

Ansatz (for power series)

$$
\begin{aligned}
&a_0(\varepsilon,N)\Big[F_0(N)+F_1(N)\varepsilon+F_2(N)\varepsilon^2+\dots\Big]\\
+&a_1(\varepsilon,N)\Big[F_0(N+1)+F_1(N+1)\varepsilon+F_2(N+1)\varepsilon^2+\dots\Big]\\
+&\\
\vdots&\\
+&a_d(\varepsilon,N)\Big[F_0(N+d)+F_1(N+d)\varepsilon+F_2(N+d)\varepsilon^2+\dots\Big]\\
&\qquad\qquad\qquad\qquad\qquad\qquad= h_0(N)+h_1(N)\varepsilon+h_1(N)\varepsilon^2+\dots
\end{aligned}
$$

$\Downarrow$ constant terms must agree

$$
\boxed{a_0(0,N)F_0(N)+a_1(0,N)F_0(N+1)+\dots+a_d(0,N)F_0(N+d)=h_0(N)}
$$

REC solver: Using the initial values $F_0(1), F_0(2), \dots$ determines $F_0(N)$ in terms of indefinite nested sums and products.

Ansatz (for power series)

$$
\begin{aligned}
&a_0(\varepsilon,N)\Big[F_0(N)+F_1(N)\varepsilon+F_2(N)\varepsilon^2+\dots\Big]\\
+&a_1(\varepsilon,N)\Big[F_0(N+1)+F_1(N+1)\varepsilon+F_2(N+1)\varepsilon^2+\dots\Big]\\
+&\\
&\vdots\\
+&a_d(\varepsilon,N)\Big[F_0(N+d)+F_1(N+d)\varepsilon+F_2(N+d)\varepsilon^2+\dots\Big]\\
&\qquad\qquad\qquad\qquad = h_0(N)+h_1(N)\varepsilon+h_1(N)\varepsilon^2+\dots
\end{aligned}
$$

$\Downarrow$ constant terms must agree

$$
\boxed{a_0(0,N)F_0(N)+a_1(0,N)F_0(N+1)+\dots+a_d(0,N)F_0(N+d)=h_0(N)}
$$

Ansatz (for power series)

$$
\begin{aligned}
& a_0(\varepsilon,N)\Big[F_0(N)+F_1(N)\varepsilon+F_2(N)\varepsilon^2+\dots\Big]\\
+& a_1(\varepsilon,N)\Big[F_0(N+1)+F_1(N+1)\varepsilon+F_2(N+1)\varepsilon^2+\dots\Big]\\
+& \\
& \vdots\\
+& a_d(\varepsilon,N)\Big[F_0(N+d)+F_1(N+d)\varepsilon+F_2(N+d)\varepsilon^2+\dots\Big]\\
& \qquad = h_0(N)+h_1(N)\varepsilon+h_1(N)\varepsilon^2+\dots
\end{aligned}
$$

$\Downarrow$ constant terms must agree

$$\boxed{a_0(0,N)F_0(N)+a_1(0,N)F_0(N+1)+\dots+a_d(0,N)F_0(N+d)=h_0(N)}$$

$$
\begin{aligned}
&\quad a_0(\varepsilon,N)\Big[F_1(N)\varepsilon+F_2(N)\varepsilon^2+\dots\Big]\\
&+a_1(\varepsilon,N)\Big[F_1(N+1)\varepsilon+F_2(N+1)\varepsilon^2+\dots\Big]\\
&+\\
&\ \vdots\\
&+a_d(\varepsilon,N)\Big[F_1(N+d)\varepsilon+F_2(N+d)\varepsilon^2+\dots\Big]\\
&\qquad\qquad\qquad\qquad\qquad\qquad=h_0'(N)+h_1'(N)\varepsilon+h_2'(N)\varepsilon^2+\dots
\end{aligned}
$$

$$
\begin{aligned}
&a_0(\varepsilon,N)\Big[F_1(N)\varepsilon+F_2(N)\varepsilon^2+\dots\Big]\\
+&a_1(\varepsilon,N)\Big[F_1(N{+}1)\varepsilon+F_2(N{+}1)\varepsilon^2+\dots\Big]\\
+&\\
&\vdots\\
+&a_d(\varepsilon,N)\Big[F_1(N{+}d)\varepsilon+F_2(N+d)\varepsilon^2+\dots\Big]\\
&\qquad\qquad = \underbrace{h'_0(N)}_{=\mathbf{0}}+h'_1(N)\varepsilon+h'_2(N)\varepsilon^2+\dots
\end{aligned}
$$

Devide by ε

$$
\begin{aligned}
&\;a_0(\varepsilon,N)\Big[F_1(N)+F_2(N)\varepsilon+\dots\Big]\\
&+a_1(\varepsilon,N)\Big[F_1(N+1)+F_2(N+1)\varepsilon+\dots\Big]\\
&+\\
&\;\vdots\\
&+a_d(\varepsilon,N)\Big[F_1(N+d)+F_2(N+d)\varepsilon+\dots\Big]=h_1'(N)+h_2'(N)\varepsilon+\dots
\end{aligned}
$$

Now repeat for $F_1(N), F_2(N), \dots$

Remark: Works the same for Laurent series.

Blümlein, Klein, CS, Stan, J. Symbol. Comput. 2012; arXiv:1011.2656[cs.SC]
Ablinger, Blümlein, Round, CS, LL2012, arXiv:1210.1685 [cs.SC]

$$F(N)=\sum_{k=1}^{N}(-1)^k e^{-\frac{3\varepsilon\gamma}{2}}\Gamma\Big(-1-\frac{3\varepsilon}{2}\Big)B\Big(2+k,\frac{\varepsilon}{2}\Big)B(-\varepsilon+k,-\varepsilon)B\Big(1-\frac{\varepsilon}{2}+k,1+\frac{\varepsilon}{2}\Big)\binom{N}{k}$$

$\downarrow$ (summation package `Sigma.m`)

$$2(N+1)^2F(N)+\big(3\varepsilon^2+3\varepsilon N+9\varepsilon-4N^2-12N-8\big)F(N+1)$$
$$-(2\varepsilon-N-1)(\varepsilon+2N+6)F(N+2)=0\varepsilon^{-3}-\frac{16}{3}\varepsilon^{-2}+\frac{40}{3}\varepsilon^{-1}-\big(2\zeta_2-\frac{68}{3}\big)\varepsilon^0+\dots$$

$$F(1)=\frac{2}{3}\varepsilon^{-3}-\frac{11}{6}\varepsilon^{-2}+\Big(\frac{\zeta_2}{4}+\frac{79}{24}\Big)\varepsilon^{-1}+\dots$$
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$\downarrow$

$$F(N)=F_{-3}(N)\,\varepsilon^{-3}+F_{-2}(N)\,\varepsilon^{-2}+F_{-1}(N)\,\varepsilon^{-1}+\dots$$

$$F(N)=\sum_{k=1}^{N}(-1)^k e^{-\frac{3\varepsilon\gamma}{2}}\Gamma\big(-1-\frac{3\varepsilon}{2}\big)B\big(2+k,\frac{\varepsilon}{2}\big)B(-\varepsilon+k,-\varepsilon)B\big(1-\frac{\varepsilon}{2}+k,1+\frac{\varepsilon}{2}\big)\binom{N}{k}$$

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↓ (summation package `Sigma.m`)

$$F(N)=\tfrac{4N}{3(N+1)}\varepsilon^{-3}-\Big(\tfrac{2(2N+1)}{3(N+1)}S_1(N)+\tfrac{2N(2N+3)}{3(N+1)^2}\Big)\varepsilon^{-2}$$
$$\Big(\tfrac{(1-4N)}{6(N+1)}S_1(N)^2-\tfrac{N\left(N^2-2\right)}{3(N+1)^3}+\tfrac{(3N+2)(4N+5)}{3(N+1)^2}S_1(N)+\tfrac{(1-4N)}{6(N+1)}S_2(N)+\tfrac{N\zeta_2}{2(N+1)}\Big)\varepsilon^{-1}+\dots$$

Find a recurrence for the integral/sum

$$F(N)=\int_0^1\dots\int_0^1\Phi(\varepsilon,N,x_1,x_2,\dots,x_7)dx_1dx_2\dots dx_7$$
$$\stackrel{?}{=}F_{-3}(N)\varepsilon^{-3}+F_{-2}(N)\varepsilon^{-2}+F_{-1}(N)\varepsilon^{-1}+\dots$$

multivariate
Almquist/Zeilberger
(J. Ablinger)

$$a_0(\varepsilon,N)F(N)+\dots+a_d(\varepsilon,N)F(N+d)=h(\varepsilon,N)$$

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ε-recurrence solver

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ε-recurrence solver

multivariate Almquist/Zeilberger (J. Ablinger)

$$\sum_{i_1} \dots \sum_{i_7} f(\varepsilon, N, i_1, i_2, \dots, i_7)$$

Wegschaider's MultiSum Package (F. Stan)

$$a_0(\varepsilon, N)F(N) + \dots + a_d(\varepsilon, N)F(N+d) = h(\varepsilon, N)$$

Find a recurrence for the integral/sum

$$F(N)=\int_0^1\dots\int_0^1\Phi(\varepsilon,N,x_1,x_2,\dots,x_7)dx_1dx_2\dots dx_7$$
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ε-recurrence solver

multivariate Almquist/Zeilberger (J. Ablinger)

$$\sum_{i_1}\cdots\sum_{i_7}f(\varepsilon,N,i_1,i_2,\dots,i_7)$$

Wegschaider's MultiSum Package (F. Stan)

Holonomic/difference field approach (M. Round)

$$a_0(\varepsilon,N)F(N)+\dots+a_d(\varepsilon,N)F(N+d)=h(\varepsilon,N)$$

The general tactic

Feynman integrals

$\downarrow$ non-trivial transformations (DESY)

multiple sums

$\downarrow$ symbolic summation

compact expression in terms of special functions

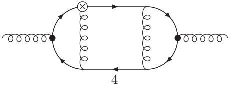
Tactic 2: Expand a recurrence in ε

Blümlein, Klein, CS, Stan, J. Symbol. Comput. 2012; arXiv:1011.2656 [cs.SC]
Ablinger, Blümlein, Round, CS, LL2012, arXiv:1210.1685 [cs.SC]

A challenging diagram and

an algorithm for coupled systems

A challenging diagram (ladder graph with 6 massive fermion lines)

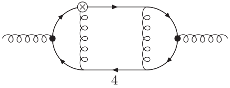
$D_4(N) =$ 

$$\stackrel{?}{=} F_{-3}(N)\varepsilon^{-3} + F_{-2}(N)\varepsilon^{-2} + F_{-1}(N)\varepsilon^{-1} + F_0(N)\varepsilon^{0} + \dots$$

Strategies:

- Symbolic summation tools: failed (so far) ☹

A challenging diagram (ladder graph with 6 massive fermion lines)

$D_4(N) =$ 

$$\stackrel{?}{=} F_{-3}(N)\varepsilon^{-3} + F_{-2}(N)\varepsilon^{-2} + F_{-1}(N)\varepsilon^{-1} + F_0(N)\varepsilon^{0} + \dots$$

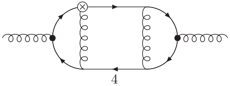
Strategies:

- Symbolic summation tools: failed (so far) ☹
- Brown's hyperlogarithm algorithm: works for the scalar version where

$$\lim_{\varepsilon\to 0} D_4(N) = F_0(N).$$

[Ablinger, Blümlein, Raab, CS, Wissbrock, Nucl. Phys. B, 2014; arXiv:1403.1137 [hep-ph]]

A challenging diagram (ladder graph with 6 massive fermion lines)

$D_4(N) =$ 

$$\stackrel{?}{=} F_{-3}(N)\varepsilon^{-3} + F_{-2}(N)\varepsilon^{-2} + F_{-1}(N)\varepsilon^{-1} + F_0(N)\varepsilon^{0} + \dots$$

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[Ablinger, Blümlein, Raab, CS, Wissbrock, Nucl. Phys. B, 2014; arXiv:1403.1137 [hep-ph]]

- New approach: for the complete diagram

De Freitas, Blümlein, CS, LL 2014, arXiv:1407.2537 [cs.SC]

Ablinger, Behring, Blümlein, De Freitas, Hasselhuhn, Manteuffel, Round, CS, Wissbrock
Nucl.Phys.B, 2014. arXiv:1406.4654

Ablinger, Behring, Blümlein, De Freitas, Manteuffel, CS, (pure singlet case) 2014. arXiv:1409.1135 [hep-ph]

Consider the power series of $D_4(N)$:

$$D_4(N) \longrightarrow \hat{D}_4(x) = \sum_{N=0}^{\infty} D_4(N) x^N$$

(holonomic closure properties)

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(holonomic closure properties)

$$\underbrace{\sum_{N=0}^{\infty} D_4(N)x^N}_{\hat{D}_4(x)}$$

IBP (extension of REDUZE_2, A.v. Manteuffel) gives

$$\underbrace{\sum_{N=0}^{\infty} D_4(N)x^N}_{\hat{D}_4(x)} = \boxed{\frac{\left(1545842\varepsilon^5 x^5 - 14325922\varepsilon^5 x^4 + \dots + 1524096x^2 - 653184x\right)}{23328\varepsilon^2(x-1)x^5}} \hat{B}_1(x)$$

$$+ \Box \hat{B}_2(x) + \dots + \Box \hat{B}_{52}(x) \dots$$

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$$+\square\hat{B}_2(x)+\dots+\square\hat{B}_{52}(x)$$

$\hat{B}_1(x),\dots,\hat{B}_{52}(x)$ can be handled with sophisticated Mellin-Barnes techniques (DESY colleagues) and symbolic summation (see first slides).

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$\hat{B}_1(x), \ldots, \hat{B}_{52}(x)$ can be handled with sophisticated Mellin-Barnes techniques (DESY colleagues) and symbolic summation (see first slides).

E.g.,

$$\hat{B}_1(x) = \sum_{N=0}^{\infty} B_1(N)x^N$$

with

$$B_1(N) = \sum_{k=1}^{N} (-1)^k e^{-\frac{3\varepsilon\gamma}{2}} \Gamma\left(\frac{-2-3\varepsilon}{2}\right) B\left(2+k, \frac{\varepsilon}{2}\right) B(-\varepsilon+k, -\varepsilon) B\left(1-\frac{\varepsilon}{2}+k, 1+\frac{\varepsilon}{2}\right) \binom{N}{k}$$

$$= \frac{4N}{3(N+1)}\varepsilon^{-3} - \left(\frac{2(2N+1)}{3(N+1)} S_1(N) + \frac{2N(2N+3)}{3(N+1)^2}\right)\varepsilon^{-2} + \ldots$$

IBP (extension of REDUZE_2, A.v. Manteuffel) gives

$$\underbrace{\sum_{N=0}^{\infty} D_4(N)x^N}_{\hat{D}_4(x)} = \boxed{\frac{\left(1545842\varepsilon^5x^5-14325922\varepsilon^5x^4+\dots+1524096x^2-653184x\right)}{23328\varepsilon^2(x-1)x^5}}\hat{B}_1(x)$$

$$+\,\square\hat{B}_2(x)+\dots+\square\hat{B}_{52}(x)$$

$$+\boxed{-\frac{\left(122\varepsilon^4x^3-2647\varepsilon^4x^2+\dots-304\varepsilon+24x^3-24x\right)}{4\varepsilon x^4}}\hat{I}_1(x)$$

$$+\boxed{\frac{\left(589\varepsilon^5x^3-20123\varepsilon^5x^2+\dots-896\varepsilon+96x^3-96x\right)}{16\varepsilon^2x^4}}\hat{I}_2(x)$$

$$+\boxed{\frac{\left(589\varepsilon^5x^3-21509\varepsilon^5x^2+\dots-1152\varepsilon+96x^3-96x\right)}{16\varepsilon^2x^4}}\hat{I}_3(x)$$

$$+\,\square\hat{I}_4(x)+\dots+\square\hat{I}_{15}(x)$$

However, $\hat{I}_1(x),\dots,\hat{I}_{15}(x)$ are hard to handle. Luckily...

... there are differential relations among the integrals. E.g.,

$$D_x \hat{I}_1(x) = -\frac{(-\varepsilon+x-1)}{(x-1)x}\hat{I}_1(x) - \frac{2}{(x-1)x}\hat{I}_2(x) + \frac{1}{(x-1)x}\hat{B}_1(x) + \dots$$

$$\begin{aligned} D_x \hat{I}_2(x) = {} & \frac{\left(3(\varepsilon+4)^2-22(\varepsilon+4)+40\right)}{4(x-1)}\hat{I}_1(x) \\ & + \frac{(-(\varepsilon+4)(3x-1)+9x-2)}{2(x-1)x}\hat{I}_2(x) - \frac{(\varepsilon+1)}{2(x-1)}\hat{I}_3(x) \\ & + \frac{1}{4}\frac{\left((\varepsilon+4)^2(6x-25)-2(\varepsilon+4)(17x-75)+48x-224\right)}{(5\varepsilon+6)(x-1)x}\hat{B}_1(x) + \dots \end{aligned}$$

$$\begin{aligned} D_x \hat{I}_3(x) = {} & -\frac{\left(3(\varepsilon+4)^2(x-2)-22(\varepsilon+4)(x-2)+40x-80\right)}{4(x-1)x}\hat{I}_1(x) \\ & + \frac{((\varepsilon+4)(3x-5)-11x+18)}{2(x-1)x}\hat{I}_2(x) - \frac{(-(\varepsilon+4)(x-2)+5x-8)}{2(x-1)x}\hat{I}_3(x) \\ & - \frac{1}{4}\frac{\left((\varepsilon+4)^2(6x-25)-2(\varepsilon+4)(17x-75)+48x-224\right)}{(5\varepsilon+6)(x-1)x}\hat{B}_1(x) + \dots \end{aligned}$$

... there are differential relations among the integrals. E.g.,

$$\begin{aligned} D_x \hat{I}_1(x) &= -\frac{(-\varepsilon+x-1)}{(x-1)x}\hat{I}_1(x) - \frac{2}{(x-1)x}\hat{I}_2(x) \\ &\quad + \frac{1}{(x-1)x}\hat{B}_1(x) + \dots \end{aligned}$$

$$\begin{aligned} D_x \hat{I}_2(x) &= \frac{\left(3(\varepsilon+4)^2-22(\varepsilon+4)+40\right)}{4(x-1)}\hat{I}_1(x) \\ &\quad + \frac{(-(\varepsilon+4)(3x-1)+9x-2)}{2(x-1)x}\hat{I}_2(x) - \frac{(\varepsilon+1)}{2(x-1)}\hat{I}_3(x) \\ &\quad + \frac{1}{4}\frac{\left((\varepsilon+4)^2(6x-25)-2(\varepsilon+4)(17x-75)+48x-224\right)}{(5\varepsilon+6)(x-1)x}\hat{B}_1(x) + \dots \end{aligned}$$

$$\begin{aligned} D_x \hat{I}_3(x) &= -\frac{\left(3(\varepsilon+4)^2(x-2)-22(\varepsilon+4)(x-2)+40x-80\right)}{4(x-1)x}\hat{I}_1(x) \\ &\quad + \frac{((\varepsilon+4)(3x-5)-11x+18)}{2(x-1)x}\hat{I}_2(x) - \frac{(-(\varepsilon+4)(x-2)+5x-8)}{2(x-1)x}\hat{I}_3(x) \\ &\quad - \frac{1}{4}\frac{\left((\varepsilon+4)^2(6x-25)-2(\varepsilon+4)(17x-75)+48x-224\right)}{(5\varepsilon+6)(x-1)x}\hat{B}_1(x) + \dots \end{aligned}$$

Step 1: From a DE system to a REC system

$$\begin{aligned} D_x \hat{I}_1(x) = &- \tfrac{(-\varepsilon+x-1)}{(x-1)x} \hat{I}_1(x) \\ &- \tfrac{2}{(x-1)x} \hat{I}_2(x) \\ &+ \tfrac{1}{(x-1)x} \hat{B}_1(x) + \dots \end{aligned}$$

Step 1: From a DE system to a REC system

$$
\begin{aligned}
D_x \sum_{N=0}^{\infty} I_1(N) x^N = &- \tfrac{(-\varepsilon+x-1)}{(x-1)x} \sum_{N=0}^{\infty} I_1(N) x^N \\
&- \tfrac{2}{(x-1)x} \sum_{N=0}^{\infty} I_2(N) x^N \\
&+ \tfrac{1}{(x-1)x} \sum_{N=0}^{\infty} B_1(N) x^N + \dots
\end{aligned}
$$

Step 1: From a DE system to a REC system

$$\sum_{N=1}^{\infty} I_1(N) N x^{N-1} = -\frac{(-\varepsilon+x-1)}{(x-1)x} \sum_{N=0}^{\infty} I_1(N) x^N$$
$$-\frac{2}{(x-1)x} \sum_{N=0}^{\infty} I_2(N) x^N$$
$$+\frac{1}{(x-1)x} \sum_{N=0}^{\infty} B_1(N) x^N + \dots$$

Step 1: From a DE system to a REC system

$$\sum_{N=1}^{\infty} I_1(N) N x^{N-1} = -\tfrac{(-\varepsilon+x-1)}{(x-1)x} \sum_{N=0}^{\infty} I_1(N) x^N - \tfrac{2}{(x-1)x} \sum_{N=0}^{\infty} I_2(N) x^N + \tfrac{1}{(x-1)x} \sum_{N=0}^{\infty} B_1(N) x^N + \dots$$

$\downarrow$ Nth coefficient

$$N I_1(N-1) - (\varepsilon + N + 1) I_1(N) + 2 I_2(N) = B_1(N) + \dots$$

... there are differential relations among the integrals. E.g.,

$$
\begin{aligned}
N I_1(N-1) - (\varepsilon + N + 1) I_1(N) + 2 I_2(N)& \\
= B_1(N) + \dots&
\end{aligned}
$$

$$
\begin{aligned}
D_x \hat{I}_2(x) =& \frac{\big(3(\varepsilon+4)^2-22(\varepsilon+4)+40\big)}{4(x-1)} \hat{I}_1(x) \\
&+ \frac{(-(\varepsilon+4)(3x-1)+9x-2)}{2(x-1)x} \hat{I}_2(x) - \frac{(\varepsilon+1)}{2(x-1)} \hat{I}_3(x) \\
&+ \frac{1}{4} \frac{\big((\varepsilon+4)^2(6x-25)-2(\varepsilon+4)(17x-75)+48x-224\big)}{(5\varepsilon+6)(x-1)x} \hat{B}_1(x) + \dots
\end{aligned}
$$

$$
\begin{aligned}
D_x \hat{I}_3(x) =& - \frac{\big(3(\varepsilon+4)^2(x-2)-22(\varepsilon+4)(x-2)+40x-80\big)}{4(x-1)x} \hat{I}_1(x) \\
&+ \frac{((\varepsilon+4)(3x-5)-11x+18)}{2(x-1)x} \hat{I}_2(x) - \frac{(-(\varepsilon+4)(x-2)+5x-8)}{2(x-1)x} \hat{I}_3(x) \\
&- \frac{1}{4} \frac{\big((\varepsilon+4)^2(6x-25)-2(\varepsilon+4)(17x-75)+48x-224\big)}{(5\varepsilon+6)(x-1)x} \hat{B}_1(x) + \dots
\end{aligned}
$$

A coupled system of difference equations

$$
\begin{aligned}
NI_1(N-1)-(\varepsilon+N+1)I_1(N)+2I_2(N)&\\
=&B_1(N)+\dots
\end{aligned}
$$

$$
\begin{aligned}
2(\varepsilon+2N+2)I_2(N)-2(3\varepsilon+2N+1)I_2(N-1)&\\
+\varepsilon(3\varepsilon+2)I_1(N-1)-2(\varepsilon+1)I_3(N-1)&\\
=&(5\varepsilon+4)B_1(N)-\frac{2(\varepsilon+1)(3\varepsilon+4)}{5\varepsilon+6}B_1(N-1)+\dots
\end{aligned}
$$

$$
\begin{aligned}
4(\varepsilon-N)I_3(N)-2\varepsilon(3\varepsilon+2)I_1(N)+\varepsilon(3\varepsilon+2)I_1(N-1)&\\
-2(3\varepsilon+1)I_2(N-1)+2(5\varepsilon+2)I_2(N)&\\
-2(\varepsilon-2N+1)I_3(N-1)&\\
=&-\frac{2(\varepsilon+1)(3\varepsilon+4)}{5\varepsilon+6}B_1(N-1)+(5\varepsilon+4)B_1(N)+\dots
\end{aligned}
$$

A coupled system of difference equations

$$
\begin{aligned}
&N I_1(N-1) - (\varepsilon + N + 1) I_1(N) + 2 I_2(N) \\
&\qquad = + \tfrac{4(N+2)}{3(N+1)} \varepsilon^{-3} + \left(\tfrac{2(2N+1)}{3(N+1)} S_1(N) - \tfrac{2\left(6N^2+13N+8\right)}{3(N+1)^2} \right) \varepsilon^{-2} + \dots
\end{aligned}
$$

$$
\begin{aligned}
&2(\varepsilon + 2N + 2) I_2(N) - 2(3\varepsilon + 2N + 1) I_2(N-1) \\
&\qquad + \varepsilon(3\varepsilon + 2) I_1(N-1) - 2(\varepsilon + 1) I_3(N-1) \\
&\qquad = \tfrac{8}{3} \varepsilon^{-3} + \left(\tfrac{4}{3} S_1(N)^2 - \tfrac{4(N+1)}{N} S_1(N) + \tfrac{4}{3} S_2(N) + \zeta_2 + 6 \right) \varepsilon^{-1} + \dots
\end{aligned}
$$

$$
\begin{aligned}
&4(\varepsilon - N) I_3(N) - 2\varepsilon(3\varepsilon + 2) I_1(N) + \varepsilon(3\varepsilon + 2) I_1(N-1) \\
&\qquad - 2(3\varepsilon + 1) I_2(N-1) + 2(5\varepsilon + 2) I_2(N) \\
&\qquad - 2(\varepsilon - 2N + 1) I_3(N-1) \\
&\qquad = - \tfrac{8}{3} \varepsilon^{-3} - \left(\tfrac{8}{3} S_1(N) - 4 \right) \varepsilon^{-2} \\
&\qquad\quad - \left(\tfrac{4}{3} S_1(N)^2 - \tfrac{4(N+1)}{N} S_1(N) + \tfrac{4}{3} S_2(N) + \zeta_2 + 6 \right) \varepsilon^{-1} + \dots
\end{aligned}
$$

Step 2: Uncouple the system

$$\begin{aligned}
\Box I_1(N-1) + \Box I_1(N) + \Box I_2(N)& \\
&= \Box\varepsilon^{-3} + \Box\varepsilon^{-2} + \Box\varepsilon^{-1} + \dots \\
\Box I_2(N) + \Box I_2(N-1) + \Box I_1(N-1) + \Box I_3(N-1)& \\
&= \Box\varepsilon^{-3} + \Box\varepsilon^{-1} + \dots \\
\Box I_3(N) + \Box I_1(N) + \Box I_1(N-1) + \Box I_2(N) + \Box I_3(N-1)& \\
&= \Box\varepsilon^{-3} + \Box\varepsilon^{-2} + \Box\varepsilon^{-1} + \dots
\end{aligned}$$

Step 2: Uncouple the system

$$\Box I_1(N-1) + \Box I_1(N) + \Box I_2(N) = \Box\varepsilon^{-3} + \Box\varepsilon^{-2} + \Box\varepsilon^{-1} + \dots$$

$$\Box I_2(N) + \Box I_2(N-1) + \Box I_1(N-1) + \Box I_3(N-1) = \Box\varepsilon^{-3} + \Box\varepsilon^{-1} + \dots$$

$$\Box I_3(N) + \Box I_1(N) + \Box I_1(N-1) + \Box I_2(N) + \Box I_3(N-1) = \Box\varepsilon^{-3} + \Box\varepsilon^{-2} + \Box\varepsilon^{-1} + \dots$$

$\downarrow$ (uncoupling algorithms[a], S. Gerhold's `OrseSys.m`)

$$\Box I_1(N) + \Box I_1(N+1) + \Box I_1(N+2) + \Box I_1(N+3) = \Box\varepsilon^{-3} + \Box\varepsilon^{-2} + \Box\varepsilon^{-1} + \dots$$

$I_2(N) =$ expression in $I_1(N)$

$I_3(N) =$ expression in $I_1(N)$

[a] We use Zürcher's uncoupling algorithm (1994)

More precisely, we get:

$$
\begin{aligned}
&-2(N+1)(N+2)(\varepsilon+N+2)I_1(N)-(N+2)\big(2\varepsilon^2-5\varepsilon N-7\varepsilon-6N^2-28N-32\big)\\
&+\big(\varepsilon^3+4\varepsilon^2N+14\varepsilon^2-4\varepsilon N^2-13\varepsilon N-3\varepsilon-6N^3-50N^2-136N-120\big)I_1(N+2)\\
-&(\varepsilon-N-2)(\varepsilon+N+4)(\varepsilon+2N+8)I_1(N+3)\\
&=-\frac{4(N+2)}{3(N+3)}\varepsilon^{-3}+\frac{2\big(4N^4+35N^3+101N^2+105N+25\big)}{3(N+1)(N+2)(N+3)^2}\varepsilon^{-2}+\dots
\end{aligned}
$$

Step 3: Solve the scalar recurrence

$$
\begin{aligned}
&-2(N+1)(N+2)(\varepsilon+N+2)I_1(N)-(N+2)\big(2\varepsilon^2-5\varepsilon N-7\varepsilon-6N^2-28N-32\big)\\
&+\big(\varepsilon^3+4\varepsilon^2N+14\varepsilon^2-4\varepsilon N^2-13\varepsilon N-3\varepsilon-6N^3-50N^2-136N-120\big)I_1(N+2)\\
&-(\varepsilon-N-2)(\varepsilon+N+4)(\varepsilon+2N+8)I_1(N+3)\\
&=-\frac{4(N+2)}{3(N+3)}\varepsilon^{-3}+\frac{2\big(4N^4+35N^3+101N^2+105N+25\big)}{3(N+1)(N+2)(N+3)^2}\varepsilon^{-2}+\dots
\end{aligned}
$$

$$
\begin{aligned}
I_1(1)&=\frac{5}{\varepsilon^3}-\frac{163}{12\varepsilon^2}+\Big(\frac{15\zeta_2}{8}+\frac{1223}{48}\Big)\varepsilon^{-1}+\dots\\
I_1(2)&=\frac{130}{27\varepsilon^3}-\frac{695}{54\varepsilon^2}+\Big(\frac{65\zeta_2}{36}+\frac{46379}{1944}\Big)\varepsilon^{-1}+\dots\\
I_1(3)&=\frac{169}{36\varepsilon^3}-\frac{395}{32\varepsilon^2}+\Big(\frac{169\zeta_2}{96}+\frac{470071}{20736}\Big)\varepsilon^{-1}+\dots
\end{aligned}
$$

using, e.g., an extension of `MATAD` (M. Steinhauser) or tools given in [arXiv:1405.4259 [hep-ph]]

Step 3: Solve the scalar recurrence

$$\begin{aligned}
&-2(N+1)(N+2)(\varepsilon+N+2)I_1(N)-(N+2)\big(2\varepsilon^2-5\varepsilon N-7\varepsilon-6N^2-28N-32\big)\\
&+\big(\varepsilon^3+4\varepsilon^2N+14\varepsilon^2-4\varepsilon N^2-13\varepsilon N-3\varepsilon-6N^3-50N^2-136N-120\big)I_1(N+2)\\
&-(\varepsilon-N-2)(\varepsilon+N+4)(\varepsilon+2N+8)I_1(N+3)\\
&=-\frac{4(N+2)}{3(N+3)}\varepsilon^{-3}+\frac{2\big(4N^4+35N^3+101N^2+105N+25\big)}{3(N+1)(N+2)(N+3)^2}\varepsilon^{-2}+\dots
\end{aligned}$$

$$I_1(1)=\frac{5}{\varepsilon^3}-\frac{163}{12\varepsilon^2}+\Big(\frac{15\zeta_2}{8}+\frac{1223}{48}\Big)\varepsilon^{-1}+\dots$$

$$I_1(2)=\frac{130}{27\varepsilon^3}-\frac{695}{54\varepsilon^2}+\Big(\frac{65\zeta_2}{36}+\frac{46379}{1944}\Big)\varepsilon^{-1}+\dots$$

$$I_1(3)=\frac{169}{36\varepsilon^3}-\frac{395}{32\varepsilon^2}+\Big(\frac{169\zeta_2}{96}+\frac{470071}{20736}\Big)\varepsilon^{-1}+\dots$$

using, e.g., an extension of `MATAD` (M. Steinhauser) or tools given in [arXiv:1405.4259 [hep-ph]]

$\downarrow$ (`Sigma.m`'s recurrence solver, see first slides)

$$\begin{aligned}
I_1(N)=&\Big(\frac{4\big(3N^2+6N+4\big)}{3(N+1)^2}+\frac{4S_1(N)}{3(N+1)}\Big)\varepsilon^{-3}\\
&-\Big(\frac{2\big(20N^3+58N^2+57N+22\big)}{3(N+1)^3}+\frac{S_1(N)^2}{N+1}+\frac{2(N+2)(2N-1)S_1(N)}{3(N+1)^2}-\frac{S_2(N)}{N+1}\Big)\varepsilon^{-2}+\dots
\end{aligned}$$

Step 4: Compute $I_2(N)$ and $I_3(N)$:

Recall: by uncoupling we expressed $I_2(N)$ and $I_3(N)$ by $I_1(N)$

Step 4: Compute $I_2(N)$ and $I_3(N)$:

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$$\begin{aligned}I_2(N) =& \square I_1(N) + \square I_1(N+1) + \square I_1(N+2)\\ &- \tfrac{2(N+2)}{3(N+1)}\varepsilon^{-3} + \left(\tfrac{6N^3+25N^2+33N+15}{3(N+1)^2(N+2)} + \tfrac{(-2N-1)}{3(N+1)}S_1(N)\right)\varepsilon^{-2} + \dots\end{aligned}$$

$$\begin{aligned}I_3(N) =& \square I_1(N) + \square I_1(N+1) + \square I_1(N+2)\\ &+ \tfrac{2(N+2)}{3(N+1)}\varepsilon^{-3} + \left(\tfrac{-2N^3-3N^2+3N+3}{3(N+1)^2(N+2)} + \tfrac{(2N+1)}{3(N+1)}S_1(N)\right)\varepsilon^{-2} + \dots\end{aligned}$$

Step 4: Compute $I_2(N)$ and $I_3(N)$:

Recall: by uncoupling we expressed $I_2(N)$ and $I_3(N)$ by $I_1(N)$, i.e.,

$$I_2(N) = \square I_1(N) + \square I_1(N+1) + \square I_1(N+2) - \frac{2(N+2)}{3(N+1)}\varepsilon^{-3} + \left(\frac{6N^3+25N^2+33N+15}{3(N+1)^2(N+2)} + \frac{(-2N-1)}{3(N+1)}S_1(N)\right)\varepsilon^{-2} + \dots$$

$$I_3(N) = \square I_1(N) + \square I_1(N+1) + \square I_1(N+2) + \frac{2(N+2)}{3(N+1)}\varepsilon^{-3} + \left(\frac{-2N^3-3N^2+3N+3}{3(N+1)^2(N+2)} + \frac{(2N+1)}{3(N+1)}S_1(N)\right)\varepsilon^{-2} + \dots$$

This yields

$$I_2(N) = \frac{4}{3\varepsilon^3} - \frac{2}{\varepsilon^2} + \left(-\tfrac{1}{3}S_1(N)^2 + \tfrac{2}{3}S_1(N) - \tfrac{1}{3}S_2(N) + \tfrac{5N+7}{3(N+1)} + \tfrac{\zeta_2}{2}\right)\varepsilon^{-1} + \dots$$

$$I_3(N) = \frac{8}{3\varepsilon^3} + \left(\frac{4(N+2)}{3(N+1)}S_1(N) - \frac{4\left(4N^2+7N+2\right)}{3(N+1)^2}\right)\varepsilon^{-2} + \left(-\frac{2\left(4N^2+11N+10\right)}{3(N+1)^2}S_1(N) + \frac{2\left(12N^3+32N^2+25N+2\right)}{3(N+1)^3} + \frac{(N-2)}{3(N+1)}S_1(N)^2 + \frac{(N-2)}{3(N+1)}S_2(N) + \zeta_2\right)\varepsilon^{-1} + \dots$$

Compute the remaining integrals

$$\underbrace{\sum_{N=0}^{\infty} D_4(N)x^N}_{\hat{D}_4(x)} = \boxed{\frac{\left(1545842\varepsilon^5x^5-14325922\varepsilon^5x^4+\dots+1524096x^2-653184x\right)}{23328\varepsilon^2(x-1)x^5}}\hat{B}_1(x)$$

$$+\Box\hat{B}_2(x)+\dots+\Box\hat{B}_{52}(x)$$

$$+\boxed{-\frac{\left(122\varepsilon^4x^3-2647\varepsilon^4x^2+\dots-304\varepsilon+24x^3-24x\right)}{4\varepsilon x^4}}\hat{I}_1(x)$$

$$+\boxed{\frac{\left(589\varepsilon^5x^3-20123\varepsilon^5x^2+\dots-896\varepsilon+96x^3-96x\right)}{16\varepsilon^2x^4}}\hat{I}_2(x)$$

$$+\boxed{\frac{\left(589\varepsilon^5x^3-21509\varepsilon^5x^2+\dots-1152\varepsilon+96x^3-96x\right)}{16\varepsilon^2x^4}}\hat{I}_3(x)$$

$$+\Box\hat{I}_4(x)+\dots+\Box\hat{I}_{15}(x)$$

Analogously, all $\hat{I}_j(x)=\sum_{N=0}^{\infty} I_j(N)x^N$, $j=1,\dots,15$ can be computed.

Final step: Insert all subresults

$$\underbrace{\sum_{N=0}^{\infty} D_4(N)x^N}_{\hat{D}_4(x)} = \boxed{\frac{\left(1545842\varepsilon^5x^5-14325922\varepsilon^5x^4+\dots+1524096x^2-653184x\right)}{23328\varepsilon^2(x-1)x^5}}\hat{B}_1(x)$$

$$+\square\hat{B}_2(x)+\dots+\square\hat{B}_{52}(x)$$

$$+\boxed{-\frac{\left(122\varepsilon^4x^3-2647\varepsilon^4x^2+\dots-304\varepsilon+24x^3-24x\right)}{4\varepsilon x^4}}\hat{I}_1(x)$$

$$+\boxed{\frac{\left(589\varepsilon^5x^3-20123\varepsilon^5x^2+\dots-896\varepsilon+96x^3-96x\right)}{16\varepsilon^2x^4}}\hat{I}_2(x)$$

$$+\boxed{\frac{\left(589\varepsilon^5x^3-21509\varepsilon^5x^2+\dots-1152\varepsilon+96x^3-96x\right)}{16\varepsilon^2x^4}}\hat{I}_3(x)$$

$$+\square\hat{I}_4(x)+\dots+\square\hat{I}_{15}(x)$$

Plugging in all expansion and extracting the N-th coefficient

(using `HarmonicSums.m`, `Sigma.m`, `EvaluateMultiSum.m`, `SumProduction.m`)

yield

$$I_4(N) = \left(\frac{64\left(N^2+N-1\right)}{3(N+1)(N+2)(N+3)(N+4)} - \frac{64S_1(N)}{3(N+3)(N+4)}\right)\varepsilon^{-3}$$

$$I_4(N) = \Big(\frac{64(N^2+N-1)}{3(N+1)(N+2)(N+3)(N+4)} - \frac{64 S_1(N)}{3(N+3)(N+4)}\Big)\varepsilon^{-3}$$

$$+ \Big(\frac{4(5N+27)}{3(N+2)(N+3)(N+4)} S_1(N)^2 - \frac{4(3N^5+68N^4+379N^3+648N^2-98N-696)}{3(N+1)(N+2)^2(N+3)^2(N+4)^2} S_1(N)$$
$$+ \frac{4(14N^6+214N^5+1179N^4+3050N^3+4097N^2+3094N+1200)}{3(N+1)^2(N+2)^2(N+3)^2(N+4)^2} + \frac{4(N+1)(4N+17)S_2(N)}{3(N+2)(N+3)(N+4)}\Big)\varepsilon^{-2}$$

$$I_4(N) = \Big(\frac{64\left(N^2+N-1\right)}{3(N+1)(N+2)(N+3)(N+4)} - \frac{64 S_1(N)}{3(N+3)(N+4)}\Big)\varepsilon^{-3}$$

$$+ \Big(\frac{4(5N+27)}{3(N+2)(N+3)(N+4)} S_1(N)^2 - \frac{4\left(3N^5+68N^4+379N^3+648N^2-98N-696\right)}{3(N+1)(N+2)^2(N+3)^2(N+4)^2} S_1(N)$$
$$+ \frac{4\left(14N^6+214N^5+1179N^4+3050N^3+4097N^2+3094N+1200\right)}{3(N+1)^2(N+2)^2(N+3)^2(N+4)^2} + \frac{4(N+1)(4N+17)S_2(N)}{3(N+2)(N+3)(N+4)}\Big)\varepsilon^{-2}$$

$$+ \Big(\frac{-23N^2-35N-176}{9(N+2)(N+3)(N+4)(N+5)} S_1(N)^3 - \frac{2\left(10N^2+53N+106\right)}{3(N+2)(N+3)(N+4)} S_{2,1}(N) + \zeta_2\Big(\frac{8\left(N^2+N-1\right)}{(N+1)(N+2)(N+3)(N+4)}$$
$$- \frac{8}{(N+3)(N+4)} S_1(N)\Big) + \frac{-8N^3-95N^2-171N-56}{3(N+2)(N+3)(N+4)(N+5)} S_2(N) S_1(N) + \frac{2(30N^3+469N^2+1873N+2542)}{9(N+2)(N+3)(N+4)(N+5)} S_3(N)$$
$$+ \frac{25N^6+213N^5+491N^4-1007N^3-7942N^2-15988N-10340}{3(N+1)(N+2)^2(N+3)^2(N+4)^2(N+5)} S_1(N)^2$$
$$+ \frac{-85N^6-1469N^5-8965N^4-23889N^3-25644N^2-3724N+5780}{3(N+1)(N+2)^2(N+3)^2(N+4)^2(N+5)} S_2(N)$$
$$- \frac{2(94N^{10}+2202N^9+22629N^8+133916N^7+505769N^6+\cdots+1817100N+563760)}{3(N+1)^2(N+2)^3(N+3)^3(N+4)^3(N+5)} S_1(N)$$
$$- \frac{2\left(44N^{11}+1696N^{10}+26555N^9+230482N^8+\cdots+4371092N+623040\right)}{3(N+1)^3(N+2)^3(N+3)^3(N+4)^3(N+5)}\Big)\varepsilon^{-1}$$

$$I_4(N) = \Big(\frac{64(N^2+N-1)}{3(N+1)(N+2)(N+3)(N+4)} - \frac{64S_1(N)}{3(N+3)(N+4)}\Big)\varepsilon^{-3}$$

$$+\Big(\frac{4(5N+27)}{3(N+2)(N+3)(N+4)}S_1(N)^2 - \frac{4(3N^5+68N^4+379N^3+648N^2-98N-696)}{3(N+1)(N+2)^2(N+3)^2(N+4)^2}S_1(N)$$
$$+\frac{4(14N^6+214N^5+1179N^4+3050N^3+4097N^2+3094N+1200)}{3(N+1)^2(N+2)^2(N+3)^2(N+4)^2} + \frac{4(N+1)(4N+17)S_2(N)}{3(N+2)(N+3)(N+4)}\Big)\varepsilon^{-2}$$

$$+\Big(\frac{-23N^2-35N-176}{9(N+2)(N+3)(N+4)(N+5)}S_1(N)^3 - \frac{2(10N^2+53N+106)}{3(N+2)(N+3)(N+4)}S_{2,1}(N) + \zeta_2\Big(\frac{8(N^2+N-1)}{(N+1)(N+2)(N+3)(N+4)}$$
$$-\frac{8}{(N+3)(N+4)}S_1(N)\Big) + \frac{-8N^3-95N^2-171N-56}{3(N+2)(N+3)(N+4)(N+5)}S_2(N)S_1(N) + \frac{2(30N^3+469N^2+1873N+2542)}{9(N+2)(N+3)(N+4)(N+5)}S_3(N)$$
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Arising objects:

$$\zeta_2, \zeta_3, (-1)^N, 2^N, S_{-3}(N), S_1(N), S_2(N), S_3(N), S_4(N), S_{-2,1}(N),$$
$$S_{2,1}(N), S_{3,1}(N)$$

[J.A.M. Vermaseren, 1998; J. Blümlein/S. Kurth, 1998]

$$I_4(N) = \Big(\frac{64(N^2+N-1)}{3(N+1)(N+2)(N+3)(N+4)} - \frac{64S_1(N)}{3(N+3)(N+4)}\Big)\varepsilon^{-3}$$

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$$S_{1,1,1,1}\left(2,\tfrac{1}{2},1,1,N\right) = \sum_{k=1}^{N}\frac{2^k\sum_{i=1}^{k}\frac{2^{-i}\sum_{j=1}^{i}\frac{\sum_{r=1}^{j}\frac{1}{r}}{j}}{i}}{k}$$

S. Moch, P. Uwer, S. Weinzierl, 2002

New algorithms for asymptotic expansions

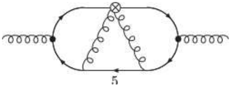
using the underlying integral representation (available in HarmonicSums.m)

- Generalized harmonic sums

$$
\begin{aligned}
S_{1,1,1,1}\left(2,\tfrac{1}{2},1,1,N\right) &= -\frac{21\zeta_2^2}{20}+\frac{1}{N}+\frac{1}{8N^2}+\frac{295}{216N^3}-\frac{1115}{96N^4}+O(N^5)\\
&+\left(\frac{1}{2N}-\frac{3}{4N^2}+\frac{19}{12N^3}-\frac{5}{N^4}+O(N^5)\right)\zeta_2\\
&+2^N\left(\frac{3}{2N}+\frac{3}{2N^2}+\frac{9}{2N^3}+\frac{39}{2N^4}+O(N^5)\right)\zeta_3\\
&+\left(\frac{1}{N}+\frac{3}{4N^2}-\frac{157}{36N^3}+\frac{19}{N^4}+O(N^5)\right)(\log(N)+\gamma)\\
&+\left(\frac{1}{2N}-\frac{3}{4N^2}+\frac{19}{12N^3}-\frac{5}{N^4}+O(N^5)\right)(\log(N)+\gamma)^2
\end{aligned}
$$

[Ablinger, Blümlein, CS, J. Math. Phys. 54, 2013, arXiv:1302.0378 [math-ph]]

An even more challenging diagram (V-type graph with five massive propagators)

$D_5(N) =$ 

$$\stackrel{?}{=} F_{-3}(N)\varepsilon^{-3} + F_{-2}(N)\varepsilon^{-2} + F_{-1}(N)\varepsilon^{-1} + F_0(N)\varepsilon^{0} + \dots$$

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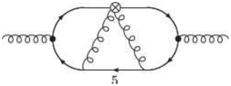
$D_5(N) =$ 

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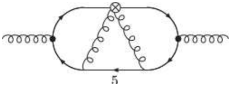
$D_5(N) =$ 

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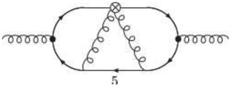

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Note: binomial sums occur in the output

New algorithms for asymptotic expansions

using the underlying integral representation (available in HarmonicSums.m)

- Nested binomial sums

$$\sum_{j=1}^{N} \frac{4^j S_1(j-1)}{\binom{2j}{j} j^2} = 7\zeta_3 + \sqrt{\pi}\sqrt{N}\Bigg\{\Bigg[-\frac{2}{N} + \frac{5}{12N^2} - \frac{21}{320N^3} - \frac{223}{10752N^4} + \frac{671}{49152N^5} + \frac{11635}{1441792N^6} - \frac{1196757}{136314880N^7} - \frac{376193}{50331648N^8} + \frac{201980317}{18253611008N^9} + O(N^{-10})\Bigg]\ln(\bar{N}) - \frac{4}{N} + \frac{5}{18N^2} - \frac{263}{2400N^3} + \frac{579}{12544N^4} + \frac{10123}{1105920N^5} - \frac{1705445}{71368704N^6} - \frac{27135463}{11164188672N^7} + \frac{197432563}{7927234560N^8} + \frac{405757489}{775778467840N^9} + O(N^{-10})\Bigg\}$$

Ablinger, Blümlein, CS, ACAT 2013, arXiv:1310.5645 [math-ph]

Ablinger, Blümlein, Raab, CS, 2014. arXiv:1407.1822 [hep-th]

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- New mathematics has being developed to explore the new function spaces (asymptotic expansions).