SYNASC 2014, West University of Timisoara, Romania

Symbolic Summation in Difference Rings with Applications in Combinatorics, Numerics, and Physics

Carsten Schneider
RISC, J. Kepler University Linz, Austria
September 25, 2014

Abstract: Symbolic summation deals with the simplification of formulas or with proving identities given in terms of (usually complicated) multiple nested sums. A first breakthrough in this research area was achieved for indefinite single sums over hypergeometric expressions (e.g., single sums over binomial coefficients or factorials). Namely, using Gospers telescoping algorithm (1978) one can decide, if such a sum can be written in terms of hypergeometric expressions. Another crucial contribution is Zeilbergers observation (1991) that Gospers algorithm can be extended to creative telescoping. This enables the user to compute linear recurrence relations for definite hypergeometric sums. Finally, Pektovseks Hyper algorithm determines all hypergeometric solutions of a given linear recurrence with polynomial coefficients. Combining these algorithms yields a method that decides if a definite hypergeometric sum can be expressed as a linear combination of hypergeometric expressions.
In this tutorial we present a generalization of these summation techniques that work not only for sums over hypergeometric expressions but for sums over indefinite nested multiple sums and products. Here we focus on the following two aspects.
* Indefinite summation: Simplify expressions in terms of indefinite nested sums and products such that the occurring sums are algebraically independent and such the obtained expressions consist of sums that have certain optimality criteria (such as minimal nesting depth or minimal degree in the denominators).
* Definite summation: Compute recurrences (based on the paradigm of Zs creative telescoping) for definite sums with summands given in terms of indefinite nested sums and products, and solve the derived recurrences in terms of indefinite nested sums and products (also called dAlembertian solutions, a subclass of Liouvillian solutions). In this way, one can decide (similarly to the hypergeometric case) if such a definite nested sum can be expressed in terms of indefinite nested sums and products.
The algorithmic framework of this approach is based on Karrs difference fields (1981) and a recent generalization to a difference ring theory. We will work out the underlying ideas and algorithms and present the current developments and improvements in this research topic. Special emphasis will be put on concrete examples arising, e.g., from combinatorial problems and numerics. In particular, we demonstrate how the symbolic summation algorithms can be utilized to evaluate whole classes of Feynman integrals arising in the context of Elementary Quantum Field Theory. The latter examples pop up within an intensive cooperation with the Theory Group (Johannes Blmlein) of the Deutsches Elektronen-Synchrotron DESY, Zeuthen.

Abstract: Symbolic summation deals with the simplification of formulas or with proving identities given in terms of (usually complicated) multiple nested sums. A first breakthrough in this research area was achieved for indefinite single sums over hypergeometric expressions (e.g., single sums over binomial coefficients or factorials). Namely, using Gospers telescoping algorithm (1978) one can decide, if such a sum can be written in terms of hypergeometric expressions. Another crucial contribution is Zeilbergers observation (1991) that Gospers algorithm can be extended to creative telescoping. This enables the user to compute linear recurrence relations for definite hypergeometric sums. Finally, Pektovseks Hyper algorithm determines all hypergeometric solutions of a given linear recurrence with polynomial coefficients. Combining these algorithms yields a method that decides if a definite hypergeometric sum can be expressed as a linear combination of hypergeometric expressions.
In this tutorial we present a generalization of these summation techniques that work not only for sums over hypergeometric expressions but for sums over indefinite nested multiple sums and products. Here we focus on the following two aspects.
* Indefinite summation: Simplify expressions in terms of indefinite nested sums and products such that the occurring sums are algebraically independent and such the obtained expressions consist of sums that have certain optimality criteria (such as minimal nesting depth or minimal degree in the denominators).
* Definite summation: Compute recurrences (based on the paradigm of Zs creative telescoping) for definite sums with summands given in terms of indefinite nested sums and products, and solve the derived recurrences in terms of indefinite nested sums and products (also called dAlembertian solutions, a subclass of Liouvillian solutions). In this way, one can decide (similarly to the hypergeometric case) if such a definite nested sum can be expressed in terms of indefinite nested sums and products.
The algorithmic framework of this approach is based on Karrs difference fields (1981) and a recent generalization to a difference ring theory. We will work out the underlying ideas and algorithms and present the current developments and improvements in this research topic. Special emphasis will be put on concrete examples arising, e.g., from combinatorial problems and numerics. In particular, we demonstrate how the symbolic summation algorithms can be utilized to evaluate whole classes of Feynman integrals arising in the context of Elementary Quantum Field Theory. The latter examples pop up within an intensive cooperation with the Theory Group (Johannes Blmlein) of the Deutsches Elektronen-Synchrotron DESY, Zeuthen.

Abstract: Symbolic summation deals with the simplification of formulas or with proving identities given in terms of (usually complicated) multiple nested sums. A first breakthrough in this research area was achieved for indefinite single sums over hypergeometric expressions (e.g., single sums over binomial coefficients or factorials). Namely, using Gospers telescoping algorithm (1978) one can decide, if such a sum can be written in terms of hypergeometric expressions. Another crucial contribution is Zeilbergers observation (1991) that Gospers algorithm can be extended to creative telescoping. This enables the user to compute linear recurrence relations for definite hypergeometric sums. Finally, Pektovseks Hyper algorithm determines all hypergeometric solutions of a given linear recurrence with polynomial coefficients. Combining these algorithms yields a method that decides if a definite hypergeometric sum can be expressed as a linear combination of hypergeometric expressions.

In this tutorial we present a generalization of these summation techniques that work not only for sums over hypergeometric expressions but for sums over indefinite nested multiple sums and products. Here we focus on the following two aspects.

* Indefinite summation: Simplify expressions in terms of indefinite nested sums and products such that the occurring sums are algebraically independent and such the obtained expressions consist of sums that have certain optimality criteria (such as minimal nesting depth or minimal degree in the denominators).

* Definite summation: Compute recurrences (based on the paradigm of Zs creative telescoping) for definite sums with summands given in terms of indefinite nested sums and products, and solve the derived recurrences in terms of indefinite nested sums and products (also called dAlembertian solutions, a subclass of Liouvillian solutions). In this way, one can decide (similarly to the hypergeometric case) if such a definite nested sum can be expressed in terms of indefinite nested sums and products.

The algorithmic framework of this approach is based on Karrs difference fields (1981) and a recent generalization to a difference ring theory. We will work out the underlying ideas and algorithms and present the current developments and improvements in this research topic. Special emphasis will be put on concrete examples arising, e.g., from combinatorial problems and numerics. In particular, we demonstrate how the symbolic summation algorithms can be utilized to evaluate whole classes of Feynman integrals arising in the context of Elementary Quantum Field Theory. The latter examples pop up within an intensive cooperation with the Theory Group (Johannes Blmlein) of the Deutsches Elektronen-Synchrotron DESY, Zeuthen.

Abstract: Symbolic summation deals with the simplification of formulas or with proving identities given in terms of (usually complicated) multiple nested sums. A first breakthrough in this research area was achieved for indefinite single sums over hypergeometric expressions (e.g., single sums over binomial coefficients or factorials). Namely, using Gospers telescoping algorithm (1978) one can decide, if such a sum can be written in terms of hypergeometric expressions. Another crucial contribution is Zeilbergers observation (1991) that Gospers algorithm can be extended to creative telescoping. This enables the user to compute linear recurrence relations for definite hypergeometric sums. Finally, Pektovseks Hyper algorithm determines all hypergeometric solutions of a given linear recurrence with polynomial coefficients. Combining these algorithms yields a method that decides if a definite hypergeometric sum can be expressed as a linear combination of hypergeometric expressions.
In this tutorial we present a generalization of these summation techniques that work not only for sums over hypergeometric expressions but for sums over indefinite nested multiple sums and products. Here we focus on the following two aspects.

* Indefinite summation: Simplify expressions in terms of indefinite nested sums and products such that the occurring sums are algebraically independent and such the obtained expressions consist of sums that have certain optimality criteria (such as minimal nesting depth or minimal degree in the denominators).
* Definite summation: Compute recurrences (based on the paradigm of Zs creative telescoping) for definite sums with summands given in terms of indefinite nested sums and products, and solve the derived recurrences in terms of indefinite nested sums and products (also called dAlembertian solutions, a subclass of Liouvillian solutions). In this way, one can decide (similarly to the hypergeometric case) if such a definite nested sum can be expressed in terms of indefinite nested sums and products.

The algorithmic framework of this approach is based on Karrs difference fields (1981) and a recent generalization to a difference ring theory. We will work out the underlying ideas and algorithms and present the current developments and improvements in this research topic. Special emphasis will be put on concrete examples arising, e.g., from combinatorial problems and numerics. In particular, we demonstrate how the symbolic summation algorithms can be utilized to evaluate whole classes of Feynman integrals arising in the context of Elementary Quantum Field Theory. The latter examples pop up within an intensive cooperation with the Theory Group (Johannes Blmlein) of the Deutsches Elektronen-Synchrotron DESY, Zeuthen.

Abstract: Symbolic summation deals with the simplification of formulas or with proving identities given in terms of (usually complicated) multiple nested sums. A first breakthrough in this research area was achieved for indefinite single sums over hypergeometric expressions (e.g., single sums over binomial coefficients or factorials). Namely, using Gospers telescoping algorithm (1978) one can decide, if such a sum can be written in terms of hypergeometric expressions. Another crucial contribution is Zeilbergers observation (1991) that Gospers algorithm can be extended to creative telescoping. This enables the user to compute linear recurrence relations for definite hypergeometric sums. Finally, Pektovseks Hyper algorithm determines all hypergeometric solutions of a given linear recurrence with polynomial coefficients. Combining these algorithms yields a method that decides if a definite hypergeometric sum can be expressed as a linear combination of hypergeometric expressions.
In this tutorial we present a generalization of these summation techniques that work not only for sums over hypergeometric expressions but for sums over indefinite nested multiple sums and products. Here we focus on the following two aspects.

* Indefinite summation: Simplify expressions in terms of indefinite nested sums and products such that the occurring sums are algebraically independent and such the obtained expressions consist of sums that have certain optimality criteria (such as minimal nesting depth or minimal degree in the denominators).
* Definite summation: Compute recurrences (based on the paradigm of Zs creative telescoping) for definite sums with summands given in terms of indefinite nested sums and products, and solve the derived recurrences in terms of indefinite nested sums and products (also called dAlembertian solutions, a subclass of Liouvillian solutions). In this way, one can decide (similarly to the hypergeometric case) if such a definite nested sum can be expressed in terms of indefinite nested sums and products.

The algorithmic framework of this approach is based on Karrs difference fields (1981) and a recent generalization to a difference ring theory. We will work out the underlying ideas and algorithms and present the current developments and improvements in this research topic. Special emphasis will be put on concrete examples arising, e.g., from combinatorial problems and numerics. In particular, we demonstrate how the symbolic summation algorithms can be utilized to evaluate whole classes of Feynman integrals arising in the context of Elementary Quantum Field Theory. The latter examples pop up within an intensive cooperation with the Theory Group (Johannes Blmlein) of the Deutsches Elektronen-Synchrotron DESY, Zeuthen.

Abstract: Symbolic summation deals with the simplification of formulas or with proving identities given in terms of (usually complicated) multiple nested sums. A first breakthrough in this research area was achieved for indefinite single sums over hypergeometric expressions (e.g., single sums over binomial coefficients or factorials). Namely, using Gospers telescoping algorithm (1978) one can decide, if such a sum can be written in terms of hypergeometric expressions. Another crucial contribution is Zeilbergers observation (1991) that Gospers algorithm can be extended to creative telescoping. This enables the user to compute linear recurrence relations for definite hypergeometric sums. Finally, Pektovseks Hyper algorithm determines all hypergeometric solutions of a given linear recurrence with polynomial coefficients. Combining these algorithms yields a method that decides if a definite hypergeometric sum can be expressed as a linear combination of hypergeometric expressions.
In this tutorial we present a generalization of these summation techniques that work not only for sums over hypergeometric expressions but for sums over indefinite nested multiple sums and products. Here we focus on the following two aspects.

* Indefinite summation: Simplify expressions in terms of indefinite nested sums and products such that the occurring sums are algebraically independent and such the obtained expressions consist of sums that have certain optimality criteria (such as minimal nesting depth or minimal degree in the denominators).
* Definite summation: Compute recurrences (based on the paradigm of Zs creative telescoping) for definite sums with summands given in terms of indefinite nested sums and products, and solve the derived recurrences in terms of indefinite nested sums and products (also called dAlembertian solutions, a subclass of Liouvillian solutions). In this way, one can decide (similarly to the hypergeometric case) if such a definite nested sum can be expressed in terms of indefinite nested sums and products.

The algorithmic framework of this approach is based on Karrs difference fields (1981) and a recent generalization to a difference ring theory. We will work out the underlying ideas and algorithms and present the current developments and improvements in this research topic. Special emphasis will be put on concrete examples arising, e.g., from combinatorial problems and numerics. In particular, we demonstrate how the symbolic summation algorithms can be utilized to evaluate whole classes of Feynman integrals arising in the context of Elementary Quantum Field Theory. The latter examples pop up within an intensive cooperation with the Theory Group (Johannes Blmlein) of the Deutsches Elektronen-Synchrotron DESY, Zeuthen.

 (in 90 minutes)

Abstract: Symbolic summation deals with the simplification of formulas or with proving identities given in terms of (usually complicated) multiple nested sums. A first breakthrough in this research area was achieved for indefinite single sums over hypergeometric expressions (e.g., single sums over binomial coefficients or factorials). Namely, using Gospers telescoping algorithm (1978) one can decide, if such a sum can be written in terms of hypergeometric expressions. Another crucial contribution is Zeilbergers observation (1991) that Gospers algorithm can be extended to creative telescoping. This enables the user to compute linear recurrence relations for definite hypergeometric sums. Finally, Pektovseks Hyper algorithm determines all hypergeometric solutions of a given linear recurrence with polynomial coefficients. Combining these algorithms yields a method that decides if a definite hypergeometric sum can be expressed as a linear combination of hypergeometric expressions.
In this tutorial we present a generalization of these summation techniques that work not only for sums over hypergeometric expressions but for sums over indefinite nested multiple sums and products. Here we focus on the following two aspects.

* **Indefinite summation**: Simplify expressions in terms of indefinite nested sums and products such that the occurring sums are algebraically independent and such the obtained expressions consist of sums that have certain optimality criteria (such as minimal nesting depth or minimal degree in the denominators).
* **Definite summation:** Compute recurrences (based on the paradigm of Zs creative telescoping) for definite sums with summands given in terms of indefinite nested sums and products, and solve the derived recurrences in terms of indefinite nested sums and products (also called dAlembertian solutions, a subclass of Liouvillian solutions). In this way, one can decide (similarly to the hypergeometric case) if such a definite nested sum can be expressed in terms of indefinite nested sums and products.

The algorithmic framework of this **approach is based on Karrs difference fields (1981)** and a **recent generalization to a difference ring theory**. We will work out the underlying ideas and algorithms and present the current developments and improvements in this research topic. Special emphasis will be put on **concrete examples arising, e.g., from combinatorial problems and numerics**. In particular, we demonstrate how the symbolic summation algorithms can be utilized to evaluate whole classes of Feynman integrals arising in the context of **Elementary Quantum Field Theory**. The latter examples pop up within an intensive cooperation with the Theory Group (Johannes Blmlein) of the Deutsches Elektronen-Synchrotron DESY, Zeuthen.

☺ (takes only 90 minutes)

Abstract: Symbolic summation deals with the simplification of formulas or with proving identities given in terms of (usually complicated) multiple nested sums. A first breakthrough in this research area was achieved for indefinite single sums over hypergeometric expressions (e.g., single sums over binomial coefficients or factorials). Namely, using Gospers telescoping algorithm (1978) one can decide, if such a sum can be written in terms of hypergeometric expressions. Another crucial contribution is Zeilbergers observation (1991) that Gospers algorithm can be extended to creative telescoping. This enables the user to compute linear recurrence relations for definite hypergeometric sums. Finally, Pektovseks Hyper algorithm determines all hypergeometric solutions of a given linear recurrence with polynomial coefficients. Combining these algorithms yields a method that decides if a definite hypergeometric sum can be expressed as a linear combination of hypergeometric expressions.
In this tutorial we present a generalization of these summation techniques that work not only for sums over hypergeometric expressions but for sums over indefinite nested multiple sums and products. Here we focus on the following two aspects.

* **Indefinite summation**: Simplify expressions in terms of indefinite nested sums and products such that the occurring sums are algebraically independent and such the obtained expressions consist of sums that have certain optimality criteria (such as minimal nesting depth or minimal degree in the denominators).
* **Definite summation:** Compute recurrences (based on the paradigm of Zs creative telescoping) for definite sums with summands given in terms of indefinite nested sums and products, and solve the derived recurrences in terms of indefinite nested sums and products (also called dAlembertian solutions, a subclass of Liouvillian solutions). In this way, one can decide (similarly to the hypergeometric case) if such a definite nested sum can be expressed in terms of indefinite nested sums and products.

The algorithmic framework of this **approach is based on Karrs difference fields (1981)** and a **recent generalization to a difference ring theory**. We will work out the underlying ideas and algorithms and present the current developments and improvements in this research topic. Special emphasis will be put on **concrete examples arising, e.g., from combinatorial problems and numerics**. In particular, we demonstrate how the symbolic summation algorithms can be utilized to evaluate whole classes of Feynman integrals arising in the context of **Elementary Quantum Field Theory**. The latter examples pop up within an intensive cooperation with the Theory Group (Johannes Blmlein) of the Deutsches Elektronen-Synchrotron DESY, Zeuthen.

(I will do my best)

For overview publications see:

- CS. Symbolic summation assists combinatorics. *Sém. Lothar. Combin.* 56(B56), 36pp, 2007.
- CS. Simplifying Multiple Sums in Difference Fields. In: Computer Algebra in Quantum Field Theory: Integration, Summation and Special Functions, J. Blümlein, C. Schneider (ed.), Texts and Monographs in Symbolic Computation, pp. 325-360. Springer, 2013.
- CS. Modern Summation Methods for Loop Integrals in Quantum Field Theory: The Packages Sigma, EvaluateMultiSums and SumProduction. In: Proc. ACAT 2013, J. Phys.: Conf. Ser.: 523/012037, pp. 1-17, 2014.

Relevant research literature: next slides

For all my contributions (including these slides) see:
http://www.risc.jku.at/home/cschneid.

Part 1: Indefinite summation

Simplify

$$\sum_{k=1}^{n} S_1(k) = \text{?}$$

where $S_1(k) = \sum_{i=1}^{k} \frac{1}{i} \quad (= H_k)$

Software

Telescoping

GIVEN $f(k) = S_1(k)$.

FIND $g(k)$:

$$\boxed{f(k) = g(k+1) - g(k)}$$

for all $1 \leq k \leq n$ and $n \geq 0$.

Telescoping

GIVEN $f(k) = S_1(k)$.

FIND $g(k)$:

$$\boxed{f(k) = g(k+1) - g(k)}$$

for all $1 \leq k \leq n$ and $n \geq 0$.

We compute

$$g(k) = (S_1(k) - 1)k.$$

Telescoping

GIVEN $f(k) = S_1(k)$.

FIND $g(k)$:

$$\boxed{f(k) = g(k+1) - g(k)}$$

for all $1 \leq k \leq n$ and $n \geq 0$.

Summing this equation over k from 1 to n gives

$$\boxed{\sum_{k=1}^{n} S_1(k)} = \boxed{g(n+1) - g(1)}$$

$$= (S_1(n+1) - 1)(n+1).$$

$$\sum_{k=0}^{a} F(k) = ?$$

$$\sum_{k=0}^{a} F(k) =?$$

$$\downarrow$$

Indefinite summation.

GIVEN $F(k)$.

FIND $G(k)$:

$$F(k) = G(k+1) - G(k)$$

and such that RHS is as "simple" as LHS.

$$\sum_{k=0}^{a} F(k) = ?$$

$$\downarrow$$

Indefinite summation.

GIVEN $F(k)$.

FIND $G(k)$:

$$F(k) = G(k+1) - G(k)$$

and such that RHS is as "simple" as LHS.

$$\downarrow$$

$$\boxed{\sum_{k=0}^{a} F(k) = G(a+1) - G(0)}$$

such that RHS is "simpler" than LHS.

$$\sum_{k=0}^{a} F(k) = ?$$

$\downarrow$ `SigmaReduce[`$\sum_{k=0}^{a} F(k)$`]`

Indefinite summation.

GIVEN nested product-sum expression $F(k)$.

FIND nested product-sum expression $G(k)$:

$$F(k) = G(k+1) - G(k)$$

and such that RHS is as "simple" as LHS.

$\downarrow$

$$\sum_{k=0}^{a} F(k) = G(a+1) - G(0)$$

such that RHS is "simpler" than LHS.

Definition. $F(k)$ is called (indefinite) nested product-sum expression w.r.t. k if it can be built by

Definition. $F(k)$ is called (indefinite) nested product-sum expression w.r.t. k if it can be built by

- k and a finite number of constants,

$$k, \quad 2, \quad \zeta_2$$

Definition. $F(k)$ is called (indefinite) nested product-sum expression w.r.t. k if it can be built by

- k and a finite number of constants,
- the four operations $(+, -, \cdot, /)$

$$\frac{k \cdot k + 2 \cdot k - \zeta_2}{(k+1)(k+2)}$$

Definition. $F(k)$ is called (indefinite) nested product-sum expression w.r.t. k if it can be built by

- k and a finite number of constants,
- the four operations $(+, -, \cdot, /)$
- indefinite nested sums and products

$$\sum_{i=l}^{k} f(i) \quad \text{or} \quad \prod_{i=l}^{k} f(i) \qquad l \in \mathbb{N}$$

$f(i)$: free of k and nested product-sum expression w.r.t. i.

$$\sum_{i=1}^{k} \frac{1}{i}, \quad \prod_{i=1}^{k} i$$

Definition. $F(k)$ is called (indefinite) nested product-sum expression w.r.t. k if it can be built by

- k and a finite number of constants,
- the four operations $(+, -, \cdot, /)$
- indefinite nested sums and products

$$\sum_{i=l}^{k} f(i) \quad \text{or} \quad \prod_{i=l}^{k} f(i) \qquad l \in \mathbb{N}$$

$f(i)$: free of k and nested product-sum expression w.r.t. i.

$$\sum_{j=1}^{k} \frac{\left(\sum_{i=1}^{j} \frac{1}{i}\right)^2}{j}$$

Definition. $F(k)$ is called (indefinite) nested product-sum expression w.r.t. k if it can be built by

- k and a finite number of constants,
- the four operations $(+,-,\cdot,/)$
- indefinite nested sums and products

$$\sum_{i=l}^{k} f(i) \quad \text{or} \quad \prod_{i=l}^{k} f(i) \qquad l \in \mathbb{N}$$

$f(i)$: free of k and nested product-sum expression w.r.t. i.

$$\sum_{j=1}^{k} \frac{\Big(\sum_{i=1}^{j} \frac{1}{i}\Big)^2}{\sum_{i=1}^{j} \frac{1}{(i^2+1)} + \prod_{i=1}^{j} \sum_{r=1}^{i} \frac{1}{r}}$$

Definition. $F(k)$ is called (indefinite) nested product-sum expression w.r.t. k if it can be built by

- k and a finite number of constants,
- the four operations $(+,-,\cdot,/)$
- indefinite nested sums and products

$$\sum_{i=l}^{k} f(i) \quad \text{or} \quad \prod_{i=l}^{k} f(i) \qquad l\in\mathbb{N}$$

$f(i)$: free of k and nested product-sum expression w.r.t. i.

```
+, −, ∗, /,   SigmaSum[f,{i,l,k}]   SigmaProduct[f,{i,l,k}]
```

$$\sum_{j=1}^{k} \frac{\Big(\sum_{i=1}^{j} \frac{1}{i}\Big)^2}{\sum_{i=1}^{j} \frac{1}{(i^2+1)} + \prod_{i=1}^{j} \sum_{r=1}^{i} \frac{1}{r}}$$

Telescoping in the given difference field

FIND a closed form for

$$\sum_{k=1}^{n} S_1(k).$$

A difference field for the summand

Consider the rational function field

$$\mathbb{F} := \mathbb{Q}$$

with the automorphism $\sigma : \mathbb{F} \to \mathbb{F}$ defined by

$$\sigma(c) = c \quad \forall c \in \mathbb{Q},$$

Telescoping in the given difference field

FIND a closed form for

$$\sum_{k=1}^{n} S_1(k).$$

A difference field for the summand

Consider the rational function field

$$\mathbb{F} := \mathbb{Q}(k)$$

with the automorphism $\sigma : \mathbb{F} \to \mathbb{F}$ defined by

$$\sigma(c) = c \quad \forall c \in \mathbb{Q},$$

$$\sigma(k) = k + 1, \qquad \mathcal{S}\, k = k + 1,$$

Telescoping in the given difference field

FIND a closed form for

$$\sum_{k=1}^{n} S_1(k).$$

A difference field for the summand

Consider the rational function field

$$\mathbb{F} := \mathbb{Q}(k)(s)$$

with the automorphism $\sigma : \mathbb{F} \to \mathbb{F}$ defined by

$$\sigma(c) = c \quad \forall c \in \mathbb{Q},$$

$$\sigma(k) = k + 1, \qquad \mathcal{S}\, k = k + 1,$$

$$\sigma(s) = s + \frac{1}{k+1}, \qquad \mathcal{S}\, S_1(k) = S_1(k) + \frac{1}{k+1}.$$

Telescoping in the given difference field

FIND $g \in \mathbb{F}$:

$$\boxed{\sigma(g) - g = s.}$$

Telescoping in the given difference field

FIND $g \in \mathbb{F}$:

$$\boxed{\sigma(g) - g = s.}$$

We compute

$$g = (s-1)k \in \mathbb{F}.$$

Telescoping in the given difference field

FIND $g \in \mathbb{F}$:

$$\boxed{\sigma(g) - g = s.}$$

We compute

$$g = (s-1)k \in \mathbb{F}.$$

This gives

$$\boxed{g(k+1) - g(k) = S_1(k)}$$

with

$$g(k) = (S_1(k) - 1)k.$$

Telescoping in the given difference field

FIND $g \in \mathbb{F}$:

$$\boxed{\sigma(g) - g = s.}$$

We compute

$$g = (s-1)k \in \mathbb{F}.$$

This gives

$$\boxed{g(k+1) - g(k) = S_1(k)}$$

with

$$g(k) = (S_1(k) - 1)k.$$

Hence,

$$(S_1(n+1) - 1)(n+1) = \sum_{k=1}^{n} S_1(k).$$

The basic summation algorithm

(a simplified version of Karr's algorithm, 1981)

C. Schneider. Fast Algorithms for Refined Parameterized Telescoping in Difference Fields. To appear in *Computer Algebra and Polynomials*, Lecture Notes in Computer Science (LNCS), Springer, 2014. arXiv:1307.7887 [cs.SC].

Exploits results from:

- M. Karr. Summation in finite terms. *J. ACM*, 28:305–350, 1981.
- M. Bronstein. On solutions of linear ordinary difference equations in their coefficient field. *J. Symbolic Comput.*, 29(6):841–877, 2000.
- C. Schneider. A collection of denominator bounds to solve parameterized linear difference equations in $\Pi\Sigma$-extensions. (SYNASC'04), *An. Univ. Timișoara Ser. Mat.-Inform.*, 42(2):163–179, 2004.

CONSTRUCT a difference field $(\mathbb{F}, \sigma)$:

- a rational function field

$$\mathbb{F} := \mathbb{K}$$

- with an automorphism

$$\sigma(c) = c \quad \forall c \in \mathbb{K}$$

CONSTRUCT a difference field $(\mathbb{F},\sigma)$:

- a rational function field

$$\mathbb{F} := \mathbb{K}(t_1)$$

- with an automorphism

$$\sigma(c) = c \quad \forall c \in \mathbb{K}$$
$$\sigma(t_1) = a_1\, t_1 + f_1, \quad a_1 \in \mathbb{K}^*, \qquad f_1 \in \mathbb{K}$$

CONSTRUCT a difference field $(\mathbb{F},\sigma)$:

- a rational function field

$$\mathbb{F} := \mathbb{K}(t_1)(t_2)$$

- with an automorphism

$$\begin{aligned}
\sigma(c) &= c \quad \forall c \in \mathbb{K} \\
\sigma(t_1) &= a_1\,t_1 + f_1, \quad a_1 \in \mathbb{K}^*, && f_1 \in \mathbb{K} \\
\sigma(t_2) &= a_2\,t_2 + f_2, \quad a_2 \in \mathbb{K}(t_1)^*, && f_2 \in \mathbb{K}(t_1)
\end{aligned}$$

CONSTRUCT a difference field $(\mathbb{F}, \sigma)$:

- a rational function field

$$\mathbb{F} := \mathbb{K}(t_1)(t_2)\dots(t_e)$$

- with an automorphism

$$\begin{aligned}
\sigma(c) &= c \quad \forall c \in \mathbb{K} \\
\sigma(t_1) &= a_1\,t_1 + f_1, \quad a_1 \in \mathbb{K}^*, && f_1 \in \mathbb{K} \\
\sigma(t_2) &= a_2\,t_2 + f_2, \quad a_2 \in \mathbb{K}(t_1)^*, && f_2 \in \mathbb{K}(t_1) \\
&\vdots \\
\sigma(t_e) &= a_e\,t_e + f_e, \quad a_e \in \mathbb{K}(t_1,\dots,t_{e-1})^*, && f_e \in \mathbb{K}(t_1,\dots,t_{e-1})
\end{aligned}$$

CONSTRUCT a difference field $(\mathbb{F},\sigma)$:

- a rational function field

$$\mathbb{F} := \mathbb{K}(t_1)(t_2)\dots(t_e)$$

- with an automorphism

$$\begin{aligned}
\sigma(c) &= c \quad \forall c\in\mathbb{K} \\
\sigma(t_1) &= a_1\,t_1 + f_1, \quad a_1\in\mathbb{K}^*, && f_1\in\mathbb{K} \\
\sigma(t_2) &= a_2\,t_2 + f_2, \quad a_2\in\mathbb{K}(t_1)^*, && f_2\in\mathbb{K}(t_1) \\
&\vdots \\
\sigma(t_e) &= a_e\,t_e + f_e, \quad a_e\in\mathbb{K}(t_1,\dots,t_{e-1})^*, && f_e\in\mathbb{K}(t_1,\dots,t_{e-1})
\end{aligned}$$

GIVEN $f\in\mathbb{F}$;
FIND $g\in\mathbb{F}$ such that

$$\sigma(g)-g=f.$$

CONSTRUCT a $\Pi\Sigma$-field $(\mathbb{F}, \sigma)$:

- a rational function field

$$\mathbb{F} := \mathbb{K}(t_1)(t_2)\dots(t_e)$$

- with an automorphism

$$\begin{aligned}
\sigma(c) &= c \quad \forall c \in \mathbb{K} \\
\sigma(t_1) &= a_1\, t_1 + f_1, \quad a_1 \in \mathbb{K}^*, && f_1 \in \mathbb{K} \\
\sigma(t_2) &= a_2\, t_2 + f_2, \quad a_2 \in \mathbb{K}(t_1)^*, && f_2 \in \mathbb{K}(t_1) \\
&\vdots \\
\sigma(t_e) &= a_e\, t_e + f_e, \quad a_e \in \mathbb{K}(t_1,\dots,t_{e-1})^*, && f_e \in \mathbb{K}(t_1,\dots,t_{e-1})
\end{aligned}$$

such that

$$\text{const}_\sigma\mathbb{F} = \{c \in \mathbb{K}(t_1)(t_2)\dots(t_e) | \sigma(c) = c\} = \mathbb{K}.$$

GIVEN $f \in \mathbb{F}$;
FIND $g \in \mathbb{F}$ such that

$$\sigma(g) - g = f.$$

Telescoping in the given difference field

FIND a closed form for

$$\sum_{k=1}^{n} S_1(k).$$

A $\Pi\Sigma^*$-field for the summand $\boxed{\text{const}_\sigma \mathbb{F} = \mathbb{Q}}$

Consider the rational function field

$$\mathbb{F} := \mathbb{Q}(k)(s)$$

with the automorphism $\sigma : \mathbb{F} \to \mathbb{F}$ defined by

$$\sigma(c) = c \quad \forall c \in \mathbb{Q},$$

$$\sigma(k) = k + 1, \qquad \mathcal{S}\, k = k + 1,$$

$$\sigma(s) = s + \frac{1}{k+1}, \qquad \mathcal{S}\, S_1(k) = S_1(k) + \frac{1}{k+1}.$$

FIND $g \in \mathbb{Q}(k)(s)$:

$$\sigma(g) - g = s.$$

FIND $g \in \mathbb{Q}(k)(s)$:

$$\sigma(g) - g = s.$$

Denominator bound: COMPUTE a polynomial $d \in \mathbb{Q}(k)[s]^*$:

$$\forall g \in \mathbb{Q}(k)(s): \quad \sigma(g) - g = s \quad \Rightarrow \quad g\,d \in \mathbb{Q}(k)[s].$$

FIND $g \in \mathbb{Q}(k)(s)$:

$$\sigma(g) - g = s.$$

Denominator bound: COMPUTE a polynomial $d \in \mathbb{Q}(k)[s]^*$:

$$\forall g \in \mathbb{Q}(k)(s): \quad \sigma(g) - g = s \quad \Rightarrow \quad g\,d \in \mathbb{Q}(k)[s].$$

FIND $g' \in \mathbb{Q}(k)[s]$ with

$$\sigma(\frac{g'}{d}) - \frac{g'}{d} = s.$$

FIND $g\in\mathbb{Q}(k)(s)$:

$$\sigma(g)-g=s.$$

Denominator bound: COMPUTE a polynomial $d\in\mathbb{Q}(k)[s]^*$:

$$d=1$$

$$\forall g\in\mathbb{Q}(k)(s):\quad \sigma(g)-g=s\quad\Rightarrow\quad g\,d\in\mathbb{Q}(k)[s].$$

FIND $g'\in\mathbb{Q}(k)[s]$ with

$$\sigma(\frac{g'}{d})-\frac{g'}{d}=s.$$

FIND $g \in \mathbb{Q}(k)(s)$:

$$\sigma(g) - g = s.$$

Denominator bound: COMPUTE a polynomial $d \in \mathbb{Q}(k)[s]^*$:

$d = 1$

$$\forall g \in \mathbb{Q}(k)(s): \quad \sigma(g) - g = s \quad \Rightarrow \quad g\,d \in \mathbb{Q}(k)[s].$$

FIND $g' \in \mathbb{Q}(k)[s]$ with

$$\sigma(\frac{g'}{d}) - \frac{g'}{d} = s.$$

Degree bound: COMPUTE $b \geq 0$:

$b = 2$

$$\forall g \in \mathbb{Q}(k)[s] \quad \sigma(g) - g = s \quad \Rightarrow \quad \deg(g) \leq b.$$

FIND $g \in \mathbb{Q}(k)(s)$:

$$\sigma(g) - g = s.$$

Denominator bound: COMPUTE a polynomial $d \in \mathbb{Q}(k)[s]^*$:

$d = 1$

$$\forall g \in \mathbb{Q}(k)(s): \quad \sigma(g) - g = s \quad \Rightarrow \quad g\,d \in \mathbb{Q}(k)[s].$$

FIND $g' \in \mathbb{Q}(k)[s]$ with

$$\sigma(\frac{g'}{d}) - \frac{g'}{d} = s.$$

Degree bound: COMPUTE $b \geq 0$:

$b = 2$

$$\forall g \in \mathbb{Q}(k)[s] \quad \sigma(g) - g = s \quad \Rightarrow \quad \deg(g) \leq b.$$

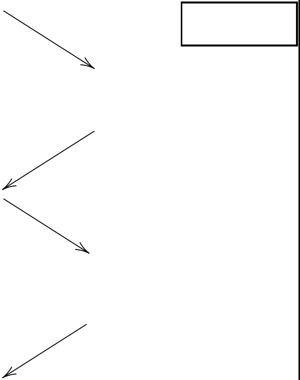
Polynomial Solution: FIND

g=s k-k

$$g = g_2\,s^2 + g_1\,s + g_0 \in \mathbb{Q}(k)[s].$$

ANSATZ $g = g_2\, s^2 + g_1\, s + g_0 \in \mathbb{Q}(k)[s]$

$$\sigma(g) - g = s$$



ANSATZ $g = g_2\, s^2 + g_1\, s + g_0 \in \mathbb{Q}(k)[s]$

$$\left[\sigma(g_2\, s^2 + g_1 s + g_0)\right] - \left[g_2\, s^2 + g_1 s + g_0\right] = s$$

ANSATZ $g = g_2\, s^2 + g_1\, s + g_0 \in \mathbb{Q}(k)[s]$

$$\left[\sigma(g_2)^2\, \sigma(s) + \sigma(g_1\, s + g_0)\right] \\ -\left[g_2\, s^2 + g_1 s + g_0\right] = s$$

ANSATZ $g = g_2\, s^2 + g_1\, s + g_0 \in \mathbb{Q}(k)[s]$

$$\Big[\sigma(g_2)\big(s + \tfrac{1}{k+1}\big)^2 + \sigma(g_1\, s + g_0)\Big] - \big[g_2\, s^2 + g_1 s + g_0\big] = s$$

ANSATZ $g = g_2\, s^2 + g_1\, s + g_0 \in \mathbb{Q}(k)[s]$

$$\left[\sigma(g_2)\left(s+\tfrac{1}{k+1}\right)^2+\sigma(g_1\, s+g_0)\right] - \left[g_2\, s^2+g_1 s+g_0\right]=s$$

coeff. comp.

$$\sigma(g_2)-g_2=0$$

ANSATZ $g = g_2\, s^2 + g_1\, s + g_0 \in \mathbb{Q}(k)[s]$

$$\left[\sigma(g_2)\left(s + \tfrac{1}{k+1}\right)^2 + \sigma(g_1\, s + g_0)\right] - \left[g_2\, s^2 + g_1 s + g_0\right] = s$$

coeff. comp.

$$\boxed{\sigma(g_2) - g_2 = 0}$$

$g_2 = c \in \mathbb{Q}$

ANSATZ $g = g_2\, s^2 + g_1\, s + g_0 \in \mathbb{Q}(k)[s]$

$$\Big[\sigma(g_2)\big(s + \tfrac{1}{k+1}\big)^2 + \sigma(g_1\, s + g_0)\Big] - \big[g_2\, s^2 + g_1 s + g_0\big] = s$$

coeff. comp.

$$\sigma(g_2) - g_2 = 0$$

$g_2 = c \in \mathbb{Q}$

$$\sigma(g_1\, s + g_0) - (g_1\, s + g_0) = s - c\Big[\tfrac{2s(k+1)+1}{(k+1)^2}\Big]$$

ANSATZ $g = g_2\, s^2 + g_1\, s + g_0 \in \mathbb{Q}(k)[s]$

$$\left[\sigma(g_2)\left(s+\tfrac{1}{k+1}\right)^2+\sigma(g_1\, s+g_0)\right] - \left[g_2\, s^2+g_1 s+g_0\right] = s$$

coeff. comp.

$$\boxed{\sigma(g_2) - g_2 = 0}$$

$g_2 = c \in \mathbb{Q}$

$$\sigma(g_1\, s + g_0) - (g_1\, s + g_0) = s - c\left[\tfrac{2s(k+1)+1}{(k+1)^2}\right]$$

coeff. comp.

$$\boxed{\sigma(g_1) - g_1 = 1 - c\,\tfrac{2}{k+1}}$$

ANSATZ $g = g_2\, s^2 + g_1\, s + g_0 \in \mathbb{Q}(k)[s]$

$$\Big[\sigma(g_2)\big(s + \tfrac{1}{k+1}\big)^2 + \sigma(g_1\, s + g_0)\Big] - \big[g_2\, s^2 + g_1 s + g_0\big] = s$$

coeff. comp.

$$\sigma(g_2) - g_2 = 0$$

$g_2 = c \in \mathbb{Q}$

$$\sigma(g_1\, s + g_0) - (g_1\, s + g_0) = s - c\Big[\tfrac{2s(k+1)+1}{(k+1)^2}\Big]$$

coeff. comp.

$$\sigma(g_1) - g_1 = 1 - c\,\frac{2}{k+1}$$

$c = 0,\quad g_1 = k + d,\ d \in \mathbb{Q}$

ANSATZ $g = g_2\, s^2 + g_1\, s + g_0 \in \mathbb{Q}(k)[s]$

$$\Big[\sigma(g_2)\big(s + \tfrac{1}{k+1}\big)^2 + \sigma(g_1\, s + g_0)\Big] - \big[g_2\, s^2 + g_1 s + g_0\big] = s$$

coeff. comp.

$$\boxed{\sigma(g_2) - g_2 = 0}$$

$g_2 = c \in \mathbb{Q}$

$$\sigma(g_1\, s + g_0) - (g_1\, s + g_0) = s - c\Big[\tfrac{2s(k+1)+1}{(k+1)^2}\Big]$$

coeff. comp.

$$\boxed{\sigma(g_1) - g_1 = 1 - c\, \tfrac{2}{k+1}}$$

$c = 0$, $g_1 = k + d$, $d \in \mathbb{Q}$

$$\boxed{\sigma(g_0) - g_0 = -1 - d\frac{1}{k+1}}$$

ANSATZ $g = g_2\, s^2 + g_1\, s + g_0 \in \mathbb{Q}(k)[s]$

$$\Big[\sigma(g_2)\big(s+\tfrac{1}{k+1}\big)^2+\sigma(g_1\,s+g_0)\Big] -\Big[g_2\,s^2+g_1 s+g_0\Big]=s$$

coeff. comp. → $\sigma(g_2) - g_2 = 0$

g=sk-k

$g_2 = c \in \mathbb{Q}$ →

$$\sigma(g_1\,s+g_0)-(g_1\,s+g_0)=s-c\Big[\tfrac{2s(k+1)+1}{(k+1)^2}\Big]$$

coeff. comp. → $\sigma(g_1) - g_1 = 1 - c\,\frac{2}{k+1}$

$c = 0$, $g_1 = k + d$, $d \in \mathbb{Q}$ →

$$\sigma(g_0) - g_0 = -1 - d\frac{1}{k+1}$$

← $g_0 = -k$, $d = 0$

Telescoping in the given difference field

FIND $g \in \mathbb{F}$:

$$\boxed{\sigma(g) - g = s.}$$

We compute

$$g = (s-1)k \in \mathbb{F}.$$

This gives

$$\boxed{g(k+1) - g(k) = S_1(k)}$$

with

$$g(k) = (S_1(k) - 1)k.$$

Hence,

$$(S_1(n+1)-1)(n+1) = \sum_{k=1}^{n} S_1(k).$$

$$\sum_{k=0}^{a} F(k) =?$$

$$\downarrow$$

Sigma `Reduce[`$\sum_{k=0}^{a} F(k)$`]`

Indefinite summation.

GIVEN nested product-sum expression $F(k)$.

FIND nested product-sum expression $G(k)$:

$$F(k) = G(k+1) - G(k)$$

and such that RHS is as “simple” as LHS.

$$\downarrow$$

$$\sum_{k=0}^{a} F(k) = G(a+1) - G(0)$$

such that RHS is “simpler” than LHS.

Simplification in $\Pi\Sigma$-fields

For algorithmic details see:

- CS. Symbolic summation with single-nested sum extensions. In J. Gutierrez, editor, *Proc. ISSAC'04*, pages 282–289. ACM Press, 2004.
- CS. Product representations in $\Pi\Sigma$-fields. *Ann. Comb.*, 9(1):75–99, 2005.
- CS. Simplifying Sums in $\Pi\Sigma$-Extensions. *J. Algebra Appl.*, 6(3):415–441, 2007.
- CS. A refined difference field theory for symbolic summation. *J. Symbolic Comput.*, 43(9):611–644, 2008. [arXiv:0808.2543v1].
- S.A. Abramov, M. Petkovšek. Polynomial ring automorphisms, rational (w, σ)-canonical forms, and the assignment problem. *J. Symbolic Comput.*, 45(6): 684–708, 2010.
- CS. Structural Theorems for Symbolic Summation. *Appl. Algebra Engrg. Comm. Comput.*, 21(1):1–32, 2010.
- CS. Fast Algorithms for Refined Parameterized Telescoping in Difference Fields. To appear in *Computer Algebra and Polynomials*, Lecture Notes in Computer Science (LNCS), Springer, 2014. arXiv:1307.7887 [cs.SC].

For special cases see:

- S.A. Abramov. On the summation of rational functions. Zh. vychisl. mat. Fiz., 11: 1071-1074, 1971.
- P. Paule. Greatest factorial factorization and symbolic summation, J. Symbolic Comput., 20(3): 235-268, 1995.

A difference field approach (M. Karr, 1981)

GIVEN a $\Pi\Sigma$-field $(\mathbb{F}, \sigma)$ with $f \in \mathbb{F}$.

FIND $g \in \mathbb{F}$:

$$\sigma(g) - g = f.$$

A symbolic summation approach (see, e.g., J. Symb. Comput 2008; arXiv:0808.2543)

1. FIND an appropriate $\Pi\Sigma$-field $(\mathbb{F}, \sigma)$ with $f \in \mathbb{F}$.

2. FIND $g \in \mathbb{F}$:

$$\sigma(g) - g = f.$$

A symbolic summation approach (see, e.g., J. Symb. Comput 2008; arXiv:0808.2543)

1. FIND an appropriate $\Pi\Sigma$-field $(\mathbb{F}, \sigma)$ with $f \in \mathbb{F}$.

2. FIND an appropriate extension $\mathbb{E} > \mathbb{F}$ with $g \in \mathbb{E}$:

$$\sigma(g) - g = f.$$

A symbolic summation approach (see, e.g., J. Symb. Comput 2008; arXiv:0808.2543)

1. FIND an appropriate $\Pi\Sigma$-field $(\mathbb{F},\sigma)$ with $f\in\mathbb{F}$.

2. FIND an appropriate extension $\mathbb{E}>\mathbb{F}$ with $g\in\mathbb{E}$:

$$\sigma(g)-g=f.$$

appropriate = degrees in denominators minimal

Example

$$\sum_{k=1}^{a}\Big(\frac{-2+k}{10(1+k^2)}+\frac{(1-4k-2k^2)S_1(k)}{10(1+k^2)(2+2k+k^2)}+\frac{(1-4k-2k^2)S_3(k)}{5(1+k^2)(2+2k+k^2)}\Big)$$
$$=\frac{a^2+4a+5}{10(a^2+2a+2)}S_1(a)-\frac{(a-1)(a+1)}{5(a^2+2a+2)}S_3(a)-\frac{2}{5}\sum_{k=1}^{a}\frac{1}{k^2}$$

A symbolic summation approach (see, e.g., J. Symb. Comput 2008; arXiv:0808.2543)

1. FIND an appropriate $\Pi\Sigma$-field $(\mathbb{F},\sigma)$ with $f \in \mathbb{F}$.

2. FIND an appropriate extension $\mathbb{E} > \mathbb{F}$ with $g \in \mathbb{E}$:

$$\sigma(g) - g = f.$$

appropriate = sum representations with optimal nesting depth

Example

Example:

$$\sum_{k=1}^{n} \frac{\sum_{j=1}^{k} \frac{\sum_{i=1}^{j} \frac{1}{i}}{j}}{k} = \frac{1}{6}\left(\sum_{i=1}^{n}\frac{1}{i}\right)^3 + \frac{1}{2}\left(\sum_{i=1}^{n}\frac{1}{i^2}\right)\left(\sum_{i=1}^{n}\frac{1}{i}\right) + \frac{1}{3}\sum_{i=1}^{n}\frac{1}{i^3}$$

depth 3 depth 1

$$\sum_{k=1}^{n}\frac{\sum_{j=1}^{k}\frac{\sum_{i=1}^{j}\frac{1}{i}}{j}}{k}$$

$$\sum_{k=1}^{n}\frac{\sum_{j=1}^{k}\frac{\sum_{i=1}^{j}\frac{1}{i}}{j}}{k}$$

S

$\Pi\Sigma$-field $(\mathbb{Q}(k)(s),\sigma)$ with

$$\sigma(k)=k+1$$
$$\sigma(s)=s+\frac{1}{k+1}$$

$$\sum_{k=1}^{n}\frac{\sum_{j=1}^{k}\frac{\sum_{i=1}^{j}\frac{1}{i}}{j}}{k}$$

t

$\Pi\Sigma$-field $(\mathbb{Q}(k)(s)(t),\sigma)$ with

$$\sigma(k) = k + 1$$

$$\sigma(s) = s + \frac{1}{k+1}$$

$$\sigma(t) = t + \frac{\sigma(s)}{k+1}$$

$$\sum_{k=1}^{n}\frac{\sum_{j=1}^{k}\frac{\sum_{i=1}^{j}\frac{1}{i}}{j}}{k} \qquad T$$

$\Pi\Sigma$-field $(\mathbb{Q}(k)(s)(t)(T),\sigma)$ with

$$\sigma(k)=k+1$$
$$\sigma(s)=s+\frac{1}{k+1}$$
$$\sigma(t)=t+\frac{\sigma(s)}{k+1}$$
$$\sigma(T)=T+\frac{\sigma(t)}{k+1}$$

No simplification ☹

$$\sum_{k=1}^{n}\frac{\sum_{j=1}^{k}\frac{\sum_{i=1}^{j}\frac{1}{i}}{j}}{k} \qquad \curvearrowright \quad S$$

$\Pi\Sigma$-field $(\mathbb{Q}(k)(s)(t)(T),\sigma)$ with

$$\sigma(k)=k+1$$
$$\sigma(s)=s+\frac{1}{k+1}$$
$$\sigma(t)=t+\frac{\sigma(s)}{k+1}$$
$$\sigma(T)=T+\frac{\sigma(t)}{k+1}$$

$\Pi\Sigma$-field $(\mathbb{Q}(k)(s),\sigma)$ with

$$\sigma(k)=k+1$$
$$\sigma(s)=s+\frac{1}{k+1}$$

$$\sum_{k=1}^{n}\frac{\sum_{j=1}^{k}\frac{\sum_{i=1}^{j}\frac{1}{i}}{j}}{k}$$

$$\frac{1}{2}(s^2+s_2)$$

$\Pi\Sigma$-field $(\mathbb{Q}(k)(s)(t)(T),\sigma)$ with

$$\sigma(k)=k+1$$
$$\sigma(s)=s+\frac{1}{k+1}$$
$$\sigma(t)=t+\frac{\sigma(s)}{k+1}$$
$$\sigma(T)=T+\frac{\sigma(t)}{k+1}$$

$\Pi\Sigma$-field $(\mathbb{Q}(k)(s)(s_2),\sigma)$ with

$$\sigma(k)=k+1$$
$$\sigma(s)=s+\frac{1}{k+1}$$
$$\sigma(s_2)=s_2+\frac{1}{(k+1)^2}$$

$$\sum_{k=1}^{n}\frac{\sum_{j=1}^{k}\frac{\sum_{i=1}^{j}\frac{1}{i}}{j}}{k} \qquad \frac{1}{6}s^3+\frac{1}{2}s_2^2 s+\frac{1}{3}s_3$$

$\Pi\Sigma$-field $(\mathbb{Q}(k)(s)(t)(T),\sigma)$ with

$$\sigma(k)=k+1$$
$$\sigma(s)=s+\frac{1}{k+1}$$
$$\sigma(t)=t+\frac{\sigma(s)}{k+1}$$
$$\sigma(T)=T+\frac{\sigma(t)}{k+1}$$

$\Pi\Sigma$-field $(\mathbb{Q}(k)(s)(s_2)(s_3),\sigma)$ with

$$\sigma(k)=k+1$$
$$\sigma(s)=s+\frac{1}{k+1}$$
$$\sigma(s_2)=s_2+\frac{1}{(k+1)^2}$$
$$\sigma(s_3)=s_3+\frac{1}{(k+1)^3}$$

Example

$$\sum_{k=1}^{n}\frac{\sum_{j=1}^{k}\frac{\sum_{i=1}^{j}\frac{1}{i}}{j}}{k}=\frac{1}{6}\left(\sum_{i=1}^{n}\frac{1}{i}\right)^3+\frac{1}{2}\left(\sum_{i=1}^{n}\frac{1}{i^2}\right)\left(\sum_{i=1}^{n}\frac{1}{i}\right)+\frac{1}{3}\sum_{i=1}^{n}\frac{1}{i^3}$$

$\Pi\Sigma$-field $(\mathbb{Q}(k)(s)(t)(T),\sigma)$ with

$$\sigma(k)=k+1$$
$$\sigma(s)=s+\frac{1}{k+1}$$
$$\sigma(t)=t+\frac{\sigma(s)}{k+1}$$
$$\sigma(T)=T+\frac{\sigma(t)}{k+1}$$

$\Pi\Sigma$-field $(\mathbb{Q}(k)(s)(s_2)(s_3),\sigma)$ with

$$\sigma(k)=k+1$$
$$\sigma(s)=s+\frac{1}{k+1}$$
$$\sigma(s_2)=s_2+\frac{1}{(k+1)^2}$$
$$\sigma(s_3)=s_3+\frac{1}{(k+1)^3}$$

A symbolic summation approach (see, e.g., J. Symb. Comput 2008; arXiv:0808.2543)

1. FIND an appropriate $\Pi\Sigma$-field $(\mathbb{F}, \sigma)$ with $f \in \mathbb{F}$.

2. FIND an appropriate extension $\mathbb{E} > \mathbb{F}$ with $g \in \mathbb{E}$:

$$\sigma(g) - g = f.$$

appropriate = sum representations with optimal nesting depth

Example

Example:

$$\sum_{k=1}^{n} \frac{\sum_{j=1}^{k} \frac{\sum_{i=1}^{j} \frac{1}{i}}{j}}{k} = \frac{1}{6}\left(\sum_{i=1}^{n} \frac{1}{i}\right)^3 + \frac{1}{2}\left(\sum_{i=1}^{n} \frac{1}{i^2}\right)\left(\sum_{i=1}^{n} \frac{1}{i}\right) + \frac{1}{3}\sum_{i=1}^{n} \frac{1}{i^3}$$

depth 3 depth 1

A symbolic summation approach (see, e.g., J. Symb. Comput 2008; arXiv:0808.2543)

1. FIND an appropriate $\Pi\Sigma$-field $(\mathbb{F}, \sigma)$ with $f \in \mathbb{F}$.

2. FIND an appropriate extension $\mathbb{E} > \mathbb{F}$ with $g \in \mathbb{E}$:

$$\sigma(g) - g = f.$$

appropriate = sum representations with minimal number of objects Example

$$\sum_{k=0}^{a}(-1)^k S_1(k)^2 \binom{n}{k}$$
$$= -\frac{1}{n}\sum_{i_1=1}^{a}\frac{(-1)^{i_1}}{i_1}\binom{n}{i_1} - (a-n)\left(n^2 S_1(a)^2 + 2n S_1(a) + 2\right)\frac{(-1)^a\binom{n}{a}}{n^3} - \frac{2}{n^2}$$

Simplification of nested product-sum expressions

$A(k)$: indefinite nested product-sum expression

$\downarrow$ `SigmaReduce[A,k]`

$B(k)$: indefinite nested product-sum expression

- such that

$$A(k) = B(k)$$

Simplification of nested product-sum expressions

$A(k)$: indefinite nested product-sum expression

$\downarrow$ `SigmaReduce[A,k]`

$B(k)$: indefinite nested product-sum expression

- such that

$$A(k) = B(k)$$

- such that all the sums in $B(k)$ are **simplified** as above

Simplification of nested product-sum expressions

$A(k)$: indefinite nested product-sum expression

$\downarrow$ `SigmaReduce[A,k]`

$B(k)$: indefinite nested product-sum expression

- such that

$$A(k) = B(k)$$

- such that all the sums in $B(k)$ are **simplified** as above
- and such that the arising sums in $B(k)$ are **algebraically independent** (i.e., they do not satisfy any polynomial relation)

Detail 1: Algebraic independence

- CS. Simplifying Multiple Sums in Difference Fields. In: Computer Algebra in Quantum Field Theory: Integration, Summation and Special Functions, J. Blümlein, C. Schneider (ed.), Texts and Monographs in Symbolic Computation, pp. 325-360. Springer, 2013.
- CS. Parameterized Telescoping Proves Algebraic Independence of Sums. *Ann. Comb.*, 14(4):533–552, 2010. [arXiv:0808.2596].

Consider the ring

$$\mathbb{Q}^{\mathbb{N}} = \{\langle a_0, a_1, \dots \rangle \,|\, a_i \in \mathbb{Q}\}$$

with component wise addition and multiplication.

Consider the ring

$$\mathbb{Q}^{\mathbb{N}} = \{\langle a_0, a_1, \dots \rangle \,|\, a_i \in \mathbb{Q}\}$$

with component wise addition and multiplication.

Homework: For a polynomial $p \in \mathbb{Q}[n]$ define the evaluation function at $i \in \mathbb{N}$:

$$\mathrm{ev}(p, i) = p|_{n \to i}$$

Consider the ring

$$\mathbb{Q}^{\mathbb{N}} = \{\langle a_0, a_1, \dots \rangle \,|\, a_i \in \mathbb{Q}\}$$

with component wise addition and multiplication.

Homework: For a polynomial $p \in \mathbb{Q}[n]$ define the evaluation function at $i \in \mathbb{N}$:

$$\mathrm{ev}(p, i) = p|_{n \to i}$$

Then $\mathbb{Q}[n] \to \mathbb{Q}^{\mathbb{N}}$ with

$$p \to \langle \mathrm{ev}(p, i) \rangle_{i \geq 0}$$

is a ring embedding.

Consider the ring

$$\mathbb{Q}^{\mathbb{N}} = \{\langle a_0, a_1, \dots \rangle \,|\, a_i \in \mathbb{Q}\}$$

with component wise addition and multiplication.

Homework: For a polynomial $p \in \mathbb{Q}[n]$ define the evaluation function at $i \in \mathbb{N}$:

$$\mathrm{ev}(p, i) = p|_{n \to i}$$

Then $\mathbb{Q}[n] \to \mathbb{Q}^{\mathbb{N}}$ with

$$p \to \langle \mathrm{ev}(p, i) \rangle_{i \geq 0}$$

is a ring embedding.

What about $\mathbb{Q}(n)$?

Refinement: the ring of sequences

"Two sequences are equal if they agree form a certain point on."

Refinement: the ring of sequences

"Two sequences are equal if they agree form a certain point on."

More precisely,

1. Define the equivalence relation

$$\langle a_i\rangle_{i\geq 0} \sim \langle b_i\rangle_{i\geq 0} \quad \Leftrightarrow \quad a_i = b_i \text{ from a certain point on}$$

Refinement: the ring of sequences

"Two sequences are equal if they agree form a certain point on."

More precisely,

1. Define the equivalence relation

$$\langle a_i\rangle_{i\geq 0} \sim \langle b_i\rangle_{i\geq 0} \quad \Leftrightarrow \quad a_i = b_i \text{ from a certain point on}$$

2. Denote the equivalence class for some $\mathbf{a} \in \mathbb{K}^{\mathbb{N}}$ by

$$[\mathbf{a}] := \{\mathbf{b} \in \mathbb{K}^{\mathbb{N}} | \, \mathbf{a} \sim \mathbf{b}\}$$

Refinement: the ring of sequences

"Two sequences are equal if they agree form a certain point on."

More precisely,

1. Define the equivalence relation
$$\langle a_i\rangle_{i\geq 0} \sim \langle b_i\rangle_{i\geq 0} \quad\Leftrightarrow\quad a_i = b_i \text{ from a certain point on}$$
2. Denote the equivalence class for some $\mathbf{a} \in \mathbb{K}^{\mathbb{N}}$ by
$$[\mathbf{a}] := \{\mathbf{b} \in \mathbb{K}^{\mathbb{N}} | \, \mathbf{a} \sim \mathbf{b}\}$$
3. We get the ring of sequences
$$\mathbb{Q}^{\mathbb{N}}/\sim = \{[\mathbf{a}] | \, \mathbf{a} \in \mathbb{Q}^{\mathbb{N}}\}$$
with term-wise addition and multiplication.

Refinement: the ring of sequences

"Two sequences are equal if they agree form a certain point on."

More precisely,

1. Define the equivalence relation

$$\langle a_i\rangle_{i\geq 0} \sim \langle b_i\rangle_{i\geq 0} \quad \Leftrightarrow \quad a_i = b_i \text{ from a certain point on}$$

2. Denote the equivalence class for some $\mathbf{a} \in \mathbb{K}^{\mathbb{N}}$ by

$$[\mathbf{a}] := \{\mathbf{b} \in \mathbb{K}^{\mathbb{N}} | \, \mathbf{a} \sim \mathbf{b}\}$$

3. We get the ring of sequences

$$\mathbb{Q}^{\mathbb{N}}/\sim = \{[\mathbf{a}] | \, \mathbf{a} \in \mathbb{Q}^{\mathbb{N}}\}$$

   with term-wise addition and multiplication.

4. Sloppy notation:

$$\langle a_i\rangle_{i\geq 0} := [\langle a_i\rangle_{i\geq 0}]$$

Homework: For a rational function $\frac{p}{q} \in \mathbb{Q}(n)$ define the evaluation function

$$\operatorname{ev}(\tfrac{p}{q}, i) = \begin{cases} 0 & \text{if } q|_{n \to i} = 0 \\ \frac{p}{q}|_{n \to i} & \text{otherwise.} \end{cases}$$

Homework: For a rational function $\frac{p}{q} \in \mathbb{Q}(n)$ define the evaluation function

$$\operatorname{ev}(\tfrac{p}{q}, i) = \begin{cases} 0 & \text{if } q|_{n\to i} = 0 \\ \frac{p}{q}|_{n\to i} & \text{otherwise.} \end{cases}$$

Then $\mathbb{Q}(n) \to \mathbb{Q}^{\mathbb{N}}/\sim$ with

$$\frac{p}{q} \to \langle \operatorname{ev}(\tfrac{p}{q}, i\rangle)_{i\geq 0}$$

is a ring embedding.

Homework: For a rational function $\frac{p}{q} \in \mathbb{Q}(n)$ define the evaluation function

$$\operatorname{ev}(\tfrac{p}{q}, i) = \begin{cases} 0 & \text{if } q|_{n\to i} = 0 \\ \frac{p}{q}|_{n\to i} & \text{otherwise.} \end{cases}$$

Then $\mathbb{Q}(n) \to \mathbb{Q}^{\mathbb{N}}/\sim$ with

$$\frac{p}{q} \to \langle \operatorname{ev}(\tfrac{p}{q}, i \rangle)_{i\geq 0}$$

is a ring embedding.

The sequences generated by $\mathbb{Q}(n)$, i.e.,

$$\{\operatorname{ev}(\tfrac{p}{q}, i))_{i\geq 0} |\, \tfrac{p}{q} \in \mathbb{Q}(n)\}$$

are called rational sequences.

Example: harmonic numbers

1. There is an embedding of the polynomial ring $\mathbb{Q}(n)[h_1, h_2, \dots]$ in $\mathbb{Q}^{\mathbb{N}}/\sim$ with

$$h_1 \longrightarrow \left\langle \sum_{i=1}^{n} \frac{1}{i} \right\rangle_{n\geq 0}, \qquad h_2 \longrightarrow \left\langle \sum_{i=1}^{n} \frac{1}{i^2} \right\rangle_{n\geq 0} \qquad \dots$$

Example: harmonic numbers

1. There is an embedding of the polynomial ring $\mathbb{Q}(n)[h_1, h_2, \dots]$ in $\mathbb{Q}^{\mathbb{N}}/\sim$ with

$$h_1 \longrightarrow \left\langle \sum_{i=1}^{n} \frac{1}{i} \right\rangle_{n \geq 0}, \qquad h_2 \longrightarrow \left\langle \sum_{i=1}^{n} \frac{1}{i^2} \right\rangle_{n \geq 0} \qquad \dots$$

$\Rightarrow$ The sequences of the generalized harmonic numbers

$$S_1(n) = \sum_{k=1}^{n} \frac{1}{k}, \quad S_2(n) = \sum_{k=1}^{n} \frac{1}{k^2}, \quad S_3(n) = \sum_{k=1}^{n} \frac{1}{k^3}, \quad \dots$$

are algebraically independent over the rational sequences.

Example: harmonic numbers

1. There is an embedding of the polynomial ring $\mathbb{Q}(n)[h_1, h_2, \ldots]$ in $\mathbb{Q}^{\mathbb{N}}/\sim$ with

$$h_1 \longrightarrow \left\langle \sum_{i=1}^{n} \frac{1}{i} \right\rangle_{n\geq 0}, \qquad h_2 \longrightarrow \left\langle \sum_{i=1}^{n} \frac{1}{i^2} \right\rangle_{n\geq 0} \qquad \ldots$$

$\Rightarrow$ The sequences of the generalized harmonic numbers

$$S_1(n) = \sum_{k=1}^{n} \frac{1}{k}, \quad S_2(n) = \sum_{k=1}^{n} \frac{1}{k^2}, \quad S_3(n) = \sum_{k=1}^{n} \frac{1}{k^3}, \quad \ldots$$

are algebraically independent over the rational sequences.

In general, a big class of $\Pi\Sigma^*$-fields (more precisely, a subring) can be embedded (algorithmically) into the ring of sequences. So far, this class covered all considered applications.

Detail 2: Constructing $\Pi\Sigma$-fields (resp. rings)

- M. Karr. Summation in finite terms. *J. ACM*, 28:305–350, 1981.
- CS. A Symbolic Summation Approach to Find Optimal Nested Sum Representations. In A. Carey, D. Ellwood, S. Paycha, and S. Rosenberg, editors, *Motives, Quantum Field Theory, and Pseudodifferential Operators*, pages 285–308. 2010.
- CS. A Difference Ring Theory for Symbolic Summation. arXiv:1408.2776 [cs.SC] pp. 1-47. 2014.

CONSTRUCT a $\Pi\Sigma$-field $(\mathbb{F}, \sigma)$:

- a rational function field

$$\mathbb{F} := \mathbb{K}$$

- with an automorphism

$$\sigma(c) = c \quad \forall c \in \mathbb{K}$$

such that

$$\text{const}_\sigma \mathbb{F} = \{c \in \mathbb{K} \qquad | \sigma(c) = c\} = \mathbb{K}.$$

CONSTRUCT a $\Pi\Sigma$-field $(\mathbb{F}, \sigma)$:

- a rational function field

$$\mathbb{F} := \mathbb{K}(t_1)$$

- with an automorphism

$$\sigma(c) = c \quad \forall c \in \mathbb{K}$$

$$\sigma(t_1) = a_1\, t_1 + f_1, \quad a_1 \in \mathbb{K}^*, \qquad\qquad f_1 \in \mathbb{K}$$

such that

$$\mathrm{const}_\sigma \mathbb{F} = \{c \in \mathbb{K}(t_1) \qquad\qquad |\sigma(c) = c\} = \mathbb{K}.$$

CONSTRUCT a $\Pi\Sigma$-field $(\mathbb{F}, \sigma)$:

- a rational function field

$$\mathbb{F} := \mathbb{K}(t_1)(t_2)$$

- with an automorphism

$$\begin{aligned} \sigma(c) &= c \quad \forall c \in \mathbb{K} \\ \sigma(t_1) &= a_1\, t_1 + f_1, \quad a_1 \in \mathbb{K}^*, \qquad f_1 \in \mathbb{K} \\ \sigma(t_2) &= a_2\, t_2 + f_2, \quad a_2 \in \mathbb{K}(t_1)^*, \qquad f_2 \in \mathbb{K}(t_1) \end{aligned}$$

such that

$$\mathrm{const}_\sigma \mathbb{F} = \{c \in \mathbb{K}(t_1)(t_2) \quad | \sigma(c) = c\} = \mathbb{K}.$$

CONSTRUCT a $\Pi\Sigma$-field $(\mathbb{F},\sigma)$:

- a rational function field

$$\mathbb{F} := \mathbb{K}(t_1)(t_2)\dots(t_e)$$

- with an automorphism

$$\begin{aligned}
\sigma(c) &= c \quad \forall c \in \mathbb{K} \\
\sigma(t_1) &= a_1\, t_1 + f_1, \quad a_1 \in \mathbb{K}^*, && f_1 \in \mathbb{K} \\
\sigma(t_2) &= a_2\, t_2 + f_2, \quad a_2 \in \mathbb{K}(t_1)^*, && f_2 \in \mathbb{K}(t_1) \\
&\vdots \\
\sigma(t_e) &= a_e\, t_e + f_e, \quad a_e \in \mathbb{K}(t_1,\dots,t_{e-1})^*, && f_e \in \mathbb{K}(t_1,\dots,t_{e-1})
\end{aligned}$$

such that

$$\text{const}_\sigma\mathbb{F} = \{c \in \mathbb{K}(t_1)(t_2)\dots(t_e) | \sigma(c) = c\} = \mathbb{K}.$$

CONSTRUCT a $\Pi\Sigma$-field $(\mathbb{F}, \sigma)$:

- a rational function field

$$\mathbb{F} := \mathbb{K}(t_1)(t_2)\dots(t_e)$$

- with an automorphism

$$\begin{aligned}
\sigma(c) &= c \quad \forall c \in \mathbb{K} \\
\sigma(t_1) &= a_1\, t_1 + f_1, \quad a_1 \in \mathbb{K}^*, && f_1 \in \mathbb{K} \\
\sigma(t_2) &= a_2\, t_2 + f_2, \quad a_2 \in \mathbb{K}(t_1)^*, && f_2 \in \mathbb{K}(t_1) \\
&\vdots \\
\sigma(t_e) &= a_e\, t_e + f_e, \quad a_e \in \mathbb{K}(t_1, \dots, t_{e-1})^*, && f_e \in \mathbb{K}(t_1, \dots, t_{e-1})
\end{aligned}$$

such that

$$\text{const}_\sigma \mathbb{F} = \{c \in \mathbb{K}(t_1)(t_2)\dots(t_e) | \sigma(c) = c\} = \mathbb{K}.$$

GIVEN $f \in \mathbb{F}$;
FIND $g \in \mathbb{F}$ such that

$$\sigma(g) - g = f.$$

Represent sums

- Let $(\mathbb{F}, \sigma)$ be a difference field with constant field

$$\text{const}_\sigma \mathbb{F} := \{k \in \mathbb{F} | \sigma(k) = k\}.$$

Represent sums

- Let $(\mathbb{F}, \sigma)$ be a difference field with constant field

$$\text{const}_\sigma\mathbb{F} := \{k \in \mathbb{F} | \sigma(k) = k\}.$$

- Adjoin a new variable t to $\mathbb{F}$ (i.e., $\mathbb{F}(t)$ is a rational function field).

Represent sums

- Let $(\mathbb{F}, \sigma)$ be a difference field with constant field

$$\text{const}_\sigma\mathbb{F} := \{k \in \mathbb{F} | \sigma(k) = k\}.$$

- Adjoin a new variable t to $\mathbb{F}$ (i.e., $\mathbb{F}(t)$ is a rational function field).
- Extend the shift operator s.t.

$$\sigma(t) = t + f \qquad \text{for some } f \in \mathbb{F}.$$

Represent sums

- Let $(\mathbb{F}, \sigma)$ be a difference field with constant field

$$\text{const}_\sigma\mathbb{F} := \{k \in \mathbb{F} | \sigma(k) = k\}.$$

- Adjoin a new variable t to $\mathbb{F}$ (i.e., $\mathbb{F}(t)$ is a rational function field).
- Extend the shift operator s.t.

$$\sigma(t) = t + f \qquad \text{for some } f \in \mathbb{F}.$$

(Karr 1981) Then $\text{const}_\sigma\mathbb{F}(t) = \text{const}_\sigma\mathbb{F}$ iff

$$\nexists g \in \mathbb{F} : \quad \boxed{\sigma(g) = g + f}$$

Represent sums

- Let $(\mathbb{F}, \sigma)$ be a difference field with constant field

$$\text{const}_\sigma\mathbb{F} := \{k \in \mathbb{F} | \sigma(k) = k\}.$$

- Adjoin a new variable t to $\mathbb{F}$ (i.e., $\mathbb{F}(t)$ is a rational function field).
- Extend the shift operator s.t.

$$\sigma(t) = t + f \qquad \text{for some } f \in \mathbb{F}.$$

(Karr 1981) Then $\text{const}_\sigma\mathbb{F}(t) = \text{const}_\sigma\mathbb{F}$ iff

$$\nexists g \in \mathbb{F}: \quad \boxed{\sigma(g) = g + f}$$

Such a difference field extension $(\mathbb{F}(t), \sigma)$ of $(\mathbb{F}, \sigma)$ is called Σ^*-extension

Represent sums

- Let $(\mathbb{F}, \sigma)$ be a difference field with constant field

$$\text{const}_\sigma\mathbb{F} := \{k \in \mathbb{F} | \sigma(k) = k\}.$$

- Adjoin a new variable t to $\mathbb{F}$ (i.e., $\mathbb{F}(t)$ is a rational function field).
- Extend the shift operator s.t.

$$\sigma(t) = t + f \qquad \text{for some } f \in \mathbb{F}.$$

(Karr 1981) Then $\text{const}_\sigma\mathbb{F}(t) = \text{const}_\sigma\mathbb{F}$ iff

$$\nexists g \in \mathbb{F}: \quad \boxed{\sigma(g) = g + f}$$

There are 2 cases:

1. $\boxed{\nexists g \in \mathbb{F}: \ \sigma(g) = g + f}$ $(\mathbb{F}(t), \sigma)$ is a Σ^*-extension of $(\mathbb{F}, \sigma)$

Represent sums

- Let $(\mathbb{F}, \sigma)$ be a difference field with constant field

$$\text{const}_\sigma \mathbb{F} := \{k \in \mathbb{F} | \sigma(k) = k\}.$$

- Adjoin a new variable t to $\mathbb{F}$ (i.e., $\mathbb{F}(t)$ is a rational function field).
- Extend the shift operator s.t.

$$\sigma(t) = t + f \qquad \text{for some } f \in \mathbb{F}.$$

(Karr 1981) Then $\text{const}_\sigma \mathbb{F}(t) = \text{const}_\sigma \mathbb{F}$ iff

$$\nexists g \in \mathbb{F} : \quad \boxed{\sigma(g) = g + f}$$

There are 2 cases:

1. $\boxed{\nexists g \in \mathbb{F} : \ \sigma(g) = g + f}$ $(\mathbb{F}(t), \sigma)$ is a Σ^*-extension of $(\mathbb{F}, \sigma)$
2. $\boxed{\exists g \in \mathbb{F} : \sigma(g) = g + f}$ No need for a Σ^*-extension!

Represent products

$$\prod_{i=1}^{k} i + \prod_{i=1}^{k}(-i)$$

Represent products

$$\prod_{i=1}^{k} i + \prod_{i=1}^{k}(-i)$$

Attempt 1: Construct

$$\mathbb{F} = \mathbb{Q}(k) \qquad \begin{aligned} \sigma(c) &= c \ \ \forall c \in \mathbb{Q} \\ \sigma(k) &= k + 1 \end{aligned}$$

where

$$\text{const}_\sigma \mathbb{Q}(k) = \mathbb{Q}$$

Represent products

$$\boxed{\prod_{i=1}^{k} i} + \prod_{i=1}^{k}(-i)$$

$$\boxed{\mathcal{S}\prod_{i=1}^{k} i = (k+1)\prod_{i=1}^{k} i}$$

Attempt 1: Construct

$$\mathbb{F} = \mathbb{Q}(k)$$

$$\sigma(c) = c \ \ \forall c \in \mathbb{Q}$$
$$\sigma(k) = k + 1$$

where

$$\text{const}_\sigma \mathbb{Q}(k) = \mathbb{Q}$$

There is no $g \in \mathbb{Q}(k)^*$ s.t.

$$\sigma(g) = (k+1)\,g$$

Represent products

$$\boxed{\prod_{i=1}^{k} i} + \prod_{i=1}^{k}(-i) \qquad \boxed{\mathcal{S}\prod_{i=1}^{k} i = (k+1)\prod_{i=1}^{k} i}$$

Attempt 1: Construct

$$\mathbb{F} = \mathbb{Q}(k)(\mathbf{t_1}) \qquad \begin{aligned} \sigma(c) &= c \ \ \forall c \in \mathbb{Q} \\ \sigma(k) &= k+1 \\ \sigma(\mathbf{t_1}) &= (k+1)\,\mathbf{t_1} \end{aligned}$$

where

$$\text{const}_\sigma \mathbb{Q}(k) = \mathbb{Q}$$

There is no $g \in \mathbb{Q}(k)^*$ s.t.

$$\sigma(g) = (k+1)\,g$$

Represent products

$$\boxed{\prod_{i=1}^{k} i} + \prod_{i=1}^{k}(-i) \qquad \boxed{\mathcal{S}\prod_{i=1}^{k} i = (k+1)\prod_{i=1}^{k} i}$$

Attempt 1: Construct

$$\mathbb{F} = \mathbb{Q}(k)(\mathbf{t_1})$$

$$\sigma(c) = c \quad \forall c \in \mathbb{Q}$$
$$\sigma(k) = k+1$$
$$\sigma(\mathbf{t_1}) = (k+1)\,\mathbf{t_1}$$

where

$$\mathrm{const}_\sigma \mathbb{Q}(k)(\mathbf{t_1}) = \mathbb{Q}$$

Represent products

$$\prod_{i=1}^{k} i + \boxed{\prod_{i=1}^{k}(-i)} \qquad \boxed{\mathcal{S}\prod_{i=1}^{k}(-i) = -(k+1)\prod_{i=1}^{k} i}$$

Attempt 1: Construct

$$\mathbb{F} = \mathbb{Q}(k)(t_1) \qquad \begin{aligned} \sigma(c) &= c \quad \forall c \in \mathbb{Q} \\ \sigma(k) &= k+1 \\ \sigma(t_1) &= (k+1)\,t_1 \end{aligned}$$

where

$$\text{const}_\sigma \mathbb{Q}(k)(t_1) = \mathbb{Q}$$

There is no $g \in \mathbb{Q}(k)(t_1)^*$ s.t.

$$\sigma(g) = -(k+1)\,g$$

Represent products

$$\prod_{i=1}^{k} i + \boxed{\prod_{i=1}^{k}(-i)} \qquad \boxed{\mathcal{S}\prod_{i=1}^{k}(-i) = -(k+1)\prod_{i=1}^{k} i}$$

Attempt 1: Construct

$$\mathbb{F} = \mathbb{Q}(k)(t_1)(\mathbf{t_2})$$

$$\begin{aligned}\sigma(c) &= c \quad \forall c \in \mathbb{Q}\\ \sigma(k) &= k+1\\ \sigma(t_1) &= (k+1)\,t_1\\ \sigma(\mathbf{t_2}) &= -(k+1)\mathbf{t_2}\end{aligned}$$

where

$$\mathrm{const}_\sigma \mathbb{Q}(k)(t_1) = \mathbb{Q}$$

There is no $g \in \mathbb{Q}(k)(t_1)^*$ s.t.

$$\sigma(g) = -(k+1)\,g$$

Represent products

$$\prod_{i=1}^{k} i + \boxed{\prod_{i=1}^{k}(-i)}$$

$$\boxed{\mathcal{S}\prod_{i=1}^{k}(-i) = -(k+1)\prod_{i=1}^{k} i}$$

Attempt 1: Construct

$$\mathbb{F} = \mathbb{Q}(k)(t_1)(\mathbf{t_2})$$

$$\begin{aligned}\sigma(c) &= c \quad \forall c \in \mathbb{Q}\\ \sigma(k) &= k+1\\ \sigma(t_1) &= (k+1)\,t_1\\ \sigma(\mathbf{t_2}) &= -(k+1)\mathbf{t_2}\end{aligned}$$

where

$$\text{const}_\sigma \mathbb{Q}(k)(t_1)(\mathbf{t_2}) \neq \mathbb{Q}$$

Represent products

$$\prod_{i=1}^{k} i + \boxed{\prod_{i=1}^{k}(-i)}$$

Attempt 2: Construct

$$\mathbb{F} = \mathbb{Q}(k)(t_1)$$

$$\sigma(c) = c \quad \forall c \in \mathbb{Q}$$
$$\sigma(k) = k + 1$$
$$\sigma(t_1) = (k+1)\,t_1$$

where

$$\text{const}_\sigma \mathbb{Q}(k)(t_1) = \mathbb{Q}$$

Represent products

$$\prod_{i=1}^{k} i + \boxed{(-1)^k \prod_{i=1}^{k} i} \qquad \boxed{\mathcal{S}\,(-1)^k = -(-1)^k}$$

Attempt 2: Construct

$$\mathbb{F} = \mathbb{Q}(k)(t_1) \qquad \begin{aligned} \sigma(c) &= c \;\; \forall c \in \mathbb{Q} \\ \sigma(k) &= k+1 \\ \sigma(t_1) &= (k+1)\,t_1 \end{aligned}$$

where

$$\mathrm{const}_\sigma \mathbb{Q}(k)(t_1) = \mathbb{Q}$$

Represent products

$$\prod_{i=1}^{k} i + \boxed{(-1)^k \prod_{i=1}^{k} i} \qquad \boxed{\mathcal{S}\,(-1)^k = -(-1)^k}$$

Attempt 2: Construct

$$\mathbb{F} = \mathbb{Q}(k)(t_1)[\mathbf{x}] \qquad \begin{aligned} \sigma(c) &= c \ \ \forall c \in \mathbb{Q} \\ \sigma(k) &= k+1 \\ \sigma(t_1) &= (k+1)\,t_1 \\ \sigma(\mathbf{x}) &= -\mathbf{x}, \quad \mathbf{x}^2 = 1 \end{aligned}$$

where

$$\text{const}_\sigma \mathbb{Q}(k)(t_1)[\mathbf{x}] = \mathbb{Q}$$

Represent products

$$\prod_{i=1}^{k} i + \boxed{(-1)^k \prod_{i=1}^{k} i}$$

$$\boxed{\mathcal{S}\,(-1)^k = -(-1)^k}$$

Attempt 2: Construct

$$\mathbb{F} = \mathbb{Q}(k)(t_1)[\mathbf{x}]$$

$$\begin{aligned} \sigma(c) &= c \;\; \forall c \in \mathbb{Q} \\ \sigma(k) &= k+1 \\ \sigma(t_1) &= (k+1)\,t_1 \\ \sigma(\mathbf{x}) &= -\mathbf{x}, \quad \mathbf{x}^2 = 1 \end{aligned}$$

where

$$\text{const}_\sigma \mathbb{Q}(k)(t_1)[\mathbf{x}] = \mathbb{Q}$$

By construction we have $g = \mathbf{x}t_1$ s.t.

$$\sigma(g) = -(k+1)\,g$$

Represent products

$$\prod_{i=1}^{k} i + \boxed{(-1)^k \prod_{i=1}^{k} i} \qquad\qquad \boxed{\mathcal{S}\,(-1)^k = -(-1)^k}$$

Attempt 2: Construct

$$\mathbb{F} = \mathbb{Q}(k)(t_1)[\mathbf{x}] \qquad \begin{aligned} \sigma(c) &= c \ \ \forall c \in \mathbb{Q} \\ \sigma(k) &= k+1 \\ \sigma(t_1) &= (k+1)\,t_1 \\ \sigma(\mathbf{x}) &= -\mathbf{x}, \quad \mathbf{x}^2 = 1 \end{aligned}$$

where

$$\mathrm{const}_\sigma \mathbb{Q}(k)(t_1)[\mathbf{x}] = \mathbb{Q}$$

But we introduce zero-divisors:

$$(\mathbf{x}+1)(\mathbf{x}-1) = 0 \qquad\qquad \text{field}$$

$$\Updownarrow \qquad\qquad \Downarrow$$

$$((-1)^k+1)((-1)^k-1) = 0 \qquad\qquad \text{ring}$$

Represent indefinite nested sums/products in a difference ring $(\mathbb{A}, \sigma)$:

$$\mathbb{A} := \mathbb{K} = \mathbb{Q}(n_1)\dots(n_l)$$

Finite no. of products over $\mathbb{K}(k)$
(hypergeometric terms)

Represent indefinite nested sums/products in a difference ring $(\mathbb{A}, \sigma)$:

$$\mathbb{A} := \mathbb{K}(k)$$

$\sigma(c) = c \quad \forall c \in \mathbb{K}$

$\sigma(k) = k + 1$

Finite no. of products over $\mathbb{K}(k)$ (hypergeometric terms)

such that

$\text{const}_\sigma \mathbb{A} = \{c \in \mathbb{K}(k) \qquad |\sigma(c) = c\} = \mathbb{K}.$

Represent indefinite nested sums/products in a difference ring $(\mathbb{A}, \sigma)$:

$$\mathbb{A} := \mathbb{K}(k)(t_1)$$

$$\sigma(c) = c \quad \forall c \in \mathbb{K}$$
$$\sigma(k) = k + 1$$
$$\sigma(t_1) = a_1\, t_1, \qquad a_1 \in \mathbb{K}(k)^*$$

Finite no. of products over $\mathbb{K}(k)$
(hypergeometric terms)

such that

$$\mathrm{const}_\sigma \mathbb{A} = \{c \in \mathbb{K}(k)(t_1) \qquad |\sigma(c) = c\} = \mathbb{K}.$$

Represent indefinite nested sums/products in a difference ring $(\mathbb{A}, \sigma)$:

$$\mathbb{A} := \mathbb{K}(k)(t_1)(t_2)$$

$$\sigma(c) = c \quad \forall c \in \mathbb{K}$$
$$\sigma(k) = k + 1$$
$$\sigma(t_1) = a_1\, t_1, \qquad a_1 \in \mathbb{K}(k)^*$$
$$\sigma(t_2) = a_2\, t_2, \qquad a_2 \in \mathbb{K}(k)^*$$

Finite no. of products over $\mathbb{K}(k)$ (hypergeometric terms)

such that

$$\text{const}_\sigma\mathbb{A} = \{c \in \mathbb{K}(k)(t_1)(t_2) \quad | \sigma(c) = c\} = \mathbb{K}.$$

Represent indefinite nested sums/products in a difference ring $(\mathbb{A}, \sigma)$:

$$\mathbb{A} := \mathbb{K}(k)(t_1)(t_2)\dots(t_e)$$

$$\begin{aligned}
\sigma(c) &= c \quad \forall c \in \mathbb{K} \\
\sigma(k) &= k+1 \\
\sigma(t_1) &= a_1\, t_1, && a_1 \in \mathbb{K}(k)^* \\
\sigma(t_2) &= a_2\, t_2, && a_2 \in \mathbb{K}(k)^* \\
&\vdots \\
\sigma(t_e) &= a_e\, t_e, && a_e \in \mathbb{K}(k)^*
\end{aligned}$$

Finite no. of products over $\mathbb{K}(k)$ (hypergeometric terms)

such that

$$\text{const}_\sigma \mathbb{A} = \{c \in \mathbb{K}(k)(t_1)(t_2)\dots(t_e) \quad | \sigma(c) = c\} = \mathbb{K}.$$

Represent indefinite nested sums/products in a difference ring $(\mathbb{A}, \sigma)$:

$$\mathbb{A} := \mathbb{K}(k)(t_1)(t_2)\dots(t_e)[x]$$

$$\begin{aligned}
\sigma(c) &= c \quad \forall c \in \mathbb{K} \\
\sigma(k) &= k+1 \\
\sigma(t_1) &= a_1\, t_1, & a_1 &\in \mathbb{K}(k)^* \\
\sigma(t_2) &= a_2\, t_2, & a_2 &\in \mathbb{K}(k)^* \\
&\vdots \\
\sigma(t_e) &= a_e\, t_e, & a_e &\in \mathbb{K}(k)^* \\
\sigma(x) &= -x, & x^2 &= 1
\end{aligned}$$

Finite no. of products over $\mathbb{K}(k)$ (hypergeometric terms)

such that

$$\text{const}_\sigma \mathbb{A} = \{c \in \mathbb{K}(k)(t_1)(t_2)\dots(t_e)[x] \;|\sigma(c) = c\} = \mathbb{K}.$$

Represent indefinite nested sums/products in a difference ring $(\mathbb{A}, \sigma)$:

$$\mathbb{A} := \mathbb{K}(k)(t_1)(t_2)\dots(t_e)[x](s_1)$$

$\sigma(c) = c \quad \forall c \in \mathbb{K}$ $\sigma(k) = k + 1$ $\sigma(t_1) = a_1\, t_1, \qquad a_1 \in \mathbb{K}(k)^*$ $\sigma(t_2) = a_2\, t_2, \qquad a_2 \in \mathbb{K}(k)^*$ $\vdots$ $\sigma(t_e) = a_e\, t_e, \qquad a_e \in \mathbb{K}(k)^*$ $\sigma(x) = -x, \qquad x^2 = 1$ Finite no. of products over $\mathbb{K}(k)$ (hypergeometric terms)	$\sigma(s_1) = s_1 + f_1 \quad f_1 \in \mathbb{K}(k)(t_1)\dots(t_e)[x]$ Indefinite nested sums in the numerators

such that

$$\text{const}_\sigma \mathbb{A} = \{c \in \mathbb{K}(k)(t_1)(t_2)\dots(t_e)[x][s_1] \quad | \sigma(c) = c\} = \mathbb{K}.$$

Represent indefinite nested sums/products in a difference ring $(\mathbb{A}, \sigma)$:

$$\mathbb{A} := \mathbb{K}(k)(t_1)(t_2)\dots(t_e)[x](s_1)(s_2)$$

$$\begin{aligned}
\sigma(c) &= c \quad \forall c \in \mathbb{K} \\
\sigma(k) &= k+1 \\
\sigma(t_1) &= a_1\, t_1, && a_1 \in \mathbb{K}(k)^* \\
\sigma(t_2) &= a_2\, t_2, && a_2 \in \mathbb{K}(k)^* \\
&\vdots \\
\sigma(t_e) &= a_e\, t_e, && a_e \in \mathbb{K}(k)^* \\
\sigma(x) &= -x, && x^2 = 1
\end{aligned}$$

Finite no. of products over $\mathbb{K}(k)$ (hypergeometric terms)

$$\begin{aligned}
\sigma(s_1) &= s_1 + f_1 && f_1 \in \mathbb{K}(k)(t_1)\dots(t_e)[x] \\
\sigma(s_2) &= s_2 + f_2 && f_2 \in \mathbb{K}(k)(t_1)\dots(t_e)[x][s_1]
\end{aligned}$$

Indefinite nested sums in the numerators

such that

$$\text{const}_\sigma\mathbb{A} = \{c \in \mathbb{K}(k)(t_1)(t_2)\dots(t_e)[x][s_1][s_2] \quad |\sigma(c) = c\} = \mathbb{K}.$$

Represent indefinite nested sums/products in a difference ring $(\mathbb{A}, \sigma)$:

$$\mathbb{A} := \mathbb{K}(k)(t_1)(t_2)\dots(t_e)[x](s_1)(s_2)\dots(s_r)$$

$$\begin{aligned}
\sigma(c) &= c \quad \forall c \in \mathbb{K}\\
\sigma(k) &= k+1\\
\sigma(t_1) &= a_1\, t_1, && a_1 \in \mathbb{K}(k)^*\\
\sigma(t_2) &= a_2\, t_2, && a_2 \in \mathbb{K}(k)^*\\
&\vdots\\
\sigma(t_e) &= a_e\, t_e, && a_e \in \mathbb{K}(k)^*\\
\sigma(x) &= -x, && x^2 = 1
\end{aligned}$$

Finite no. of products over $\mathbb{K}(k)$ (hypergeometric terms)

$$\begin{aligned}
\sigma(s_1) &= s_1 + f_1 && f_1 \in \mathbb{K}(k)(t_1)\dots(t_e)[x]\\
\sigma(s_2) &= s_2 + f_2 && f_2 \in \mathbb{K}(k)(t_1)\dots(t_e)[x][s_1]\\
&\vdots\\
\sigma(s_r) &= s_r + f_r && f_2 \in \mathbb{K}(k)(t_1)\dots(t_e)[x]\\
& && [s_1]\dots[s_{r-1}]
\end{aligned}$$

Indefinite nested sums in the numerators

such that

$$\text{const}_\sigma\mathbb{A} = \{c \in \mathbb{K}(k)(t_1)(t_2)\dots(t_e)[x][s_1][s_2]\dots[s_r] | \sigma(c) = c\} = \mathbb{K}.$$

Represent sums

- Let $(\mathbb{F},\sigma)$ be a difference field with constant field

$$\text{const}_\sigma\mathbb{F} := \{k\in\mathbb{F} | \sigma(k)=k\}$$

field

- Adjoin a new variable t to $\mathbb{F}$ (i.e., $\mathbb{F}(t)$ is a rational function field).
- Extend the shift operator s.t.

$$\sigma(t) = t + f \qquad \text{for some } f\in\mathbb{F}.$$

$\Downarrow$

ring

(Karr 1981) Then $\text{const}_\sigma\mathbb{F}(t) = \text{const}_\sigma\mathbb{F}$ iff

$$\nexists g\in\mathbb{F}: \quad \boxed{\sigma(g) = g + f}$$

There are 2 cases:

1. $\boxed{\nexists g\in\mathbb{F}:\ \sigma(g)=g+f}$ $(\mathbb{F}(t),\sigma)$ is a Σ^*-extension of $(\mathbb{F},\sigma)$
2. $\boxed{\exists g\in\mathbb{F}:\sigma(g)=g+f}$ No need for a Σ^*-extension!

Represent indefinite nested sums/products in a difference ring $(\mathbb{A}, \sigma)$:

$$\mathbb{A} := \mathbb{K}(k)(t_1)(t_2)\dots(t_e)[x](s_1)(s_2)\dots(s_r)$$

$\sigma(c) = c \quad \forall c \in \mathbb{K}$

$\sigma(k) = k + 1$

$\sigma(t_1) = a_1\, t_1, \qquad a_1 \in \mathbb{K}(k)^*$

$\sigma(t_2) = a_2\, t_2, \qquad a_2 \in \mathbb{K}(k)^*$

$\vdots$

$\sigma(t_e) = a_e\, t_e, \qquad a_e \in \mathbb{K}(k)^*$

$\sigma(x) = -x, \qquad x^2 = 1$

Finite no. of products over $\mathbb{K}(k)$ (hypergeometric terms)

$\sigma(s_1) = s_1 + f_1 \quad f_1 \in \mathbb{K}(k)(t_1)\dots(t_e)[x]$

$\sigma(s_2) = s_2 + f_2 \quad f_2 \in \mathbb{K}(k)(t_1)\dots(t_e)[x][s_1]$

$\vdots$

$\sigma(s_r) = s_r + f_r \quad f_2 \in \mathbb{K}(k)(t_1)\dots(t_e)[x][s_1]\dots[s_{r-1}]$

Indefinite nested sums in the numerators

For generalizations see arXiv:1408.2776 [cs.SC]

such that

$$\text{const}_\sigma\mathbb{A} = \{c \in \mathbb{K}(k)(t_1)(t_2)\dots(t_e)[x][s_1][s_2]\dots[s_r] | \sigma(c) = c\} = \mathbb{K}.$$

Simplification of nested product-sum expressions

$A(k)$: indefinite nested product-sum expression

$\downarrow$ `SigmaReduce[A,k]`

$B(k)$: indefinite nested product-sum expression

- such that
$$A(k) = B(k)$$
- such that all the sums in $B(k)$ are **simplified** as above
- and such that the arising sums in $B(k)$ are **algebraically independent** (i.e., they do not satisfy any polynomial relation)

Simplification of nested product-sum expressions

$A(k)$: indefinite nested product-sum expression

$\downarrow$ `SigmaReduce[A,k]`

$B(k)$: indefinite nested product-sum expression

- such that

$$A(k) = B(k)$$

- such that all the sums in $B(k)$ are **simplified** as above
- and such that the arising sums in $B(k)$ are **algebraically independent** (i.e., they do not satisfy any polynomial relation)

Examples

Complete algorithm (implemented in Sigma) if

- the products are hypergeometric (or q-hypergeometric/mixed)
- the sums occur only in the numerator

Part 2: Definite summation

Application 1: Number theory/analysis of algorithms

R. Pemantle and C. Schneider. When is 0.999... equal to 1? Amer. Math. Monthly. 114(4): 344–350, 2007.

You've Got Mail (7/2004)

```
From: Doron Zeilberger
To: Robin Pemantle, Herbert Wilf
CC:Carsten Schneider



Robin and Herb,

I am willing to bet that Carsten Schneider's SIGMA package
for handling sums with harmonic numbers (among others)
can do it in a jiffy. I am Cc-ing this to Carsten.

Carsten: please do it, and Cc- the  answer to me.
             -Doron
```

The problem

```
From: Robin Pemantle [University of Pennsylvania]
To: herb wilf; doron zeilberger

Herb, Doron,

I have a sum that, when I evaluate numerically, looks suspiciously
like it comes out to exactly 1.
Is there a way I can automatically decide this?
The sum may be written in many ways, but one is:
```

$$\sum_{n,k=1}^{\infty}\frac{S_1(k)(S_1(n+1)-1)}{kn(n+1)(k+n)};\qquad S_1(k):=\sum_{i=1}^{k}\frac{1}{i}\ (=H_k).$$

[Arose in the analysis of the simplex algorithm on the Klee-Minty cube]

$$S=\sum_{n=1}^{\infty}\frac{S_1(n+1)-1}{n(n+1)}\boxed{\sum_{k=1}^{\infty}\frac{S_1(k)}{k(k+n)}}$$

where $$S_1(k)=\sum_{i=1}^{k}\frac{1}{i}.$$

GIVEN

$$\text{A'}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k)}{k(k+n)}}_{=:\, f(n,k)}.$$

Telescoping

GIVEN

$$\text{A'}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k)}{k(k+n)}}_{=:\, f(n,k)}.$$

FIND $g(n,k)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{f(n,k)}$$

for all $n, k \geq 1$.

Telescoping

GIVEN

$$\text{A'}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k)}{k(k+n)}}_{=:\, f(n,k)} .$$

FIND $g(n,k)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{f(n,k)}$$

for all $n, k \geq 1$.

$$\boxed{g(n,a+1) - g(n,1)} = \boxed{\sum_{k=1}^{a} f(n,k)}$$

Telescoping

GIVEN

$$\text{A'}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k)}{k(k+n)}}_{=:\, f(n,k)}.$$

FIND $g(n,k)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{f(n,k)}$$

for all $n, k \geq 1$.

no solution ☹

Zeilberger's creative telescoping paradigm

GIVEN

$$\text{A'}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k)}{k(k+n)}}_{=:\, f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n), c_1(n)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n) f(n,k) + c_1(n)\, f(n+1,k)}$$

for all $n, k \geq 1$.

no solution ☹

Zeilberger's creative telescoping paradigm

GIVEN

$$\text{A'}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k)}{k(k+n)}}_{=:\, f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n), c_1(n), c_2(n)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n) f(n,k) + c_1(n)\, f(n+1,k) + c_2(n)\, f(n+2,k)}$$

for all $n, k \geq 1$.

solution ☺

Zeilberger's creative telescoping paradigm

GIVEN

$$\text{A'}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k)}{k(k+n)}}_{=: f(n,k)} .$$

FIND $g(n,k)$ and $c_0(n), c_1(n), c_2(n)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n) f(n,k) + c_1(n)\, f(n+1,k) + c_2(n)\, f(n+2,k)}$$

for all $n, k \geq 1$.

Sigma *computes*: $c_0(n) = n^2$, $c_1(n) = -(n+1)(2n+1)$, $c_2(n) = (n+1)(n+2)$

and

$$g(n,k) := -\frac{k S_1(k) + n + k}{(n+k)(n+k+1)},$$

$$g(n,k+1) := -\frac{(1+n) S_1(k) + n + k + 2}{(n+k+1)(n+k+2)}.$$

Zeilberger's creative telescoping paradigm

GIVEN

$$\text{A'}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k)}{k(k+n)}}_{=:\, f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n), c_1(n), c_2(n)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n)f(n,k) + c_1(n)\,f(n+1,k) + c_2(n)\,f(n+2,k)}$$

for all $n, k \geq 1$.

Summing this equation over k from 1 to a gives:

$$\boxed{g(n,a+1) - g(n,1)} = \boxed{\sum_{k=1}^{a}\Big[c_0(n)f(n,k)+c_1(n)f(n+1,k)+c_2(n)f(n+2,k)\Big]}$$

Zeilberger's creative telescoping paradigm

GIVEN

$$\text{A'}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k)}{k(k+n)}}_{=:\, f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n), c_1(n), c_2(n)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n) f(n,k) + c_1(n)\, f(n+1,k) + c_2(n)\, f(n+2,k)}$$

for all $n, k \geq 1$.

Summing this equation over k from 1 to a gives:

$$\boxed{g(n,a+1) - g(n,1)} = \boxed{\sum_{k=1}^{a} c_0(n) f(n,k) + \sum_{k=1}^{a} c_1(n) f(n+1,k) + \sum_{k=1}^{a} c_2(n) f(n+2,k)}$$

Zeilberger's creative telescoping paradigm

GIVEN

$$\text{A'}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k)}{k(k+n)}}_{=:\, f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n), c_1(n), c_2(n)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n) f(n,k) + c_1(n)\, f(n+1,k) + c_2(n)\, f(n+2,k)}$$

for all $n, k \geq 1$.

Summing this equation over k from 1 to a gives:

$$\boxed{g(n,a+1) - g(n,1)} = \boxed{c_0(n)\sum_{k=1}^{a} f(n,k) + c_1(n)\sum_{k=1}^{a} f(n+1,k) + c_2(n)\sum_{k=1}^{a} f(n+2,k)}$$

Zeilberger's creative telescoping paradigm

GIVEN

$$\text{A'}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k)}{k(k+n)}}_{=:\, f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n), c_1(n), c_2(n)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n) f(n,k) + c_1(n)\, f(n+1,k) + c_2(n)\, f(n+2,k)}$$

for all $n, k \geq 1$.

Summing this equation over k from 1 to a gives:

$$\boxed{g(n,a+1) - g(n,1)} = \boxed{c_0(n)\text{A'}(n) + c_1(n)\text{A'}(n+1) + c_2(n)\text{A'}(n+2)}$$

Zeilberger's creative telescoping paradigm

GIVEN

$$\mathsf{A'}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k)}{k(k+n)}}_{=:\, f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n), c_1(n), c_2(n)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n) f(n,k) + c_1(n)\, f(n+1,k) + c_2(n)\, f(n+2,k)}$$

for all $n, k \geq 1$.

Summing this equation over k from 1 to a gives:

$$\boxed{g(n,a{+}1) - g(n,1)} = \boxed{c_0(n)\mathsf{A'}(n) + c_1(n)\mathsf{A'}(n+1) + c_2(n)\mathsf{A'}(n+2)}$$

$$||\qquad\qquad\qquad\qquad ||$$

$$\frac{a}{(n{+}1)(a{+}n{+}1)} - \frac{(a{+}1)S_1(a)}{(a{+}n{+}1)(a{+}n{+}2)} \qquad n^2\mathsf{A'}(n) - (n{+}1)(2n{+}1)\mathsf{A'}(n+1) + (n{+}1)(n{+}2)\mathsf{A'}(n{+}2)$$

Summation principles

$$n^2\mathbf{A}(n) - (n+1)(2n+1)\mathbf{A}(n+1) + (n+1)(n+2)\mathbf{A}(n+2) = \frac{1}{n+1}$$

Recurrence finder

$$\mathbf{A}(n) = \sum_{k=1}^{\infty} \frac{S_1(k)}{k(k+n)}$$

Summation principles

$$n^2\mathbf{A}(n) - (n+1)(2n+1)\mathbf{A}(n+1) + (n+1)(n+2)\mathbf{A}(n+2) = \frac{1}{n+1}$$

Recurrence solver

$$\mathbf{A}(n) = \sum_{k=1}^{\infty} \frac{S_1(k)}{k(k+n)} \in \left\{ c_1 \frac{nS_1(n)-1}{n^2} + c_2 \frac{1}{n} + \frac{nS_1(n)^2 - 2S_1(n) + nS_2(n)}{2n^2} \,\middle|\, c_1, c_2 \in \mathbb{R} \right\}$$

where

$$S_2(n) = \sum_{i=1}^{n} \frac{1}{i^2}$$

Summation principles

$$n^2\mathbf{A}(n)-(n+1)(2n+1)\mathbf{A}(n+1)+(n+1)(n+2)\mathbf{A}(n+2)=\frac{1}{n+1}$$

Summation package Sigma

$$\mathbf{A}(n)=\sum_{k=1}^{\infty}\frac{S_1(k)}{k(k+n)} = \boxed{0\frac{nS_1(n)-1}{n^2}+\zeta_2\frac{1}{n}+\frac{nS_1(n)^2-2S_1(n)+nS_2(n)}{2n^2}}$$

where

$$S_2(n)=\sum_{i=1}^{n}\frac{1}{i^2}\qquad \zeta_z=\sum_{i=1}^{\infty}\frac{1}{i^z}\ (=\zeta(z))$$

In[1]:= $<<$ **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= $$\mathbf{mySum} = \sum_{k=1}^{a} \frac{S_1(k)}{k(k+n)}$$

In[1]:= $<<$ **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= $$\mathbf{mySum} = \sum_{k=1}^{a} \frac{S_1(k)}{k(k+n)}$$

In[3]:= **rec = GenerateRecurrence[mySum, n][[1]]**

Out[3]= $$n^2\mathrm{SUM}[n] - (n+1)(2n+1)\mathrm{SUM}[n+1] + (n+1)(n+2)\mathrm{SUM}[n+2] == \frac{(-a-1)S_1(a)}{(a+n+1)(a+n+2)} + \frac{a}{(n+1)(a+n+1)}$$

In[1]:= $<<$ **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= $$\mathbf{mySum} = \sum_{k=1}^{a} \frac{S_1(k)}{k(k+n)}$$

In[3]:= **rec = GenerateRecurrence[mySum, n][[1]]**

Out[3]= $$n^2\mathrm{SUM}[n] - (n+1)(2n+1)\mathrm{SUM}[n+1] + (n+1)(n+2)\mathrm{SUM}[n+2] == \frac{(-a-1)S_1(a)}{(a+n+1)(a+n+2)} + \frac{a}{(n+1)(a+n+1)}$$

In[4]:= **rec = LimitRec[rec, SUM[n], {n}, a]**

Out[4]= $$n^2\mathrm{SUM}[n] - (n+1)(2n+1)\mathrm{SUM}[n+1] + (n+1)(n+2)\mathrm{SUM}[n+2] = \frac{1}{n+1}$$

In[1]:= $<<$ **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= $$\mathbf{mySum} = \sum_{k=1}^{a} \frac{S_1(k)}{k(k+n)}$$

In[3]:= $\mathbf{rec = GenerateRecurrence[mySum, n][[1]]}$

Out[3]= $$n^2\mathrm{SUM}[n] - (n+1)(2n+1)\mathrm{SUM}[n+1] + (n+1)(n+2)\mathrm{SUM}[n+2] == \frac{(-a-1)S_1(a)}{(a+n+1)(a+n+2)} + \frac{a}{(n+1)(a+n+1)}$$

In[4]:= $\mathbf{rec = LimitRec[rec, SUM[n], \{n\}, a]}$

Out[4]= $$n^2\mathrm{SUM}[n] - (n+1)(2n+1)\mathrm{SUM}[n+1] + (n+1)(n+2)\mathrm{SUM}[n+2] = \frac{1}{n+1}$$

In[5]:= $\mathbf{recSol = SolveRecurrence[rec, SUM[n]]}$

Out[5]= $$\Big\{\{0, \frac{1}{n}\}, \{0, \frac{\sum_{i=1}^{n}\frac{1}{i}}{n} - \frac{1}{n^2}\}, \{1, \frac{\left(\sum_{i=1}^{n}\frac{1}{i}\right)^2}{2n} - \frac{\sum_{i=1}^{n}\frac{1}{i}}{n^2} + \frac{\sum_{i=1}^{n}\frac{1}{i^2}}{2n}\}\Big\}$$

In[1]:= $<<$ **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= $\mathbf{mySum} = \sum_{k=1}^{a} \frac{S_1(k)}{k(k+n)}$

In[3]:= $\mathbf{rec} = \mathbf{GenerateRecurrence}[\mathbf{mySum}, \mathbf{n}][[1]]$

Out[3]= $n^2 \mathrm{SUM}[n] - (n+1)(2n+1)\mathrm{SUM}[n+1] + (n+1)(n+2)\mathrm{SUM}[n+2] == \frac{(-a-1)S_1(a)}{(a+n+1)(a+n+2)} + \frac{a}{(n+1)(a+n+1)}$

In[4]:= $\mathbf{rec} = \mathbf{LimitRec}[\mathbf{rec}, \mathbf{SUM}[\mathbf{n}], \{\mathbf{n}\}, \mathbf{a}]$

Out[4]= $n^2 \mathrm{SUM}[n] - (n+1)(2n+1)\mathrm{SUM}[n+1] + (n+1)(n+2)\mathrm{SUM}[n+2] = \frac{1}{n+1}$

In[5]:= $\mathbf{recSol} = \mathbf{SolveRecurrence}[\mathbf{rec}, \mathbf{SUM}[\mathbf{n}]]$

Out[5]= $\{\{0, \frac{1}{n}\}, \{0, \frac{\sum_{i=1}^{n} \frac{1}{i}}{n} - \frac{1}{n^2}\}, \{1, \frac{(\sum_{i=1}^{n} \frac{1}{i})^2}{2n} - \frac{\sum_{i=1}^{n} \frac{1}{i}}{n^2} + \frac{\sum_{i=1}^{n} \frac{1}{i^2}}{2n}\}\}$

In[6]:= $\mathbf{FindLinearCombination}[\mathbf{recSol}, \{1, \{\zeta_2, 1/2 + \zeta_2/2\}\}, \mathbf{n}, 2]$

Out[6]= $-\frac{\sum_{i=1}^{n} \frac{1}{i}}{n^2} + \frac{(\sum_{i=1}^{n} \frac{1}{i})^2}{2n} + \frac{\sum_{i=1}^{n} \frac{1}{i^2}}{2n} + \frac{\zeta_2}{n}$

$$S = \sum_{n=1}^{\infty} \frac{S_1(n+1)-1}{n(n+1)} \quad \underbrace{\boxed{\sum_{k=1}^{\infty} \frac{S_1(k)}{k(k+n)}}}$$

$$= \frac{\zeta_2}{n} + \frac{nS_1(n)^2 - 2S_1(n) + nS_2(n)}{2n^2}$$

$$S=\sum_{n=1}^{\infty}\frac{S_1(n+1)-1}{n(n+1)}\quad\underbrace{\boxed{\sum_{k=1}^{\infty}\frac{S_1(k)}{k(k+n)}}}$$

$$=\frac{\zeta_2}{n}+\frac{nS_1(n)^2-2S_1(n)+nS_2(n)}{2n^2}$$

$$=-4\zeta_2+(\zeta_2-1)\sum_{i=1}^{\infty}\frac{S_1(i)}{i^2}-\sum_{i=1}^{\infty}\frac{S_1(i)^2}{i^3}$$

$$+\frac{1}{2}\sum_{i=1}^{\infty}\frac{S_1(i)^3}{i^2}+\frac{1}{2}\sum_{i=1}^{\infty}\frac{S_1(i)S_2(i)}{i^2}$$

$$S=\sum_{n=1}^{\infty}\frac{S_1(n+1)-1}{n(n+1)} \qquad \underbrace{\boxed{\sum_{k=1}^{\infty}\frac{S_1(k)}{k(k+n)}}}$$

$$=\frac{\zeta_2}{n}+\frac{nS_1(n)^2-2S_1(n)+nS_2(n)}{2n^2}$$

$$=-4\zeta_2+(\zeta_2-1)\sum_{i=1}^{\infty}\frac{S_1(i)}{i^2}-\sum_{i=1}^{\infty}\frac{S_1(i)^2}{i^3}$$

$$+\frac{1}{2}\sum_{i=1}^{\infty}\frac{S_1(i)^3}{i^2}+\frac{1}{2}\sum_{i=1}^{\infty}\frac{S_1(i)S_2(i)}{i^2}$$

$$=-4\zeta_2-2\zeta_3+4\zeta_2\zeta_3+2\zeta_5=0.999222...$$

J.M. Borwein and R. Girgensohn. Evaluation of triple Euler sums. *Electron. J. Combin.*, 3:1–27, 1996.
P. Flajolet and B. Salvy. Euler sums and contour integral representations. *Experim. Math.*, 7(1):15–35, 1998.

$$S=\sum_{n=1}^{\infty}\frac{S_1(n+1)-1}{n(n+1)} \qquad \underbrace{\boxed{\sum_{k=1}^{\infty}\frac{S_1(k)}{k(k+n)}}}$$

$$=\frac{\zeta_2}{n}+\frac{nS_1(n)^2-2S_1(n)+nS_2(n)}{2n^2}$$

$$=-4\zeta_2+(\zeta_2-1)\sum_{i=1}^{\infty}\frac{S_1(i)}{i^2}-\sum_{i=1}^{\infty}\frac{S_1(i)^2}{i^3}$$

$$+\frac{1}{2}\sum_{i=1}^{\infty}\frac{S_1(i)^3}{i^2}+\frac{1}{2}\sum_{i=1}^{\infty}\frac{S_1(i)S_2(i)}{i^2}$$

$$=-4\zeta_2-2\zeta_3+4\zeta_2\zeta_3+2\zeta_5=0.999222...$$

J.M. Borwein and R. Girgensohn. Evaluation of triple Euler sums. *Electron. J. Combin.*, 3:1–27, 1996.
P. Flajolet and B. Salvy. Euler sums and contour integral representations. *Experim. Math.*, 7(1):15–35, 1998.
J. Blümlein and D. J. Broadhurst and J. A. M. Vermaseren, The Multiple Zeta Value Data Mine, *Comput. Phys. Commun.*, 181:582–625, 2010.

The definite summation toolbox

C. Schneider. Symbolic summation assists combinatorics. *Sém. Lothar. Combin.* 56(B56), 36pp, 2007.

depends heavily on:

- M. Karr. Summation in finite terms. *J. ACM*, 28:305–350, 1981.
- S.A. Abramov, M. Petkovšek. D'Alembertian solutions of linear differential and difference equations, Proc. ISSAC'94, 169-174, 1994.
- P. A. Hendriks and M. F. Singer. Solving difference equations in finite terms. *J. Symbolic Comput.*, 27(3):239–259, 1999.
- M. Bronstein. On solutions of linear ordinary difference equations in their coefficient field. *J. Symbolic Comput.*, 29(6):841–877, 2000.
- CS. Symbolic summation in difference fields. J. Kepler University, May 2001. PhD Thesis.
- CS. Solving parameterized linear difference equations in terms of indefinite nested sums and products. *J. Differ. Equations Appl.*, 11(9):799–821, 2005.
- S.A. Abramov, M. Bronstein, M. Petkovšek, CS, in preparation

and generalizes the hypergeometric summation paradigms presented in

M. Petkovšek, H. S. Wilf, and D. Zeilberger. $A = B$. A. K. Peters, Wellesley, MA, 1996.

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$A(n) = \sum_{k=1}^{n} f(n,k);$$

$f(n,k)$: indefinite nested product-sum in k;
n: extra parameter

Find a recurrence for $A(n)$

Back to creative telescoping

Given
$$f(n,k)=\frac{S_1(k)}{k(k+n)};$$
Find $g(n,k)$ and $c_0(n),c_1(n),c_2(n)$:

$$\boxed{g(n,k+1)-g(n,k)}=\boxed{c_0(n)\,f(n,k)+c_1(n)\,f(n+1,k)+c_2(n)\,f(n+2,k)}$$

Back to creative telescoping

Given

$$f(n,k)=\frac{S_1(k)}{k(k+n)};$$

Find $g(n,k)$ and $c_0(n),c_1(n),c_2(n)$:

$$\boxed{g(n,k+1)-g(n,k)}=\boxed{c_0(n)\,f(n,k)+c_1(n)\,f(n+1,k)+c_2(n)\,f(n+2,k)}$$

A difference field for the **summand:**

Construct a rational function field

$$\mathbb{F}$$

and a field automorphism $\sigma:\mathbb{F}\to\mathbb{F}$ defined by

Back to creative telescoping

Given
$$f(n,k)=\frac{S_1(k)}{k(k+n)};$$
Find $g(n,k)$ and $c_0(n), c_1(n), c_2(n)$:

$$\boxed{g(n,k+1)-g(n,k)}=\boxed{c_0(n)\,f(n,k)+c_1(n)\,f(n+1,k)+c_2(n)\,f(n+2,k)}$$

A difference field for the **summand:**

Construct a rational function field

$$\mathbb{F}:=\mathbb{Q}(\mathbf{n})$$

and a field automorphism $\sigma:\mathbb{F}\to\mathbb{F}$ defined by

$$\sigma(\mathbf{c})=\mathbf{c}\quad\forall\mathbf{c}\in\mathbb{Q}(\mathbf{n}),$$

Back to creative telescoping

Given

$$f(n,k)=\frac{S_1(k)}{k(k+n)};$$

Find $g(n,k)$ and $c_0(n), c_1(n), c_2(n)$:

$$\boxed{g(n,k+1)-g(n,k)} = \boxed{c_0(n)\,f(n,k)+c_1(n)\,f(n+1,k)+c_2(n)\,f(n+2,k)}$$

A difference field for the **summand:**

Construct a rational function field

$$\mathbb{F}:=\mathbb{Q}(n)(\mathbf{k})$$

and a field automorphism $\sigma:\mathbb{F}\to\mathbb{F}$ defined by

$$\sigma(c)=c \quad \forall c\in\mathbb{Q}(n),$$

$$\sigma(\mathbf{k})=\mathbf{k}+\mathbf{1}, \qquad\qquad \mathcal{S}\,k=k+1,$$

Back to creative telescoping

Given

$$f(n,k)=\frac{S_1(k)}{k(k+n)};$$

Find $g(n,k)$ and $c_0(n), c_1(n), c_2(n)$:

$$\boxed{g(n,k+1)-g(n,k)}=\boxed{c_0(n)\,f(n,k)+c_1(n)\,f(n+1,k)+c_2(n)\,f(n+2,k)}$$

A difference field for the **summand:**

Construct a rational function field

$(\mathbb{F},\sigma)$ is a $\Pi\Sigma$-field

$$\mathbb{F}:=\mathbb{Q}(n)(k)(\mathbf{s})$$

Karr 1981

and a field automorphism $\sigma:\mathbb{F}\to\mathbb{F}$ defined by

$$\sigma(c)=c \quad \forall c\in\mathbb{Q}(n),$$

$$\sigma(k)=k+1, \qquad\qquad \mathcal{S}\,k=k+1,$$

$$\sigma(\mathbf{s})=\mathbf{s}+\frac{\mathbf{1}}{\mathbf{k+1}}, \qquad\qquad \mathcal{S}\,S_1(k)=S_1(k)+\frac{1}{k+1},$$

Back to creative telescoping

Given
$$f(n,k)=\frac{S_1(k)}{k(+n)};$$
Find $g(n,k)$ and $c_0(n),c_1(n),c_2(n)$:
$$\boxed{g(n,k+1)-g(n,k)}=\boxed{c_0(n)\,f(n,k)+c_1(n)\,f(n+1,k)+c_2(n)\,f(n+2,k)}$$

FIND $g\in\mathbb{F}$ and $c_0,c_1,c_2\in\mathbb{Q}(n)$:
$$\boxed{\sigma(g)-g}=\boxed{c_0\,\frac{s}{k(k+n)}+c_1\,\frac{s}{k(k+n+1)}+c_2\,\frac{s}{k(k+n+2)}}$$

Back to creative telescoping

Given

$$f(n,k)=\frac{S_1(k)}{k(+n)};$$

Find $g(n,k)$ and $c_0(n), c_1(n), c_2(n)$:

$$\boxed{g(n,k+1)-g(n,k)}=\boxed{c_0(n)\,f(n,k)+c_1(n)\,f(n+1,k)+c_2(n)\,f(n+2,k)}$$

FIND $g\in\mathbb{F}$ and $c_0, c_1, c_2 \in \mathbb{Q}(n)$:

$$\boxed{\sigma(g)-g}=\boxed{c_0\,\frac{s}{k(k+n)}+c_1\,\frac{s}{k(k+n+1)}+c_2\,\frac{s}{k(k+n+2)}}$$

$$\downarrow$$

$$c_0=n^2,\ c_1=-(n+1)(2n+1),\ c_2=(n+1)(n+2)$$
$$g=-\frac{ks+n+k}{(n+k)(n+k+1)}$$

Zeilberger's creative telescoping paradigm

GIVEN

$$A'(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k)}{k(k+n)}}_{=:\, f(n,k)} .$$

FIND $g(n,k)$ and $c_0(n), c_1(n), c_2(n)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n) f(n,k) + c_1(n)\, f(n+1,k) + c_2(n)\, f(n+2,k)}$$

for all $n, k \geq 1$.

Sigma *computes*: $c_0(n) = n^2$, $c_1(n) = -(n+1)(2n+1)$, $c_2(n) = (n+1)(n+2)$

and

$$g(n,k) := -\frac{kS_1(k) + n + k}{(n+k)(n+k+1)},$$

$$g(n,k+1) := -\frac{(1+n)S_1(k) + n + k + 2}{(n+k+1)(n+k+2)}.$$

Zeilberger's creative telescoping paradigm

GIVEN

$$\text{A'}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k)}{k(k+n)}}_{=:\, f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n), c_1(n), c_2(n)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n) f(n,k) + c_1(n)\, f(n+1,k) + c_2(n)\, f(n+2,k)}$$

for all $n, k \geq 1$.

Summing this equation over k from 1 to a gives:

$$\boxed{g(n,a+1) - g(n,1)} = \boxed{\sum_{k=1}^{a} \Big[c_0(n) f(n,k) + c_1(n) f(n+1,k) + c_2(n) f(n+2,k)\Big]}$$

Zeilberger's creative telescoping paradigm

GIVEN

$$\text{A'}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k)}{k(k+n)}}_{=:\,f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n), c_1(n), c_2(n)$:

$$\boxed{g(n,k+1)-g(n,k)} = \boxed{c_0(n)f(n,k)+c_1(n)\,f(n+1,k)+c_2(n)\,f(n+2,k)}$$

for all $n,k\geq 1$.

Summing this equation over k from 1 to a gives:

$$\boxed{g(n,a+1)-g(n,1)} = \boxed{\sum_{k=1}^{a}c_0(n)f(n,k)+\sum_{k=1}^{a}c_1(n)f(n+1,k)+\sum_{k=1}^{a}c_2(n)f(n+2,k)}$$

Zeilberger's creative telescoping paradigm

GIVEN

$$\text{A'}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k)}{k(k+n)}}_{=:\, f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n), c_1(n), c_2(n)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n) f(n,k) + c_1(n)\, f(n+1,k) + c_2(n)\, f(n+2,k)}$$

for all $n, k \geq 1$.

Summing this equation over k from 1 to a gives:

$$\boxed{g(n,a+1) - g(n,1)} = \boxed{c_0(n)\sum_{k=1}^{a} f(n,k) + c_1(n)\sum_{k=1}^{a} f(n+1,k) + c_2(n)\sum_{k=1}^{a} f(n+2,k)}$$

Zeilberger's creative telescoping paradigm

GIVEN

$$\mathsf{A'}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k)}{k(k+n)}}_{=:\,f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n), c_1(n), c_2(n)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n) f(n,k) + c_1(n)\, f(n+1,k) + c_2(n)\, f(n+2,k)}$$

for all $n, k \geq 1$.

Summing this equation over k from 1 to a gives:

$$\boxed{g(n,a+1) - g(n,1)} = \boxed{c_0(n)\mathsf{A'}(n) + c_1(n)\mathsf{A'}(n+1) + c_2(n)\mathsf{A'}(n+2)}$$

Zeilberger's creative telescoping paradigm

GIVEN

$$\text{A'}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k)}{k(k+n)}}_{=:\, f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n), c_1(n), c_2(n)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n) f(n,k) + c_1(n)\, f(n+1,k) + c_2(n)\, f(n+2,k)}$$

for all $n, k \geq 1$.

Summing this equation over k from 1 to a gives:

$$\boxed{g(n,a+1) - g(n,1)} = \boxed{c_0(n)\text{A'}(n) + c_1(n)\text{A'}(n+1) + c_2(n)\text{A'}(n+2)}$$

$$\frac{a}{(n+1)(a+n+1)} - \frac{(a+1)S_1(a)}{(a+n+1)(a+n+2)} \quad || \qquad || \quad n^2\text{A'}(n) - (n+1)(2n+1)\text{A'}(n+1) + (n+1)(n+2)\text{A'}(n+2)$$

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$A(n)=\sum_{k=1}^{n} f(n,k);$$

$f(n,k)$: indefinite nested product-sum in k;
n: extra parameter

Find a recurrence for $A(n)$

2. Recurrence solving

GIVEN a recurrence

$a_0(n),\dots,a_d(n),h(n)$:
indefinite nested product-sum expressions in n.

$$a_0(n)A(n)+\dots+a_d(n)A(n+d)=h(n);$$

Find all solutions expressible by indefinite nested products/sums in n.
(Abramov/Bronstein/Petkovšek/CS, in preparation)

Recurrence solving

Special case: homogeneous recurrences with $a_i(n) \in \mathbb{K}[n]$

$$a_d(n)A(n+d) + a_{d-1}(n)A(n+d-1) + \cdots + a_0(n)A(n) = 0$$

Recurrence solving

Special case: homogeneous recurrences with $a_i(n) \in \mathbb{K}[n]$

$$a_d(n)A(n+d) + a_{d-1}(n)A(n+d-1) + \cdots + a_0(n)A(n) = 0$$
$$||$$
$$\Big[a_d(n)S^d + a_{d-1}(n)S^{d-1} + \cdots + a_0(n)I\Big]A(n)$$

Recurrence solving

Special case: homogeneous recurrences with $a_i(n) \in \mathbb{K}[n]$

$$a_d(n)A(n+d) + a_{d-1}(n)A(n+d-1) + \dots + a_0(n)A(n) = 0$$

$$||$$

$$\Big[a_d(n)S^d + a_{d-1}(n)S^{d-1} + \dots + a_0(n)I\Big]A(n)$$

Hyper (Petkovšek)

$$\prod_{j=\lambda}^{n} b_1(j-1)$$

Recurrence solving

Special case: homogeneous recurrences with $a_i(n) \in \mathbb{K}[n]$

$$a_d(n)A(n+d) + a_{d-1}(n)A(n+d-1) + \cdots + a_0(n)A(n) = 0$$
$$||$$
$$\Big[\Big(\tilde{a}_{d-1}(n)S^{d-1} + \tilde{a}_{d-2}(n)S^{d-2} + \cdots + \tilde{a}_0(n)I\Big)\Big(S - b_1(n)\Big)\Big]A(n)$$

$$\prod_{j=\lambda}^{n} b_1(j-1)$$

Recurrence solving

Special case: homogeneous recurrences with $a_i(n) \in \mathbb{K}[n]$

$$a_d(n)A(n+d) + a_{d-1}(n)A(n+d-1) + \dots + a_0(n)A(n) = 0$$

$$||$$

$$\Big[\Big(\tilde{a}_{d-1}(n)S^{d-1} + \tilde{a}_{d-2}(n)S^{d-2} + \dots + \tilde{a}_0(n)I\Big)\Big(S - b_1(n)\Big)\Big]A(n)$$

Hyper (Petkovšek)

$$\prod_{j=\lambda}^{n} b_2(j-1)$$

Recurrence solving

Special case: homogeneous recurrences with $a_i(n) \in \mathbb{K}[n]$

$$a_d(n)A(n+d) + a_{d-1}(n)A(n+d-1) + \cdots + a_0(n)A(n) = 0$$

$$\|$$

$$\Big[\Big(\tilde{a}_{d-2}(n)S^{d-2} + \tilde{a}_{d-3}(n)S^{d-3} + \cdots + \tilde{a}_0(n)I\Big)\Big(S - b_2(n)\Big)\Big(S - b_1(n)\Big)\Big]A(n)$$

$$\prod_{j=\lambda}^{n} b_2(j-1)$$

Recurrence solving

Special case: homogeneous recurrences with $a_i(n) \in \mathbb{K}[n]$

$$a_d(n)A(n+d) + a_{d-1}(n)A(n+d-1) + \dots + a_0(n)A(n) = 0$$

$$||$$

$$c(n)\Big(S - b_d(n)\Big)\dots\Big(S - b_2(n)\Big)\Big(S - b_1(n)\Big)A(n)$$

Recurrence solving

Special case: homogeneous recurrences with $a_i(n) \in \mathbb{K}[n]$

$$a_d(n)A(n+d) + a_{d-1}(n)A(n+d-1) + \cdots + a_0(n)A(n) = 0$$

$$\|$$

$$c(n)\Big(S - b_d(n)\Big)\dots\Big(S - b_2(n)\Big)\Big(S - b_1(n)\Big)A(n)$$

$$L_1(n) = \prod_{j=\lambda}^{n} b_1(j-1)$$

Recurrence solving

Special case: homogeneous recurrences with $a_i(n) \in \mathbb{K}[n]$

$$a_d(n)A(n+d) + a_{d-1}(n)A(n+d-1) + \dots + a_0(n)A(n) = 0$$
$$||$$
$$c(n)\Big(S - b_d(n)\Big)\dots\Big(S - b_2(n)\Big)\Big(S - b_1(n)\Big)A(n)$$

$$L_1(n) = \prod_{j=\lambda}^{n} b_1(j-1)$$

$$L_2(n) = \prod_{j=\lambda}^{n} b_1(j-1) \sum_{i_1=\lambda}^{n-1} \frac{\prod_{j=\lambda}^{i_1} b_2(j-1)}{\prod_{j=\lambda}^{i1+1} b_1(j-1)}$$

Recurrence solving

Special case: homogeneous recurrences with $a_i(n) \in \mathbb{K}[n]$

$$a_d(n)A(n+d) + a_{d-1}(n)A(n+d-1) + \dots + a_0(n)A(n) = 0$$

$$||$$

$$c(n)\Big(S - b_d(n)\Big)\dots\Big(S - b_2(n)\Big)\Big(S - b_1(n)\Big)A(n)$$

$$L_1(n) = \prod_{j=\lambda}^{n} b_1(j-1)$$

d linearly independent solutions

$$L_2(n) = \prod_{j=\lambda}^{n} b_1(j-1) \sum_{i_1=\lambda}^{n-1} \frac{\prod_{j=\lambda}^{i_1} b_2(j-1)}{\prod_{j=\lambda}^{i1+1} b_1(j-1)}$$

$$\vdots$$

$$L_d(n) = \prod_{j=\lambda}^{n} b_1(j-1) \sum_{i_1=\lambda}^{n-1} \frac{\prod_{j=\lambda}^{i_1} b_2(j-1)}{\prod_{j=\lambda}^{i_1+1} b_1(j-1)} \dots \sum_{i_{d-1}=\lambda}^{i_{d-2}-1} \frac{\prod_{j=\lambda}^{i_{d-1}} b_d(j-1)}{\prod_{j=\lambda}^{i_{d-1}+1} b_{d-1}(j-1)}$$

Recurrence solving

Special case: homogeneous recurrences with $a_i(n) \in \mathbb{K}[n]$

$$a_d(n)A(n+d) + a_{d-1}(n)A(n+d-1) + \dots + a_0(n)A(n) = 0$$
$$||$$
$$c(n)\Big(S - b_d(n)\Big)\dots\Big(S - b_2(n)\Big)\Big(S - b_1(n)\Big)A(n)$$

$$L_1(n) = \prod_{j=\lambda}^{n} b_1(j-1)$$

d linearly independent solutions

$$L_2(n) = \prod_{j=\lambda}^{n} b_1(j-1) \sum_{i_1=\lambda}^{n-1} \frac{\prod_{j=\lambda}^{i_1} b_2(j-1)}{\prod_{j=\lambda}^{i1+1} b_1(j-1)}$$

Example

$$\vdots$$

$$L_d(n) = \prod_{j=\lambda}^{n} b_1(j-1) \sum_{i_1=\lambda}^{n-1} \frac{\prod_{j=\lambda}^{i_1} b_2(j-1)}{\prod_{j=\lambda}^{i_1+1} b_1(j-1)} \dots \sum_{i_{d-1}=\lambda}^{i_{d-2}-1} \frac{\prod_{j=\lambda}^{i_{d-1}} b_d(j-1)}{\prod_{j=\lambda}^{i_{d-1}+1} b_{d-1}(j-1)}$$

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$A(n) = \sum_{k=1}^{n} f(n,k);$$

$f(n,k)$: indefinite nested product-sum in k;
n: extra parameter

Find a recurrence for $A(n)$

2. Recurrence solving

GIVEN a recurrence

$a_0(n), \dots, a_d(n), h(n)$:
indefinite nested product-sum expressions in n.

$$a_0(n)A(n) + \cdots + a_d(n)A(n+d) = h(n);$$

Find all solutions expressible by indefinite nested products/sums in n.
(Abramov/Bronstein/Petkovšek/CS, in preparation)

Note: the sum solutions are highly nested;
they can be expressed in our difference rings

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$A(n)=\sum_{k=1}^{n} f(n,k);$$

$f(n,k)$: indefinite nested product-sum in k;
n: extra parameter

Find a recurrence for $A(n)$

2. Recurrence solving

GIVEN a recurrence

$a_0(n),\dots,a_d(n),h(n)$:
indefinite nested product-sum expressions in n.

$$a_0(n)A(n)+\dots+a_d(n)A(n+d)=h(n);$$

Find all solutions expressible by indefinite nested products/sums in n.
(Abramov/Bronstein/Petkovšek/CS, in preparation)

3. Simplify the solutions (using difference ring theory) s.t.

- the sums are algebraic independent
- the sums can be given in terms of special functions

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$A(n) = \sum_{k=1}^{n} f(n,k);$$

$f(n,k)$: indefinite nested product-sum in k;
n: extra parameter

Find a recurrence for $A(n)$

2. Recurrence solving

GIVEN a recurrence

$a_0(n),\dots,a_d(n),h(n)$:
indefinite nested product-sum expressions in n.

$$a_0(n)A(n)+\dots+a_d(n)A(n+d)=h(n);$$

Find all solutions expressible by indefinite nested products/sums in n.
(Abramov/Bronstein/Petkovšek/CS, in preparation)

4. Find a "closed form"

A(n)=combined solutions in terms of indefinite nested sums in n.

Application 2: Padé Approximation to $\log(x)$ at $x = 1$

K. Driver, H. Prodinger, C. Schneider, and J.A.C. Weideman. Padé approximations to the logarithm II: Identities, recurrences, and symbolic computation. *Ramanujan J.* 11(2):139–158, 2006.

Padé Approximation to $\log(x)$ at $x = 1$

FIND

$$r_m(x) = \sum_{k=0}^{m} a_k x^k, \qquad s_m(x) = \sum_{k=0}^{m} b_k x^k$$

s.t.

$$-\frac{s_m(x)}{r_m(x)} \equiv log(x) \mod (x-1)^{2m+1}$$

"Approximate $\log(x)$ around 1 with a rational function"

Padé Approximation to $\log(x)$ at $x = 1$

FIND

$$r_m(x) = \sum_{k=0}^{m} a_k x^k, \qquad s_m(x) = \sum_{k=0}^{m} b_k x^k$$

s.t.

$$-\frac{s_m(x)}{r_m(x)} \equiv log(x) \mod (x-1)^{2m+1}$$

$$\Updownarrow$$

$$r_m(x)\,\log(x) + s_m(x) \equiv 0 \mod (x-1)^{2m+1}$$

Padé Approximation to $\log(x)$ at $x = 1$

FIND

$$r_m(x) = \sum_{k=0}^{m} a_k x^k, \qquad s_m(x) = \sum_{k=0}^{m} b_k x^k$$

s.t.

$$-\frac{s_m(x)}{r_m(x)} \equiv log(x) \mod (x-1)^{2m+1}$$

$$\Updownarrow$$

$$r_m(x)\,\log(x) + s_m(x) \equiv 0 \mod (x-1)^{2m+1}$$

$$\Updownarrow$$

$$\boxed{r_m(x)\,\log(x) + s_m(x) = O((x-1)^{2m+1})}$$

- Linear Padé

FIND $r_m(x)=\sum_{k=0}^{m} a_k x^k, s_m(x)=\sum_{k=0}^{m} b_k x^k$:

$$\boxed{r_m(x)\,\log(x)+s_m(x)=O((x-1)^{2m+1})}$$

- Linear Padé

FIND $r_m(x)=\sum_{k=0}^{m} a_k x^k, s_m(x)=\sum_{k=0}^{m} b_k x^k$:

$$\boxed{r_m(x)\ \log(x)+s_m(x)=O((x-1)^{2m+1})}$$

- Quadratic Padé

FIND $r_m(x)=\sum_{k=0}^{m} a_k x^k, s_m(x)=\sum_{k=0}^{m} b_k x^k, t_m(x)=\sum_{k=0}^{m} c_k x^k$:

$$\boxed{r_m(x)\,(\log x)^2+s_m(x)\ \log(x)+t_m(x)=O((x-1)^{3m+2})}$$

- Linear Padé

FIND $r_m(x)=\sum_{k=0}^{m} a_k x^k,\ s_m(x)=\sum_{k=0}^{m} b_k x^k$:

$$\boxed{r_m(x)\ \log(x)+s_m(x)=O((x-1)^{2m+1})}$$

- Quadratic Padé

FIND $r_m(x)=\sum_{k=0}^{m} a_k x^k,\ s_m(x)=\sum_{k=0}^{m} b_k x^k,\ t_m(x)=\sum_{k=0}^{m} c_k x^k$:

$$\boxed{r_m(x)\,(\log x)^2+s_m(x)\ \log(x)+t_m(x)=O((x-1)^{3m+2})}$$

- Higher Order Padé ($n\geq 1$)

FIND $r_m(x)=\sum_{k=0}^{m} a_k x^k,\ s_m(x)=\sum_{k=0}^{m} b_k x^k,\dots,t_m(x)=\sum_{k=0}^{m} c_k x^k$:

$$\boxed{r_m(x)\,(\log x)^n+s_m(x)\,(\log x)^{n-1}+\dots+t_m(x)=O((x-1)^{(n+1)(m+1)-1})}$$

Quadratic Padé

Note: $r_m(x), s_m(x), t_m(x)$ with

$$\boxed{r_m(x)\,(\log x)^2 + s_m(x)\,\log(x) + t_m(x) = O((x-1)^{3m+2})}$$

are uniquely defined (up to a constant factor).

Quadratic Padé

Note: $r_m(x), s_m(x), t_m(x)$ with

$$\boxed{r_m(x)\,(\log x)^2 + s_m(x)\ \log(x) + t_m(x) = O((x-1)^{3m+2})}$$

are uniquely defined (up to a constant factor).

DEFINE

$$\boxed{R_m(x) := r_m(x)\,(\log x)^2 + s_m(x)\ \log(x) + t_m(x)}$$

Quadratic Padé

Note: $r_m(x), s_m(x), t_m(x)$ with

$$\boxed{r_m(x)\,(\log x)^2 + s_m(x)\,\log(x) + t_m(x) = O((x-1)^{3m+2})}$$

are uniquely defined (up to a constant factor).

DEFINE

$$\boxed{R_m(x) := r_m(x)\,(\log x)^2 + s_m(x)\,\log(x) + t_m(x)}$$

THEN $R_m(x)$ is a solution of

$$\boxed{\left[x(\delta - m)^3 - \delta^3\right] y(x) = 0}$$

for the operator $\delta := x\,\frac{d}{dx}$

$$\boxed{\left[x(\delta-m)^3-\delta^3\right]y(x)=0}$$

$$\downarrow$$

Frobenius' method

$$\downarrow$$

General solution:

$$\boxed{y(x)=c_1\,y_1(x)+c_2\,y_2(x)+c_3\,y_3(x)}$$

with

$$y_1(x)=\sum_{k=0}^{m}\binom{m}{k}^3(-x)^k,\qquad y_2(x)=y_1(x)\log(x)+\sum_{k=0}^{m}\left[\frac{d}{dk}\binom{m}{k}^3\right](-x)^k,$$

$$y_3(x)=y_1(x)\log^2(x)+2\log(x)\sum_{k=0}^{m}\left[\frac{d}{dk}\binom{m}{k}^3\right](-x)^k+\sum_{k=0}^{m}\left[\frac{d^2}{dk^2}\binom{m}{k}^3\right](-x)^k.$$

Hence

$$\begin{aligned} R_m(x) &= \boxed{r_m(x)}\,(\log x)^2 + \boxed{s_m(x)}\,\log(x) + \boxed{t_m(x)} \\ &= c_1\,y_1(x) + c_2\,y_2(x) + c_3\,y_3(x) \end{aligned}$$

with

$$R_m(1) = 0, \quad R'_m(1) = 0.$$

Hence

$$\begin{aligned}R_m(x) &= \boxed{r_m(x)}\,(\log x)^2 + \boxed{s_m(x)}\,\log(x) + \boxed{t_m(x)}\\ &= c_1\,y_1(x) + c_2\,y_2(x) + c_3\,y_3(x)\end{aligned}$$

with

$$R_m(1) = 0, \quad R'_m(1) = 0.$$

Computer experiments:

$$c_1 = \pi^2, c_2 = 0, c_3 = 1$$

Hence

$$R_m(x) = \boxed{r_m(x)}\,(\log x)^2 + \boxed{s_m(x)}\,\log(x) + \boxed{t_m(x)}$$
$$= c_1\,y_1(x) + c_2\,y_2(x) + c_3\,y_3(x)$$

with

$$R_m(1) = 0, \quad R_m'(1) = 0.$$

Computer experiments:

$$c_1 = \pi^2, c_2 = 0, c_3 = 1$$

$$R_m(x) = \boxed{\sum_{k=0}^{m} \binom{m}{k}^3 (-x)^k}\,(\log x)^2 + \boxed{2\sum_{k=0}^{m} \left[\frac{d}{dk}\binom{m}{k}^3\right](-x)^k}\,\log(x)$$
$$+ \boxed{\sum_{k=0}^{m} \left[\frac{d^2}{dk^2}\binom{m}{k}^3 + \pi^2\binom{m}{k}^3\right](-x)^k}$$

Computer experiments

$$R_m(1) = 0, \quad R'_m(1) = 0 \text{ with } c_1 = \pi^2, c_2 = 0, c_3 = 1$$

Computer experiments

$$R_m(1) = 0, \quad R'_m(1) = 0 \text{ with } c_1 = \pi^2, c_2 = 0, c_3 = 1$$

$$\Updownarrow$$

A. Weideman

$$\sum_{k=0}^{m}(-1)^k \left(\frac{d^2}{dk^2} + \pi^2\right)\left[k^\ell \binom{m}{k}^3\right] = 0, \qquad \ell = 0, 1$$

Computer experiments

$$R_m(1) = 0, \quad R'_m(1) = 0 \text{ with } c_1 = \pi^2, c_2 = 0, c_3 = 1$$

$\Updownarrow$ *A. Weideman*

$$\sum_{k=0}^{m} (-1)^k \left(\frac{d^2}{dk^2} + \pi^2\right) \left[k^\ell \binom{m}{k}^3\right] = 0, \qquad \ell = 0, 1$$

$\Updownarrow$ *H. Prodinger*

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k}^3 \Big[3(S_1(m-k) - S_1(k))^2 + S_2(m-k) + S_2(k)\Big] = 0,$$

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k}^3 \Big[k(3(S_1(m-k) - S_1(k))^2 + S_2(m-k) + S_2(k)) + 2(S_1(m-k) - S_1(k))\Big] = 0$$

Sigma

In[7]:= $\mathbf{mySum} = \sum_{k=0}^{m}(-1)^k \binom{m}{k}^3 (3(S_1(m-k) - S_1(k))^2 + S_2(m-k) + S_2(k));$

In[8]:= $\mathbf{GenerateRecurrence[mySum, n]}$

Out[8]= $\{3(3m+2)(3m+4)\mathrm{SUM}(m) + (m+2)^2\mathrm{SUM}(m+2) = 0\}$

In[7]:= $\mathbf{mySum} = \sum_{k=0}^{m} (-1)^k \binom{m}{k}^3 (3(S_1(m-k) - S_1(k))^2 + S_2(m-k) + S_2(k));$

In[8]:= $\mathbf{GenerateRecurrence[mySum, n]}$

Out[8]= $\{3(3m+2)(3m+4)\mathrm{SUM}(m) + (m+2)^2\mathrm{SUM}(m+2) = 0\}$

Theorem (Sigma, 2005). For $n \geq 0$,

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k}^3 \left[3(S_1(m-k) - S_1(k))^2 + S_2(m-k) + S_2(k)\right]$$

- Linear Padé

$$r_m(x)\,\log(x) + s_m(x) = O((x-1)^{2m+1})$$

$$\sum_{k=0}^{m} \binom{m}{k}^2 \Big[1 + 2k(\mathsf{H}_{m-k} - \mathsf{H}_k)\Big] = 0$$

- Linear Padé

$$r_m(x)\ \log(x) + s_m(x) = O((x-1)^{2m+1})$$

$$\sum_{k=0}^{m} \binom{m}{k}^2 \left[1 + 2k(\mathsf{H}_{m-k} - \mathsf{H}_k)\right] = 0$$

- Quadratic Padé

$$r_m(x)\,(\log x)^2 + s_m(x)\ \log(x) + t_m(x) = O((x-1)^{3m+2})$$

$$\sum_{k=0}^{m}(-1)^k\binom{m}{k}^3\left[3(S_1(m-k)-S_1(k))^2+S_2(m-k)+S_2(k)\right]=0,$$

$$\sum_{k=0}^{m}(-1)^k\binom{m}{k}^3\Big[k(3(S_1(m-k)-S_1(k))^2+S_2(m-k)+S_2(k)) \\ +2(S_1(m-k)-S_1(k))\Big]=0$$

- Cubic Padé

$$r_m(x)\,(\log x)^3 + s_m(x)\,(\log x)^2 + t_m(x)\,\log(x) + u_m(x) = O((x-1)^{4m+3})$$

$$\begin{aligned}\sum_{k=0}^{m}\binom{m}{k}^4\Big[&3(S_1(m-k)-S_1(k))^2+S_2(m-k)+S_2(k)\\&+4k(S_1(m-k)-S_1(k))^3\\&+6(S_1(m-k)-S_1(k))(S_2(m-k)+S_2(k))\\&+S_3(m-k)-S_3(k)\Big]=0\end{aligned}$$

- Padé of order 4

$$
\begin{aligned}
r_m(x)\,(\log x)^4 + s_m(x)\,(\log x)^3 + t_m(x)\,(\log x)^2 \\
+ u_m(x)\,\log(x) + v_m = O((x-1)^{5m+4})
\end{aligned}
$$

$$
\begin{aligned}
\sum_{k=0}^{m}(-1)^k\binom{m}{k}^5\Big[&125\big(S_1(k)-S_1(m-k)\big)^4\\
&+150\big(S_1(k)-S_1(m-k)\big)^2\big(S_2(k)+S_2(m-k)\big)\\
&+15\big(S_2(k)+S_2(m-k)\big)^2\\
&+40\big(S_1(k)-S_1(m-k)\big)\big(S_3(k)-S_3(m-k)\big)\\
&+6S_4(k)+6S_4(m-k)\Big]=0
\end{aligned}
$$

$$\sum_{k=0}^{m}(-1)^k\binom{m}{k}^5\Big[-60\big(S_1(k)-S_1(m-k)\big)\big(S_2(k)+S_2(m-k)\big)+4\big(25\big(-S_1(k)+S_1(m-k)\big)^3-2S_3(k)+2S_3(m$$
$$+5k\big(-S_1(k)+S_1(m-k)\big)\big(25\big(-S_1(k)+S_1(m-k)\big)^3-8S_3(k)+8S_3(m-k)\big)$$
$$+3k\big(5\big(S_2(k)+S_2(m-k)\big)\big(10\big(S_1(k)-S_1(m-k)\big)^2+S_2(k)+S_2(m-k)\big)+2\big(H_k^{(4)}+H_{m-k}^{(4)}\big)\big)\Big]=0$$

$$\sum_{k=0}^{m}(-1)^k\binom{m}{k}^5\Big[125(-1+k)kS_1(k)^4+100(-1+2k)S_1(m-k)^3+125(-1+k)kS_1(m-k)^4+100S_1(k)^3(1-2k$$
$$-5(-1+k)kS_1(m-k))+30S_1(m-k)^2(2+5(-1+k)k(S_2(k)+S_1(m-k)^{(2)}))+30S_1(k)^2(2+5((-2+4k)S_1(m-k$$
$$+5(-1+k)kS_1(m-k)^2+(-1+k)k(S_2(k)+S_1(m-k)^{(2)})))+20S_1(k)((15-30k)S_1(m-k)^2-25(-1+k)kS_1(m$$
$$+(3-6k)S_2(k)+3S_1(m-k)^{(2)}-3S_1(m-k)(2+5(-1+k)k(S_2(k)+S_1(m-k)^{(2)}))+2k(-3S_1(m-k)^{(2)}+(-1+k)($$
$$-S_1(m-k)^{(3)})))+20S_1(m-k)((-3+6k)S_2(k)+(-3+6k)S_1(m-k)^{(2)}-2(-1+k)k(S_3(k)-S_1(m-k)^{(3)}))$$
$$+4(3S_2(k)+3S_1(m-k)^{(2)}+2S_3(k)-2S_1(m-k)^{(3)})+k(15(-1+k)S_2(k)^2+30(-1+k)S_2(k)S_1(m-k)^{(2)}$$
$$+15(-1+k)S_1(m-k)^{(2)\,2}+2(-8S_3(k)+8S_1(m-k)^{(3)}+3(-1+k)(S_4(k)+S_1(m-k)^{(4)})))\Big]=0$$

$$\sum_{k=0}^{m}(-1)^k\binom{m}{k}^5\Big[125(-2+k)(-1+k)kS_1(k)^4+100(2+3(-2+k)k)S_1(m-k)^3+125(-2+k)(-1+k)kS_1(m-k$$
$$100S_1(k)^3(-2-3(-2+k)k-5(-2+k)(-1+k)kS_1(m-k))+30(-1+k)S_1(m-k)^2(6+5(-2+k)k(S_2(k)+S_2(m-k$$
$$+30S_1(k)^2(10(2+3(-2+k)k)S_1(m-k)+25(-2+k)(-1+k)kS_1(m-k)^2+(-1+k)(6+5(-2+k)k(S_2(k)+S_2(m-$$
$$+4S_1(m-k)(6+15(2+3(-2+k)k)S_2(k)+5(3(2+3(-2+k)k)S_2(m-k)-2(-2+k)(-1+k)k(S_3(k)-S_3(m-k))))$$

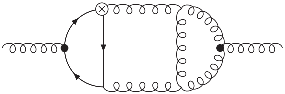
Application 3: Particle physics

For exciting methods/calculations see:

- I. Bierenbaum, J. Blümlein, S. Klein, and CS. Two–Loop Massive Operator Matrix Elements for Unpolarized Heavy Flavor Production to $O(\epsilon)$. *Nucl.Phys. B* 803(1-2):1–41, 2008.
- J. Blümlein, S. Klein, CS, F. Stan. A Symbolic Summation Approach to Feynman Integral Calculus. J. Symbolic Comput. 47: 1267-1289, 2012.
- J. Ablinger, J. Blümlein, S. Klein, CS, F. Wissbrock. The $O(\alpha_s^3)$ Massive Operator Matrix Elements of $O(n_f)$ for the Structure Function $F_2(x,Q^2)$ and Transversity. Nucl. Phys. B, 844: 26-54, 2011.
- J. Ablinger, J. Blümlein, A. Hasselhuhn, S. Klein, CS, F. Wissbrock. Massive 3-loop Ladder Diagrams for Quarkonic Local Operator Matrix Elements. Nuclear Physics B. 864: 52-84, 2012.
- J. Blümlein, A. Hasselhuhn, S. Klein, CS. The $O(\alpha_s^3 n_f T_F^2 C_{A,F})$ Contributions to the Gluonic Massive Operator Matrix Elements. Nuclear Physics B: 866: 196-211, 2013.
- J. Ablinger, J. Blümlein, A. De Freitas A. Hasselhuhn, A. von Manteuffel, M. Round, C. Schneider, F. Wissbrock. The Transition Matrix Element $A_{gq}(N)$ of the Variable Flavor Number Scheme at $O(\alpha_s^3)$. Nuclear Physics B 882, pp. 263-288. 2014.
- J. Ablinger, J. Blümlein, C. Raab, CS, F. Wissbrock. Calculating Massive 3-loop Graphs for Operator Matrix Elements by the Method of Hyperlogarithms. Nuclear Physics B 885, pp. 409-447. 2014.
- J. Ablinger, J. Blümlein, A. De Freitas, A. Hasselhuhn, A. von Manteuffel, M. Round, CS. The $O(\alpha_s^3 T_F^2)$ Contributions to the Gluonic Operator Matrix Element. Nuclear Physics B 885, pp. 280-317. 2014.
- J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. Hasselhuhn, A. von Manteuffel, M. Round, CS, F. Wissbrock. The 3-Loop Non-Singlet Heavy Flavor Contributions and Anomalous Dimensions for the Structure Function $F_2(x,Q^2)$ and Transversity. Nuclear Physics B 886, pp. 733-823. 2014.
- J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. von Manteuffel, CS. The 3-Loop Pure Singlet Heavy Flavor Contributions to the Structure Function $F_2(x,Q^2)$ and the Anomalous Dimension. submitted, pp. 1-85. 2014. arXiv:1409.1135 [hep-ph].

Evaluation of Feynman diagrams

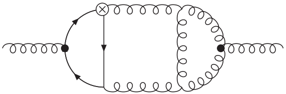
(long term project with J. Blümlein, Deutsches Elektronen–Synchrotron)



Behavior of particles

Evaluation of Feynman diagrams

(long term project with J. Blümlein, Deutsches Elektronen–Synchrotron)



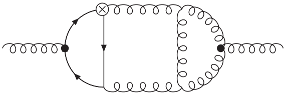
Behavior of particles

$$\longrightarrow \qquad \int \Phi(n, \epsilon, x)dx$$

Feynman integrals

Evaluation of Feynman diagrams

(long term project with J. Blümlein, Deutsches Elektronen–Synchrotron)



Behavior of particles

$$\int \Phi(n,\epsilon,x)dx$$

Feynman integrals

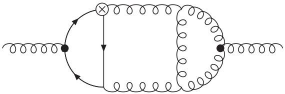
DESY

$$\sum f(n,\epsilon,k)$$

multi sums

Evaluation of Feynman diagrams

(long term project with J. Blümlein, Deutsches Elektronen–Synchrotron)



Behavior of particles → $\int \Phi(n, \epsilon, x) dx$ Feynman integrals

↓ **DESY**

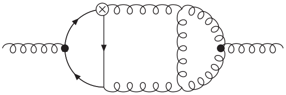
$\sum f(n, \epsilon, k)$ multi sums

symbolic summation

simple sum expressions ← multi sums

Evaluation of Feynman diagrams

(long term project with J. Blümlein, Deutsches Elektronen–Synchrotron)



Behavior of particles

$$\int \Phi(n,\epsilon,x)dx$$

Feynman integrals

DESY

$$\sum f(n,\epsilon,k)$$

multi sums

symbolic summation

simple sum expressions

processable by physicists

Evaluations required for the LHC experiment at CERN

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$A(n)=\sum_{k=1}^{n}f(n,k);$$

$f(n,k)$: indefinite nested product-sum in k;
n: extra parameter

Find a recurrence for $A(n)$

2. Recurrence solving

GIVEN a recurrence

$a_0(n),\dots,a_d(n),h(n)$:
indefinite nested product-sum expressions in n.

$$a_0(n)A(n)+\dots+a_d(n)A(n+d)=h(n);$$

Find all solutions expressible by indefinite nested products/sums in n.
(Abramov/Bronstein/Petkovšek/CS, in preparation)

4. Find a "closed form"

A(n)=combined solutions in terms of indefinite nested sums in n.

Iterative application from inside to outside transforms

definite multi-sums

$$\downarrow$$

indefinite nested sums

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+s-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+s-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\boxed{\sum_{s=0}^{n-j+s-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}}$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+s-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\boxed{\sum_{s=0}^{n-j+s-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}}$$

$$||$$

$$\boxed{\binom{j+1}{r}\Big(\frac{(-1)^r(-j+n-2)!r!}{(n+1)(-j+n+r-1)(-j+n+r)!}+\frac{(-1)^{n+r}(j+1)!(-j+n-2)!(-j+n-1)_r r!}{(n-1)n(n+1)(-j+n+r)!(-j-1)_r(2-n)_j}\Big)}$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+s-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$\|$$

$$\boxed{\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\binom{j+1}{r}\Big(\frac{(-1)^r(-j+n-2)!r!}{(n+1)(-j+n+r-1)(-j+n+r)!}+\frac{(-1)^{n+r}(j+1)!(-j+n-2)!(-j+n-1)_r r!}{(n-1)n(n+1)(-j+n+r)!(-j-1)_r(2-n)_j}\Big)}$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+s-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$\|$$

$$\boxed{\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\binom{j+1}{r}\Big(\frac{(-1)^r(-j+n-2)!r!}{(n+1)(-j+n+r-1)(-j+n+r)!}+\frac{(-1)^{n+r}(j+1)!(-j+n-2)!(-j+n-1)_r r!}{(n-1)n(n+1)(-j+n+r)!(-j-1)_r(2-n)_j}\Big)}$$

$$\|$$

$$\boxed{\Big(\frac{n^2-n+1}{(n-1)^2n^2(n+1)(2-n)_j}+\frac{\sum_{i=1}^{j}\frac{(2-n)_i}{(-i+n-1)^2(i+1)!}}{(n+1)(2-n)_j}+\frac{(-1)^{j+n}(-j-2)(-j+n-2)!}{(j-n+1)(n+1)^2n!}\Big)(j+1)!-\frac{1}{(n+1)^2(-j+n-1)}}$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+s-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\sum_{j=0}^{n-2}\Bigg(\Big(\frac{n^2-n+1}{(n-1)^2n^2(n+1)(2-n)_j}+\frac{\sum_{i=1}^{j}\frac{(2-n)_i}{(-i+n-1)^2(i+1)!}}{(n+1)(2-n)_j}+\frac{(-1)^{j+n}(-j-2)(-j+n-2)!}{(j-n+1)(n+1)^2n!}\Big)(j+1)!-\frac{1}{(n+1)^2(-j+n-1)}\Bigg)$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+s-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\sum_{j=0}^{n-2}\Bigg(\Bigg(\frac{n^2-n+1}{(n-1)^2n^2(n+1)(2-n)_j}+\frac{\sum_{i=1}^{j}\frac{(2-n)_i}{(-i+n-1)^2(i+1)!}}{(n+1)(2-n)_j}+$$
$$\frac{(-1)^{j+n}(-j-2)(-j+n-2)!}{(j-n+1)(n+1)^2n!}\Bigg)(j+1)!-\frac{1}{(n+1)^2(-j+n-1)}\Bigg)$$

$$||$$

$$\frac{-n^2-n-1}{n^2(n+1)^3}+\frac{(-1)^n\left(n^2+n+1\right)}{n^2(n+1)^3}-\frac{2S_{-2}(n)}{n+1}+\frac{S_1(n)}{(n+1)^2}+\frac{S_2(n)}{-n-1}$$

Note: $S_a(n)=\sum_{i=1}^{n}\frac{\mathsf{sign}(a)^i}{i^{|a|}}$, $a\in\mathbb{Z}\setminus\{0\}$.

Automatic toolbox:

In[9]:= $<<$ **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[10]:= $<<$ **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

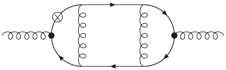
In[11]:= $<<$ **EvaluateMultiSums.m**

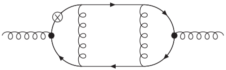
EvaluateMultiSums by Carsten Schneider © RISC-Linz

In[12]:= $$\textbf{EvaluateMultiSum}\Big[\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!},$$

{{s,0,n-j+r-2},{r,0,j+1},{j,0,n-2}]

Out[12]= $$\frac{-n^2-n-1}{n^2(n+1)^3}+\frac{(-1)^n\left(n^2+n+1\right)}{n^2(n+1)^3}-\frac{2S_{-2}(n)}{n+1}+\frac{S_1(n)}{(n+1)^2}+\frac{S_2(n)}{-n-1}$$


(massive 3–loop ladder graph with operator insertion)

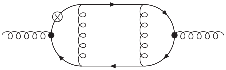


J. Blümlein
A. Hasselhuhn

$$= \frac{C_3}{(n+1)(n+2)} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{l=2}^{n+2} \binom{n+2}{l} \sum_{j=2}^{l} \binom{l}{j} \Bigg\{$$

$$\sum_{k=1}^{j} \binom{j}{k} \sum_{r=0}^{l-k} \binom{l-k}{r} (-1)^{l+j+k+r} \frac{B\left(k, m+1+\frac{\varepsilon}{2}\right)\Gamma(k+r+m+n+\frac{\varepsilon}{2})}{\Gamma(m+1)\Gamma(n+1)\Gamma(k+r+\frac{\varepsilon}{2})} \frac{B\left(r+l-1, n+1+\frac{\varepsilon}{2}\right)}{(n+3-j)} \frac{B\left(k+m-\frac{\varepsilon}{2}, r+1+n-\right.}{(k+r+1+m+n-\varepsilon}$$

$$+ \sum_{r=0}^{l-j} \binom{l-j}{r} (-1)^{l+j+r} \frac{B\left(r+l-1, n+1+\frac{\varepsilon}{2}\right)\Gamma(j+r+m+n+\frac{\varepsilon}{2})}{\Gamma(m+1)\Gamma(n+1)\Gamma(j+r+\frac{\varepsilon}{2})} \frac{B\left(j, m+1+\frac{\varepsilon}{2}\right) B\left(j+m-\frac{\varepsilon}{2}, r+1+n-\frac{\varepsilon}{2}\right)}{(j+r+1+m+n-\varepsilon)(n+3-j)} \Bigg\}$$



J. Blümlein
A. Hasselhuhn
$$=\frac{C_3}{(n+1)(n+2)}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{l=2}^{n+2}\binom{n+2}{l}\sum_{j=2}^{l}\binom{l}{j}\Bigg\{$$

$$\sum_{k=1}^{j}\tbinom{j}{k}\sum_{r=0}^{l-k}\tbinom{l-k}{r}(-1)^{l+j+k+r}\frac{B\left(k,m+1+\frac{\varepsilon}{2}\right)\Gamma(k+r+m+n+\frac{\varepsilon}{2})}{\Gamma(m+1)\Gamma(n+1)\Gamma(k+r+\frac{\varepsilon}{2})}\frac{B\left(r+l-1,n+1+\frac{\varepsilon}{2}\right)}{(n+3-j)}\frac{B\left(k+m-\frac{\varepsilon}{2},r+1+n-\right.}{(k+r+1+m+n-\varepsilon}$$

$$+\sum_{r=0}^{l-j}\tbinom{l-j}{r}(-1)^{l+j+r}\frac{B\left(r+l-1,n+1+\frac{\varepsilon}{2}\right)\Gamma(j+r+m+n+\frac{\varepsilon}{2})}{\Gamma(m+1)\Gamma(n+1)\Gamma(j+r+\frac{\varepsilon}{2})}\frac{B\left(j,m+1+\frac{\varepsilon}{2}\right)B\left(j+m-\frac{\varepsilon}{2},r+1+n-\frac{\varepsilon}{2}\right)}{(j+r+1+m+n-\varepsilon)(n+3-j)}\Bigg\}$$

|| symbolic summation (Sigma,...)

$$\frac{C_3}{(n+1)(n+2)(n+3)}\Bigg\{\frac{1}{6}S_1^3(n)+\frac{n^2+12n+16}{2(n+1)(n+2)}S_1(n)^2+\frac{4(2n+3)}{(n+1)^2(n+2)}S_1(n)$$

$$+2\Big[-2^{n+3}+3-(-1)^n\Big]\zeta_3+\Big[\frac{3n^2+40n+56}{2(n+1)(n+2)}-\frac{1}{2}S_1(n)\Big]S_2(n)$$

$$-(-1)^nS_{-3}(n)+\frac{8(2n+3)}{(n+1)^3(n+2)}-\frac{3n+17}{3}S_3(n)-2(-1)^nS_{-2,1}(n)-(n+3)S_{2,1}(n)$$

$$+2^{n+4}S_{1,2}\left(\frac{1}{2},1;n\right)+2^{n+3}\boxed{S_{1,1,1}\left(\frac{1}{2},1,1;n\right)}\Bigg\}+O(\varepsilon)$$

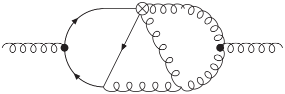
$$\boxed{S_{1,1,1}\left(\frac{1}{2},1,1;n\right)}=\sum_{i=1}^{n}\frac{\left(\frac{1}{2}\right)^i\sum_{j=1}^{i}\frac{\sum_{k=1}^{j}\frac{1}{k}}{j}}{i}$$

$$\boxed{S_{1,1,1}\left(\frac{1}{2},1,1;n\right)} = \sum_{i=1}^{n} \frac{\left(\frac{1}{2}\right)^i \sum_{j=1}^{i} \frac{\sum_{k=1}^{j} \frac{1}{k}}{j}}{i}$$

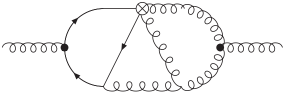
|| asymptotic expansion (HarmonicSums package)

$$2^{-n}\left(+\frac{541}{n^6}-\frac{75}{n^5}+\frac{13}{n^4}-\frac{3}{n^3}+\frac{1}{n^2}-\frac{1}{2n}\right)(\ln(n)+\gamma)^2$$
$$+2^{-n-3}\left(-\frac{114686}{5n^6}+\frac{44099}{15n^5}-\frac{1372}{3n^4}+\frac{266}{3n^3}-\frac{20}{n^2}\right)(\ln(n)+\gamma)$$
$$+2^{-n}\left(+\frac{541}{n^6}-\frac{75}{n^5}+\frac{13}{n^4}-\frac{3}{n^3}+\frac{1}{n^2}-\frac{1}{2n}\right)\zeta_2+\frac{3\zeta_3}{4}$$
$$+2^{-n-9}\left(\frac{69280576}{45n^6}-\frac{1582096}{9n^5}+\frac{69184}{3n^4}-\frac{3264}{n^3}+\frac{256}{n^2}\right)+O\left(\frac{1}{2^n n^7}\right)$$

(J. Ablinger, J. Blümlein, CS; arXiv:1302.0378 [math-ph])



$$= F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + F_{-1}(n)\varepsilon^{-1} + \boxed{F_0(n)} + \dots$$



$$= F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + F_{-1}(n)\varepsilon^{-1} + \boxed{F_0(n)} + \dots$$

Simplify

$$||$$

$$\sum_{j=0}^{n-3}\sum_{k=0}^{j}\sum_{l=0}^{k}\sum_{q=0}^{-j+n-3}\sum_{s=1}^{-l+n-q-3}\sum_{r=0}^{-l+n-q-s-3}(-1)^{-j+k-l+n-q-3}\times$$

$$\times\frac{\binom{j+1}{k+1}\binom{k}{l}\binom{n-1}{j+2}\binom{-j+n-3}{q}\binom{-l+n-q-3}{s}\binom{-l+n-q-s-3}{r}r!(-l+n-q-r-s-3)!(s-1)!}{(-l+n-q-2)!(-j+n-1)(n-q-r-s-2)(q+s+1)}$$

$$\Big[4S_1(-j+n-1)-4S_1(-j+n-2)-2S_1(k)$$

$$-(S_1(-l+n-q-2)+S_1(-l+n-q-r-s-3)-2S_1(r+s))$$

$$+2S_1(s-1)-2S_1(r+s)\Big]$$ **+ 3 further 6–fold sums**

$$\boxed{F_0(n)} =$$

$$\frac{7}{12}S_1(n)^4 + \frac{(17n+5)S_1(n)^3}{3n(n+1)} + \Big(\frac{35n^2-2n-5}{2n^2(n+1)^2} + \frac{13S_2(n)}{2} + \frac{5(-1)^n}{2n^2}\Big)S_1(n)^2$$
$$+\Big(-\frac{4(13n+5)}{n^2(n+1)^2} + \Big(\frac{4(-1)^n(2n+1)}{n(n+1)} - \frac{13}{n}\Big)S_2(n) + \Big(\frac{29}{3} - (-1)^n\Big)S_3(n)$$
$$+\big(2+2(-1)^n\big)S_{2,1}(n) - 28S_{-2,1}(n) + \frac{20(-1)^n}{n^2(n+1)}\Big)S_1(n) + \Big(\frac{3}{4} + (-1)^n\Big)S_2(n)^2$$
$$-2(-1)^nS_{-2}(n)^2 + S_{-3}(n)\Big(\frac{2(3n-5)}{n(n+1)} + \big(26+4(-1)^n\big)S_1(n) + \frac{4(-1)^n}{n+1}\Big)$$
$$+\Big(\frac{(-1)^n(5-3n)}{2n^2(n+1)} - \frac{5}{2n^2}\Big)S_2(n) + S_{-2}(n)\Big(10S_1(n)^2 + \Big(\frac{8(-1)^n(2n+1)}{n(n+1)}$$
$$+\frac{4(3n-1)}{n(n+1)}\Big)S_1(n) + \frac{8(-1)^n(3n+1)}{n(n+1)^2} + \big(-22+6(-1)^n\big)S_2(n) - \frac{16}{n(n+1)}\Big)$$
$$+\Big(\frac{(-1)^n(9n+5)}{n(n+1)} - \frac{29}{3n}\Big)S_3(n) + \Big(\frac{19}{2} - 2(-1)^n\Big)S_4(n) + \big(-6+5(-1)^n\big)S_{-4}(n)$$
$$+\Big(-\frac{2(-1)^n(9n+5)}{n(n+1)} - \frac{2}{n}\Big)S_{2,1}(n) + \big(20+2(-1)^n\big)S_{2,-2}(n) + \big(-17+13(-1)^n\big)S_{3,1}(n)$$
$$-\frac{8(-1)^n(2n+1)+4(9n+1)}{n(n+1)}S_{-2,1}(n) - \big(24+4(-1)^n\big)S_{-3,1}(n) + \big(3-5(-1)^n\big)S_{2,1,1}(n)$$
$$+32S_{-2,1,1}(n) + \left(\frac{3}{2}S_1(n)^2 - \frac{3S_1(n)}{n} + \frac{3}{2}(-1)^nS_{-2}(n)\right)\zeta_2$$

Abstract: Symbolic summation deals with the simplification of formulas or with proving identities given in terms of (usually complicated) multiple nested sums. A first breakthrough in this research area was achieved for indefinite single sums over hypergeometric expressions (e.g., single sums over binomial coefficients or factorials). Namely, using Gospers telescoping algorithm (1978) one can decide, if such a sum can be written in terms of hypergeometric expressions. Another crucial contribution is Zeilbergers observation (1991) that Gospers algorithm can be extended to creative telescoping. This enables the user to compute linear recurrence relations for definite hypergeometric sums. Finally, Pektovseks Hyper algorithm determines all hypergeometric solutions of a given linear recurrence with polynomial coefficients. Combining these algorithms yields a method that decides if a definite hypergeometric sum can be expressed as a linear combination of hypergeometric expressions.
In this tutorial we present a generalization of these summation techniques that work not only for sums over hypergeometric expressions but for sums over indefinite nested multiple sums and products. Here we focus on the following two aspects.

* Indefinite summation: Simplify expressions in terms of indefinite nested sums and products such that the occurring sums are algebraically independent and such the obtained expressions consist of sums that have certain optimality criteria (such as minimal nesting depth or minimal degree in the denominators).
* Definite summation: Compute recurrences (based on the paradigm of Zs creative telescoping) for definite sums with summands given in terms of indefinite nested sums and products, and solve the derived recurrences in terms of indefinite nested sums and products (also called dAlembertian solutions, a subclass of Liouvillian solutions). In this way, one can decide (similarly to the hypergeometric case) if such a definite nested sum can be expressed in terms of indefinite nested sums and products.

The algorithmic framework of this approach is based on Karrs difference fields (1981) and a recent generalization to a difference ring theory. We will work out the underlying ideas and algorithms and present the current developments and improvements in this research topic. Special emphasis will be put on concrete examples arising, e.g., from combinatorial problems and numerics. In particular, we demonstrate how the symbolic summation algorithms can be utilized to evaluate whole classes of Feynman integrals arising in the context of Elementary Quantum Field Theory. The latter examples pop up within an intensive cooperation with the Theory Group (Johannes Blmlein) of the Deutsches Elektronen-Synchrotron DESY, Zeuthen.

Abstract: Symbolic summation deals with the simplification of formulas or with proving identities given in terms of (usually complicated) multiple nested sums. A first breakthrough in this research area was achieved for indefinite single sums over hypergeometric expressions (e.g., single sums over binomial coefficients or factorials). Namely, using Gospers telescoping algorithm (1978) one can decide, if such a sum can be written in terms of hypergeometric expressions. Another crucial contribution is Zeilbergers observation (1991) that Gospers algorithm can be extended to creative telescoping. This enables the user to compute linear recurrence relations for definite hypergeometric sums. Finally, Pektovseks Hyper algorithm determines all hypergeometric solutions of a given linear recurrence with polynomial coefficients. Combining these algorithms yields a method that decides if a definite hypergeometric sum can be expressed as a linear combination of hypergeometric expressions.

In this tutorial we present a generalization of these summation techniques that work not only for sums over hypergeometric expressions but for sums over indefinite nested multiple sums and products. Here we focus on the following two aspects.

* **Indefinite summation**: Simplify expressions in terms of indefinite nested sums and products such that the occurring sums are algebraically independent and such the obtained expressions consist of sums that have certain optimality criteria (such as minimal nesting depth or minimal degree in the denominators).

* **Definite summation:** Compute recurrences (based on the paradigm of Zs creative telescoping) for definite sums with summands given in terms of indefinite nested sums and products, and solve the derived recurrences in terms of indefinite nested sums and products (also called dAlembertian solutions, a subclass of Liouvillian solutions). In this way, one can decide (similarly to the hypergeometric case) if such a definite nested sum can be expressed in terms of indefinite nested sums and products.

The algorithmic framework of this **approach is based on Karrs difference fields (1981)** and a **recent generalization to a difference ring theory**. We will work out the underlying ideas and algorithms and present the current developments and improvements in this research topic. Special emphasis will be put on **concrete examples arising, e.g., from combinatorial problems and numerics**. In particular, we demonstrate how the symbolic summation algorithms can be utilized to evaluate whole classes of Feynman integrals arising in the context of **Elementary Quantum Field Theory**. The latter examples pop up within an intensive cooperation with the Theory Group (Johannes Blmlein) of the Deutsches Elektronen-Synchrotron DESY, Zeuthen.

☺ (I hope I did my best)