MÖBIUS PHOTOGRAMMETRY

MATTEO GALLET, GEORG NAWRATIL, AND JOSEF SCHICHO

Abstract. Motivated by results on the mobility of mechanical devices called pentapods, this paper deals with a mathematically freestanding problem, which we call *Möbius Photogrammetry*. Unlike traditional photogrammetry, which tries to recover a set of points in three–dimensional space from a finite set of central projection, we consider the problem of reconstructing a vector of points in $\mathbb{R}^3$ starting from its orthogonal parallel projections. Moreover, we assume that we have partial information about these projections, namely that we know them only up to Möbius transformations. The goal in this case is to understand to what extent we can reconstruct the starting set of points, and to prove that the result can be achieved if we allow some uncertainties in the answer. Eventually, the techniques developed in the paper allow us to show that for a pentapod with mobility at least two, either some anchor points are collinear, or platform and base are similar, or they are planar and affine equivalent.

1. Introduction

In this paper we consider the following problem: given a vector of 5 points $\vec{A} = (A_1, \ldots, A_5)$ in $\mathbb{R}^3$, suppose we have partial information about its orthogonal projections along all directions in $\mathbb{R}^3$; in particular, we suppose to know each of them only up to Möbius transformations of the plane. Then we ask if and to what extent we can extract information on $\vec{A}$ starting from this partial knowledge about its orthogonal projections.

In order to deal with this question, in Section 2 we start formalizing it in the following way:

- the set of directions of $\mathbb{R}^3$ is identified with the unit sphere S^2;
- a set of 5 points in the plane, considered up to Möbius transformations, gives a point in the moduli space M_5 of five points in $\mathbb{P}^1_{\mathbb{C}}$.

So all the information about orthogonal projections of a vector $\vec{A}$ of points can be encoded in one function $f_{\vec{A}} : S^2 \longrightarrow M_5$, which we call a *Möbius camera*. In Section 3 we give a formal definition of the map $f_{\vec{A}}$ and we explain how it can be thought as a map between projective varieties. We explore some properties of the Möbius camera, in particular we relate the degree of the image of $f_{\vec{A}}$ (which is always a curve if the points are not all aligned) with the geometric configuration of the points $\{A_i\}$. The main result of this section is Theorem 3.11:

1991 *Mathematics Subject Classification.* 53A17 (Kinematics), 14L35 (Classical groups), 14P99 (Real algebraic geometry).

Key words and phrases. Bond Theory, n-pods, self-motion.

Theorem. *Let $\vec{A}$ and $\vec{B}$ be two 5-tuples of points in $\mathbb{R}^3$ such that no 4 of them are collinear. Assume that $f_{\vec{A}}(S^2)$ and $f_{\vec{B}}(S^2)$ are equal as curves in M_5. If $\vec{A}$ is coplanar, then $\vec{B}$ is also coplanar and affine equivalent to $\vec{A}$. If $\vec{A}$ is not coplanar, then $\vec{B}$ is similar to $\vec{A}$.*

Eventually, Section 4 presents an application of the theory developed so far to pentapods with mobility greater than or equal to 2. In fact, and this is where the authors took one of the motivations for this paper, Theorem 3.19 of [2] gives a necessary condition for mobility 2 of n-pods in the form of a disjunction of several statements, one of which being: "*there are infinitely many pairs (L, R) of elements of S^2 such that the points $\pi_L(p_1), \ldots, \pi_L(p_n)$ and $\pi_R(P_1), \ldots, \pi_R(P_n)$ differ by an inversion or a similarity*". We focus on this case, and using Theorem 3.11 we show that base and platform points are either similar or planar and affine equivalent.

2. Setting up the Möbius photogrammetry problem

We are going to consider the following problem: given a vector $\vec{A} = (A_1, \ldots, A_5)$ of 5 points in $\mathbb{R}^3$, we want to define a map $f_{\vec{A}}$, which we will call *Möbius camera*, associating to each direction $\varepsilon \in S^2$ the orthogonal projection of $\vec{A}$ along ε, considered up to Möbius transformations. Moreover, starting from the collection of all orthogonal projections of $\vec{A}$, we want to understand to what extent we can reconstruct $\vec{A}$.

In order to set up this photogrammetric problem in a formal way, first of all we have to make clear what do we mean by "consider up to Möbius transformations". Recall that a *Möbius transformation* is a map g of the complex projective line $\mathbb{P}^1_{\mathbb{C}}$ to itself of the form

$$g : \begin{cases} (1:z) & \mapsto \left(1 : \frac{a\,z+b}{c\,z+d}\right) \quad \text{if } z \neq -d/c \\ (1:-d/c) & \mapsto (0:1) \\ (0:1) & \mapsto (1: a/c) \end{cases} \qquad \text{where } ad - bc \neq 0$$

with the convention that $g(0:1) = (0:1)$ if $c = 0$. If we are given two n-tuples $(m_1, \ldots, m_n)$ and $(n_1, \ldots, n_n)$ of points in the plane $\mathbb{R}^2$, we say that they are *Möbius equivalent* if, once we identify $\mathbb{R}^2$ with $\mathbb{C} \hookrightarrow \mathbb{P}^1_{\mathbb{C}}$, there is a Möbius transformation g sending m_i to n_i for every $i \in \{1, \ldots, n\}$.

The rest of this and the next section are aimed to define the concept of *Möbius camera*. We will clarify what are the domain and the codomain of this map, and what is its explicit formulation.

We start discussing the domain of our desired map: as described before, it should be S^2, thought as the set of directions in $\mathbb{R}^3$. On the other hand, we would like it to be an algebraic variety. We are going to see now that not only S^2 can be considered as a complex projective curve, but it also naturally carries the structure of a real variety. This property will be crucial in the proofs of Subsection 3.3.

Definition 2.1. A *real structure* on a complex variety is a pair (X, α), where X is a complex variety and α is an anti-holomorphic involution (see [11], Chapter 1, Proposition 1.3).

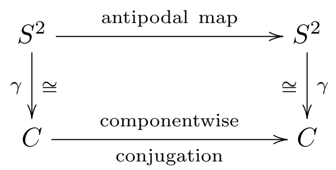
Remark 2.2. An example of a real structure is given by the complex projective space $\mathbb{P}^n_{\mathbb{C}}$ together with componentwise complex conjugation. One can prove that there exist exactly two real structures on $\mathbb{P}^1_{\mathbb{C}}$ (up to isomorphism), and they are given by the following two involutions:

$$(s,t) \;\mapsto\; (\overline{s},\overline{t}) \quad \text{and} \quad (s:t) \;\mapsto\; (-\overline{t}:\overline{s})$$

In particular, the fixed points of the first involution are precisely the closed points of $\mathbb{P}^1_{\mathbb{R}}$, while the second one does not have any fixed point. Moreover there is a natural bijection between $\mathbb{P}^1_{\mathbb{C}}$ and S^2, and under this bijection the second involution corresponds to the antipodal map. Hence we can think of S^2 as a real algebraic variety, whose anti-holomorphic involution is given by the antipodal map.

The following result provides another identification of S^2 with an algebraic curve which is simply a different projective embedding of $\mathbb{P}^1_{\mathbb{C}}$, but which enables us to perform the computations needed to define the Möbius camera in a simpler way.

Lemma 2.3. *There is a bijection* $\gamma : S^2 \longrightarrow C = \{x^2+y^2+z^2=0\} \subseteq \mathbb{P}^2_{\mathbb{C}}$ *such that the following diagram commutes:*



Proof. Let $\varepsilon \in S^2$, then pick $\varepsilon', \varepsilon''$ in the orthogonal space $\langle\varepsilon\rangle^{\perp}$ such that

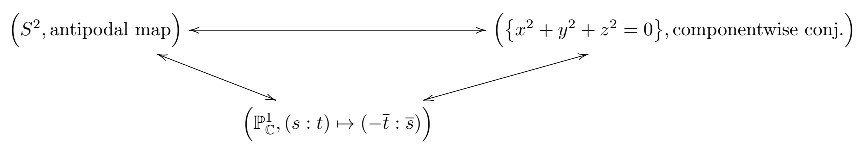
- $\varepsilon', \varepsilon''$ form an orthonormal basis of $\langle\varepsilon\rangle^{\perp}$;
- $\varepsilon, \varepsilon', \varepsilon''$ form a right basis of $\mathbb{R}^3$, namely $\det\left(\,\varepsilon\;\varepsilon'\;\varepsilon''\,\right) > 0$.

If $\varepsilon' = (\lambda', \mu', \nu')$ and $\varepsilon'' = (\lambda'', \mu'', \nu'')$, then we consider the vector

$$\varepsilon' + i\,\varepsilon'' \;=\; (\lambda' + i\,\lambda'', \mu' + i\,\mu'', \nu' + i\,\nu'') \in \mathbb{C}^3$$

By a direct computation one can check that the point in $\mathbb{P}^2_{\mathbb{C}}$ given by $\varepsilon' + i\,\varepsilon''$ lies on C. We notice that a different choice of ε' and ε'' leads to the same point in $\mathbb{P}^2_{\mathbb{C}}$. Then the map $\gamma : \varepsilon \mapsto \varepsilon' + i\,\varepsilon''$ is well-defined and satisfies the requirements of the thesis. □

Remark 2.4. Recalling Remark 2.2, we have that S^2 is in bijection with $\mathbb{P}^1_{\mathbb{C}}$, and via the previous map γ we obtain an isomorphism of real algebraic curves between $\mathbb{P}^1_{\mathbb{C}}$ and C given by homogeneous polynomials of degree 2. We get the following triangle of bijections and involutions:



The identification of S^2 with the conic $C \subseteq \mathbb{P}^2_{\mathbb{C}}$ becomes very useful when we want to deal with orthogonal projections.

Definition 2.5. Given a unit vector $\varepsilon \in S^2$, we say that a linear map $\pi_\varepsilon : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ is an *orthogonal projection along* ε if $\ker \pi_\varepsilon = \langle \varepsilon \rangle$ and π_ε is an isometry on $\langle \varepsilon \rangle^\perp$. Moreover we ask that the preimages of the standard bases of $\mathbb{R}^2$ lying on $\langle \varepsilon \rangle^\perp$ form, together with ε, a positively oriented bases. Note that in this way π_ε is well-defined only up to direct Euclidean isometry of the image, namely rotations around the origin.

Remark 2.6. From the definition of γ given in the proof of Lemma 2.3 we see that the vectors ε' and ε'' we form starting from $\varepsilon \in S^2$ satisfy $\left(\begin{smallmatrix} \varepsilon & \varepsilon' & \varepsilon'' \end{smallmatrix} \right) \in \mathrm{SO}(3, \mathbb{R})$. One can check that $\left(\begin{smallmatrix} \varepsilon' & \varepsilon'' \end{smallmatrix} \right)^t$ gives the matrix of π_ε, the orthogonal projection along ε. Identifying $\mathbb{R}^2$ with the complex plane $\mathbb{C}$ one finds that if $\gamma(\varepsilon) = (x : y : z)$ and $A = (p, q, r)$ is a point in $\mathbb{R}^3$, then $\pi_\varepsilon(A)$ is given, as a point in $\mathbb{C}$, by $p\,x + q\,y + r\,z$. If now we change the representative of $\gamma(\varepsilon)$, this modifies the image under the orthogonal projection by possibly a rotation and a dilation.

Hence taking into account Remark 2.6 we can realize the orthogonal projection π_ε as the dot product

$$\langle (x, y, z), \cdot \rangle : \mathbb{R}^3 \longrightarrow \mathbb{C}$$

where $(x : y : z)$ is any representative of $\gamma(\varepsilon)$ with Hermitian norm equal to $\sqrt{2}$. Thus we can view any orthogonal projection of n points $\vec{A} = (A_1, \ldots, A_n)$ as an n-tuple of points in $\mathbb{C}$. Then $\pi_\varepsilon(\vec{A})$ can be seen as one single point in $\mathbb{C}^n$. We can use the embedding $\mathbb{C} \hookrightarrow \mathbb{P}^1_\mathbb{C}$ sending z to $(z : 1)$ to identify $\pi_\varepsilon(\vec{A})$ with a point in $\left(\mathbb{P}^1_\mathbb{C}\right)^n$. We will extensively use this fact in the definition of the Möbius camera, and for proving some of its properties.

In order to understand what should be the codomain of our photographic map we start with the following known result (see, for example, [5], Chapter 2, Sections 2.1 and 2.2):

Proposition 2.7. $\big\{\text{Möbius transformations}\big\} \cong \mathbb{P}\mathrm{GL}(2, \mathbb{C}) \cong \mathrm{Aut}(\mathbb{P}^1_\mathbb{C})$.

Due to Proposition 2.7 we can use the natural action of $\mathbb{P}\mathrm{GL}(2, \mathbb{C})$ on $\left(\mathbb{P}^1_\mathbb{C}\right)^n$ to express that two orthogonal projections are Möbius equivalent. This leads us to the following definition, which we denote as temporary because unfortunately, despite the fact that it looks all-embracing and clean, it will not be very useful for us.

Temporary Definition. Let $\vec{A}$ be a finite set of distinct points in $\mathbb{R}^3$ and $\varepsilon \in S^2$. The *Möbius picture of* $\vec{A}$ *along* ε is the equivalence class under the action of $\mathbb{P}\mathrm{GL}(2, \mathbb{C})$ of any orthogonal parallel projection of $\vec{A}$ along the direction ε, considered as an n-tuple of points in $\mathbb{P}^1_\mathbb{C}$.

Remark 2.8. We notice that the concept of Möbius picture is well-defined. In fact, although the choice of different orthogonal projections along the same direction determines different points of $\left(\mathbb{P}^1_\mathbb{C}\right)^n$, they all differ by a Möbius transformation (given by a rotation, as mentioned in the end of Definition 2.5), so their equivalence class is the same.

Remark 2.9. Since the group of Möbius transformations is 3-transitive, then all configurations of n points in $\mathbb{R}^2$ are Möbius equivalent for $1 \leq n \leq 3$, so Möbius photogrammetry can be reasonably approached only if $n \geq 4$.

The problem with the Temporary Definition is that it involves an object, the quotient of $\left(\mathbb{P}^1_{\mathbb{C}}\right)^n$ by the action of $\mathbb{P}\mathrm{GL}(2, \mathbb{C})$, which does not have good geometric properties, namely it does not have a natural structure of an algebraic variety. In Subsection 3.1 we will see that we can obtain a much better object at a fair price, and we will focus on the case which is most interesting for us, namely $n = 5$. On the other hand, this operation has a cost: we will not be able to define Möbius pictures for any arbitrary configuration of points (see Footnote 3), but we need to put some restrictions. However, we will see that the concept of Möbius camera is meaningful for any configuration of points (see Remark 3.5).

3. The Möbius camera

3.1. **A projective embedding of the moduli space of 5 points in $\mathbb{P}^1_{\mathbb{C}}$.** Geometric Invariant Theory tells us that, in order to obtain our desired set of equivalence classes under the action of $\mathbb{P}\mathrm{GL}(2, \mathbb{C})$, which we denote by M_5[1], we cannot consider the equivalence classes of all 5-tuples, but we have to restrict to an open subset of $\left(\mathbb{P}^1_{\mathbb{C}}\right)^5$ (for an introduction to this topic, see [1], in particular Chapter 6, or [8], in particular Chapters 0 and 1). Therefore we will be forced to impose some conditions on the vector $\vec{A}$ of points in $\mathbb{R}^3$, in order to ensure that it is possible to define its Möbius picture along a given direction. After accepting this limitation one can construct an embedding of the quotient in projective space defined by invariants of the 5-tuple[2]. It is possible to embed M_5 as a quintic surface in $\mathbb{P}^5_{\mathbb{C}}$: in [4] it is explained a possible way to determine this surface and the quotient map $\left(\mathbb{P}^1_{\mathbb{C}}\right)^5 \dashrightarrow M_5$. We briefly describe the procedure:

1. Consider a convex pentagon P in the plane, and construct all plane undirected multigraphs without loops whose set of vertices coincides with the set of vertices of P, and which satisfy the following conditions:
   · edges are given by segments;
   · any two edges do not intersect;
   · the valency of every vertex is 2;

   There are exactly 6 of these graphs, showed in Figure 1.
2. Associate to each graph $G = (E, V)$ a homogeneous polynomial in the coordinates $\{(a_i : b_i)\}$ of $\left(\mathbb{P}^1_{\mathbb{C}}\right)^5$ according to the following rules:
   i. for every edge $e \in E$, $e = (i, j)$ with $i < j$, define

$$\varphi_e = a_i\, b_j - a_j\, b_i$$

[1] In the literature this object is usually denoted by $\mathcal{M}_{0,5}$, since it is the moduli space of genus 0 smooth curves with 5 marked points, but here we will always omit the index 0.

[2] For $n = 4$, there are two invariants, defining an open embedding $M_4 \hookrightarrow \mathbb{P}^1_{\mathbb{C}}$; the quotient of the two projective invariants is an absolute invariant, the cross ratio. This is in accordance with the fact that the projective equivalence of two 4-tuples of points in $\mathbb{P}^1_{\mathbb{C}}$ is completely determined by their cross ratio.

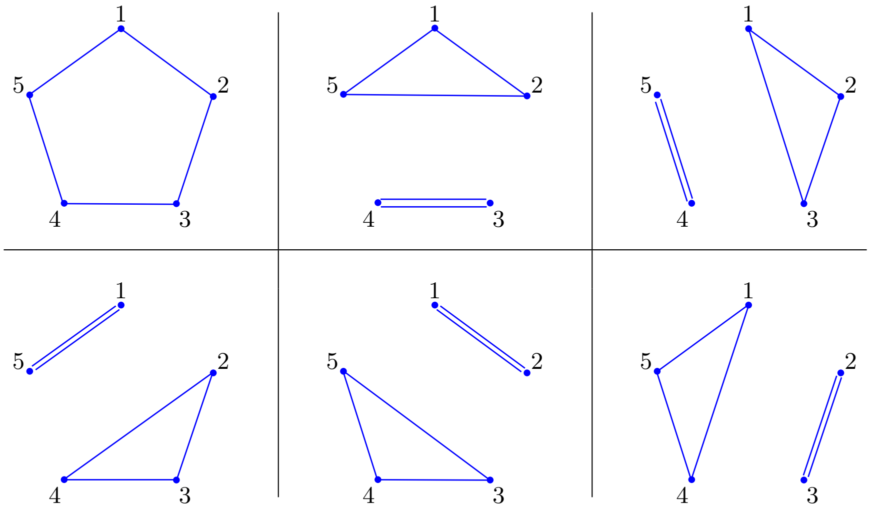


FIGURE 1. The only six planar undirected multigraphs without loops with vertices on a regular pentagon, valency 2 and non-intersecting edges.

ii. set

$$\varphi_G = \prod_{e \in E} \varphi_e$$

For example the polynomial associated to the first graph in Figure 1 is

$$\varphi_0 = (a_1\, b_2 - a_2\, b_1)(a_2\, b_3 - a_3\, b_2)(a_3\, b_4 - a_4\, b_3)(a_4\, b_5 - a_5\, b_5)(a_1\, b_5 - a_5\, b_1)$$

3. These polynomials determine a rational map $\varphi : \left(\mathbb{P}^1_{\mathbb{C}}\right)^5 \dashrightarrow \mathbb{P}^5_{\mathbb{C}}$.
4. Consider the open set

$$\mathcal{U} = \left\{(m_1, \ldots, m_5) \,:\, \text{no three of the } m_i \text{ coincide}\right\} \subseteq \left(\mathbb{P}^1_{\mathbb{C}}\right)^5$$

5. We have that image $\left(\varphi_{|\mathcal{U}}\right) = M_5$. It turns out that if we take coordinates $t, x_1, \ldots, x_5$ in $\mathbb{P}^5_{\mathbb{C}}$, then the equations for M_5 are:

$$x_{i-2}\, x_{i+2} = t\, x_i + t^2 \qquad \forall i \in \{1, \ldots, 5\}$$

where the indices are taken modulo 5.

Remark 3.1. It is known that M_5 is a Del Pezzo surface of degree 5: exactly 10 lines lie on such a surface and they correspond to equivalence classes of 5-tuples $(m_1, \ldots, m_5)$ for which at least two points coincide. We denote by L_{ij} the line in M_5 corresponding to classes for which $m_i = m_j$.

Remark 3.2. The variety M_5 has a canonical real structure, inherited from the real structure of the projective line. An equivalence class is real (namely, it is a fixed point for the anti-holomorphic involution on M_5) if and only if it can be represented by a 5-tuple of real points in $\mathbb{P}^1_{\mathbb{C}}$. Equivalently, the points are colinear or cocircular; in fact, we recall these two facts about automorphisms of $\mathbb{P}^1_{\mathbb{C}}$:

· Möbius transformations map lines and circles to lines and circles;

· the action of Möbius transformations is transitive on lines and circles.

Since a 5-tuple is given by real points in $\mathbb{P}^1_{\mathbb{C}}$ if and only if the corresponding points in $\mathbb{R}^2$ lie on a line, the claim follows from the previous two considerations.

3.2. **Definition of the Möbius camera.** From this construction we infer the condition we have to impose on the vector $\vec{A}$ of points in $\mathbb{R}^3$ so that we can speak of a Möbius picture along an arbitrary direction[3]: no 3 points among the $\{A_i\}$ should be aligned. In this way for every $\varepsilon \in S^2$ we will have that $\pi_\varepsilon(\vec{A})$ lies on $\mathcal{U}$, hence its equivalence class is a well-defined element of M_5.

Definition 3.3. Let $\vec{A}$ be a vector of 5 points in $\mathbb{R}^3$ and $\varepsilon \in S^2$. Suppose that no three points of $\vec{A}$ lie on a line parallel to ε. The *Möbius picture of* $\vec{A}$ *along* ε is the point in M_5 given by the equivalence class (under the action of $\mathbb{P}\mathrm{GL}(2, \mathbb{C})$) of any orthogonal parallel projection of $\vec{A}$ along the direction ε, considered as an n-tuple of points in $\mathbb{P}^1_{\mathbb{C}}$.

Our next task is to define the notion of Möbius camera, namely a function which, once fixed a vector of points, takes a direction $\varepsilon \in S^2$ and associates to it the Möbius picture of the points along that direction.

Definition 3.4. Let $\vec{A}$ be a vector of 5 points in $\mathbb{R}^3$. The *Möbius camera*, or *photographic map*, for $\vec{A}$ is the morphism of varieties given by:

$$\begin{array}{rccc} f_{\vec{A}}: & C & \longrightarrow & M_5 \subseteq \mathbb{P}^5_{\mathbb{C}} \\ & c & \mapsto & \varphi\Big(\big(\pi_\varepsilon(A_1) : 1\big), \ldots, \big(\pi_\varepsilon(A_5) : 1\big) \Big) \\ & & & = \varphi\Big(\big(\langle c, A_1\rangle : 1\big), \ldots, \big(\langle c, A_5\rangle : 1\big) \Big) \end{array}$$

where C is the curve $\{x^2 + y^2 + z^2 = 0\} \subseteq \mathbb{P}^2_{\mathbb{C}}$, and $c \leftrightarrow \varepsilon$ under the bijection γ established in Lemma 2.3; moreover the map $\varphi : \left(\mathbb{P}^1_{\mathbb{C}}\right)^5 \dashrightarrow M_5$ is the quotient map defined in Subsection 3.1. We denote by $\widetilde{f}_{\vec{A}}$ the precomposition of $f_{\vec{A}}$ by the parametrization of C described in Remark 2.4.

Remark 3.5. The Möbius picture of $\vec{A}$ cannot be defined for those $c \in C$ such that there exist three points in $\vec{A}$ lying on a line parallel to the direction defined by c. Since the points $c \in C$ for which the Möbius picture is defined form an open subset of C, then the map $f_{\vec{A}}$ is a priori a rational one. However, since C is a smooth curve, then $f_{\vec{A}}$ extends to a regular map, namely it is defined also on the points which do not admit a Möbius picture. In algebraic terms, the polynomials defining the function have a common factor vanishing at those points, which can be canceled.

Remark 3.6. The map φ is given by homogeneous polynomials of degree 5 in the coordinates $\{(a_i : b_i)\}$ of $\left(\mathbb{P}^1_{\mathbb{C}}\right)^5$. Hence $\widetilde{f}_{\vec{A}}$ is given by homogeneous polynomials of degree 10 in the coordinates $(s : t)$ of $\mathbb{P}^1_{\mathbb{C}}$.

[3] This is the condition on configurations mentioned at the end of Section 2.

3.3. **Properties of the Möbius camera.** The following lemmata describe the behavior of the image of a Möbius camera depending on the geometry of the vector $\vec{A}$.

Lemma 3.7. *Let $\vec{A} = (A_1, \ldots, A_5)$ be a 5-tuple of coplanar points which are not collinear. Then the photographic map $f_{\vec{A}} : C \longrightarrow M_5$ is $2:1$ to a rational curve of degree 5, 4, 3, or 2 in M_5.*

Proof. Suppose that the A_i are coplanar: then, after a suitable change of coordinates, we can assume that $A_i = (p_i, q_i, 0)$ for every i, and in this case the photographic map $f_{\vec{A}}$ factors through the restriction to C of the projection $\tau_{x,y} : \mathbb{P}^2_{\mathbb{C}} \longrightarrow \mathbb{P}^1_{\mathbb{C}}$ sending $(x : y : z) \mapsto (x : y)$, which is a $2:1$ map. Hence we get

$$\begin{array}{ccc} C & \xrightarrow{\quad f_{\vec{A}} \quad} & M_5 \\ & {\scriptstyle \tau_{x,y}} \searrow \quad \nearrow {\scriptstyle g_{\vec{A}}} & \\ & \mathbb{P}^1_{\mathbb{C}} & \end{array}$$

If we show that $g_{\vec{A}}$ is birational, then $f_{\vec{A}}$ is $2:1$. The map $g_{\vec{A}}$ is given by 6 components, each of which is the product of five linear polynomials in x and y. Each of these polynomials is of the form $G_{ij} = x(p_i - p_j) + y(q_i - q_j)$. Hence the components of $g_{\vec{A}}$ have the following structure:

$$\begin{array}{ccccccc} (g_{\vec{A}})_0 & = & G_{12} & G_{23} & G_{34} & G_{45} & G_{15} \\ (g_{\vec{A}})_1 & = & G_{12} & G_{25} & G_{15} & G_{34} & G_{34} \\ (g_{\vec{A}})_2 & = & G_{12} & G_{23} & G_{13} & G_{45} & G_{45} \\ (g_{\vec{A}})_3 & = & G_{23} & G_{34} & G_{24} & G_{15} & G_{15} \\ (g_{\vec{A}})_4 & = & G_{34} & G_{45} & G_{35} & G_{12} & G_{12} \\ (g_{\vec{A}})_5 & = & G_{14} & G_{45} & G_{15} & G_{23} & G_{23} \end{array} \tag{1}$$

We notice that if the lines $\overrightarrow{A_iA_j}$ and $\overrightarrow{A_hA_k}$ are parallel, then G_{ij} and G_{hk} only differ by a scalar multiple. Since the 5 points are not collinear, only four configurations are allowed (after possibly relabeling the points), as shown in Figure 2.

Case (a): The components of $g_{\vec{A}}$ do not have factors in common, so

$$\deg\left(g_{\vec{A}}(\mathbb{P}^1_{\mathbb{C}})\right) \cdot \deg\left(g_{\vec{A}}\right) = 5$$

Hence either $g_{\vec{A}}$ is a birational map to a curve of degree 5, or it is a $5:1$ map to a line. If the image of $g_{\vec{A}}$ were a line, then because of Remark 3.1 that line would coincide with one of the 10 lines of M_5. This would mean that whatever direction we use, the projections of two points always coincide in any Möbius picture, and this is not possible. Hence this possibility must be ruled out, obtaining that $g_{\vec{A}}$ is birational.

Case (b): Here G_{12}, G_{23} and G_{13} are equal up to scalar multiplication, so all the components have one factor in common, which can be removed. Hence

$$\deg\left(g_{\vec{A}}(\mathbb{P}^1_{\mathbb{C}})\right) \cdot \deg\left(g_{\vec{A}}\right) = 4$$

this leading to three possibilities: $\deg\left(g_{\vec{A}}\right) = 1$, 2 or 4. The case when $\deg\left(g_{\vec{A}}\right)$ is 4 can be discarded as in Case (a), so in order to prove the thesis

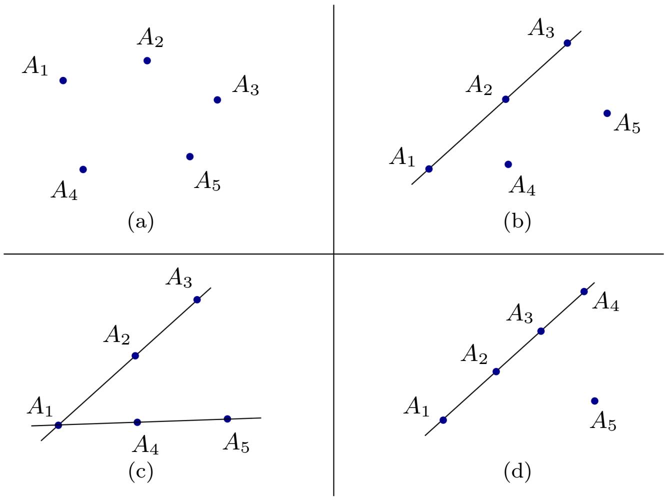


FIGURE 2. Possible configurations of 5 points in the plane: (a) no 3 points are aligned, (b) exactly 3 points are aligned, (c) $3+3$ points are collinear, (d) exactly 4 points are collinear.

we only have to consider the situation $\deg(g_{\vec{A}}) = 2$. In this case the image of $g_{\vec{A}}$ would be a conic, but from the general theory of Del Pezzo surfaces we have the following:

Claim. $g_{\vec{A}}(\mathbb{P}^1_{\mathbb{C}})$ cannot be a conic.

Proof. It is well known that the surface M_5 contains 5 families of conics, and every irreducible conic belongs exactly to one of them. These families arise in the following way: fix an index $i \in \{1, \ldots, 5\}$ and consider the map $M_5 \longrightarrow M_4 \cong \mathbb{P}^1_{\mathbb{C}}$ sending the equivalence class of $(m_1, \ldots, m_5)$ to the equivalence class of $(m_1, \ldots, m_{i-1}, m_{i+1}, \ldots, m_5)$, namely we remove the i-th point; the fibers of this map give one family of conics. From this description, recalling the definition of the lines L_{ij} (see Remark 3.1), we see that the i-th family of conics intersects only 4 lines, namely L_{ij} for $j \neq i$ (recall that $L_{ij} = L_{ji}$). On the other hand, by inspecting our current situation, we see that the image $g_{\vec{A}}(\mathbb{P}^1_{\mathbb{C}})$ has to intersect the lines L_{14}, L_{15}, L_{24}, L_{25}, L_{34} and L_{35} (in general it will also intersect the line L_{45}, but this does not happen if $\overrightarrow{A_1A_3}$ and $\overrightarrow{A_4A_5}$ are parallel). Thus $g_{\vec{A}}(\mathbb{P}^1_{\mathbb{C}})$ cannot be one of the conics in M_5.

Hence $g_{\vec{A}}$ can only be birational to a curve of degree 4.

Case (c): Here G_{12}, G_{23} and G_{13} are equal up to scalar multiplication and the same for G_{14}, G_{45} and G_{15}. One can check that all components have two factors in common. Thus, considering what we did in Case (a), the only possible situation is the one in which $g_{\vec{A}}$ is birational to a curve of degree 3.

Case (d): In this case G_{12}, G_{23}, G_{13}, G_{24}, G_{34} and G_{14} are equal up to scalar multiplication. One deduces that all components have three factors in common and so analogously as in Case (c) we have that $g_{\vec{A}}$ is birational to a curve of degree 2. □

Lemma 3.8. *Let* $\vec{A} = (A_1, \ldots, A_5)$ *be a* 5*-tuple of points. If the* $\{A_i\}$ *are not coplanar, then the photographic map* $f_{\vec{A}} : C \longrightarrow M_5$ *is birational to a rational curve of degree* 10 *or* 8 *in* M_5.

Proof. We argue as in the proof of Lemma 3.7: if we write H_{ij} for the linear polynomial $x(p_i - p_j) + y(q_i - q_j) + z(r_i - r_j)$, then the components of $f_{\vec{A}}$ have the same structure as described by Equation (1), where we replace G_{ij} by H_{ij}. Since the $\{A_i\}$ are not coplanar, we can have only three possibilities (after a possible relabeling of the points), showed in Figure 3.

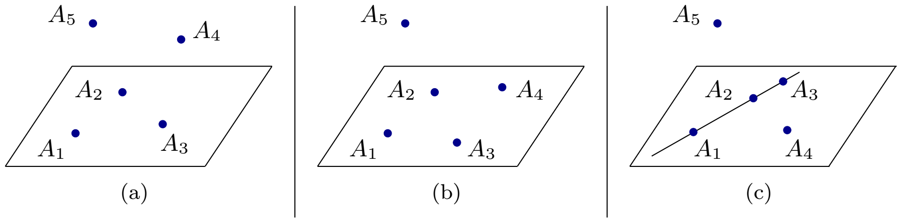


FIGURE 3. Possible configurations of 5 non coplanar points in the space: (a) no 4 points are coplanar, (b) 4 coplanar points, no 3 of them aligned, (c) 3 aligned points.

Case (a/b): In this situation the components $\left(f_{\vec{A}}\right)_i$ of $f_{\vec{A}}$ do not have any common factor, hence either $f_{\vec{A}}$ is a birational map with image a degree 10 curve, or $f_{\vec{A}}$ is $2:1$ to a curve of degree 5. We prove that in the second case the points should be coplanar, so this cannot happen. If we suppose that the map $f_{\vec{A}}$ is $2:1$, we have the following:

Claim. It is possible to define a regular map $r_{\vec{A}} : C \longrightarrow C$ which respects the real structure on C and such that

$$r_{\vec{A}}^2 = \mathrm{id} \quad \text{and} \quad f_{\vec{A}}\left(r_{\vec{A}}(\varepsilon)\right) = f_{\vec{A}}(\varepsilon)$$

Proof. Suppose in fact that we are given a finite map $f : C \longrightarrow D$ where C is a smooth curve and f is generically $2:1$. If $\widetilde{D}$ is the normalization of D, we can lift f to a finite map $\widetilde{f} : C \longrightarrow \widetilde{D}$ which is also generically $2:1$. Then we define set-theoretically an involution $r : C \longrightarrow C$ in the following way: pick a point $P \in C$; in particular, P is a prime divisor of C, so we map it to $f^*(f_*(P)) - P$[4], which is also a prime divisor of C, namely a point (we passed to the normalization in order to have good functorial properties of divisors; generically this map swaps the two elements in a fiber of f). In order to prove that this map is regular, since the map f is generically $2:1$ we can suppose that locally it is given by the canonical injection

[4]Here $f_* : \mathrm{Div}(C) \longrightarrow \mathrm{Div}(D)$ and $f^* : \mathrm{Div}(D) \longrightarrow \mathrm{Div}(C)$ denote respectively the *pushforward* and the *pullback* induced by f between the groups of divisors of the curves C and D. For definitions and properties of these notions see, for example, [3] Appendix A.

$R \longrightarrow {R[x]}\big/_{(x^2+bx+c)} = S$, where $\mathrm{Spec}(R)$ is an open set in $\widetilde{D}$ and $\mathrm{Spec}(S)$ is an open set in C. Hence r is locally given by the homomorphism $S \longrightarrow S$ sending $x \mapsto -b-x$, which exchanges the two roots of x^2+bx+c. In this way we see that r is regular. Moreover, if C is a real variety and f is a real map, then also r is a real map.

If we think of C as the unit sphere S^2, because of its properties $r_{\vec{A}}$ has to be a rotation of S^2 of 180° along an axis, which also proves that $r_{\vec{A}}$ has two fixed points (the intersections of S^2 with the axis of rotation). Recall the definition of the lines L_{ij} in M_5 (see Remark 3.1). Then we get that

$$f_{\vec{A}}^{-1}(L_{ij}) = \left\{ \frac{A_i - A_j}{\|A_i - A_j\|}, \frac{A_j - A_i}{\|A_i - A_j\|} \right\}$$

On the other hand, if $\varepsilon \in f_{\vec{A}}^{-1}(L_{ij})$, then also $r_{\vec{A}}(\varepsilon) \in f_{\vec{A}}^{-1}(L_{ij})$, so there are only two options:

i. either $r_{\vec{A}}(\varepsilon) = -\varepsilon$, meaning that ε lies on a great circle of S^2 (the one orthogonal to the axis determined by $r_{\vec{A}}$) since $r_{\vec{A}}$ coincides with the antipodal map only on this great circle;

ii. or $r_{\vec{A}}(\varepsilon) = \varepsilon$, meaning that ε is one of the two fixed points of $r_{\vec{A}}$.

If possibility i. happens for every L_{ij}, this means that the direction of all lines $\overrightarrow{A_iA_j}$ lie on a great circle of S^2, this implying that the points $\{A_i\}$ are coplanar. If this were not the case, since in our configuration no three A_i are collinear in this case we have that possibility ii. can happen only for one line L_{ij}. Let us suppose that this line is L_{12}: this would imply that the points A_2, A_3, A_4 and A_5 are coplanar (in Case (a) here we would have already reached a contradiction) and the line $\overrightarrow{A_1A_2}$ is orthogonal to the plane on which the other points lie. On the other hand, the fact that all lines but L_{12} fall on possibility i. implies that also A_1, A_2, A_3 and A_4 are coplanar. Hence all points are coplanar. But this is in contradiction with our assumption that the points $\{A_i\}$ are not coplanar.

We have shown that in this case the only situation which is left possible is that the map $f_{\vec{A}}$ is birational to a degree 10 curve.

Case (c): Here we have that H_{12}, H_{23} and H_{13} are equal up to a scalar factor, so the components of $f_{\vec{A}}$ have one factor in common, which can be removed. Thus four situations are possible: either $f_{\vec{A}}$ is birational to a curve of degree 8, or it is $2:1$ to a curve of degree 4, or it is $4:1$ to a conic, or it is $8:1$ to a line. Arguments similar to the ones of Case (a) in the proof of Lemma 3.7 rule out the last two situations. In order to prove that the $2:1$ situation is not possible, we proceed as in Case (a): here the image $f_{\vec{A}}(C)$ does not meet all the lines L_{ij}, but from the configuration of the points A_i it is ensured that the curve intersects L_{14}, L_{24}, L_{34}, L_{15}, L_{25}, L_{35} and L_{45}, which is enough to prove that the points are coplanar. □

Remark 3.9. We notice that if all the 5 points of $\vec{A}$ are aligned, then all the 6 components of the photographic map are proportional, hence $f_{\vec{A}}$ is a constant map.

Remark 3.10. Lichtblau stated the following conjecture (see Conjecture 2 of [6]; this was later proved by him in [7], Proposition 3 and Theorem 4): *there can only exist an infinite number of cylinders of revolution passing through five distinct points in* $\mathbb{R}^3$ *if the points are located on two parallel lines.* Under the assumption that infinitely many circular cylinders are real we can use Lemma 3.8 to give an alternative proof to the one of Lichtblau. In fact, if a 5-tuple has such a property, then the image of its photographic map will have infinitely many real points (see Remark 3.2). Since S^2 does not have real points, it follows that the photographic map cannot be birational, hence the points have to be coplanar. From this it is well known that the points actually have to lie on two parallel lines. Therefore only the question if this condition is also necessary for the existence of infinitely many circular cylinders over $\mathbb{C}$ passing through five real distinct points remains open.

Now we state and prove the main result of this section.

Theorem 3.11. *Let* $\vec{A}$ *and* $\vec{B}$ *be two* 5*-tuples of points in* $\mathbb{R}^3$ *such that no* 4 *of them are collinear. Assume that* $f_{\vec{A}}(C)$ *and* $f_{\vec{B}}(C)$ *are equal as curves in* M_5*. If* $\vec{A}$ *is coplanar, then* $\vec{B}$ *is also coplanar and affine equivalent to* $\vec{A}$*. If* $\vec{A}$ *is not coplanar, then* $\vec{B}$ *is similar to* $\vec{A}$*.*

Proof. Suppose that $\vec{A}$ is not coplanar. Then by Lemma 3.8 we know that $f_{\vec{A}}$ is birational to a curve of degree 10 or 8. From Lemma 3.7 we have that $\vec{B}$ is also not coplanar, since otherwise we would have a curve of different degree as the image of $f_{\vec{B}}$. Thus $f_{\vec{B}}$ is birational, and by composing $\widetilde{f}_{\vec{A}}$ and $\widetilde{f}_{\vec{B}}^{-1}$ we get an isomorphism $\rho : \mathbb{P}^1_{\mathbb{C}} \xrightarrow{\cong} \mathbb{P}^1_{\mathbb{C}}$ which respects the real structure, since both $\widetilde{f}_{\vec{A}}$ and $\widetilde{f}_{\vec{B}}$ do so. Thus ρ is a rotation of S^2. If we apply the rotation ρ to $\vec{A}$ we obtain a vector of points $\vec{A}'$ so that the diagram

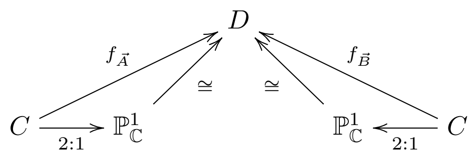
$$\begin{array}{ccc} & D & \\ \nearrow^{\widetilde{f}_{\vec{A}'}} & & \nwarrow^{\widetilde{f}_{\vec{B}}} \\ \mathbb{P}^1_{\mathbb{C}} & \xrightarrow[\text{id}]{} & \mathbb{P}^1_{\mathbb{C}} \end{array}$$

commutes, namely $\widetilde{f}_{\vec{A}}$ and $\widetilde{f}_{\vec{B}}$ coincide as maps. The goal now is to show that the direction $\overrightarrow{A'_iA'_j}$ and $\overrightarrow{B_iB_j}$ coincide for every i and j, this proving that $\vec{A}'$ and $\vec{B}$ are similar, from which we derive the thesis. Let us consider the situation when D has degree 10. Recall the definition of the lines L_{ij} in M_5 (see Remark 3.1). Analogously as in the proof of Lemma 3.8 we have that

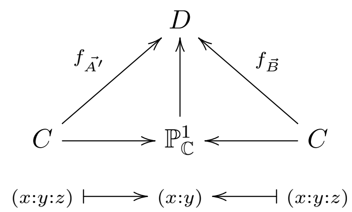
$$f_{\vec{A}'}^{-1}(L_{ij}) \;=\; \left\{ \frac{A'_i - A'_j}{\|A'_i - A'_j\|}, \frac{A'_j - A'_i}{\|A'_i - A'_j\|} \right\}$$

and similarly for $f_{\vec{B}}^{-1}(L_{ij})$. Since the two maps $\widetilde{f}_{\vec{A}'}$ and $\widetilde{f}_{\vec{B}}$ coincide our claim is proved. In the situation when D has degree 8 the argument is the same, but in this case D does not intersect all the lines L_{ij}; however, knowing that $f_{\vec{A}'}^{-1}(D \cap L_{ij})$ and $f_{\vec{B}}^{-1}(D \cap L_{ij})$ are equal for $ij \in \{14, 24, 34, 15, 25, 35, 45\}$ (see Case (c) of Lemma 3.8) gives already enough information for proving that $\vec{A}'$ and $\vec{B}$ are similar.

Suppose that $\vec{A}$ is coplanar, then from Lemma 3.7 the map $f_{\vec{A}}$ is $2:1$ to a curve of degree 5, 4 or 3 (we avoid the conic case, since no 4 points are collinear by hypothesis; the reason for this is clarified in Remark 3.13). Hence the only possibility is that also $\vec{B}$ is coplanar, because otherwise from Lemma 3.8 we would get a curve of degree 10 or 8 as the image of $f_{\vec{B}}$. As in the proof of Lemma 3.7, we know that both $f_{\vec{A}}$ and $f_{\vec{B}}$ factor through a $2:1$ map to $\mathbb{P}^1_{\mathbb{C}}$ followed by a birational map. By a change of coordinates we can suppose that this $2:1$ map is given by sending $(x:y:z) \mapsto (x:y)$. The picture of the situation is:



Thus we get an isomorphism $\mathbb{P}^1_{\mathbb{C}} \xrightarrow{\cong} \mathbb{P}^1_{\mathbb{C}}$ which makes the previous diagram commute. If M is the invertible 2×2 matrix representing it, and we denote by A' the vector of points obtained by applying the affinity associated to M to $\vec{A}$, then the following diagram commutes:



In this way we reached the point where $f_{\vec{A}'}$ and $f_{\vec{B}}$ are equal as maps, thus we can proceed as in the non planar case, proving that A' and B are similar, so $\vec{A}$ and $\vec{B}$ are affine equivalent. □

Remark 3.12. We can describe an algorithm which takes as an input the image of the photographic map of a vector of points $\vec{A}$ satisfying the conditions of Theorem 3.11 and gives back a vector of points $\vec{C}$ which is similar to $\vec{A}$. In Algorithm 1 we describe the procedure in the case of non planar points, when the degree of D is 10. This is the easiest situation, because we have information about all the directions of the lines passing through the points of $\vec{A}$.

We notice that we can always perform Steps 5, 6 and 7, namely, the involved lines always intersect. This is ensured by the fact that we start from an existing configuration of points.

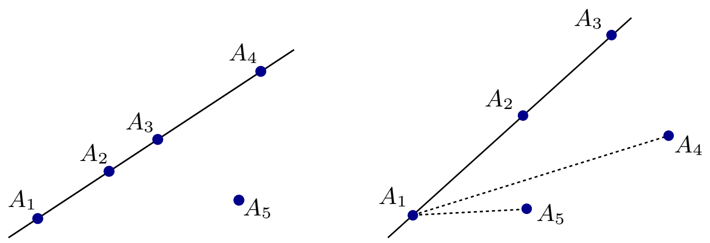
When the curve D has degree 8, 5, 4 or 3 the algorithm is almost the same, we just have to take into account that D will not intersect all the lines L_{ij}: the ones which are disjoint from the image of $f_{\vec{A}}$ reveal which points in $\vec{C}$ will be collinear, and the others can be used to identify the whole configuration.

Algorithm 1 Non planar point reconstruction

Input: $D \subseteq M_5$, a degree 10 curve such that $f_{\vec{A}}(S^2) = D$.
Output: $\vec{C}$ such that it is similar to $\vec{A}$.

1: **Parametrize** D via φ respecting the real structure of D.
2: **Compute** $\{\varepsilon_{ij}, -\varepsilon_{ij}\} = \varphi^{-1}(L_{ij})$ for all i, j.
3: **Set** $C_1 = (0, 0, 0)$.
4: **Pick** C_2 arbitrary on the line $\{C_1 + t\,\varepsilon_{12} \,:\, t \in \mathbb{R}\}$.
5: **Construct** C_3 as the intersection of the lines $\{C_1 + t\,\varepsilon_{13}\}$ and $\{C_2 + t\,\varepsilon_{23}\}$.
6: **Construct** C_4 using ε_{24} and ε_{34} as in Step 5.
7: **Construct** C_5 using ε_{35} and ε_{45} as in Step 5.
8: **Return** $\vec{C} = (C_1, \dots, C_5)$.

Remark 3.13. We notice that we have to avoid the case when 4 points are collinear (namely when the degree of the image of the photographic map is 2), because in that case it is not possible to reconstruct the direction $\overrightarrow{A_1A_4}$. In fact, $f_{\vec{A}}(C) \cap L_{14} = \emptyset$ since projecting in that direction would give a configuration where four points coincide, which is not allowed in M_5. In this case one can show that the images of two photographic maps $f_{\vec{A}}(C)$ and $f_{\vec{B}}(C)$ are equal if and only if the cross ratios of the two 4-tuples of collinear points are equal. On the other hand, also when we only have three aligned points the image of the photographic map does not intersect the line L_{13}, but in this case we can reconstruct the whole configuration regardless the knowledge of $\overrightarrow{A_1A_3}$, since we can use $\overrightarrow{A_1A_4}$ and $\overrightarrow{A_1A_5}$ to determine A_1 starting from A_4 and A_5, and do the same for A_2 and A_3 — this procedure cannot be applied to the previous configuration. The two situations are described in Figure 4.



(a) (b)

FIGURE 4. In the case of (a) four collinear points, the reconstruction algorithm does not work, since it is not possible to recover the direction of the line on which the four points lie. Instead, if we only allow (b) three collinear points, then the algorithm succeeds since we can reconstruct the aligned points using the other ones.

Eventually, it is possible to extend the consequences of Theorem 3.11 to tuples of n points when $n > 5$. In order to do this, starting from such an n-tuple $\vec{A}$ one can define a photographic map $f_{\vec{A}} : C \longrightarrow M_n$, where M_n is the moduli space of n

points in $\mathbb{P}^1_{\mathbb{C}}$, in the same way as we did in this paper. Then for every sub-tuple of 5 elements of $\vec{A}$, say $(A_1, \dots, A_5)$, one has a commutative diagram:

$$\begin{array}{ccc} C & \xrightarrow{\;f_{\vec{A}}\;} & M_n \\ & {\scriptstyle f_{(A_1,\dots,A_5)}}\searrow \qquad \swarrow{\scriptstyle \delta_{(1,\dots,5)}} & \\ & M_5 & \end{array} \tag{2}$$

where $\delta_{(1,\dots,5)}$ associates the equivalence class of the n-tuple $(m_1, \dots, m_n)$ to the equivalence class of the 5-tuple $(m_1, \dots, m_5)$ (this is a rational map).

Corollary 3.14. *Theorem 3.11 holds true also if we take $\vec{A}$ and $\vec{B}$ to be two n-tuples of points in $\mathbb{R}^3$ where no $n-1$ points are collinear, provided that $n \geq 5$.*

Proof. We prove the statement by reducing to the $n = 5$ case and applying Theorem 3.11. Suppose that $\vec{A}$ is not coplanar; we want to prove that $\vec{A}$ and $\vec{B}$ are similar. After possibly relabeling the points, we can suppose that $A_1, \dots, A_4$ are not coplanar. By hypothesis we have that $f_{\vec{A}}(C) = f_{\vec{B}}(C)$, so from Diagram 2 we can infer that for every $k \geq 5$ we have $f_{(A_1,\dots,A_4,A_k)}(C) = f_{(B_1,\dots,B_4,B_k)}(C)$. Thus by Theorem 3.11 we get that for every $k \geq 5$, the two 5-tuples $(A_1, \dots, A_4, A_k)$ and $(B_1, \dots, B_4, B_k)$ are not coplanar and similar. Now, since in this case there exists a unique similarity sending $(A_1, \dots, A_4)$ to $(B_1, \dots, B_4)$, from what we said we have that the same similarity sends A_k to B_k for all $k \geq 5$. Hence $\vec{A}$ and $\vec{B}$ are similar. If $\vec{A}$ is coplanar, then from the commutativity of Diagram 2 and by Theorem 3.11 we obtain that also $\vec{B}$ is coplanar. Now we can proceed as before to get the thesis, but here in order to be able to use Theorem 3.11 we have to make sure that we can choose $A_1, \dots, A_4$ so that for every $k \geq 5$ there are no 4 collinear points among $A_1, \dots, A_4, A_k$. This is ensured by the hypothesis that no $n-1$ among the $\{A_i\}$ are collinear, since the latter is the only case when this choice cannot be made. Hence we can conclude as before, since an affinity is completely determined by the image of 3 non collinear points. □

4. A necessary condition for pentapods with mobility 2

We can finally apply the theory we developed so far to get necessary conditions for mobility of pentapods. The geometry of this kind of mechanical manipulators is defined by the coordinates of the 5 platform anchor points $p_1, \dots, p_5 \in \mathbb{R}^3$ and of the 5 base anchor points $P_1, \dots, P_5 \in \mathbb{R}^3$ in one of their possible configurations. All pairs of points (p_i, P_i) are connected by a rigid body, called *leg*, so that for all possible configurations the distance $d_i = \|p_i - P_i\|$ is preserved. The dimension of the set of possible configurations of a pentapod is called its *mobility* (for a formal definition of this concept, see [2], Section 3, Definition 3.2).

In [2] we proved the following conditions for n-pods (replace 5 by n in the previous paragraph):

Theorem 4.1. *Let Π be an n-pod with mobility 2 or higher. Then one of the following holds:*

(a) *there are infinitely many pairs* (L, R) *of elements of* S^2 *such that the points* $\pi_L(p_1), \ldots, \pi_L(p_n)$ *and* $\pi_R(P_1), \ldots, \pi_R(P_n)$ *differ by an inversion or a similarity;*
(b) *there exists* $m \leq n$ *such that* $p_1, \ldots, p_m$ *are collinear and* $P_{m+1} = \ldots = P_n$*, up to permutation of indices and interchange between base and platform;*
(c) *there exists* $m \leq n$ *with* $1 < m < n-1$ *such that* $p_1, \ldots, p_m$ *lie on a line* $g \subseteq \mathbb{R}^3$ *and* $p_{m+1}, \ldots, p_n$ *are located on a line* $g' \subseteq \mathbb{R}^3$ *parallel to* g*; moreover* $P_1, \ldots, P_m$ *lie on a line* $G \subseteq \mathbb{R}^3$ *and* $P_{m+1}, \ldots, P_n$ *are located on a line* $G' \subseteq \mathbb{R}^3$ *parallel to* G*, up to permutation of indices.*

For $n = 5$ we can use our Möbius Photogrammetry technique to reformulate condition (a) above in a more geometric fashion.

Theorem 4.2. *Let* Π *a pentapod with mobility* 2 *or higher. Then one of the following conditions holds:*

(a) *the platform and the base are similar;*
(b) *the platform and the base are planar and affine equivalent;*
(c) *there exists* $m \leq 5$ *such that* $p_1, \ldots, p_m$ *are collinear and* $P_{m+1}, \ldots, P_5$ *coincide, up to permutation of indices and interchange of platform and base;*
(d) *the points* p_1, p_2, p_3 *lie on a line* $g \subseteq \mathbb{R}^3$ *and* p_4, p_5 *lie on a line* $g' \subseteq \mathbb{R}^3$ *parallel to* g*, and* P_1, P_2, P_3 *lie on a line* $G \subseteq \mathbb{R}^3$ *and* P_4, P_5 *lie on a line* $G' \subseteq \mathbb{R}^3$ *parallel to* G*, up to permutation of indices.*

Proof. Since Π has mobility at least 2, then by Theorem 4.1 either we are in cases (c) or (d), or there are infinitely many pairs (L, R) of elements of S^2 such that the points $\pi_L(p_1), \ldots, \pi_L(p_5)$ and $\pi_R(P_1), \ldots, \pi_R(P_5)$ differ by an inversion or a similarity. Let us consider then this last case. Since we can suppose that no 4 point of the base or platform are aligned (otherwise we are in case (c) or (d)), we have in particular that the photographic maps $f_{\vec{P}}$ and $f_{\vec{p}}$ of base and platform points of Π are not constant. Hence, if we re-interpret the assumption in the language we developed in this paper, we have that the images $f_{\vec{P}}(C)$ and $f_{\vec{p}}(C)$ have infinitely points in common. Since both are irreducible algebraic curves, they must coincide, and we get (a) or (b) by Theorem 3.11. □

Remark 4.3. For quadropods the analogous statement of Theorem 4.2 does not hold. In fact, all quadropods have mobility at least 2, but the general quadropod does not fulfill any of the conditions (a)–(d) of the theorem. For tripods the statement is trivially true, since conditions (b) and (c) are always fulfilled.

For n-pods with $n > 5$ one can prove a statement analogous to Theorem 4.2 by using Corollary 3.14.

Based on Theorem 4.2 a complete classification of pentapods with mobility 2 was given in [9] and [10].

Acknowledgments

The first and third author's research is supported by the Austrian Science Fund (FWF): W1214-N15/DK9 and P26607 - "Algebraic Methods in Kinematics: Motion Factorisation and Bond Theory". The second author's research is funded by the Austrian Science Fund (FWF): P24927-N25 - "Stewart Gough platforms with self-motions".

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