BOND THEORY FOR PENTAPODS AND HEXAPODS

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Abstract. This paper deals with the old and classical problem of determining necessary conditions for the overconstrained mobility of some mechanical device. In particular, we show that the mobility of pentapods/hexapods implies either a collinearity condition on the anchor points, or a geometric condition on the normal projections of base and platform points. The method is based on a specific compactification of the group of direct isometries of $\mathbb{R}^3$.

1. Introduction

The objects we will focus on in this paper are the so–called *n–pods*. For $n = 5$ they are referred as pentapods and for $n = 6$ as hexapods, which are also known as *Stewart Gough platforms*. As described in [10], the geometry of this kind of mechanical manipulators is defined by the coordinates of the n platform anchor points $p_i = (a_i, b_i, c_i) \in \mathbb{R}^3$ and of the n base anchor points $P_i = (A_i, B_i, C_i) \in \mathbb{R}^3$ in one of their possible configurations. All pairs of points (p_i, P_i) are connected by a rigid body, called *leg*, so that for all possible configurations the distance $d_i = \|p_i - P_i\|$ is preserved.

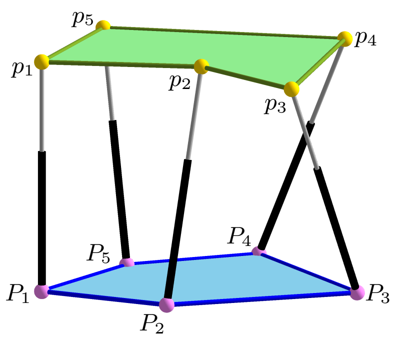

Figure 1. Sketch of an n-pod for $n = 5$, a pentapod.

Notation. We will think of an n–pod Π as a triple

$$\Pi = \Big((p_1, \ldots, p_n), (P_1, \ldots, P_n), (d_1, \ldots, d_n) \Big)$$

where p_i, P_i and d_i are defined as above.

We are interested in describing the *self–motions* of a given n–pod Π, namely which direct isometries σ of $\mathbb{R}^3$ satisfy the condition (which is called the *spherical condition*)

$$\|\sigma(p_i) - P_i\| \;=\; d_i \qquad \text{for all } i \in \{1, \ldots, n\} \tag{1}$$

In particular we want to understand what is the *dimension* of the set of these isometries, namely the *mobility* of Π (we will make these concepts precise at the beginning of Section 3), and what conditions we have to impose on the base and platform points to reach a prescribed mobility.

In order to do so we first study the group of direct isometries of $\mathbb{R}^3$, and in particular we embed and compactify it in a projective space in a way specifically tuned for n–pods. The key idea behind all of this is that, if we introduce suitable coordinates in the space of direct isometries and we consider the condition given by Equation (1) then the latter becomes linear in these coordinates. These coordinates provide the desired compactification, and Section 2 is devoted to the study of some of its geometric properties, which play an important role in the proofs of the results of the subsequent section. In particular, we focus our attention on the natural action of direct isometries on this compactification and on its boundary. In Section 3 we use this information to establish some results which can be framed in the so–called *bond theory* (see [6], [10]): the presence of some boundary point in the projective closure of the set of self–motions of an n–pod implies precise geometric constraints on base and platform points. The set of boundary points is a 5–dimensional complex algebraic variety, but we give geometric interpretation of its complex points, which form a 10–dimensional real algebraic variety. These results allow us to provide some necessary conditions for the mobility of n–pods. Eventually we consider the case of n–pods with high mobility (namely with strictly more than one degree of freedom) and we also provide necessary conditions on the geometry of these devices.

From the kinematic point of view this paper contains the following main results (see Corollary 3.17 and Theorem 3.19):

Result 1 *If an n–pod is mobile, then one of the following conditions holds:*

(i) *There exists at least one pair of orthogonal projections π_L and π_R such that the projections of the platform points $p_1, \ldots, p_n$ by π_L and of the base points $P_1, \ldots, P_n$ by π_R differ by an inversion or a similarity.*

(ii) *There exists $m \leq n$ such that $p_1, \ldots, p_m$ are collinear and $P_{m+1}, \ldots, P_n$ are collinear, up to permutation of indices.*

In the following we only give one example for each of the two cases of Result 1, as a full listing of all examples known in the literature is beyond the scope of this paper (for some of them, see for example [5], [7] and [8]).

Example ad (i):

It was proven by Bricard (cf. Chapter VI of [3]) that there is exactly one type of non–trivial[1] motions, where all points have spherical paths. Moreover it is well known (cf. page 324–325 of [1]) that this motion–type is a composition of a rotation about a fixed axis and a translation parallel to this axis. Without loss of generality we can assume that this axis is the z–axis. We can take any number of points $p_1, \ldots, p_n$ as platform anchor points, and the centers $P_1, \ldots, P_n$ of the spheres

[1]The trivial motions with this property are translations with spherical trajectories, and the rotation of the moving system about a fixed axis or fixed point, respectively.

containing their paths as base anchor points. For $n \geq 6$ and generic choice of $p_1, \ldots, p_n$, we will get an n–pod with mobility 1. If we project the points p_i and P_i onto the xy–plane, then the resulting points $\pi_z(p_i)$ and $\pi_z(P_i)$ are coupled by an inversion ι followed by a rotation ρ and therefore the condition (i) is fulfilled.

Based on this example we also want to show that the condition (i) is not sufficient for the existence of a self-motion. To do so, we add an extra leg (p_{n+1}, P_{n+1}) where P_{n+1} is not the center of the sphere holding the path of p_{n+1}, but another point with the same xy–coordinates. Then the hypothesis of (i) in Result 1 is fulfilled, but the mobility of the new $(n+1)$–pod is zero.

Example ad (ii):

On the contrary, condition (ii) is sufficient for the existence of a self–motion. If the platform is located in a way that the carrier line of $p_1, \ldots, p_m$ coincides with the carrier line of $P_{m+1}, \ldots, P_n$, then the platform can rotate freely about this line. Therefore we get a 1–dimensional set of so–called *butterfly–motions.*

Note that n–pods which fulfill the condition (ii) can also have further self–motions beside these butterfly–motions, even if they do not possess the property of item (i). Good examples for this fact are the three types of Bircard's flexible octahedra [2], as they can be interpreted as hexapods.

Result 2 *Let Π be an n–pod with mobility 2 or higher. Then one of the following holds:*

- (a) *there are infinitely many pairs (L, R) of elements of S^2 such that the points $\pi_L(p_1), \ldots, \pi_L(p_n)$ and $\pi_R(P_1), \ldots, \pi_R(P_n)$ are equivalent by an inversion or a similarity;*
- (b) *there exists $m \leq n$ such that $p_1, \ldots, p_m$ are collinear and $P_{m+1} = \ldots = P_n$, up to permutation of indices and interchange between base and platform;*
- (c) *there exists $m \leq n$ with $1 < m < n-1$ such that $p_1, \ldots, p_m$ lie on a line $g \subseteq \mathbb{R}^3$ and $p_{m+1}, \ldots, p_n$ lie on a line $g' \subseteq \mathbb{R}^3$ parallel to g, and $P_1, \ldots, P_m$ lie on a line $G \subseteq \mathbb{R}^3$ and $P_{m+1}, \ldots, P_n$ lie on a line $G' \subseteq \mathbb{R}^3$ parallel to G, up to permutation of indices.*

This last result, in particular condition (a), is the starting point of further investigations on pentapods with mobility 2, which are carried on in [4], relying on a new technique called *Möbius Photogrammetry.*

2. Compactification of SE_3

We start our discussion in Subsection 2.1 by introducing a new compactification of the group of direct isometries of $\mathbb{R}^3$ in a projective space. Then in Subsection 2.2 we study the natural action of isometries on this compactification, and we prove that it is given by linear changes of coordinates. At last in Subsection 2.3 we describe the boundary of the compactification.

2.1. A new compactification.

We study the 6-dimensional algebraic group SE_3 of direct isometries of affine 3–space $\mathbb{R}^3$ into itself. One can embed SE_3 as an open subset of a quadric hypersurface in $\mathbb{P}^7_{\mathbb{R}}$, called the *Study quadric*. This compactification of SE_3 turns out to be extremely useful in the study of mobility properties of objects coming from robotics and kinematics (see for example [6], [10]). However, in our situation we will see that a different compactification will lead us to a better comprehension of the phenomena which can arise.

Any isometry of $\mathbb{R}^3$ can be written as a pair (M, y), where $M \in \mathrm{SO}_3$ is the linear contribution and $y \in \mathbb{R}^3$ is the image of the origin $o \in \mathbb{R}^3$. We define $x := -M^t y = -M^{-1} y$ and $r := \langle x, x\rangle = \langle y, y\rangle$, where $\langle\cdot,\cdot\rangle$ is the Euclidean scalar product. The isometry (M, y) is considered as a point in $\mathbb{P}^{16}_{\mathbb{R}}$ with coordinates

- $m_{11}, \ldots, m_{33}$ (the entries of the matrix),
- $x_1, \ldots, x_3$ (the coordinates of x),
- $y_1, \ldots, y_3$ (the coordinates of y),
- r and h (a homogenization coordinate; for group elements we can assume that it is equal to 1).

The group SE_3 is defined by the inequality $h \neq 0$ and equations

$$MM^t = M^t M = h^2 \cdot \mathrm{id}_{\mathbb{R}^3}, \;\; \det(M) = h^3,$$

$$M^t y + hx = 0, \;\; Mx + hy = 0,$$

$$\langle x, x\rangle = \langle y, y\rangle = rh$$

(not all equations are needed, for instance $M^t y + hx = 0$ is a consequence of the other equations and the inequality). We define $X_{\mathbb{R}}$ as the Zariski closure of SE_3, i.e. the zero set of the set of all equations vanishing at SE_3. Using computer algebra, a Gröbner basis for this set of equations can be computed[2]. The degree of $X_{\mathbb{R}}$ can also be computed using computer algebra by the leading monomials of the Gröbner basis: it is 40.

Remark 2.1. In the first version of this paper we constructed a projectively isomorphic compactification of SE_3 using Study parameters. However it turned out that the construction above is more direct, computationally simpler and easier to generalize to higher dimensions.

We consider the spherical conditions $\|\sigma(p_i) - P_i\| = d_i$ we want to impose to rigid motions in $\mathbb{R}^3$; if we set $h = 1$, this can be expressed by:

$$\begin{aligned} d_i^2 &= \langle Mp_i + y - P_i, Mp_i + y - P_i\rangle \\ &= \langle Mp_i, Mp_i\rangle + 2\langle Mp_i, y\rangle + r + \langle P_i, P_i\rangle - 2\langle Mp_i, P_i\rangle - 2\langle y, P_i\rangle \\ &= \langle p_i, p_i\rangle - \langle P_i, P_i\rangle + r + 2\langle p_i, M^t y\rangle - 2\langle Mp_i, P_i\rangle - 2\langle y, P_i\rangle \\ &= \langle p_i, p_i\rangle - \langle P_i, P_i\rangle + r - 2\langle p_i, x\rangle - 2\langle y, P_i\rangle - 2\langle Mp_i, P_i\rangle. \end{aligned} \tag{2}$$

Remark 2.2. After homogenization, Equation (2) becomes linear in the projective coordinates of $\mathbb{P}^{16}_{\mathbb{R}}$.

[2]This can be done, for example, by adjoining a temporary variable u, computing a Gröbner basis of the equations above and equation $hu - 1$ with an elimination order that eliminates u, and then taking the subset of the basis of elements that have degree 0 in u.

By introducing this compactification of SE_3 we reduced the problem of dealing with Equation (1) to the problem of understanding linear equations on $X_{\mathbb{R}} \subseteq \mathbb{P}^{16}_{\mathbb{R}}$. In order to fully use the techniques from algebraic geometry and to be able to set up bond theory, we need to extend our ground field to the complex numbers.

Definition 2.3. From now on we work with the complexification of $X_{\mathbb{R}}$, denoted by X. In order to do this we simply take the equations defining X (which have real coefficients), and we think of them as polynomials over $\mathbb{C}$. Hence what we get is a projective variety in $\mathbb{P}^{16}_{\mathbb{C}}$ of complex dimension 6 and degree 40 whose real points are in bijection with the points of $X_{\mathbb{R}}$. Inside X we can consider the complexification of SE_3, which we will denote by $\mathrm{SE}_{3,\mathbb{C}}$, hence we have an injective map $\Phi : \mathrm{SE}_3 \hookrightarrow \mathrm{SE}_{3,\mathbb{C}} \subseteq X$.

2.2. **Action of** SE_3 **on** X**.** In this subsection we want to extend the natural actions of SE_3 on itself, given by composition on the left and on the right, to actions of SE_3 on $\mathbb{P}^{16}_{\mathbb{C}}$ which restrict to actions on X. This will be useful in the proofs of Section 3, because it will allow us to exploit the symmetries of X.

Let $\sigma_1 : v \mapsto M_1 v + y_1$ and $\sigma_2 : v \mapsto M_2 v + y_2$ be isometries. Then the product $\sigma_{12} = \sigma_1\sigma_2$ maps

$$v \ \mapsto \ (M_1M_2)v + (M_1y_2 + y_1).$$

We set $M_{12} := M_1M_2$ and $y_{12} := M_1y_2 + y_1$. The remaining affine coordinates of σ_{12} are

$$\begin{aligned}
x_{12} &= -M_{12}^t y_{12} = -M_2^t M_1^t M_1 y_2 - M_2^t M_1^t y_1 \\
&= -M_2^t y_2 - M_2^t M_1^t y_1 = x_2 + M_2^t x_1, \\
r_{12} &= \langle y_{12}, y_{12}\rangle = \langle y_1, y_1\rangle + \langle M_1y_2, M_1y_2\rangle + 2\langle M_1y_2, y_1\rangle \\
&= r_1 + r_2 - 2\langle x_1, y_2\rangle.
\end{aligned}$$

This product becomes bilinear after homogenization, and the projective coordinates of σ_{12} are

$$\text{(3)} \qquad \big(\underbrace{h_1h_2}_{h_{12}} : \underbrace{M_1M_2}_{M_{12}} : \underbrace{M_2^t x_1 + h_1x_2}_{x_{12}} : \underbrace{h_2y_1 + M_1y_2}_{y_{12}} : \underbrace{h_2r_1 + h_1r_2 - 2\langle x_1, y_2\rangle}_{r_{12}}\big).$$

(The matrices and vectors appearing in the above coordinates should be replaced by their entries.)

Proposition 2.4. *Let* $(h_1 : M_1 : x_1 : y_1 : r_1), (h_2 : M_2 : x_2 : y_2 : r_2) \in X$*. Then the above product is defined if at least one of* h_1*,* h_2 *is not equal to zero.*

Proof. Assume that the product is undefined, which means that all entries are zero. In particular, $h_1h_2 = 0$. We assume $h_1 = 0$ (the other case $h_2 = 0$ can be treated analogously). Assume, indirectly, that $h_2 \neq 0$. Since $\det(M_2) = h_2^3$, it follows that M_2 is invertible. Since $M_1M_2 = 0$, it follows that $M_1 = 0$. Since $M_2^t x_1 + h_1x_2 = 0$, it follows that $x_1 = 0$. Since $h_2r_1 + h_1r_2 - 2\langle x_1, y_2\rangle = 0$, it follows that $r_1 = 0$. Then all coordinates of the first element vanish, a contradiction. $\square$

Remark 2.5. Since the h–coordinate of any element $\sigma \in \Phi(\mathrm{SE}_3)$ is always different from zero, then Proposition 2.4 ensures that left and right multiplication by σ, which a priori are maps from $\Phi(\mathrm{SE}_3)$ to itself, extend to linear maps $X \longrightarrow X$.

Remark 2.6. We specialize Equation (3) to the cases of left and right multiplication by translations or rotations along the origin. We fix $\sigma \in X$ and we suppose that it has coordinates $\sigma = (h : M : x : y : r)$.

a) Given a vector $s \in \mathbb{R}^3$, the translation by s gives the following element $\sigma' \in \Phi(\mathrm{SE}_3)$:
$$\sigma' = \bigl(1 : \mathrm{id} : -s : s : \langle s, s\rangle\bigr).$$
Then left multiplication by σ' provides
$$\sigma' \, \sigma \;=\; \bigl(h : M : -M^t s + x : hs + y : h\,\langle s, s\rangle + r + 2\langle s, y\rangle\bigr),$$
while right multiplication by σ' provides
$$\sigma \, \sigma' \;=\; \bigl(h : M : x - hs : y + Ms : r + h\,\langle s, s\rangle - 2\langle x, s\rangle\bigr).$$

b) Given an orthogonal matrix $M' \in \mathrm{SO}_3$, the rotation around the origin by M' gives the following element $\sigma' \in \Phi(\mathrm{SE}_3)$:
$$\sigma' \;=\; (1 : M : 0 : 0 : 0).$$
Then left multiplication by σ' provides
$$\sigma' \, \sigma \;=\; \bigl(h : M'M : x : M'y : r\bigr)$$
while right multiplication by σ' provides
$$\sigma \, \sigma' \;=\; \bigl(h : MM' : M'x : y : r\bigr)$$

2.3. **Boundary of X.**

Definition 2.7. The *boundary* of X is defined as $B := X \setminus \mathrm{SE}_{3,\mathbb{C}}$. It is the closed subset of X cut out by the linear equation $h = 0$.

For any point $(h : M : x : y : r) \in B$, we have
$$MM^t \;=\; M^t M \;=\; 0, \qquad M^t y \;=\; Mx \;=\; 0,$$
$$\langle x, x\rangle \;=\; \langle y, y\rangle \;=\; 0.$$
The first equation shows that $\mathrm{rank}(M) \leq 1$. Hence $M = vw^t$ for two suitable vectors $v, w \in \mathbb{C}^3$. It should be noted that v and w are not unique: one may multiply v by a nonzero complex number and w by its inverse. We have two cases:

i. if $M = 0$, we can take both v and w to be zero;

ii. if $M \neq 0$, then again by the same equation, it follows that $\langle v, v\rangle = \langle w, w\rangle = 0$. The second equation implies $\langle x, w\rangle = \langle y, v\rangle = 0$. Then the subspace spanned by x and w is totally isotropic with respect to $\langle \cdot, \cdot\rangle$, and this implies it has dimension 1, so x and w are linearly dependent. Similarly, also y and v are linearly dependent.

We can partition the boundary in five subsets. The nomenclature of the various types of points will become clear in Section 3.

2.3.1. *Vertex.* For any real point in B, we have $v = w = x = y = 0$. The only nonzero coordinate is r, so we have a unique real point $v_0 = (0 : \ldots : 0 : 1)$ in B, called the *vertex*. A computer algebra computation shows that this is a point of multiplicity 20 on X, but we do not need this fact.

2.3.2. *Inversion Points.* Consider the matrix $N := rM + 2yx^t$ (it will be clear later in the discussion why we choose this expression for N). A boundary point β with $M \neq 0$ and $N \neq 0$ is called an *inversion point*. In this case we have $x = \lambda w$ and $y = \mu v$ with $\lambda, \mu \in \mathbb{C}$. Hence the coordinates of an inversion point can be written as $(0 : vw^t : \lambda w : \mu v : r)$. Since v and w satisfy the quadratic equation $\langle v, v\rangle = \langle w, w\rangle = 0$ (called the equation of the *absolute conic* in $\mathbb{P}^2_{\mathbb{C}}$), the complex dimension of the set of inversion points is 5 (one for v, one for w, one for λ, one for μ, one for r). One can show that these are smooth points of the boundary, but we do not need this fact.

In order to compute normal forms, we first apply rotations. Multiplication from the right by a rotation of matrix M' gives (see Remark 2.6)

$$(0 : vw^t M' : \lambda\, M' w : \mu\, v : r),$$

so it leaves v fixed. Being M' an orthogonal matrix, it is in particular unitary, so it preserves both the scalar product and the Hermitian norm of w, and the action is transitive on vectors with $\langle w, w\rangle = 0$ and of the same Hermitian norm. Hence w can be taken to a vector of the form $\delta\,(1, i, 0)^t$, where $\delta \in \mathbb{C}^*$. Multiplication from the left acts analogously on v. Hence by suitable rotations from both sides we obtain $v = \gamma\,(1, i, 0)^t$ and $w = \delta\,(1, i, 0)^t$ with both γ and δ different from zero since $M \neq 0$. Then projectively we can suppose that $M = \left(\begin{smallmatrix} 1 & i & 0 \\ i & -1 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)$.

The action of left multiplication by a translation with vector $s \in \mathbb{R}^3$ on the boundary point β gives (see Remark 2.6)

$$\bigl(0 : vw^t : (-\langle v, s\rangle + \lambda)\, w : \mu\, v : r + 2\mu\,\langle v, s\rangle\bigr),$$

and similarly the action by right multiplication with vector $t \in \mathbb{R}^3$ gives

$$\bigl(0 : vw^t : \lambda\, w : (\langle w, t\rangle + \mu)\, v : r - 2\lambda\,\langle w, t\rangle\bigr).$$

This shows that we can achieve $\lambda = \mu = 0$ by multiplication by translations from both sides (for example, since we reduced to the situation $v = (1, i, 0)^t$, one can take $s_1 = \mathrm{Re}\lambda$, $s_2 = \mathrm{Im}\lambda$ and s_3 to be arbitrary, where $s = (s_1, s_2, s_3)^t$, and similarly for t). It also shows that the matrix N is invariant under translations (this was the reason why we chose N in this way). So by translations from both sides, we obtain $x = y = 0$. The value of r cannot be changed by any rotation that fixes $x = y = 0$, but we still can apply a rotation of the form $\left(\begin{smallmatrix} c & d & 0 \\ -d & c & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$, with $c^2 + d^2 = 1$, from the left. The effect on M is multiplication by $(c + id)$, and we have no effect on r. Projectively, this is the same as leaving M untouched and multiplying r by $(c + id)^{-1}$. Hence we can reach the situation where $r \in \mathbb{R}_{>0}$. We notice that r cannot be zero, otherwise we would have $N = 0$. So inversion points have the following normal forms:

$$\beta = (0 : \underbrace{1 : i : 0 : i : -1 : 0 : 0 : 0 : 0}_{M} : \underbrace{0 : 0 : 0}_{x} : \underbrace{0 : 0 : 0}_{y} : r),$$

with $r \in \mathbb{R}_{>0}$.

2.3.3. *Butterfly Points.* A boundary point β with $M \neq 0$ and $N = 0$ is called a *butterfly point*. The complex dimension of the set of butterfly points is 4: as before, we can choose v and w on the absolute conic curve, and $\lambda, \mu \in \mathbb{C}^*$. The normal form is constructed similarly as above. In this case, when we obtain $x = y = 0$, the fact that $M \neq 0$ and $N = 0$ forces r to be zero. In this case, we have only a single normal form, namely

$$\beta = (0 : \underbrace{1 : i : 0 : i : -1 : 0 : 0 : 0 : 0}_{M} : \underbrace{0 : 0 : 0}_{x} : \underbrace{0 : 0 : 0}_{y} : 0).$$

2.3.4. *Similarity Points.* The points $\beta = (0 : M : x : y : r) \in B$ such that $M = 0$, $x \neq 0$ and $y \neq 0$ are called *similarity points*. Since x and y are on the absolute conic, the complex dimension of the set of similarity points is 4.

To compute normal forms of similarity points, we first apply rotations. As we saw in Subsection 2.3.2, right multiplication fixes y and r and can transform x to $\gamma\,(1, i, 0)^t$, and left multiplication fixes x and r and can transform y to $\delta\,(1, i, 0)^t$, with both γ and δ in $\mathbb{C}^*$. Hence projectively we can always suppose that $\delta = 1$, so we can reduce any similarity point to one such that $x = \gamma\,(1, i, 0)^t$ and $y = (1, i, 0)^t$. Then translations act transitively on r, thus we may get to the situation with $r = 0$. Eventually, as we have already seen in Subsection 2.3.2, we can perform rotations so that we can ensure that γ is a real positive number. So we get normal forms of the following kind

$$\beta = (0 : \underbrace{0 : 0 : 0 : 0 : 0 : 0 : 0 : 0 : 0}_{M} : \underbrace{\gamma : i\gamma : 0}_{x} : \underbrace{1 : i : 0}_{y} : 0),$$

with $\gamma \in \mathbb{R}_{>0}$.

2.3.5. *Collinearity Points.* For the last group of points β in B we have $M = 0$ and either $x = 0$, $y \neq 0$ or $x \neq 0$, $y = 0$. These points are called *collinearity points*. There are two subsets of collinearity points, one with $x = 0$ and one with $y = 0$. Both subsets have complex dimension 2 (since there is still a free value for r to choose).

By rotations, we can achieve either $x = (1 : i : 0)^t$ or $y = (1 : i : 0)^t$. Translations act transitively on r, so we get two normal forms, namely

$$\beta = (0 : \underbrace{0 : 0 : 0 : 0 : 0 : 0 : 0 : 0 : 0}_{M} : \underbrace{1 : i : 0}_{x} : \underbrace{0 : 0 : 0}_{y} : 0),$$

$$\beta = (0 : \underbrace{0 : 0 : 0 : 0 : 0 : 0 : 0 : 0 : 0}_{M} : \underbrace{0 : 0 : 0}_{x} : \underbrace{1 : i : 0}_{y} : 0).$$

We conclude the section about the boundary of X by showing that we can associate to each inversion, butterfly and similarity point a pair (L, R) of elements of S^2, namely oriented directions in $\mathbb{R}^3$. This piece of information will play an important role in the main results of Section 3.

We recall that in the case of inversion and butterfly points the matrix M is of rank 1, since it is non zero and has rank ≤ 1, as implied by the boundary condition $h = 0$. Hence M is of the form vw^t for two non zero vectors whose coordinates satisfy

$$v_1^2 + v_2^2 + v_3^2 \;=\; 0 \quad \text{and} \quad w_1^2 + w_2^2 + w_3^2 \;=\; 0$$

Then we can think of v and w as points of the conic $C = \left\{\alpha^2 + \beta^2 + \gamma^2 = 0\right\}$ in $\mathbb{P}^2_{\mathbb{C}}$. Notice that, although v and w are not unique, they always give the same pair of points on the conic. We would like to think of v and w as directions in $\mathbb{R}^3$, and in order to do this we provide an identification between C and S^2. This identification is accomplished in two steps, namely first we consider an isomorphism $C \cong \mathbb{P}^1_{\mathbb{C}}$ and then we take the bijection between $\mathbb{P}^1_{\mathbb{C}}$ and S^2 given by the stereographic projection[3]. The isomorphism $C \cong \mathbb{P}^1_{\mathbb{C}}$ is given by the parametrization

$$\mathbb{P}^1_{\mathbb{C}} \ni (s,t) \;\mapsto\; \big((s^2 - t^2) : i(s^2 + t^2) : 2st\big) \in C$$

and its inverse

$$\begin{cases} (\alpha : \beta : \gamma) & \mapsto \quad (\alpha - i\beta : \gamma) \qquad \text{if } (i\alpha + \beta, \gamma) \neq (0,0) \\ (\alpha : \beta : \gamma) & \mapsto \quad (\gamma : -\alpha - i\beta) \quad \text{otherwise} \end{cases}$$

The identification between $\mathbb{P}^1_{\mathbb{C}}$ and S^2 by stereographic projection is provided by the following equations:

$$\begin{cases} (0,0,1) & \mapsto \quad (0:1) \in \mathbb{P}^1_{\mathbb{C}} \\[4pt] (\lambda, \mu, \nu) & \mapsto \quad \left(1 : \frac{\lambda + i\mu}{1-\nu}\right) \in \mathbb{P}^1_{\mathbb{C}} \quad \text{for all } (\lambda,\mu,\nu) \in S^2 \setminus \big\{(0,0,1)\big\} \end{cases}$$

$$\begin{cases} (0:1) & \mapsto \quad (0,0,1) \in S^2 \\[4pt] (1 : a + ib) & \mapsto \quad \left(\frac{2a}{a^2+b^2+1}, \frac{2b}{a^2+b^2+1}, \frac{a^2+b^2-1}{a^2+b^2+1}\right) \quad \text{for all } a, b \in \mathbb{R} \end{cases}$$

For example if $v = (1 : i : 0) \in C$, then the corresponding element of S^2 is the South pole $(0,0,-1)$.

In this way it is possible to assign to each inversion or butterfly point a pair (L, R) of elements in S^2. We would like to do the same for similarity points. There is a glaring obstruction in doing this, namely the fact that for similarity points both the h and the m_{ij}–coordinates are zero. On the other hand for all boundary points the two matrices M and xy^t are linear dependent, and in the case of similarity points xy^t is different from zero. Moreover x and y satisfy $\langle x, x\rangle = \langle y, y\rangle = 0$. So we can associate to a similarity point the pair of elements of S^2 coming from the vectors x and y.

[3]These identifications are very special ones, since they become isomorphisms of real varieties when we consider, respectively, componentwise complex conjugation on C, the map $(s,t) \mapsto (-\overline{t}, \overline{s})$ on $\mathbb{P}^1_{\mathbb{C}}$ and the antipodal map on S^2 as real structures. This can be understood as a hint why these particular choices work well, but we do not use this property in our investigations.

Definition 2.8. Via these identifications we can associate to every inversion, butterfly or similarity point β in B a pair (L, R) of elements of S^2, which are respectively called the *left* and the *right vector* of β.

3. Geometric Interpretation of Bonds

This section represents an instance of a more general technique called *bond theory*: the goal is to extract information from boundary points which arise as limits of self–motions of an n–pod. Boundary points do not represent direct isometries of $\mathbb{R}^3$, but nevertheless we can give them geometric meaning, since their presence as limits of self–motion determine geometric conditions the base and platform points have to satisfy.

Recall from Section 2 that $\Phi(\mathrm{SE}_3)$ is an embedding of SE_3 in $\mathbb{P}^{16}_{\mathbb{C}}$, that we denoted by $\mathrm{SE}_{3,\mathbb{C}}$ its complexification and that we defined X as the Zariski closure of $\mathrm{SE}_{3,\mathbb{C}}$. Moreover, recall that we think of an n–pod as a triple

$$\Pi = \Big((p_1, \ldots, p_n), (P_1, \ldots, P_n), (d_1, \ldots, d_n)\Big)$$

Eventually, recall that in the new coordinates of $\mathbb{P}^{16}_{\mathbb{C}}$ the spherical condition given by Equation (1) reads as

$$d_i^2\, h \;=\; \big(\langle p_i, p_i\rangle - \langle P_i, P_i\rangle\big)\, h + r - 2\langle p_i, x\rangle - 2\langle y, P_i\rangle - 2\langle M p_i, P_i\rangle, \tag{4}$$

which gives a linear form l_i on $\mathbb{P}^{16}_{\mathbb{C}}$.

Remark 3.1. Let Π be an n–pod, then the real points of the intersection

$$\mathrm{SE}_{3,\mathbb{C}} \cap \big\{l_i \,=\, 0\big\}$$

are in bijective correspondence with the set of all $\sigma \in \mathrm{SE}_3$ such that the distance between $\sigma(p_i)$ and P_i is d_i, namely the set of direct isometries satisfying the spherical condition for p_i and P_i.

Definition 3.2. Let Π be an n–pod, then the intersection of X with the hyperplanes defined by $\big\{l_i = 0\big\}$ for $i \in \{1, \ldots, n\}$ is called the *complex configuration set* of Π and denoted by K_Π; the real points of this intersection are called the *real configuration set* of Π. The complex dimension of $K_\Pi \cap \mathrm{SE}_{3,\mathbb{C}}$ as a complex algebraic variety is called the *mobility* of Π. If the mobility of Π is greater than or equal to 1, then Π is said to be *mobile*.

Remark 3.3. For a generic hexapod, the complex configuration set is finite of cardinality 40. This has been shown by [14]. It also follows from the fact $\deg(X) = 40$ that was mentioned in Subsection 2.1: in fact in the coordinates of $\mathbb{P}^{16}_{\mathbb{C}}$ the spherical equation (2) becomes linear, hence every leg of an n–pod imposes a linear condition on X. For a generic hexapod Π, its complex configuration set K_Π is given by the intersection of 6 generic hyperplanes in $\mathbb{P}^{16}_{\mathbb{C}}$ with X. The intersection of the hyperplanes gives a generic codimension 6 linear space H_Π. Now we use the following general fact from projective geometry: the intersection of a complex projective variety of dimension r and degree d with a generic linear space of codimension r consists of d points. Since X has dimension 6, its intersection with H_Π, namely K_Π, is given by a finite number of points whose cardinality equals the degree of X. Hence in the generic case K_Π is constituted by 40 points.

Definition 3.4. Let Π be an n–pod, we define its set of *bonds* B_Π as the intersection of K_Π and the hyperplane $\{h = 0\}$, namely B_Π is the intersection of K_Π with the boundary B of X, as defined in Subsection 2.3.

Remark 3.5. For bonds, Equation (4) reduces to

$$r - 2\langle p_i, x\rangle - 2\langle y, P_i\rangle - 2\langle Mp_i, P_i\rangle \;=\; 0. \tag{5}$$

Definition 3.6. We call the condition imposed by Equation (5) the *pseudo spherical condition* for the points (p_i, P_i) at the bond $(0 : M : x : y : r)$.

Remark 3.7. Recall that the vertex v_0 is the only real point of B (see Subsection 2.3.1). Since v_0 can never be a bond of an n–pod (in fact, by instantiating v_0 in Equation (5) we would get the contradiction $1 = 0$), then B_Π has no real points.

Remark 3.8. If an n–pod Π is mobile, then by definition $\dim K_\Pi \cap \mathrm{SE}_{3,\mathbb{C}} \geq 1$, so $\dim K_\Pi \geq 1$. Since B_Π is an hyperplane section of K_Π, it follows that the dimension decreases at most by 1, so B_Π is not empty. By the same argument we have that if the mobility is greater than, or equal to 2, then Π admits infinitely many bonds.

Before coming to the main results of this section, recall that at the end of Subsection 2.3 we associated to each inversion, butterfly and similarity points a pair of directions in S^2, called the left and right vector of the boundary point.

Definition 3.9. Given a unit vector $\varepsilon \in S^2$, we denote by $\pi_\varepsilon : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ the *orthogonal projection along* ε, namely for every $p = (a, b, c) \in \mathbb{R}^3$, the point $\pi_\varepsilon(p)$ is the orthogonal projection of p on the affine plane orthogonal to ε, passing through the origin.

We are ready to state and prove the main results of this section.

Theorem 3.10. *There is a one-to-one correspondence between inversion/similarity points β with both left and right vectors L and R equal to the South pole $(0, 0, -1) \in S^2$ and inversions/similarities κ of the plane such that for any pair of points (p, P) in $\mathbb{R}^3$ the pseudo spherical condition for (p, P) at β is equivalent to the fact that $\kappa(q) = Q$ where $q = \pi_L(p)$ and $Q = \pi_R(P)$.*

Proof. We first treat the case of inversion points. Suppose that $\beta_0 \in B$ is an inversion point with $L = R = (0, 0, -1)$. Suppose furthermore that β_0 is in the normal form (see Subsection 2.3.2):

$$\beta_0 \;=\; (0 : \underbrace{1 : i : 0 : i : -1 : 0 : 0 : 0 : 0}_{M} : \underbrace{0 : 0 : 0}_{x} : \underbrace{0 : 0 : 0}_{y} : r),$$

with $r \in \mathbb{R}_{>0}$. We get that $\pi_L = \pi_R$ are the projection on the first two coordinates. Thus if $p = (a, b, c)$ and $P = (A, B, C)$, then $q = (a, b)$ and $Q = (A, B)$. If we instantiate the pseudo spherical condition for (p, P) at β_0 given by Equation (5) we get the relations:

$$\begin{cases} aA - bB = r/2 \\ bA + aB = 0 \end{cases} \tag{6}$$

which define an inversion κ_0 mapping q to Q. Conversely, suppose we are given an inversion κ_0 described by Equation (6). Then going backwards in the previous argument we can see that we obtain an inversion point in normal form as in the thesis.

Suppose now that the $\beta \in B$ is an inversion point with $L = R = (0, 0, -1)$, but not necessarily in normal form. Then, as we saw in Subsection 2.3.2, we can find two isometries $\sigma_1, \sigma_2 \in \mathrm{SE}_3$ which fix left and right vectors such that $\sigma_1 \beta \sigma_2 = \beta_0$ is in normal form (here $\sigma_1 \beta \sigma_2$ denotes the element of X obtained by left action by σ_1 on β and then by right action of σ_2). Moreover σ_1 and σ_2 induce isometries τ_1 and τ_2 of $\mathbb{R}^2$ such that the following two diagrams commute:

$$\begin{tikzcd} \mathbb{R}^3 \arrow[r, "\sigma_1"] \arrow[d, "\pi_L"'] & \mathbb{R}^3 \arrow[d, "\pi_L"] \\ \mathbb{R}^2 \arrow[r, "\tau_1"] & \mathbb{R}^2 \end{tikzcd} \qquad \begin{tikzcd} \mathbb{R}^3 \arrow[r, "\sigma_2"] \arrow[d, "\pi_R"'] & \mathbb{R}^3 \arrow[d, "\pi_R"] \\ \mathbb{R}^2 \arrow[r, "\tau_2"] & \mathbb{R}^2 \end{tikzcd}$$

If κ_0 is the inversion associated to β_0, then we define $\kappa = \tau_1 \kappa_0 \tau_2$, and one can check that the association $\beta \leftrightarrow \kappa$ satisfies the requirements of the thesis.

We consider now the case of similarity points. Suppose that $\beta_0 \in B$ is a similarity point with $L = R = (0, 0, -1)$. Suppose furthermore that β_0 is in the normal form (see Subsection 2.3.4):

$$\beta = (0 : \underbrace{0 : 0 : 0 : 0 : 0 : 0 : 0 : 0 : 0}_{M} : \underbrace{\gamma : i\gamma : 0}_{x} : \underbrace{1 : i : 0}_{y} : 0),$$

with $\gamma \in \mathbb{R}_{>0}$. Again for this kind of points π_L and π_R are both the projection on the first two coordinates. Performing analogous computations as before we get the relations:

$$\begin{cases} A = -\gamma\, a \\ B = -\gamma\, b \end{cases} \tag{7}$$

These define a similarity κ_0 mapping q to Q. Conversely, and in the general case of points not in normal form, we argue as for inversion points. □

Remark 3.11. As pointed out in Subsection 2.3.2, the complex dimension of the set of inversion points is 5. Theorem 3.10 allows, as remarked in the Introduction, to associate to it a real dimension, which can be computed as follows: 2 degrees of freedom for choosing the vector L and the same for R, and 6 degrees of freedom for specifying an inversion from $\mathbb{R}^2$ to itself. So, in total, we get 10. We can argue analogously for similarity points: here the real dimension is 8.

Corollary 3.12. *Assume that $\beta \in B_\Pi$ is an inversion/similarity bond of Π. Let $L, R \in S^2$ be the left and right vector of β. For $i = 1, \ldots, n$, set $q_i = \pi_L(p_i)$ and $Q_i = \pi_R(P_i)$. Then there is an inversion/similarity of $\mathbb{R}^2$ mapping $q_1, \ldots, q_n$ to $Q_1, \ldots, Q_n$.*

Conversely, let $L, R \in S^2$ be two unit vectors such that the images of $(p_1, \ldots, p_n)$ under π_L and of $(P_1, \ldots, P_n)$ under π_R differ by an inversion/ similarity. Then Π has an inversion/similarity bond with left vector L and right vector R.

Proof. In both cases of inversion and similarity points we can apply suitable isometries in order to put β in normal form. Then it is enough to apply Theorem 3.10. $\square$

Theorem 3.13. *There is a one-to-one correspondence between butterfly points β and pairs (g_L, g_R) of oriented lines in $\mathbb{R}^3$ such that for any pair of points (p, P) in $\mathbb{R}^3$ the pseudo spherical condition for (p, P) at β is equivalent to the fact that $p \in g_L$ or $P \in g_R$.*

Proof. Suppose that $\beta_0 \in B$ is a butterfly point in the normal form (see Subsection 2.3.3):

$$\beta_0 = (0 : \underbrace{1 : i : 0 : i : -1 : 0 : 0 : 0 : 0}_{M} : \underbrace{0 : 0 : 0}_{x} : \underbrace{0 : 0 : 0}_{y} : 0).$$

In this case we associate to β_0 the lines $g_L = g_R = \{z\text{−axis}\}$, both oriented to the South pole $(0, 0, -1) \in S^2$. If we instantiate the pseudo spherical condition for (p, P) at β given by Equation (5) we get the relations:

$$\left\{ \begin{array}{l} aA - bB = 0 \\ aB + bA = 0 \end{array} \right. \tag{8}$$

Equation (8) can be interpreted as: the vector (a, b) is parallel both to the vector $(A, -B)$ and to the vector (B, A). This is possible if and only if either $(a, b) = (0, 0)$ or $(A, B) = (0, 0)$. Hence either p is of the form $(0, 0, c)$ (namely it lies on g_L) or P is of the form $(0, 0, C)$ (namely it lies on g_R).

If $\beta \in B$ is an arbitrary butterfly point, then from Subsection 2.3.3 we know that there exist isometries $\sigma_1, \sigma_2 \in \mathrm{SE}_3$ such that $\sigma_1 \beta \sigma_2 = \beta_0$ is in normal form. Then we associate to β the pair of lines

$$(g_L, g_R) = \Big((\sigma_1)^{-1} \big(\{z\text{−axis}\}\big), (\sigma_2)^{-1} \big(\{z\text{−axis}\}\big) \Big)$$

with orientation given by the left and right vectors of β. One can check that the equivalence in the thesis holds. Conversely, given two oriented lines g_L and g_R we can find isometries $\sigma_1, \sigma_2 \in \mathrm{SE}_3$ such that $g_L = \sigma_1\big(\{z\text{−axis}\}\big)$ and $g_R = \sigma_2\big(\{z\text{−axis}\}\big)$, both oriented to the South pole $(0, 0, -1) \in S^2$. Then we associate to (g_L, g_R) the butterfly point $\sigma_1 \beta \sigma_2$. $\square$

Corollary 3.14. *Assume that $\beta \in B_\Pi$ is a butterfly bond of Π. Let $L, R \in S^2$ be the left and right vector of β. Then, up to permutation of indices $1, \ldots, n$, there exists $m \le n$ such that $p_1, \ldots, p_m$ are collinear on a line parallel to L, and $P_{m+1}, \ldots, P_n$ are collinear on a line parallel to R.*

Conversely, let $L, R \in S^2$ be two unit vectors such that $p_1, \ldots, p_m$ are collinear on a line parallel to L, and $P_{m+1}, \ldots, P_n$ are collinear on a line parallel to R. Then Π has a butterfly bond with left vector L and right vector R.

Notation. Recall from Subsection 2.3.5 that the set of collinearity points is partitioned into two subsets: if the y–coordinate of a collinearity point is zero we call it a *left collinearity point*, while if the x–coordinate is zero we call it a *right collinearity point*.

Theorem 3.15. *There is a one-to-one correspondence between left (resp. right) collinearity points β and oriented lines g in $\mathbb{R}^3$ such that for any pair of points (p, P) in $\mathbb{R}^3$ the pseudo spherical condition for (p, P) at β is equivalent to the fact that $p \in g$ (resp. $P \in g$).*

Proof. Suppose that $\beta_0 \in B$ is a left collinearity point and suppose that it is in normal form (see Subsection 2.3.5):

$$\beta = (0 : \underbrace{0 : 0 : 0 : 0 : 0 : 0 : 0 : 0 : 0}_{M} : \underbrace{1 : i : 0}_{x} : \underbrace{0 : 0 : 0}_{y} : 0),$$

We associate to β_0 the line $g = \{z\text{−axis}\}$, directed to the South pole $(0, 0, -1) \in S^2$. If we instantiate the pseudo spherical condition for (p, P) at β_0 given by Equation (5) we get the relations:

$$0 \;=\; -2(a + ib) \quad \Leftrightarrow \quad a = b = 0$$

which is equivalent to $p \in g$.

If $\beta \in B$ is an arbitrary left collinearity point we proceed as in the proof of Theorem 3.13. Analogous arguments prove the statement about right collinearity points. □

Corollary 3.16. *Assume that $\beta \in B_\Pi$ is a collinearity bond of Π. Then either $p_1, \ldots, p_n$ are collinear or $P_1, \ldots, P_n$ are collinear (or both).*

Conversely, if $p_1, \ldots, p_n$ are collinear or $P_1, \ldots, P_n$ are collinear (or both), then Π has a collinearity bond.

The following Corollary gives a necessary criterion for mobility of n–pods. For the fist time (to the authors' knowledge) a necessary criterion for the mobility of hexapods can be defined by the invariant linkage parameters, irrespective of a specific configuration. (The well-known criterion for infinitesimal mobility, see [9], refers to an explicit relative pose of the platform with respect to the base.)

Corollary 3.17. *If an n–pod is mobile, then one of the following conditions holds:*

- *(i) There exists at least one pair of orthogonal projections π_L and π_R such that the projections of the platform points $p_1, \ldots, p_n$ by π_L and of the base points $P_1, \ldots, P_n$ by π_R differ by an inversion or a similarity.*
- *(ii) There exists $m \leq n$ such that $p_1, \ldots, p_m$ are collinear and $P_{m+1}, \ldots, P_n$ are collinear, up to permutation of indices.*

Proof. Since by hypothesis $K_\Pi \cap \mathrm{SE}_{3,\mathbb{C}}$ has dimension ≥ 1, it follows that B_Π is not empty (see Remark 3.8). Hence there is at least one inversion/similarity/collinearity/butterfly bond, and then the result follows from Corollaries (3.12), (3.14) and (3.16). □

Remark 3.18. As the bonds are determined by the invariant linkage parameters, they are independent of the leg lengths. As a consequence a hexapod, which has 40 solutions for the direct kinematics over $\mathbb{C}$ (see Remark 3.3), is free of bonds and therefore also free of self-motions. Due to the fact that condition (i) of Corollary 3.17 is not sufficient, the converse is not true; i.e. there exist hexapods with less than 40 solutions for the direct kinematics, which are free of self-motions (e.g. hexapods where the platform and base are planar and projective — but not affine — equivalent [11]).

We conclude stating our last result, concerning constraints on base and platform points of n–pods with higher mobility.

Theorem 3.19. *Let Π be an n–pod with mobility 2 or higher. Then one of the following holds:*

(a) *there are infinitely many pair (L, R) of elements of S^2 such that the points $\pi_L(p_1), \ldots, \pi_L(p_n)$ and $\pi_R(P_1), \ldots, \pi_R(P_n)$ differ by an inversion or a similarity;*

(b) *there exists $m \leq n$ such that $p_1, \ldots, p_m$ are collinear and $P_{m+1} = \ldots = P_n$, up to permutation of indices and interchange between base and platform;*

(c) *there exists $m \leq n$ with $1 < m < n-1$ such that $p_1, \ldots, p_m$ lie on a line $g \subseteq \mathbb{R}^3$ and $p_{m+1}, \ldots, p_n$ lie on a line $g' \subseteq \mathbb{R}^3$ parallel to g, and $P_1, \ldots, P_m$ lie on a line $G \subseteq \mathbb{R}^3$ and $P_{m+1}, \ldots, P_n$ lie on a line $G' \subseteq \mathbb{R}^3$ parallel to G, up to permutation of indices.*

Proof. Since Π has mobility at least 2, it has infinitely many bonds (see Remark 3.8). Assume that Π admits one collinearity bond, then we have b) with $m = n$. Assume that it admits infinitely many butterfly points, then in particular by Corollary 3.14 there exists $m \leq n$ such that $p_1, \ldots, p_m$ are collinear and $P_{m+1}, \ldots, P_n$ lie on infinitely many different lines, and therefore we have (b). Hence we are left with the case when we have infinitely many inversion or similarity bonds. If these bonds provide infinitely many different left and right vectors, we are in case (a). Otherwise we have that there are infinitely many inversion or similarity points with the same left and right vectors (L, R). We want to argue that in this case both sets $\mathcal{U} = \{\pi_L(p_1), \ldots, \pi_L(p_n)\}$ and $\mathcal{V} = \{\pi_R(P_1), \ldots, \pi_R(P_n)\}$ consist of two points. In fact by Corollary 3.12 the inversion/similarity associated to these bonds maps $\pi_L(p_i)$ to $\pi_R(P_i)$, so $\mathcal{U}$ and $\mathcal{V}$ have the same cardinality; on the other hand any inversion or similarity is completely specified if we prescribe the image of three points, so if the cardinality of $\mathcal{U}$ were greater than 2 then we would have only one inversion or similarity. Moreover we can exclude the case when both $\mathcal{U}$ and $\mathcal{V}$ are given by one point, since this falls in case (b). Hence $p_1, \ldots, p_n$ are arranged on two parallel lines, and the same holds for $P_1, \ldots, P_n$. From this and the fact that the inversions/similarities should map $\pi_L(p_i)$ to $\pi_R(P_i)$ it follows that the only possible configurations are the ones described in (c). As a side remark, since two points fix a similarity it follows that in this case we have just one similarity point and infinitely many inversion points. □

As already mentioned in the Introduction, we can also formulate some geometric conditions on base and platform points in case (a) of Theorem 3.19. This can be done by a new technique, called *Möbius Photogrammetry*, which is developed by the authors in [4]. Moreover it should be noted that based on the results obtained with this method a complete classification of pentapods with mobility 2 was achieved in [12, 13].

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