CONGRUENCES MODULO SQUARES OF PRIMES FOR FU'S k DOTS BRACELET PARTITIONS

CRISTIAN-SILVIU RADU AND JAMES A. SELLERS

Dedicated to George Andrews on the occasion of his 75^{th} *birthday*

ABSTRACT. In 2007, Andrews and Paule introduced the family of functions $\Delta_k(n)$ which enumerate the number of broken k–diamond partitions for a fixed positive integer k. In that paper, Andrews and Paule proved that, for all $n \geq 0$, $\Delta_1(2n+1) \equiv 0 \pmod 3$ using a standard generating function argument. Soon after, Shishuo Fu provided a combinatorial proof of this same congruence. Fu also utilized this combinatorial approach to naturally define a generalization of broken k–diamond partitions which he called *k dots bracelet partitions*. He denoted the number of k dots bracelet partitions of n by $\mathfrak{B}_k(n)$ and proved various congruence properties for these functions modulo primes and modulo powers of 2. In this note, we extend the set of congruences proven by Fu by proving the following congruences: For all $n \geq 0$,

$$\begin{aligned} \mathfrak{B}_5(10n+7) &\equiv 0 \pmod{5^2}, \\ \mathfrak{B}_7(14n+11) &\equiv 0 \pmod{7^2}, \quad \text{and} \\ \mathfrak{B}_{11}(22n+21) &\equiv 0 \pmod{11^2} \end{aligned}$$

We also conjecture an infinite family of congruences modulo powers of 7 which are satisfied by the function $\mathfrak{B}_7$.

1. INTRODUCTION

Broken k-diamond partitions were introduced in 2007 by Andrews and Paule [1]. These are constructed in such a way that the generating functions of their counting sequences $(\Delta_k(n))_{n\geq 0}$ are closely related to modular forms. Namely,

$$\begin{aligned} \sum_{n=0}^{\infty} \Delta_k(n) q^n &= \prod_{n=1}^{\infty} \frac{(1-q^{2n})(1-q^{(2k+1)n})}{(1-q^n)^3(1-q^{(4k+2)n})} \\ &= q^{(k+1)/12} \frac{\eta(2\tau)\eta((2k+1)\tau)}{\eta(\tau)^3\eta((4k+2)\tau)}, \quad k \geq 1, \end{aligned}$$

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where we recall the Dedekind eta function

$$\eta(\tau) := q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n) \quad (q = e^{2\pi i \tau}). \tag{1.1}$$

In their original work, Andrews and Paule proved that, for all $n \geq 0$,

$$\Delta_1(2n+1) \equiv 0 \pmod{3} \tag{1.2}$$

by utilizing generating function manipulations. Soon after, Hirschhorn and Sellers [4] reproved (1.2) by finding an explicit representation of the generating function for $\Delta_1(2n+1)$ which implied (1.2), and Mortenson [5] developed a statistic on the partitions enumerated by $\Delta_1(2n+1)$ which naturally breaks these partitions into three subsets of equal size (thus proving (1.2) combinatorially).

More recently, Shishuo Fu [2] proved (1.2) via a combinatorial argument as well. In the process, he generalized the notion of broken k-diamond partitions to combinatorial objects which he termed *k dots bracelet partitions*. Fu [2] denoted the number of k dots bracelet partitions of n by $\mathfrak{B}_k(n)$. He then proved the following congruence properties satisfied by these functions (the first of which Fu termed "a natural generalization of" (1.2)).

Theorem 1.1. *For $n \geq 0$, $k \geq 3$, if $k = p^r$ is a prime power, then*

$$\mathfrak{B}_k(2n+1) \equiv 0 \pmod{p}.$$

Theorem 1.2. *For any $k \geq 3$, s an integer between 1 and $p-1$ such that $12s+1$ is a quadratic nonresidue modulo p, and any $n \geq 0$, if $p \mid k$ for some prime $p \geq 5$, then*

$$\mathfrak{B}_k(pn+s) \equiv 0 \pmod{p}.$$

Theorem 1.3. *For $n \geq 0$, $k \geq 3$ even, say $k = 2^m l$, where l is odd, we have*

$$\mathfrak{B}_k(2n+1) \equiv 0 \pmod{2^m}.$$

Our primary goal in this brief note is to prove the following theorem, thus extending the set of congruences mentioned above for k dots bracelet partitions.

Theorem 1.4. *For all $n \geq 0$,*

$$\begin{aligned}
\mathfrak{B}_5(10n+7) &\equiv 0 \pmod{5^2}, \\
\mathfrak{B}_7(14n+11) &\equiv 0 \pmod{7^2}, \quad \textit{and} \\
\mathfrak{B}_{11}(22n+21) &\equiv 0 \pmod{11^2}.
\end{aligned}$$

2. Proof of Theorem 1.4

For $p = 5, 7, 11$, let

$$F_p(\tau) := \eta(2p\tau)^{12-p}\eta(2\tau) \times \left(\frac{\eta^p(\tau)}{\eta(p\tau)}\right)^{p-1}.$$

We will see below that this is a natural choice since the generating function for $\mathfrak{B}_p(n)$ is given by

$$\sum_{n=0}^{\infty} \mathcal{B}_p(n)q^n = \prod_{n=1}^{\infty} \frac{(1-q^{2n})(1-q^{pn})}{(1-q^n)^p(1-q^{2pn})}.$$

We observe that

$$F_p(\tau) = q^{p+\frac{1-p^2}{12}} \prod_{n=1}^{\infty}(1-q^{2pn})^{12-p}(1-q^{2n})\left(\frac{(1-q^n)^p}{1-q^{pn}}\right)^{p-1}.$$

Set

$$\sum_{n=0}^{\infty} a_p(n)q^n := \prod_{n=1}^{\infty}(1-q^{2pn})^{12-p}(1-q^{2n})\left(\frac{(1-q^n)^p}{1-q^{pn}}\right)^{p-1}.$$

Then

$$\begin{aligned} U_{2p}(F_p(\tau)) &= U_{2p}\left(q^{p+\frac{1-p^2}{12}} \sum_{n=0}^{\infty} a_p(n)q^n\right) \\ &= \sum_{n=0}^{\infty} a_p\left(2pn + p + \frac{p^2-1}{12}\right) \end{aligned}$$

where U_{2p} is the "standard" U-operator [6, p. 28]. From Ono [6, Theorems 1.64 and 1.65] we find that $\left(\frac{\eta^p(\tau)}{\eta(p\tau)}\right)^{p-1}$ is a modular form for the group $\Gamma_0(p)$ of weight $(p-1)^2/2$. Similarly, we find that $\eta(2p\tau)^{12-p}\eta(2\tau)$ is a modular form of weight $k(p) := (13-p)/2$ with character $\chi_p(d) := \left(\frac{(-1)^{k(p)}p}{d}\right)$ for the group $\Gamma_0(4p)$. Consequently, $F_p(\tau)$ is a modular form of weight $w_p := (13-p)/2 + (p-1)^2/2$ and character $\chi_p(d)$ for the group $\Gamma_0(4p)$. Then because of [6, Prop. 2.22] also $U_{2p}(F_p(\tau))$ is a modular form of weight w_p and character $\chi_p(d)$ for the group $\Gamma_0(4p)$. Using a variant of Sturm's theorem (see Ono [6, Theorem 2.58]) we find that

$$\sum_{n=0}^{\infty} a_p\left(2pn + p + \frac{p^2-1}{12}\right) = U_{2p}(F_p(\tau)) \equiv 0 \pmod{p^2}$$

iff

$$a_p\left(2pn + p + \frac{p^2-1}{12}\right) \equiv 0 \pmod{p^2}$$

for the finite sequence of values $n = 0, 1, \ldots, \frac{w_p}{24}[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(4p)]$ or equivalently, using the notation of [6], $\mathrm{ord}_{\mathfrak{m}}(U_{2p}(F_p(\tau)) > \frac{w_p}{24}[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(4p)]$ with $\mathfrak{m} := \{i \in \mathbb{Z} : p^2|i\} = \langle p^2 \rangle$. Using Ono [6, Proposition 1.7], we find that

$$\frac{w_p}{24}[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(4p)] = \frac{p+1}{8}(13 - p + (p-1)^2).$$

We have verified that this finite set of congruences hold, and therefore

$$U_{2p}(F_p(\tau)) \equiv 0 \pmod{p^2}$$

for $p = 5, 7, 11$.

Next note that

$$U_{2p}(F_p(\tau)) \equiv 0 \pmod{p^2}$$

implies that

$$\prod_{n=1}^{\infty}(1-q^n)^{p-13}U_{2p}(F_p(\tau)) \equiv 0 \pmod{p^2}.$$

However,

$$\prod_{n=1}^{\infty}(1-q^n)^{p-13}U_{2p}(F_p(\tau)) = U_{2p}\left(\prod_{n=1}^{\infty}(1-q^{2pn})^{p-13}F_p(\tau)\right)$$

and

$$U_{2p}\left(\prod_{n=1}^{\infty}(1-q^{2pn})^{p-13}F_p(\tau)\right) \equiv U_{2p}\left(\prod_{n=1}^{\infty}(1-q^{2pn})^{p-13}F_p(\tau)\left(\frac{(1-q^n)^p}{1-q^{pn}}\right)^{-p}\right) \pmod{p^2}.$$

This implies that

$$U_{2p}\left(\prod_{n=1}^{\infty}(1-q^{2pn})^{p-13}F_p(\tau)\prod_{n=1}^{\infty}\left(\frac{(1-q^n)^p}{1-q^{pn}}\right)^{-p}\right) \equiv 0 \pmod{p^2}.$$

From the definition of $F_p(\tau)$ we know

$$\begin{aligned}\prod_{n=1}^{\infty}(1-q^{2pn})^{p-13}F_p(\tau)\prod_{n=1}^{\infty}\left(\frac{(1-q^n)^p}{1-q^{pn}}\right)^{-p} &= q^{p+\frac{1-p^2}{12}}\prod_{n=1}^{\infty}\frac{(1-q^{2n})(1-q^{pn})}{(1-q^n)^p(1-q^{2pn})} \\ &= q^{p+\frac{1-p^2}{12}}\sum_{n=0}^{\infty}\mathcal{B}_p(n)q^n.\end{aligned}$$

Hence,

$$U_{2p}\left(q^{p+\frac{1-p^2}{12}}\sum_{n=0}^{\infty}\mathcal{B}_p(n)q^n\right) = \sum_{n=0}^{\infty}\mathcal{B}_p\left(2pn+p+\frac{p^2-1}{12}\right) \equiv 0 \pmod{p^2}.$$

This completes the proof of Theorem 1.4.

3. Concluding Remarks

We close with two comments. First, given the combinatorial genesis of the definition of $\mathfrak{B}_k(n)$, it would be nice to have a combinatorial proof of Theorem 1.4. Secondly, we state the following conjectured infinite family of congruences:

Conjecture: For all $n \geq 0$ and all $\alpha \geq 1$,

$$\mathfrak{B}_7(7^\alpha n + \lambda_\alpha) \equiv 0 \pmod{7^{\lceil \frac{\alpha-1}{2} \rceil}}$$

where $\lambda_\alpha = \frac{1+7^\alpha}{2}$. This is an intriguing family of congruences given the similarity to the infinite family of congruences modulo powers of 7 which holds for the ordinary partition function $p(n)$ (as was originally proved by Watson [7] in 1938 and later proved in a more elementary fashion by Garvan [3]).

References

[1] G. E. Andrews and P. Paule. MacMahon's Partition Analysis XI: Broken Diamonds and Modular forms. *Acta Arithmetica*, 126:281–294, 2007.

[2] S. Fu. Combinatorial Proof of One Congruence for the Broken 1–Diamond Partition and a Generalization. *Int. J. of Number Theory*, 7(1):133–144, 2011.

[3] F. Garvan. A simple proof of watson's partition congruences for powers of 7. *J. Austral. Math. Soc. Ser. A*, 36(3):316–334, 1984.

[4] M. D. Hirschhorn and J. A. Sellers. On Recent Congruence Results of Andrews and Paule. *Bulletin of the Australian Mathematical Society*, 75:121–126, 2007.

[5] E. Mortenson. On the broken 1-diamond partition. *Int. J. Number Theory*, 4(2):199–218, 2008.

[6] K. Ono. *The Web of Modularity: Arithmetic of the Coefficientsof the Modular Forms and q-series.* Number 102 in CBMS Regional Conference Series in Mathematics. AMS, 2004.

[7] G. N. Watson. Ramanujan's Vermutung über Zerfällungsanzahlen. *Journal für die Reine und Angewandte Mathematik*, 179:97–128, 1938.

Research Institute for Symbolic Computation (RISC), Johannes Kepler University, A-4040 Linz, Austria, sradu@risc.uni-linz.ac.at

Department of Mathematics, Penn State University, University Park, PA 16802, USA, sellersj@psu.edu