An Algorithmic Approach to Ramanujan-Kolberg Identities

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Abstract

Let M be a given positive integer and $r = (r_\delta)_{\delta|M}$ a sequence indexed by the positive divisors δ of M. In this paper we present an algorithm that takes as input a generating function of the form $\sum_{n=0}^{\infty} a_r(n)q^n := \prod_{\delta|M}\prod_{n=1}^{\infty}(1-q^{\delta n})^{r_\delta}$ and positive integers m, N and $t \in \{0, \ldots, m-1\}$. Given this data we compute a set $P_{m,r}(t)$ which contains t and is uniquely defined by m, r and t. Next we decide if there exists a sequence $(s_\delta)_{\delta|N}$ indexed by the positive divisors δ of N, and modular functions $b_1, \ldots, b_k$ on $\Gamma_0(N)$ (where each b_j equals the product of finitely many terms from $\{q^{\delta/24}\prod_{n=1}^{\infty}(1-q^{\delta n}) : \delta|N\}$), such that:

$$q^\alpha \prod_{\delta|N}\prod_{n=1}^{\infty}(1-q^{\delta n})^{s_\delta} \times \prod_{t' \in P_{m,r}(t)} \sum_{n=0}^{\infty} a(mn+t')q^n = c_1 b_1 + \cdots + c_k b_k$$

for some $c_1, \cdots, c_k \in \mathbb{Q}$ and $\alpha := \frac{\sum_{\delta|N} \delta s_\delta}{24} + \sum_{t' \in P_{m,r}(t)} \frac{24t' + \sum_{\delta|M} \delta r_\delta}{24m}$. Our algorithm builds on work by Rademacher [9], Newman [7], and Kolberg [5].

Key words: partition identities, number theoretic algorithm, modular forms

Introduction and Basic Notions

Let $p(n)$ denote the number of partitions of n. Ramanujan [12] discovered that for all $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$:

$$p(5n+4) \equiv 0 \pmod 5, \tag{1}$$
$$p(7n+5) \equiv 0 \pmod 7, \tag{2}$$
$$p(11n+6) \equiv 0 \pmod{11}. \tag{3}$$

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Congruence (1) follows from Ramanujan's "most beautiful" identity (cf. Hardy [14, xxi-xxxvi]) eq. (17) in [11]:

$$\sum_{m=0}^{\infty} p(5m+4)q^m = 5\prod_{n=1}^{\infty} \frac{(1-q^{5n})^5}{(1-q^n)^6}. \tag{4}$$

Congruence (2) follows from another identity of Ramanujan, eq. (18) in [11]:

$$\sum_{m=0}^{\infty} p(7m+5)q^m = 7\prod_{n=1}^{\infty} \frac{(1-q^{7n})^3}{(1-q^n)^4} + 49q\prod_{n=1}^{\infty} \frac{(1-q^{7n})^7}{(1-q^n)^8}. \tag{5}$$

Our algorithm finds such identities automatically. To the best of our knowledge, so far there exists no identity similar to (4) and (5) from which (3) follows. But our algorithm enables an automatic derivation of such an identity; see Section 4.

For $N \in \mathbb{N}^* = \{1, 2, \ldots\}$, we define $R(N)$ to be the set of integer tuples $(r_\delta)_{\delta|N}$ indexed by the positive divisors δ of N.

For $r = (r_\delta)_{\delta|N} \in R(N)$ we define

$$w(r) := \sum_{\delta|N} r_\delta, \quad \sigma_\infty(r) := \sum_{\delta|N} \delta r_\delta, \quad \sigma_0(r) := \sum_{\delta|N} \frac{N}{\delta} r_\delta,$$

and

$$\Pi(r) := \prod_{\delta|N} \delta^{|r_\delta|}.$$

We define for $i, j, k, l \in \mathbb{N}$,

$$R(N,i,j,k,l) := \left\{ r \in R(N) : \begin{array}{lcll} w(r) & = & i, & \\ \sigma_\infty(r) & \equiv & j & \pmod{24}, \\ \sigma_0(r) & \equiv & k & \pmod{24}, \\ \sqrt{l\,\Pi(r)} & \in & \mathbb{N}. & \end{array} \right\}$$

and

$$R^*(N) := R(N, 0, 0, 0, 1).$$

As usual, we denote by $\eta(\tau)$ the Dedekind eta function for which

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1-q^n), \quad (q = e^{2\pi i \tau}). \tag{6}$$

Define a set of eta quotients by

$$E(N) := \left\{ q^{\frac{1}{24}\sigma_\infty(r)} \prod_{\delta|N} \prod_{n=1}^{\infty} (1-q^{\delta n})^{r_\delta} : (r_\delta)_{\delta|N} \in R^*(N) \right\}.$$

Note that

$$E(N) = \left\{ \prod_{\delta|N} \eta^{r_\delta}(\delta\tau) : (r_\delta)_{\delta|N} \in R^*(N) \right\}.$$

Denote by $\langle E(N) \rangle_{\mathbb{Q}}$ the vector space over $\mathbb{Q}$ generated by the elements of $E(N)$. Note that the constant functions are elements of this space i.e.; $1 \in \langle E(N) \rangle_{\mathbb{Q}}$. For $r \in R(N)$

we define the sequence $(a_r(n))_{n\geq 0}$ of integers by:

$$\sum_{n=0}^{\infty} a_r(n)q^n = \prod_{\delta|N}\prod_{n=1}^{\infty}(1-q^{\delta n})^{r_\delta}.$$

We call (A,B) a Ramanujan identity, if there exist $M,N\in\mathbb{N}^*$, $r=(r_\delta)_{\delta|M}\in R(M)$, $s=(s_\delta)_{\delta|N}\in R(N)$, integers $m>t\geq 0$ with $24t+\sigma_\infty(r)\equiv 0 \pmod{m}$ such that:

- $A = q^{\frac{\sigma_\infty(s)}{24}+\frac{24t+\sigma_\infty(r)}{24m}}\prod_{\delta|N}\prod_{n=1}^{\infty}(1-q^{\delta n})^{s_\delta}\sum_{n=0}^{\infty}a_r(mn+t)q^n$;
- $B\in\langle E(N)\rangle_{\mathbb{Q}}$;
- $A=B$.

With this notation it is easily seen that after multiplying (4) by $q\prod_{n=1}^{\infty}(1-q^{5n})$ on both sides, we obtain a Ramanujan identity. Similarly, multiplying (5) by $q\prod_{n=1}^{\infty}(1-q^{7n})$ we obtain a Ramanujan identity. We will present in Section 4 a Ramanujan identity (A,B) with

$$A = q^{-14}\prod_{n=1}^{\infty}\frac{(1-q^n)^{10}(1-q^{2n})^2(1-q^{11n})^{11}}{(1-q^{22n})^{22}}\sum_{n=0}^{\infty}p(11n+6)q^n,$$

where $s=(10,2,11,-22)\in R(22)$ and $r=(-1)\in R(1)$.

Kolberg [5] proved that

$$\begin{aligned}\Big(\sum_{m=0}^{\infty}p(5m+1)q^m\Big)\Big(\sum_{m=0}^{\infty}p(5m+2)q^m\Big)\\ = 2\prod_{n=1}^{\infty}\frac{(1-q^{5n})^4}{(1-q^n)^6}+25q\prod_{n=1}^{\infty}\frac{(1-q^{5n})^{10}}{(1-q^n)^{12}},\end{aligned}\tag{7}$$

and

$$\begin{aligned}\Big(\sum_{m=0}^{\infty}p(5m)q^m\Big)\Big(\sum_{m=0}^{\infty}p(5m+3)q^m\Big)\\ =3\prod_{n=1}^{\infty}\frac{(1-q^{5n})^4}{(1-q^n)^6}+25q\prod_{n=1}^{\infty}\frac{(1-q^{5n})^{10}}{(1-q^n)^{12}}.\end{aligned}\tag{8}$$

We call (A,B) a Ramanujan-Kolberg identity, if there exist $M,N\in\mathbb{N}^*$, $r=(r_\delta)_{\delta|M}\in R(M)$, $s=(s_\delta)_{\delta|N}\in R(N)$ and integers $m>t\geq 0$ such that

- A has the form:

$$A = q^{\alpha(m,t,r,s)}\prod_{\delta|N}\prod_{n=1}^{\infty}(1-q^{\delta n})^{s_\delta}\prod_{t'\in P_{m,r}(t)}\sum_{n=0}^{\infty}a_r(mn+t)q^n,\tag{9}$$

where, with the definition of the set $P_{m,r}(t)$ as in (32),

$$\alpha(m,t,r,s) := \frac{\sigma_\infty(s)}{24}+\sum_{t'\in P_{m,r}(t)}\frac{24t'+\sigma_\infty(r)}{24m};$$

- $B\in\langle E(N)\rangle_{\mathbb{Q}}$;
- $A=B$.

One may easily verify that (7) and (8) can be viewed as Ramanujan-Kolberg identities. As we will see from definition (32), a Ramanujan identity is a special case of a Ramanujan-Kolberg identity.

In this paper we present an algorithm that takes as input $M, N \in \mathbb{N}^*$, $r \in R(M)$ and integers $m > t \geq 0$. If there exist $s \in R(N)$ and $B \in \langle E(N)\rangle_{\mathbb{Q}}$ such that $A = B$ where A, defined as in (9), is a modular function for $\Gamma_0(N)$ (see Definition 1), then the algorithm output will be (s, B). If no such (s, B) exists, then the algorithm will stop and return no output.

The organization of this article is as follows: In Section 1 we present the general notions needed in the paper. In Section 2 we describe an algorithm that computes a module basis for the ring generated by some given modular functions having only poles at infinity. In Section 3 we present our main theorem and the algorithm that solves the problem described above. Furthermore, at the end of Section 3 we give a counterexample to a conjecture by Morris Newman related to eta quotients. In Section 4 we give examples of infinite product identities that can be found and proved with the algorithm in this paper. In Section 5 we describe an algorithm, obtained as a by-product, that computes all relations among given eta quotients $m_1, \ldots, m_s \in E(N)$.

1. Further Preliminaries and Definitions

We need the groups

$$\mathrm{SL}_2(\mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\},$$

$$\mathrm{SL}_2(\mathbb{Z})_\infty := \left\{ \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \mid h \in \mathbb{Z} \right\},$$

and for $N \in \mathbb{N}^*$,

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid N|c \right\};$$

in addition, we define the subset

$$\Gamma_0(N)^* := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) | a > 0, c > 0, \gcd(a, 6) = 1 \right\}.$$

Note that $\mathrm{SL}_2(\mathbb{Z})^* = \Gamma_0(1)^*$.

Let $\mathbb{H} := \{\tau \in \mathbb{C} : \mathrm{Im}(\tau) > 0\}$. We need two standard group actions of $\mathrm{SL}_2(\mathbb{Z})$: for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ and $\tau \in \mathbb{H}$ we define $\gamma\tau := \frac{a\tau+b}{c\tau+d}$; for $f : \mathbb{H} \to \mathbb{C}$ we define $f|\gamma : \mathbb{H} \to \mathbb{C}$ by $(f|\gamma)(\tau) := f(\gamma\tau)$.

Definition 1. A modular function for the group $\Gamma_0(N)$ is a function $f : \mathbb{H} \to \mathbb{C}$ with the properties:

(i) f is holomorphic on $\mathbb{H}$;

(ii) $f|\gamma = f$ for all $\gamma \in \Gamma_0(N)$;

(iii) for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ there exists an expansion

$$(f|\gamma)(\tau) = \sum_{n=n_\gamma}^{\infty} a_\gamma(n) e^{2\pi i n\tau \gcd(c^2,N)/N}, \quad \tau \in \mathbb{H} \tag{10}$$

where $a_\gamma(n_\gamma) \neq 0$. We define $\mathrm{ord}_\gamma^N(f) := n_\gamma$. If $\mathrm{ord}_{\mathrm{id}}^N(f) < 0$, then one says f has a pole of order $-\mathrm{ord}_{\mathrm{id}}^N(f)$ at infinity.

We will write (10) in the form

$$(f|\gamma)(\tau) = \sum_{n=n_\gamma}^{\infty} a_\gamma(n) q^{n \gcd(c^2,N)/N},$$

where $q := e^{2\pi i\tau}$, and we refer to this unique Laurent expansion of $(f|\gamma)(\tau)$ in powers of $q^{\gcd(c^2,N)/N}$ as the q-expansion of $f|\gamma$.

The next fact is not difficult to prove.

Lemma 2. *For γ_1, $\gamma_2 \in \mathrm{SL}_2(\mathbb{Z})$ such that $\gamma_1 \in \Gamma_0(N)\gamma_2\mathrm{SL}_2(\mathbb{Z})_\infty$ and f a modular function for $\Gamma_0(N)$:*

$$\mathrm{ord}_{\gamma_1}^N(f) = \mathrm{ord}_{\gamma_2}^N(f).$$

Definition 3. We denote the set of all modular functions for $\Gamma_0(N)$ by $K(N)$. By $K^\infty(N)$ we denote the set of modular functions having a (multiple) pole, if any, at infinity only; this is the set of all $f \in K(N)$ such that

$$\mathrm{ord}_\gamma^N(f) \geq 0 \text{ for all } \gamma \in \mathrm{SL}_2(\mathbb{Z}) \setminus \Gamma_0(N). \tag{11}$$

Remark 4. Note that to prove that $f : \mathbb{H} \to \mathbb{C}$ is in $K(N)$, it suffices to prove *(i)*-*(ii)* of Definition 1 and the existence of a $\kappa_\gamma \in \mathbb{N}^*$ such that

$$(f|\gamma)(\tau) = \sum_{n=m_\gamma}^{\infty} b_\gamma(n) e^{\frac{2\pi i n\tau}{\kappa_\gamma}}, \tag{12}$$

for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$. This implies *(iii)*: first, observe that $(f|\gamma)(\tau + N/\gcd(c^2,N)) = (f|\gamma)(\tau)$ (which is a consequence of *(ii)*); this, together with *(i)* implies that $(f|\gamma)(\tau)$ has a Laurent expansion in powers of $q^{\frac{\gcd(c^2,N)}{N}}$. Finally, by (12) and uniqueness of Laurent expansion, this Laurent expansion has a finite principal part (as required by *(iii)*). As we will see later in the proof of our main theorem it is more convenient for us to prove that $(f|\gamma)(\tau)$ is a Laurent series in powers of q^{1/κ_γ} with finite principal part (for some κ_γ), rather than to prove that $(f|\gamma)(\tau)$ is a Laurent series in powers of $q^{\frac{\gcd(c^2,N)}{N}}$ with finite principal part.

Lemma 5. *Let $f \in K(N)$ such that $\mathrm{ord}_\gamma^N(f) \geq 0$ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. Then f is constant.*

Proof. Let $\gamma_1, \dots, \gamma_n$ (with γ_1 the identity) be a complete set of representatives of the right cosets of $\Gamma_0(N)$ in $\mathrm{SL}_2(\mathbb{Z})$. Let c_i be defined by $\gamma_i = \begin{pmatrix} * & * \\ c_i & * \end{pmatrix}$. Define $a_i(n)$ by

$$f(\gamma_i \tau) = \sum_{n=n_i}^{\infty} a_i(n) q^{n/t_i},$$

where $t_i := N/\gcd(c_i^2, N)$. Since $\mathrm{ord}_{\gamma}^{N}(f) \geq 0$ (by assumption) it follows that $n_i \geq 0$. Set $g(\tau) := f(\tau) - a_1(0)$.

Then $F(\tau) := \prod_{i=1}^{n} g(\gamma_i \tau)$ satisfies $F(\gamma\tau) = F(\tau)$ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. Furthermore, $F(\tau) = \sum_{n=1}^{\infty} a(n) q^n$, because of $\mathrm{ord}_{\gamma_1}^{N}(g) > 0$ by construction. Hence $F \in K(1)$. By [15, Ch. VII, Th. 3] it follows that if $F \neq 0$ then $\mathrm{ord}_{\mathrm{id}}^{N}(F) \leq 0$. This gives a contradiction to $\mathrm{ord}_{\mathrm{id}}^{N}(F) > 0$. Therefore $F = 0$, and consequently $g(\gamma_i \tau) = 0$ for i from 1 to n. Hence $g(\tau) = f(\tau) - a_1(0) = 0$. □

Corollary 6. *Let $f \in K^{\infty}(N)$ such that $\mathrm{ord}_{id}^{N}(f) \geq 0$, then f is constant.*

2. An Algorithm for Describing $\langle \boldsymbol{E(N)} \rangle_{\mathbb{Q}} \cap \boldsymbol{K^{\infty}(N)}$ as a Finitely Generated $\mathbb{Q}[\boldsymbol{t}]$-Module

2.1. Algorithms for Finding Bases

The functions in $K(N)$, resp. $K^{\infty}(N)$, form a commutative ring with 1. Since these rings contain all constant functions, they are also vector spaces over $\mathbb{C}$. In this section we describe an algorithm that computes a module basis for the ring generated by some given modular functions having only poles at infinity.

Definition 7. For $f \in K(N)$ we define the pole order at infinity as $\mathrm{pord}(f) := -\mathrm{ord}_{\mathrm{id}}^{N}(f)$.

Definition 8. We call the sequence $z_1, \dots, z_r \in K^{\infty}(N)$ "reduced" iff

$$0 < \mathrm{pord}(z_1) < \cdots < \mathrm{pord}(z_r).$$

Definition 9. We call the sequence $z_1, \dots, z_r \in K^{\infty}(N)$ reduced with respect to a nonzero $t \in K^{\infty}(N)$" (in short: "t-reduced") iff it is reduced and for all $i, j \in \{1, \dots, r\}$, $i \neq j$:

$$\mathrm{pord}(z_i) \not\equiv \mathrm{pord}(z_j) \pmod{\mathrm{pord}(t)}$$

and

$$\mathrm{pord}(z_i) \not\equiv 0 \pmod{\mathrm{pord}(t)}.$$

Definition 10. Let R be a ring (with 1) and S a subring of R. For $m_1, \dots, m_r \in R$ we define an S-module by

$$\langle m_1, \dots, m_r \rangle_S := \{s_0 + s_1 m_1 + \cdots + s_r m_r : s_0, s_1, \dots s_r \in S\}.$$

Definition 11. For $f \in K^{\infty}(N)$ let $\sum_{n=m}^{\infty} a(n) q^n$, $a(m) \neq 0$, be the q-expansion of f. We define $\mathrm{lc}(f) := a(m)$.

Let $c \in \mathbb{C}$. For $f : \mathbb{H} \to \mathbb{C}$ with $f(\tau) = c$ for all $\tau \in \mathbb{H}$ we write $f = c$.

Lemma 12. *Suppose the sequence $z_1, \dots, z_r \in K^\infty(N)$ is reduced with respect to $t \in K^\infty(N)$. Then $M := \langle z_1, \dots, z_r\rangle_{\mathbb{C}[t]}$ is a free $\mathbb{C}[t]$-module with basis $1, z_1, \dots, z_r$. Furthermore, for $z_0 := 1$ and $0 \neq u \in M$ with $u = \sum_{i=0}^r p_i(t) z_i$, $p_i(t) \in \mathbb{C}[t]$, we have that*

$$\operatorname{pord}(u) = \operatorname{pord}(z_i) + \deg(p_i)\operatorname{pord}(t)$$

for some $i \in \{0, \dots, r\}$.

Proof. Let $I := \{i : p_i \neq 0\}$. Note that for $i \in I$: $\operatorname{pord}(p_i(t)z_i) = \operatorname{pord}(z_i) + \deg(p_i)\operatorname{pord}(t)$. Since $\operatorname{pord}(z_i) \not\equiv \operatorname{pord}(z_j) \pmod{\operatorname{pord}(t)}$, for all $i, j \in \{0, \dots, r\}$ with $i \neq j$, it follows that $\operatorname{pord}(p_i(t)z_i) \neq \operatorname{pord}(p_j(t)z_j)$ for all $i, j \in I$ with $i \neq j$. This implies that

$$\operatorname{pord}(u) = \max_{i \in I} \operatorname{pord}(p_i(t)z_i).$$

Now we prove that M is free with $(1, z_1, \dots, z_r)$ as a module basis. This is equivalent to showing that the polynomials $p_0(t), p_1(t), \dots, p_r(t)$ for u are unique. Assume that $u = \sum_{i=0}^r q_i(t) z_i$. Then $0 = \sum_{i=0}^r h_i(t) z_i$ with $h_i(t) = p_i(t) - q_i(t)$. If there exists an i such that $h_i(t) \neq 0$, then by similar arguments as above,

$$\operatorname{pord}\Big(\sum_{i=0}^r h_i(t) z_i\Big) > 0,$$

which is a contradiction to $\sum_{i=0}^r h_i(t) z_i = 0$. $\square$

Algorithm MC (Membership Check)
Input: $u, t \in K^\infty(N)$ and a t-reduced sequence $z_1, \dots, z_r \in K^\infty(N)$.
Output: "True" if $u \in \langle z_1, \dots, z_r\rangle_{\mathbb{Q}[t]}$ and "False" otherwise.

(1) $z_0 := 1$;
(2) while there exists $i \in \{0, \dots, r\}$ such that $\operatorname{pord}(u) \equiv \operatorname{pord}(z_i) \pmod{\operatorname{pord}(t)}$ do
  (a) $j := \frac{\operatorname{pord}(u) - \operatorname{pord}(z_i)}{\operatorname{pord}(t)}$;
  (b) if $j \geq 0$ then $u := u - z_i t^j \frac{\operatorname{lc}(u)}{\operatorname{lc}(z_i t^j)}$;

Note 1. We used in Algorithm MC that by Lemma 12 $0 \neq u \in \langle z_1, \dots, z_r\rangle_{\mathbb{Q}[t]}$ only if $\operatorname{pord}(u) \equiv \operatorname{pord}(z_i) \pmod{\operatorname{pord}(t)}$ for some $i \in \{0, \dots, r\}$.

It is straightforward to refine the Algorithm MC.

Algorithm MW (Membership Witness)
Input: $u, t \in K^\infty(N)$ and a t-reduced sequence $z_1, \dots, z_r \in K^\infty(N)$.
Output: If Algorithm MC returns True, $(p_0, \dots, p_r) \in \mathbb{Q}[t]^{r+1}$ such that $u = \sum_{i=0}^r p_i(t) z_i$.

Remark 13. If we choose t to be nonzero constant, then $\mathbb{Q}[t] = \mathbb{Q}$ and the $\mathbb{Q}[t]$-module turns into a $\mathbb{Q}$-vector space $\langle z_1, \dots, z_r\rangle_{\mathbb{Q}} (= \langle 1, z_1, \dots, z_r\rangle_{\mathbb{Q}})$.

More generally, we have the

Algorithm VB (Vector Space Basis)
Input: $\{m_1, \dots, m_s\} \subset K^\infty(N)$
Output: A reduced sequence $z_1, \dots, z_r \in K^\infty(N)$ which is a vector space basis such that in the sense of Definition 10,

$$\langle m_1, \dots, m_s\rangle_{\mathbb{Q}} = \langle z_1, \dots, z_r\rangle_{\mathbb{Q}}.$$

To obtain the desired output we carry out the same Gaussian elimination algorithm as used for triangularization of a matrix M. In our setting, row j of the matrix M consists of the coefficients of the principal part of the q-series of m_j. The row i of the resulting triangular matrix Z consists of the coefficients of the principal part of z_i. If S is the invertible matrix coding the elimination steps; i.e., $S \cdot M = Z$, then $S(m_1, \dots, m_s)^T = (z_r, \dots, z_1, \underbrace{c_1, \dots, c_{s-r}}_{s-r \text{ constants in } \mathbb{C}})^T$.

Example 14. We apply the Algorithm VB to:

$$(m_1, m_2, m_3) := (q^{-3} + q^{-1} + 2 + O(q), q^{-3} + q^{-2} + 3 + O(q), q^{-2} - q^{-1} + 8 + O(q)).$$

Then

$$M := \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \quad \text{and by G. elim. algorithm} \quad S := \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix};$$

so that

$$S \cdot M = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$S \cdot \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} = \begin{pmatrix} q^{-3} + q^{-1} + 2 + O(q) \\ q^{-2} - q^{-1} + 1 + O(q) \\ 7 \end{pmatrix} = \begin{pmatrix} z_2 \\ z_1 \\ 7 \end{pmatrix}.$$

Lemma 15. *Suppose that the sequence $z_1, \dots, z_r \in K^\infty(N)$ is the output of Algorithm VB on input $\{m_1, \dots, m_s\} \subseteq K^\infty(N)$. Then for $i \in \{1, \dots, r\}$:*

$$\mathrm{pord}(z_1) \leq \mathrm{pord}(z_i).$$

We use this minimality of $\mathrm{pord}(z_1)$ in the Algorithm MB. More generally, we need a module basis:

Algorithm MB (Module Basis)
Input: $\{m_1, \dots, m_s\} \subset K^\infty(N)$.
Output: $t \in K^\infty(N)$ non-constant, and a t-reduced sequence $z_1, \dots, z_r \in K^\infty(N)$ such that

$$\langle m_1, \dots, m_s \rangle_{\mathbb{Q}[t]} = \langle z_1, \dots, z_r \rangle_{\mathbb{Q}[t]}.$$

(1) Let $L := \{\}$ and $M := \{m_1, \dots, m_s\}$;
(2) while $M \neq L$ do
   (a) apply Algorithm VB to M and let $z_1, \dots, z_r$ be the output;
   (b) $L := \{z_1, \dots, z_r\}$, $M := L$;
   (c) $t := z_1$; (in view of Lemma 15)
   (d) for $i, j \in \{1, \dots, r\}$ with $j > i$ do
      if $\mathrm{pord}(z_i) \equiv \mathrm{pord}(z_j) \pmod{\mathrm{pord}(t)}$, then

(A) $v := z_j - \frac{\mathrm{lc}(z_j)}{\mathrm{lc}(z_i)\,\mathrm{lc}(t)^k} z_i t^k$, $k := \frac{\mathrm{pord}(z_j)-\mathrm{pord}(z_i)}{\mathrm{pord}(t)}$;
(B) $M := L - \{z_j\}$;
(C) if $v \neq 0$, then $M := L \cup \{v\}$;
(D) break the for loop;
(3) return $z_2, \ldots, z_r$ and t.

Algorithm AB (Algebra Basis)
Input: $\{m_1, \ldots, m_s\} \subset K^\infty(N)$
Output: $t \in K^\infty(N)$ and a t-reduced sequence $z_1, \ldots, z_r$ such that

$$\mathbb{Q}[m_1, \ldots, m_s] = \langle z_1, \ldots, z_r \rangle_{\mathbb{Q}[t]}.$$

(1) let $L := \{\}$ and $M := \{m_1, \ldots, m_r\}$;
(2) while $L \neq M$ do
  (a) apply Algorithm MB to M and let t and $z_1, \ldots, z_r$ be the output
  (b) $L := \{z_1, \cdots, z_r\}$, $M := L$;
  (c) for $i, j \in \{1, \ldots, r\}$ do
    if $z_i z_j \notin \langle z_1, \ldots, z_r \rangle_{\mathbb{Q}[t]}$, then
    (A) $M := L \cup \{t, z_i z_j\}$;
    (B) break the for loop;
(3) return L and t.

For an example of this algorithm see subsection 3.3.

Theorem 16. *The Algorithms MB and AB terminate.*

Proof. We prove termination of Algorithm MB. Because of $\mathrm{pord}(t) > 0$, after a finite number of steps we reach a minimal $k > 0$ such that $\mathrm{pord}(t) = k$. We define

$$v_s(L, k) := \min\{\mathrm{pord}(f) : f \in L, \mathrm{pord}(f) \equiv s \pmod{k}\}.$$

The only thing that prevents the algorithm from terminating is that $M \neq L$. However, if this is the case, then for some $s \in \{1, \ldots, k-1\}$, $v_s(L, k)$ has decreased. Because of Corollary 6 we have $v_s(L, k) \geq 0$. This implies that after a finite numbers of steps for each given s, $v_s(L, k)$ will no longer change which implies that $M = L$ and hence Algorithm MB will terminate. The proof that Algorithm AB terminates is analogous. □

2.2. Applying the Basis Algorithms to Eta Products

Lemma 17. *For all $\delta \in \mathbb{N}^*$ with $\delta | N$, $\left(\frac{\eta(\delta\tau)}{\eta(\tau)}\right)^{24} \in K(N)$.*

Proof. By [15, Ch. VII, Th. 6] we have

$$\eta^{24}\Big(\frac{a\tau + b}{c\tau + d}\Big) = (c\tau + d)^{12} \eta^{24}(\tau) \tag{13}$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$. Let $\begin{pmatrix} a & b \\ cN & d \end{pmatrix} \in \Gamma_0(N)$. Then

$$\eta^{24}\Big(\delta \frac{a\tau + b}{cN\tau + d}\Big) = \eta^{24}\Big(\frac{a(\delta\tau) + \delta b}{\frac{cN}{\delta}(\delta\tau) + d}\Big) = (\frac{cN}{\delta}(\delta\tau) + d)^{12} \eta(\delta\tau)^{24}. \tag{14}$$

This implies that

$$\frac{\eta\left(\delta\frac{a\tau+b}{cN\tau+d}\right)^{24}}{\eta\left(\frac{a\tau+b}{cN\tau+d}\right)^{24}}=\frac{\eta(\delta\tau)^{24}}{\eta(\tau)^{24}}.$$

Noting that the η function has no zeros or poles in $\mathbb{H}$ implies that $\left(\frac{\eta(\delta\tau)}{\eta(\tau)}\right)^{24}$ is analytic on $\mathbb{H}$. This proves property *(ii)* of Definition 1.

In order to prove property *(iii)* of Definition 1 we need to show that for $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ there exists $k \in \mathbb{N}^*$ such that the function $\left(\frac{\eta(\delta\gamma\tau)}{\eta(\gamma\tau)}\right)^{24}$ can be expressed as a Laurent series in powers of $q^{1/k}$ with finite principal part. We may write $\delta\gamma\tau = A\tau$ where A is an integer matrix with determinant δ. Furthermore, one may write $A = \gamma' T$ where $\gamma' \in \mathrm{SL}_2(\mathbb{Z})$ and T is a triangular matrix. Then

$$\left(\frac{\eta(\delta\gamma\tau)}{\eta(\gamma\tau)}\right)^{24}=\left(\frac{\eta(A\tau)}{\eta(\gamma\tau)}\right)^{24}=\underbrace{\left(\frac{\eta(\gamma' T\tau)}{\eta(\gamma\tau)}\right)^{24}=c\left(\frac{\eta(T\tau)}{\eta(\tau)}\right)^{24}}_{\text{because of (13)}}$$

for some $c \in \mathbb{C}$. Because of the triangular shape of T, $T = \begin{pmatrix} u & v \\ 0 & w \end{pmatrix}$ say, then $\eta(T\tau)$ can be expressed as a Laurent series (with finite principal part) in powers of $q^{\frac{1}{24w}}$; consequently so can $c\left(\frac{\eta(T\tau)}{\eta(\tau)}\right)^{24}$. $\square$

Lemma 17 is a special case of a much more general result of Gordon, Hughes, and Newman [8, Thm. 1.64].

Lemma 18. *Let $a, b \in \mathbb{N}$ with $a, b \geq 2$ and $a \neq b$. Let $A := \left(\frac{\eta(a\tau)}{\eta(\tau)}\right)^{24(b-1)}$ and $B := \left(\frac{\eta(b\tau)}{\eta(\tau)}\right)^{24(a-1)}$. Then we have*

$$\gcd(\mathrm{pord}(A-B), \mathrm{pord}(A)) = 1.$$

Proof. We have

$$\begin{aligned} A &= q^{(a-1)(b-1)} \prod_{n=1}^{\infty}\left(\frac{1-q^{an}}{1-q^n}\right)^{24(b-1)} \\ &= q^{(a-1)(b-1)}(1+24(b-1)q+O(q^2)) \end{aligned}$$

and, by symmetry,

$$B = q^{(a-1)(b-1)}(1+24(a-1)q+O(q^2)).$$

In particular,

$$A - B = q^{(a-1)(b-1)}(24(b-a)q+O(q^2)).$$

This implies $\mathrm{pord}(A-B) = -(a-1)(b-1)-1$ and $\mathrm{pord}(A) = -(a-1)(b-1)$. $\square$

Definition 19. For $N \in \mathbb{N}^*$ we consider the multiplicative monoid of eta quotients contained in $K^\infty(N)$ and denoted by

$$E^\infty(N) := E(N) \cap K^\infty(N).$$

Lemma 20. *For each $N \in \mathbb{N}^*$ there exists $\mu \in E^\infty(N)$ such that $\operatorname{ord}_\gamma^N(\mu) > 0$ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z}) \setminus \Gamma_0(N)$.*

Proof. The case N square free has been proven by Newman [6]. In case N is not square free let M be the largest square free integer dividing N, and let $\mu^* \in E^\infty(M)$ be such that $\operatorname{ord}_\gamma^M(\mu^*) > 0$ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z}) \setminus \Gamma_0(M)$. Then one can verify that $\mu(\tau) := \mu^*(N\tau/M)$ satisfies the desired property. $\square$

Corollary 21. *Let N be a positive integer which is not a prime. Then there exist $F, G \in \langle E^\infty(N)\rangle_{\mathbb{Q}}$ such that*

$$\gcd(\operatorname{pord}(F), \operatorname{pord}(G)) = 1.$$

Proof. By Lemma 20 there exists a function $\mu \in E^\infty(N)$ such that

$$\operatorname{ord}_\gamma^N(\mu) > 0 \text{ for all } \gamma \in \mathrm{SL}_2(\mathbb{Z}) \setminus \Gamma_0(N). \tag{15}$$

Since N is not a prime there exists a, b dividing N with $a, b \geq 2$ and $a \neq b$. Then by Lemma 17 the functions $A := \left(\frac{\eta(a\tau)}{\eta(\tau)}\right)^{24(b-1)}$ and $B := \left(\frac{\eta(b\tau)}{\eta(\tau)}\right)^{24(a-1)}$ are in $K(N)$, and by Lemma 18 they satisfy $\gcd(\operatorname{pord}(A-B), \operatorname{pord}(A)) = 1$. Furthermore, because of (15) one can choose $k \in \mathbb{N}^*$ sufficiently big such that

$$A\mu^{-k \operatorname{pord}(A)}, B\mu^{-k \operatorname{pord}(A)} \in E^\infty(N).$$

Setting $F := A\mu^{-k \operatorname{pord}(A)} - B\mu^{-k \operatorname{pord}(A)}$ implies $\gcd(\operatorname{pord}(F), \operatorname{pord}(A)) = 1$. Applying the same reasoning again we see that there exists a $j \in \mathbb{N}^*$ such that $A\mu^{j \operatorname{pord}(F)} \in E^\infty(N)$. Defining $G := A\mu^{j \operatorname{pord}(F)}$ we have $\gcd(\operatorname{pord}(F), \operatorname{pord}(G)) = 1$. $\square$

Lemma 22. *For $N \in \mathbb{N}^*$ and every $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ there exists $\gamma_N = \begin{pmatrix} A & B \\ Nx & y \end{pmatrix} \in \Gamma_0(N)$ such that $\gamma_N \gamma = \begin{pmatrix} * & * \\ \delta & * \end{pmatrix}$, where $\delta := \gcd(N, c)$.*

Proof. We need to find $x, y \in \mathbb{Z}$ and $A, B \in \mathbb{Z}$ such that $\begin{pmatrix} A & B \\ Nx & y \end{pmatrix} \in \Gamma_0(N)$ and

$$\begin{pmatrix} A & B \\ Nx & y \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} * & * \\ \delta & * \end{pmatrix}.$$

This means that we need to solve the equation

$$aNx + cy = \delta.$$

This equation always has a solution (x_0, y_0) because $\gcd(aN, c) = \delta$. The general solution is of the form

$$(x, y) = (x_0, y_0) + (kc/\delta, -kNa/\delta).$$

One can prove that k can be chosen appropriately such that $\gcd(Nx, y) = 1$. When this condition is satisfied there exist $A, B \in \mathbb{Z}$ such that $Ay - BNx = 1$, implying that $\begin{pmatrix} A & B \\ Nx & y \end{pmatrix} \in \Gamma_0(N)$. $\square$

As as a consequence of Lemma 22, given $f \in K(N)$, to know $\mathrm{ord}_\gamma^N(f)$, $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, we can restrict to consider $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $c|N$. It is easy to see that w.l.o.g. we can additionally assume that a and c are positive integers. If $f \in E(N)$ there is a well-known formula, usually named after Ligozat; see e.g. [8, Th. 1.64].

Theorem 23. *Let $f \in E(N)$ with $f = \prod_{\delta|N} \eta(\delta\tau)^{r_\delta}$ where $(r_\delta)_{\delta|N} \in R^*(N)$. Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ with $c > 0$ and $c|N$. Then*

$$\mathrm{ord}_\gamma^N(f) = \frac{N/c}{24\gcd(N/c, c)} \sum_{\delta|N} r_\delta \frac{\gcd(c,\delta)^2}{\delta}.$$

Lemma 24. *We have:*

$$E^\infty(N) = \Big\{ \prod_{\delta|N} \left(\frac{\eta(\delta\tau)}{\eta(\tau)}\right)^{r_\delta} : (r_\delta) \in R^*(N) \Big\},$$

and for all positive $c|N$, $c \neq 1$:

$$\frac{c}{24\gcd(c, N/c)} \sum_{\delta|N} r_\delta \frac{\gcd^2(\delta, N/c) - \delta}{\delta} \geq 0 \Big\},$$

Proof. It is easy too see that any $f \in E(N)$ can be written in the requested form. Because of $E(N) \subseteq K(N)$ ([8, Th. 1.64]) we only need to check the non-negativity of the orders $\mathrm{ord}_\gamma^N(f)$ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z}) \backslash \Gamma_0(N)$; this means, for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $c > 0$, $c|N$ and $c \neq N$. But this is implied by Theorem 23 taking c/N instead of c. The proof is completed by noting that then then the condition $c \neq N$ turns into $c \neq 1$. $\square$

Lemma 25. *The set $E^\infty(N)$ is a finitely generated monoid.*

Proof. Let $\delta_1, \ldots, \delta_n$ be the divisors of N greater than 1. Define

$$A_N := (a(i,j))_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n}},$$

where

$$a(i,j) := \frac{\delta_i}{24\gcd(\delta_i, N/\delta_i)} \frac{\gcd^2(\delta_j, N/\delta_i) - \delta_j}{\delta_j}.$$

Then Lemma 24 may be expressed as:

$$E^{\infty}(N) = \Big\{ \prod_{\delta|N} \left(\frac{\eta(\delta\tau)}{\eta(\tau)} \right)^{r_\delta} : r = (r_\delta) \in R^*(N), A_N(r_{\delta_1}, \dots, r_{\delta_n})^T \geq 0 \Big\}.$$

Since Newman [6] proved that A_N is invertible we may write:

$$E^{\infty}(N) = \left\{ \prod_{j=1}^{n} \left(\frac{\eta(\delta_j\tau)}{\eta(\tau)} \right)^{r_{\delta_j}} : \begin{array}{c} (-\sum_{i=1}^{n} r_{\delta_i}, r_{\delta_1}, \dots, r_{\delta_n}) \in R^*(N) \text{ s.t.} \\ \exists (x_1, \dots, x_n) \in \mathbb{N}^n \text{ with} \\ (r_{\delta_1}, \dots, r_{\delta_n})^T = A_N^{-1}(x_1, \dots, x_n)^T. \end{array} \right\}. \tag{16}$$

Given such a matrix A_N, we can choose $\kappa_i \in \mathbb{N}^*$ big enough (to make $\kappa_i A_N^{-1}$ an integer matrix and, if needed, to have them as multiples of 24)

$$(r^{(i)}_{\delta_1}, \dots, r^{(i)}_{\delta_n})^T := \kappa_i A_N^{-1} e_i, \text{ we have}$$

$$r^{(i)} := (-\sum_{i=1}^{n} r^{(i)}_{\delta_i}, r^{(i)}_{\delta_1}, \dots, r^{(i)}_{\delta_n}) \in R^*(N).$$

where e_i is the i-th column in the $n \times n$ identity matrix. Consequently, owing to (16),

$$F_i := \prod_{j=1}^{n} \left(\frac{\eta(\delta_j\tau)}{\eta(\tau)} \right)^{r^{(i)}_{\delta_j}} \in E^{\infty}(N).$$

From (16) we see that we have a one to one correspondence between the monoid consisting of $(x_1, \dots, x_n)$ satisfying the conditions in (16) and the monoid consisting of the elements of $E^{\infty}(N)$. Hence we need to find the generating set of the monoid

$$S_N := \Big\{ (x_1, \dots, x_n) \in \mathbb{N}^n : \begin{array}{c} \exists (-\sum_{i=1}^{n} r_{\delta_i}, r_{\delta_1}, \dots, r_{\delta_n}) \in R^*(N) \\ \text{s.t. } (r_{\delta_1}, \dots, r_{\delta_n})^T = A_N^{-1}(x_1, \dots, x_n)^T \end{array} \Big\}.$$

Note that $(\kappa_i e_i)^T \in S_N$, and if $a, b \in S_N$ and $a - b \geq 0$, then $a - b \in S_N$. This implies that if for $a \in S_N$ there exists $c_i \in \mathbb{Z}$ such that $b := a - \sum_{i=1}^{N} c_i(\kappa_i e_i)^T \geq 0$, then $b \in S_N$. This implies that any $a = (a_1, \dots, a_n)$ can be reduced to a $b = (b_1, \dots, b_n)$ where $b_i < \kappa_i$. Hence a finite generating set for S_N is given by

$$\left\{ (x_1, \dots, x_n) \in \mathbb{N}^n : \begin{array}{c} \exists (-\sum_{i=1}^{n} r_{\delta_i}, r_{\delta_1}, \dots, r_{\delta_n}) \in R^*(N) \\ \text{s.t. } (r_{\delta_1}, \dots, r_{\delta_n})^T = A_N^{-1}(x_1, \dots, x_n)^T \\ \text{and } x_i < \kappa_i \text{ for all } 1 \leq i \leq n \end{array} \right\} \cup \{\kappa_1 e_1, \dots, \kappa_n e_n\}.$$

□

Remark 26. We observe that in the above proof we also obtain an explicit algorithm on how to produce the generating set in question. There are more general algorithms to solve this problem, e.g. see [4] or methods from discrete geometry [2].

Example 27. Let $p \geq 5$ be a prime. The matrix A_{p^2} in Lemma 25 above is given by

$$A_{p^2} = \frac{p-1}{24} \begin{pmatrix} 1 & 0 \\ -p & -(p+1) \end{pmatrix}.$$

(Note that $p^2 - 1$ is divisible by 24.) Then

$$A_{p^2}^{-1} = \frac{24}{p^2-1} \begin{pmatrix} p+1 & 0 \\ -p & -1 \end{pmatrix}$$

and we can choose $\kappa_1 = \kappa_2 = \frac{p^2-1}{24}$. This implies that $\kappa_1 A_{p^2}^{-1} e_1 = (p+1, -p)$, $\kappa_2 A_{p^2}^{-1} e_2 = (0, -1)$ and $r^{(1)} = (-1, p+1, -p)$ resp. $r^{(2)} = (1, 0, -1)$ are in $R^*(p^2)$. Next we need to find all $(x_1, x_2) \in \mathbb{N}^2$ with $0 < x_1, x_2 < \kappa = \frac{p^2-1}{24}$ such that

$$A_{p^2}^{-1} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} =: \begin{pmatrix} r_p \\ r_{p^2} \end{pmatrix} \tag{17}$$

satisfies $(-r_p - r_{p^2}, r_p, r_{p^2}) \in R^*(p^2)$. The relation (17) implies

$$(p^2-1) \mid 24(p+1)x_1, \text{ and} \tag{18}$$
$$(p^2-1) \mid 24(px_1 + x_2). \tag{19}$$

From (18) we obtain that $x_1 = k\frac{p-1}{\gcd(p-1,24)}$ for some integer $0 < k < \frac{(p+1)\gcd(p-1,24)}{24}$. (Note that this upper bound for k is an integer.) Substituting into (19) we obtain $x_2 = t\frac{p-1}{\gcd(p-1,24)}$ for some integer $0 < t < \frac{(p+1)\gcd(p-1,24)}{24}$. Substituting this form of x_2 into (19) we obtain that

$$t \equiv k \pmod{\frac{(p+1)\gcd(p-1,24)}{24}}.$$

In case $\frac{(p+1)\gcd(p-1,24)}{24} = 1$ we have no solutions to (18)-(19). In case $\frac{(p+1)\gcd(p-1,24)}{24} > 1$ we have the choices $(k,t) = j(1,1)$ for $0 < j < \frac{(p+1)\gcd(p-1,24)}{24}$. This corresponds to the solutions

$$(x_1, x_2) = j\frac{p-1}{\gcd(p-1,24)}(1,1)$$

for which

$$A_{p^2}^{-1} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = j \begin{pmatrix} \frac{24}{\gcd(p-1,24)} \\ -\frac{24}{\gcd(p-1,24)} \end{pmatrix}.$$

We observe that

$$\frac{24}{\gcd(p-1,24)}(0, 1, -1) \in R^*(p^2)$$

iff $p \not\equiv 1 \pmod{24}$. Hence, in case $p \not\equiv 1 \pmod{24}$ we have that

$$\Big(\frac{p-1}{\gcd(p-1,24)}, \frac{p-1}{\gcd(p-1,24)}\Big) \in S_{p^2}.$$

If $p \equiv 1 \pmod{24}$ we deduce that

$$2\Big(\frac{p-1}{\gcd(p-1,24)}, \frac{p-1}{\gcd(p-1,24)}\Big) \in S_{p^2}.$$

Summarizing, a set of generators for the additive monoid S_{p^2} is given by

$$\{\kappa e_1, \kappa e_2\}$$

if $\frac{(p+1)\gcd(p-1,24)}{24} = 1$; that is in case $p = 5, 11$. In this case the multiplicative monoid $E^\infty(p^2)$ is generated by

$$\{\frac{\eta^{p+1}(p\tau)}{\eta(\tau)\eta^p(p^2\tau)}, \frac{\eta(\tau)}{\eta(p^2\tau)}\}.$$

In case $p \not\equiv 1 \pmod{24}$ and $\frac{(p+1)\gcd(p-1,24)}{24} > 1$ a set of generators for S_{p^2} is given by

$$\{\kappa e_1, \kappa e_2, \frac{p-1}{\gcd(p-1,24)}(1,1)\}.$$

In this case $E^\infty(p^2)$ is generated by

$$\{\frac{\eta^{p+1}(p\tau)}{\eta(\tau)\eta^p(p^2\tau)}, \frac{\eta(\tau)}{\eta(p^2\tau)}, \Big(\frac{\eta(p\tau)}{\eta(p^2\tau)}\Big)^{\frac{24}{\gcd(p-1,24)}}\}.$$

Finally, in case $p \equiv 1 \pmod{24}$ and $\frac{(p+1)\gcd(p-1,24)}{24} > 1$ a set of generators for S_{p^2} is given by

$$\{\kappa e_1, \kappa e_2, \frac{2(p-1)}{\gcd(p-1,24)}(1,1)\}.$$

In this case $E^\infty(p^2)$ is generated by

$$\{\frac{\eta^{p+1}(p\tau)}{\eta(\tau)\eta^p(p^2\tau)}, \frac{\eta(\tau)}{\eta(p^2\tau)}, \Big(\frac{\eta(p\tau)}{\eta(p^2\tau)}\Big)^{\frac{48}{\gcd(p-1,24)}}\}.$$

□

Our goal is to find a basis of the $\mathbb{Q}[t]$-module $E := \langle E^\infty(N)\rangle_{\mathbb{Q}}$, because the set E viewed as a $\mathbb{Q}$-module does not have a finite basis. For this reason we view it as a $\mathbb{Q}[t]$-module for some $t \in E$, then the $\mathbb{Q}[t]$-module E indeed has a finite basis. We construct this finite basis in the following way. First we find a finite set of generators $x_1, \dots, x_n$ of the monoid $E^\infty(N)$ (this is guaranteed by Lemma 25). Then $E = \mathbb{Q}[x_1, \dots, x_n]$. Next applying the Algorithm AB we obtain finitely many generators of E viewed as a $\mathbb{Q}[t]$-module for some $t \in E$ produced by the algorithm. E can now be finitely described and, most important we are now able to check membership in E by using the Algorithm MC. More precisely we may check if a given element $f \in K^\infty(N)$ may be expressed as a sum of eta quotients from $E^\infty(N)$. Our next goal is to decide whether $f \in K^\infty(N)$ belongs to $\langle E(N)\rangle_{\mathbb{Q}} \cap K^\infty(N)$; i.e., if $f \in K^\infty(N)$ may be expressed as a sum of eta quotients from $E(N)$.

We do not know if in general

$$\langle E(N)\rangle_{\mathbb{Q}} \cap K^\infty(N) = \langle E^\infty(N)\rangle_{\mathbb{Q}}?$$

However we will show the following lemma:

Lemma 28. *Given $\mu(\tau) \in E^\infty(N)$ as in Lemma 20. Then there exists a positive integer k such that*

$$\mu^k(\tau)\big(\langle E(N)\rangle_{\mathbb{Q}} \cap K^\infty(N)\big) \subseteq \langle E^\infty(N)\rangle_{\mathbb{Q}}.$$

Furthermore, k is computable in finitely many steps.

Before we prove Lemma 28 we need the following lemma:

Lemma 29. *Assume that*

$$\langle z_1, \ldots, z_r\rangle_{\mathbb{Q}[t]} = \langle E^\infty(N)\rangle_{\mathbb{Q}}$$

where the sequence $z_1, \ldots, z_r$ is reduced with respect to $t \in \langle E^\infty(N)\rangle_{\mathbb{Q}}$. Let $\mu \in E^\infty(N)$ be as in Lemma 20. Let $\nu := \operatorname{pord}(\mu) - 1$. Then there exist $x_1, \ldots, x_\nu \in \langle E^\infty(N)\rangle_{\mathbb{Q}}$ reduced with respect to μ such that

$$\langle x_1, \ldots, x_\nu\rangle_{\mathbb{Q}[\mu]} = \langle z_1, \ldots, z_r\rangle_{\mathbb{Q}[t]} \tag{20}$$

and for $i \in \{1, \ldots, \nu\}$

$$\operatorname{pord}(x_i) \equiv i \pmod{\nu + 1}. \tag{21}$$

Proof. Set $z_0 := 1$. Because of Corollary 21 we observe that for each $0 \leq i \leq \nu$ there exists unique $(a, b) \in \{0, \ldots, r\} \times \mathbb{N}$ such that $\operatorname{pord}(z_a t^b)$ is minimal with the property $\operatorname{pord}(z_a t^b) \equiv i \pmod{\nu + 1}$. We define $x_i := z_a t^b$. Set $x_0 := 1$. Then for every $a \in \{0, \ldots, r\}$ and $b \in \mathbb{N}$, there exists $i \in \{0, \ldots, \nu\}$ and $j \in \mathbb{N}$ such that $\operatorname{pord}(z_a t^b) = \operatorname{pord}(x_i \mu^j)$. By construction $x_i \in \langle z_1, \ldots, z_r\rangle_{\mathbb{Q}[t]}$ for all $i \in \{1, \ldots, \nu\}$ and since $\mu \in E^\infty(N) \subset \langle z_1, \ldots, z_r\rangle_{\mathbb{Q}[t]}$ it follows that

$$\langle x_1, \ldots, x_\nu\rangle_{\mathbb{Q}[\mu]} \subseteq \langle z_1, \ldots, z_r\rangle_{\mathbb{Q}[t]} = \langle E^\infty(N)\rangle_{\mathbb{Q}}.$$

To prove the other direction let

$$f \in \langle z_1, \ldots, z_r\rangle_{\mathbb{Q}[t]}, \quad f \notin \langle x_1, \ldots, x_\nu\rangle_{\mathbb{Q}[\mu]}$$

with $\operatorname{pord}(f)$ minimal. Then there exists $z_a t^b$ and $c \in \mathbb{Q}$ such that $\operatorname{pord}(f) > \operatorname{pord}(f - c z_a t^b)$. In particular there exists $x_i \mu^j$ such that $\operatorname{pord}(z_a t^b) = \operatorname{pord}(x_i \mu^j)$. This implies that for some $c' \in \mathbb{Q}$ we have $\operatorname{pord}(f) > \operatorname{pord}(f - c' x_i \mu^j)$. This contradicts the minimality of $\operatorname{pord}(f)$. □

Proof of Lemma 28: Let $m_1, \ldots, m_s$ be some generators of the monoid $E^\infty(N)$. We input these generators to the algorithm "Algebra Basis". Let t, $z_1, \ldots, z_r$ be the output of the algorithm. Let $\mu \in E^\infty(N)$ be as in Lemma 20. By Lemma 29 there exist $x_1, \ldots, x_\nu \in \langle E^\infty(N)\rangle_{\mathbb{Q}}$ such that (20) and (21).

Assume that $f \in (\langle E(N)\rangle_{\mathbb{Q}} \cap K^\infty(N)) \backslash \langle E_\infty(N)\rangle_{\mathbb{Q}}$. Then for some <u>minimal</u> positive integer k we have $f\mu^k \in \langle E^\infty(N)\rangle_{\mathbb{Q}}$.

Then there exists $a_i \in \langle x_1, \ldots, x_\nu\rangle_{\mathbb{Q}}$ for $i = 0, 1, \ldots, d$ such that

$$f\mu^k = a_d\mu^d + a_{d-1}\mu^{d-1} + \cdots + a_0\mu^0.$$

In particular since $k \geq 1$ by assumption we have $f\mu^{k-1} \in K^\infty(N)$. Furthermore,

$$f\mu^{k-1} = a_d\mu^{d-1} + a_{d-1}\mu^{d-2} + \cdots + a_0\mu^{-1}.$$

implies that $a_0\mu^{-1} \in K^\infty(N)$. Note that $a_0 \neq 0$ because otherwise k is not minimal. There exists x_j such that

$$\operatorname{pord}(x_j) \equiv \operatorname{pord}(a_0\mu^{-1}) \pmod{\nu+1}.$$

We define

$$(x_1^{(1)}, \dots, x_j^{(1)}, \dots, x_\nu^{(1)}) := (x_1, \dots, a_0\mu^{-1}, \dots, x_\nu). \tag{22}$$

Then there exists $a_i^{(1)} \in \langle x_1^{(1)}, \dots, x_\nu^{(1)} \rangle_{\mathbb{Q}}$ for $i = 0, 1, \dots, d_1$ such that

$$f\mu^{k-1} = a_{d_1}^{(1)}\mu^{d_1} + a_{d_1-1}^{(1)}\mu^{d_1-1} + \cdots + a_0^{(1)}\mu^0.$$

If $k-1 \geq 1$ we divide μ out and we find that $a_0^{(1)}\mu^{-1} \in K^\infty(N)$. Set $j_1 := \operatorname{pord}(a_0^{(1)}\mu^{-1})$, then

$$(x_1^{(2)}, \dots, x_{j_1}^{(2)}, \dots, x_\nu^{(2)}) := (x_1^{(1)}, \dots, a_0^{(1)}\mu^{-1}, \dots, x_\nu^{(1)}).$$

Now $f\mu^{k-2} = a_{d_2}^{(2)}\mu^{d_2} + \cdots + a_0^{(2)}\mu^0$ for some $a_i^{(2)} \in \langle x_1^{(2)}, \dots, x_\nu^{(2)} \rangle_{\mathbb{Q}}$. Continuing this procedure we obtain at the j-th step (with $j \leq k$)

$$f\mu^{k-j} = a_{d_j}^{(j)}\mu^{d_j} + a_{d_j-1}^{(j)}\mu^{d_j-1} + \cdots + a_0^{(j)}\mu^0$$

where $a_i^{(j)} \in \langle x_1^{(j)}, \dots, x_\nu^{(j)} \rangle_{\mathbb{Q}}$. We must have $a_0^{(j)} \neq 0$ because in case not then

$$f\mu^{k-j} = \mu(a_{d_j}^{(j)}\mu^{d_j} + a_{d_j-1}^{(j)}\mu^{d_j-1} + \cdots + a_1^{(j)}\mu^0)$$

and by multiplying both sides by μ^{j-1} we obtain

$$f\mu^{k-1} = (\mu^j a_{d_j}^{(j)})\mu^{d_j} + (\mu^j a_{d_j-1}^{(j)})\mu^{d_j-1} + \cdots + (\mu^j a_1^{(j)})\mu^0$$

and $\mu^j a_i^{(j)} \in \langle x_1, \dots, x_\nu \rangle_{\mathbb{Q}[\mu]}$ because of $\mu^j x_i^{(j)} \in \langle x_1, \dots, x_\nu \rangle_{\mathbb{Q}[\mu]}$. Hence $f\mu^{k-1} \in \langle E^\infty(N) \rangle_{\mathbb{Q}}$ contradicting the minimality of k.

Note that if the old basis $x_1^{(j)}, \dots, x_\nu^{(j)}$ satisfies

$$(\operatorname{pord}(x_1^{(j)}), \dots, \operatorname{pord}(x_\nu^{(j)})) = (a_1, \dots, a_\nu),$$

then for some $u \in \{1, \dots, \nu\}$, the new basis $x_1^{(j+1)}, \dots, x_\nu^{(j+1)}$ satisfies

$$(\operatorname{pord}(x_1^{(j+1)}), \dots, \operatorname{pord}(x_\nu^{(j+1)})) = (a_1, \dots, a_u - \operatorname{pord}(\mu), \dots, a_\nu).$$

Defining q_i by the relation $\operatorname{pord}(x_i) = q_i \operatorname{pord}(\mu) + r_i$, with $r_i < \operatorname{pord}(\mu)$, it follows that after at most $\sum_{j=1}^{\nu} q_j$ steps this procedure must stop because for all $i, j \in \mathbb{N}$: $x_i^{(j)} \in K^\infty(N)$. This implies that $\operatorname{pord}(x_j^{(i)}) \geq 0$ because of Corollary 6. This proves that $k \leq \sum_{j=1}^{\nu} q_j$. $\square$

3. The Main Theorem and the Algorithm

In this section we present our main theorem and the algorithm that solves the problem described in the abstract. Furthermore, at the end of this section we give a counterexample to a conjecture by Morris Newman related to eta products.

Definition 30. For $a \in \mathbb{Z}$ and n an odd integer we denote by $\left(\frac{a}{n}\right)$ the Legendre-Jacobi symbol.

Lemma 31. *Let $n > 0$ be an odd integer, then the following relation holds for integers a and b:*

$$\left(\frac{a}{n}\right)\left(\frac{b}{n}\right) = \left(\frac{ab}{n}\right). \tag{23}$$

Proof. See [13], page 71. $\square$

For $\gamma \in \mathrm{SL}_2(\mathbb{Z})^*$ the following formula was proven by Newman:

$$\eta(\gamma\tau) = (-i(c\tau + d))^{\frac{1}{2}}\,\epsilon(a, b, c, d)\eta(\tau) \tag{24}$$

where

$$\epsilon(a, b, c, d) := \left(\frac{c}{a}\right) e^{-\frac{a\pi i}{12}(c-b-3)}. \tag{25}$$

Lemma 32 (Newman)**.** *Let $N \in \mathbb{N}^*$, and $f : \mathbb{H} \to \mathbb{C}$ be a function such that for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)^*$ we have $f(\gamma\tau) = f(\tau)$. Then for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$ we have $f(\gamma\tau) = f(\tau)$.*

Definition 33. For $m, M \in \mathbb{N}^*$, $t \in \{0, \dots, m-1\}$ and $r = (r_\delta) \in R(M)$ we define:

$$f(\tau, r) := \prod_{\delta|M} \prod_{n=0}^{\infty} (1 - q^{\delta n})^{r_\delta} = \sum_{n=0}^{\infty} a_r(n) q^n, \tag{26}$$

and

$$g_{m,t}(\tau, r) := q^{\frac{24t+\sigma_\infty(r)}{24m}} \sum_{n=0}^{\infty} a_r(mn + t) q^n. \tag{27}$$

We will sometimes write $g_{m,t}(\tau)$ instead of $g_{m,t}(\tau, r)$.

Definition 34. We denote by Δ the set of all $(m, M, N, (r_\delta)) \in (\mathbb{N}^*)^3 \times R(M)$ such that for every prime p:

$$p|m \text{ implies } p|N, \tag{28}$$

and such that for every $\delta|M$ with $r_\delta \neq 0$,

$$\delta|M \text{ implies } \delta|mN.$$

Definition 35. We denote by Δ^* the set of all $(m, M, N, t, r = (r_\delta))$ with $(m, M, N, (r_\delta)) \in \Delta$ and $t \in \{0, \dots, m-1\}$ such that for $\kappa := \gcd(1 - m^2, 24)$, $\kappa \geq 1$:

$$\kappa \frac{mN^2}{M} \sigma_0(r) \equiv_{24} 0;$$

$$\kappa N w(r) \equiv_8 0;$$

$$\frac{24m}{\gcd(\kappa(-24t - \sigma_\infty(r)), 24m)} \mid N;$$

and

$$\text{if } 2|m, \text{ then } \left(\kappa N \equiv_4 0 \text{ and } 8|Ns\right) \text{ or } \left(2|s \text{ and } 8|N(1-j)\right),$$

where $j, s \in \mathbb{Z}$, j odd, are such that $\prod_{\delta|M} \delta^{|r_\delta|} = 2^s j$.

Definition 36. Let $m, M \in \mathbb{N}^*$ and $r = (r_\delta) \in R(M)$. Define the operation $\odot_r$: $\Gamma_0(N)^* \times \{0, \dots, m-1\} \mapsto \{0, \dots, m-1\}$, $(\gamma, t) \mapsto \gamma \odot_r t$, where for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ the image $\gamma \odot_r t$ is uniquely defined by the relation

$$\gamma \odot_r t \equiv ta^2 + \frac{a^2-1}{24}\sigma_\infty(r) \pmod{m}. \tag{29}$$

Theorem 37 ([10], Th. 2.14)**.** *Let $(m, M, N, t, r = (r_\delta)) \in \Delta^*$, $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)^*$ and define*

$$\beta = \beta(\gamma, r) := \prod_{\delta | M} \left(\frac{mc\delta}{a}\right)^{|r_\delta|} e^{-\frac{\pi i a}{12}\left(\frac{mc}{M}\sigma_0(r) - mb\sigma_\infty(r) - 3w(r)\right)}, \tag{30}$$

then the following formula holds for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)^*$:

$$g_{m,t}(\gamma\tau) = \beta\left(-i(c\tau+d)\right)^{\frac{w(r)}{2}} e^{2\pi i \frac{ab(1-m^2)(24t+\sigma_\infty(r))}{24m}} \cdot g_{m,\gamma\odot_r t}(\tau). \tag{31}$$

Definition 38. Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, $m, M \in \mathbb{N}^*$ and $r = (r_\delta) \in R(M)$, then we define

$$p(\gamma, r) := \min_{\lambda \in \{0,\dots,m-1\}} \frac{1}{24} \sum_{\delta | M} r_\delta \frac{\gcd^2(\delta(a + k\lambda c), mc)}{\delta m},$$

and for $N \in \mathbb{N}^*, s = (s_\delta)_{\delta | N} \in R(N)$:

$$p^*(\gamma, s) := \frac{1}{24} \sum_{\delta | N} \frac{s_\delta \gcd^2(\delta, c)}{\delta}.$$

By [10, Lem. 3.4 and Lem. 3.5]:

Theorem 39. *Let $(m, M, N, r = (r_\delta)) \in \Delta$, $t \in \{0, \dots, m-1\}$, $s = (s_\delta)_{\delta|N} \in R(N)$, $\gamma_0 \in \mathrm{SL}_2(\mathbb{Z})$, O a subset of $\{0, \dots, m-1\}$, $(n_{t'})$ a sequence of nonnegative integers indexed by the elements t' of O. Let $\gamma \in \Gamma_0(N)\gamma_0\mathrm{SL}_2(\mathbb{Z})_\infty$ and $g : \mathbb{H} \to \mathbb{C}$ defined by*

$$\begin{aligned} g(\tau) :=& (c\tau+d)^{-\left(\frac{1}{2}\left(\sum_{t'\in O} n_{t'}\right)w(r) + \frac{1}{2}w(s)\right)} \\ & \cdot q^{-\left(\sum_{t'\in O} n_{t'}\right)p(\gamma_0, r) + p^*(\gamma_0, s)} \prod_{t'\in O} g_{m,t'}^{n_{t'}}(\gamma\tau, t) \prod_{\delta | N} \eta^{s_\delta}(\delta\gamma\tau), \end{aligned}$$

for all $\tau \in \mathbb{H}$. Then $g(\tau)$ has a Fourier expansion in (nonnegative) powers of $e^{2\pi i\tau/k}$ for some $k \in \mathbb{N}^*$.

Definition 40. Let n be a positive integer. For $x \in \mathbb{Z}$ we denote by $[x]_n$ the residue class of x modulo n. We define

$$\mathbb{Z}_n^* := \{[x]_n \in \mathbb{Z}_n \mid \gcd(x,n) = 1\},$$

and

$$\mathbb{S}_n := \{y^2 \mid y \in \mathbb{Z}_n^*\}.$$

Definition 41. For $m \in \mathbb{N}^*$, $r = (r_\delta) \in R(N)$ and $t \in \{0, \dots, m-1\}$ we define the map $\overline{\odot}_r : \mathbb{S}_{24m} \times \{0, \dots, m-1\} \to \{0, \dots, m-1\}$ where $[s]_{24m}\overline{\odot}_r t$ is uniquely determined by the relation

$$[s]_{24m}\overline{\odot}_r t \equiv_m ts + \frac{s-1}{24}\sigma_\infty(r).$$

Definition 42. For $m, M \in \mathbb{N}^*$, $r \in R(M)$ and $t \in \{0, \dots, m-1\}$ we define

$$P_{m,r}(t) := \{\gamma \odot_r t | \gamma \in \Gamma_0(N)^*\}, \tag{32}$$

and

$$\chi_{m,r}(t) := \prod_{t' \in P_{m,r}(t)} e^{2\pi i \frac{(1-m^2)(24t' + \sigma_\infty(r))}{24m}}. \tag{33}$$

Lemma 43 ([10], Lem. 3.11)**.** *Given $m, M, N \in \mathbb{N}^*$, $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, $r = (r_\delta) \in R(M)$ and $t \in \{0, \dots, m-1\}$ such that* (28) *holds. Then:*

(i) $\gamma \odot_r t = [a^2]_{24m}\overline{\odot}_r t$;

(ii) *For $[s]_{24m} \in \mathbb{S}_{24m}$ we have $P_{m,r}(t) = \{[s]_{24m}\overline{\odot}_r t' | t' \in P_{m,r}(t)\}$;*

(iii) $\chi_{m,r}(t) = e^{\frac{2\pi i \nu}{24}}$ *for some integer ν.*

Definition 44. Given m, M, $N \in \mathbb{N}^*$, $t \in \{0, \dots, m-1\}$, $r = (r_\delta) \in R(M)$ and $s = (s_\delta)_{\delta|N} \in R(N)$, we define

$$F(s, r, m, t)(\tau) := \prod_{\delta|N} \eta^{s_\delta}(\delta\tau) \prod_{t' \in P_{m,r}(t)} g_{m,t'}(\tau, r).$$

We are ready to state our main theorem:

Theorem 45 ("Main Theorem")**.** *Let $(m, M, N, t, r = (r_\delta)) \in \Delta^*$, $s = (s_\delta) \in R(N)$ and ν an integer such that $\chi_{m,r}(t) = e^{\frac{2\pi i \nu}{24}}$. Then $F(s, r, m, t) \in K(N)$ iff all the conditions* (34)-(37) *hold:*

$$|P_{m,r}(t)|w(r) + w(s) = 0; \tag{34}$$

$$\nu + |P_{m,r}(t)|m\sigma_\infty(r) + \sigma_\infty(s) \equiv 0 \pmod{24}; \tag{35}$$

$$|P_{m,r}(t)|\frac{mN}{M}\sigma_0(r) + \sigma_0(s) \equiv 0 \pmod{24}; \tag{36}$$

$$\left(\prod_{\delta|M}(m\delta)^{|r_\delta|}\right)^{|P_{m,r}(t)|} \Pi(s) \textit{ is a square.} \tag{37}$$

Proof. Let β be as in Theorem 37 and define

$$\xi := \beta^{|P_{m,r}(t)|}\chi_{m,r}^{ab}(t)e^{-\frac{\pi i a}{12}\left(\frac{c}{N}\sigma_0(s) - b\sigma_\infty(s) - 3w(s)\right)} \prod_{\delta|N}\left(\frac{c/\delta}{a}\right)^{s_\delta}.$$

For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)^*$ holds:

$$\begin{aligned}
&F(s,r,m,t)(\gamma\tau)\\
=&\left(\prod_{t'\in P_{m,r}(t)} g_{m,t'}(\gamma\tau)\right)\prod_{\delta|N}\eta^{s_\delta}(\delta\gamma\tau)\\
=&(-i(c\tau+d))^{\frac{|P_{m,r}(t)|w(r)+w(s)}{2}}\xi\prod_{t'\in P_{m,r}(t)} g_{m,\gamma\odot_r t'}(\tau)\cdot\prod_{\delta|N}\eta^{s_\delta}(\delta\tau)
\end{aligned}$$

(because of the relation (14) and $(\eta(\delta\gamma\tau) = (-i(c\tau+d))^{\frac{1}{2}}\epsilon(a,\delta b,\frac{c}{\delta},d)\eta(\delta\tau)$, $\gamma \in \Gamma_0(\delta)^*$, by (24), together with Theorem 37)

$$=(-i(c\tau+d))^{\frac{|P_{m,r}(t)|w(r)+w(s)}{2}}\xi\left(\prod_{t'\in P_{m,r}(t)} g_{m,t'}(\tau)\right)\prod_{\delta|N}\eta^{s_\delta}(\delta\tau)$$

(because of *(i)*-*(ii)* in Lemma 43).

$$\tag{38}$$

Assume (34)-(37). Then because of (34), the last line of (38) reduces to

$$\xi\left(\prod_{t'\in P_{m,r}(t)} g_{m,t'}(\tau)\right)\prod_{\delta|N}\eta^{s_\delta}(\delta\tau).$$

Next we prove $\xi = 1$.

First we note that for arbitrary positive integers N and δ with $\delta|N$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)^*$ one has

$$\left(\frac{c/\delta}{a}\right) = \left(\frac{c/\delta}{a}\right)\left(\frac{\delta^2}{a}\right) = \left(\frac{c\delta}{a}\right). \tag{39}$$

This follows from (23). We have that

$$\begin{aligned}
\xi = &\left(\prod_{\delta|M}\left(\frac{mc\delta}{a}\right)^{|r_\delta|} e^{-\frac{\pi i a}{12}\left(\frac{mc}{M}\sigma_0(r)-mb\sigma_\infty(r)-3w(r)\right)}\right)^{|P_{m,r}(t)|}\\
&\times e^{\frac{2\pi i ab\nu}{24}} e^{-\frac{\pi i a}{12}\left(\frac{c}{N}\sigma_0(s)-b\sigma_\infty(s)-3w(s)\right)}\prod_{\delta|N}\left(\frac{c\delta}{a}\right)^{s_\delta}
\end{aligned}$$

(by (30), and by $\chi_{m,r}^{ab}(t) = e^{\frac{2\pi iab\nu}{24}}$)

$$\begin{aligned}
=& e^{-\frac{\pi ia}{12}\left(|P_{m,r}(t)|\frac{mc}{M}\sigma_0(r)+\frac{c}{N}\sigma_0(s)\right)} \\
&\times e^{\frac{\pi iab}{12}(\nu+|P_{m,r}(t)|m\sigma_\infty(r)+\sigma_\infty(s))} \\
&\times e^{\frac{\pi ia}{4}(|P_{m,r}(t)|w(r)+w(s))} \\
&\times \left(\prod_{\delta|M}\left(\frac{mc\delta}{a}\right)^{|r_\delta|}\right)^{|P_{m,r}(t)|}\prod_{\delta|N}\left(\frac{c\delta}{a}\right)^{s_\delta} \\
=& \left(\prod_{\delta|M}\left(\frac{mc\delta}{a}\right)^{|r_\delta|}\right)^{|P_{m,r}(t)|}\prod_{\delta|N}\left(\frac{c\delta}{a}\right)^{s_\delta}
\end{aligned}$$

(by (34), (35) and (36))

$$=\left(\frac{c}{a}\right)^{|P_{m,r}(t)|w(r)+w(s)}\left(\frac{\left(\prod_{\delta|M}m\delta^{|r_\delta|}\right)^{|P_{m,r}(t)|}\Pi(s)}{a}\right)$$

(by (23), and because $\left(\frac{i}{j}\right)^k = \left(\frac{i}{j}\right)^{|k|}$ for all integers i, j, k, with $\gcd(i, j) = 1$ and such that $\left(\frac{i}{j}\right)$ is defined.)

$$=1$$

(by (37) and (34)).

Since we have proved that $F(\gamma\tau) = F(\tau)$ for all $\gamma \in \Gamma_0(N)^*$, we have $F(\tau) = F(\gamma\tau)$ for all $\gamma \in \Gamma_0(N)$ because of Lemma 32.

So we have proved condition *(ii)* of Definition 1. Condition *(iii)* follows from Theorem 39 and Remark 4. Condition *(i)*, follows from the fact that $\eta(\tau)$ is analytic on $\mathbb{H}$ and that $F(\tau)$ generated by finite products and sums of terms of the form $\eta((A\tau + B)/C)$ which are also analytic on $\mathbb{H}$.

Now assume that $F(\tau) \in K(N)$. Taking both sides of (38) to the 24-th power we obtain

$$F^{24}(\gamma\tau) = (c\tau + d)^{12\left(|P_{m,r}(t)|w(r)+w(s)\right)}F^{24}(\tau).$$

Since $F(\gamma\tau) = F(\tau)$ (by assumption) it follows that

$$(c\tau + d)^{12\left(|P_{m,r}(t)|w(r)+w(s)\right)} = 1,$$

for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)^*$ and $\tau \in \mathbb{H}$. This implies (34) because there exists $\gamma \in \Gamma_0(N)^*$ with $c \neq 0$ for example $\gamma = \begin{pmatrix} 1 & 1 \\ N & N+1 \end{pmatrix}$.

Consequently, $\xi = \xi(a,b,c,d)$ reduces to:

$$\begin{aligned} \xi = & \left(\prod_{\delta|M} \left(\frac{mc\delta}{a} \right)^{|r_\delta|} e^{-\frac{\pi i a}{12}\left(\frac{mc}{M}\sigma_0(r) - mb\sigma_\infty(r)\right)} \right)^{|P_{m,r}(t)|} \\ & \cdot e^{\frac{2\pi i a b \nu}{24}} e^{-\frac{\pi i a}{12}\left(\frac{c}{N}\sigma_0(s) - b\sigma_\infty(s)\right)} \prod_{\delta|N} \left(\frac{c\delta}{a} \right)^{s_\delta}. \end{aligned} \tag{40}$$

Taking $\gamma = \begin{pmatrix} 1 & 1 \\ 24N & 24N+1 \end{pmatrix}$ in (40) and using $\xi(1,1,24N,24N+1) = 1$ because of $F(\tau) = F(\gamma\tau)$ for $\gamma = \begin{pmatrix} 1 & 1 \\ 24N & 24N+1 \end{pmatrix}$, we find (35) must hold and consequently for general $\xi = \xi(a,b,c,d)$:

$$\xi = \left(\prod_{\delta|M} \left(\frac{mc\delta}{a} \right)^{|r_\delta|} e^{-\frac{\pi i a}{12}\frac{mc}{M}\sigma_0(r)} \right)^{|P_{m,r}(t)|} e^{-\frac{\pi i a}{12}\frac{c}{N}\sigma_0(s)} \prod_{\delta|N} \left(\frac{c\delta}{a} \right)^{s_\delta} \tag{41}$$

Taking $\gamma = \begin{pmatrix} 1 & 1 \\ N & N+1 \end{pmatrix}$ in (41) and using $\xi(1,1,N,N+1) = 1$, we find that (36) must hold and consequently

$$\xi = \prod_{\delta|M} \left(\frac{mc\delta}{a} \right)^{|P_{m,r}(t)||r_\delta|} \prod_{\delta|N} \left(\frac{c\delta}{a} \right)^{s_\delta},$$

which together with (34) implies that

$$\begin{aligned} \xi &= \prod_{\delta|M} \left(\frac{m\delta}{a} \right)^{|P_{m,r}(t)||r_\delta|} \prod_{\delta|N} \left(\frac{\delta}{a} \right)^{s_\delta}, \\ &= \left(\frac{\left(\prod_{\delta|M} (m\delta)^{|r_\delta|} \right)^{|P_{m,r}(t)|} \Pi(s)}{a} \right) = 1. \end{aligned}$$

Next we observe that for each $a > 0$, with $\gcd(a, 6N) = 1$, there exist $b, c, d \in \mathbb{Z}$ such that $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)^*$. Therefore to prove (37), and thus our main Theorem 45, it is enough to prove the following Lemma.

Lemma 46. *Suppose $x \in \mathbb{Z}$. If $\left(\frac{x}{a}\right) = 1$ for all $a \in \mathbb{Z}$ with* $\gcd(a, 6N) = 1$, $a \equiv 5 \pmod 8$*, then x is a square.*

Proof. Assume that x is not a square. Write x uniquely as $x = 2^\alpha y$, where $\gcd(2, y) = 1$. Then

$$\begin{aligned} 1 = \left(\frac{x}{a}\right) = \left(\frac{2}{a}\right)^\alpha \left(\frac{y}{a}\right) = (-1)^{\alpha \frac{a^2-1}{8}} \left(\frac{y}{a}\right) \\ =(-1)^\alpha (-1)^{\frac{y-1}{2}\frac{a-1}{2}} \left(\frac{a}{y}\right) = (-1)^\alpha \left(\frac{a}{y}\right). \end{aligned} \tag{42}$$

If y is a square, then by (42) $\alpha \equiv 0 \pmod 2$, consequently x is a square. Now assume that y is not a square. Let y_0 be the square free part of y. Assume that $y_0 = p_1 \cdots p_k$. Let $q_1, \ldots, q_n$ be the primes dividing N and not dividing $6y_0$. Then there exists $d \in \{0, 1, \ldots, p_1 - 1\}$ such that $\left(\frac{d}{y}\right) = (-1)^{\alpha+1}$. By Chinese remaindering we can solve the system

$$\begin{aligned} a &\equiv 5 \pmod 8, \\ a &\equiv d \pmod{p_1}, \\ a &\equiv 1 \pmod{p_i}, \quad \text{for } i = 2, \ldots, k, \\ a &\equiv 1 \pmod{q_i}, \quad \text{for } i = 1, \ldots, n, \end{aligned}$$

and if $p_1 \neq 3$, $a \equiv 1 \pmod 3$.

Hence $\gcd(a, 6N) = 1$ and $\left(\frac{x}{a}\right) = -1$, a contradiction to (42). □

Consequently x must be a square and (37) is proven. □

We do the order estimation in view of Lemma 2 together with (11).

Theorem 47. *Let $(m, M, N, (r_\delta)) \in \Delta$ and $s = (s_\delta) \in R(N)$. Assume for $F(s, r, m, t)$ as in Definition 44 that $F(s, r, m, t) \in K(N)$. Let p and p^* be as in Definition 38. Let $\{\gamma_0, \ldots, \gamma_n\} \subseteq \mathrm{SL}_2(\mathbb{Z})$ be a complete set of representatives of the double cosets $\Gamma_0(N)\backslash\mathrm{SL}_2(\mathbb{Z})/\mathrm{SL}_2(\mathbb{Z})_\infty$ with $\gamma_0 = id$.*

Then for $i = 0, \ldots, n$,

$$\mathrm{ord}^N_{\gamma_i}(F(s, r, m, t)) \geq \frac{N}{\gcd(c^2, N)} \Big(|P_{m,r}(t)| p(\gamma_i, r) + p^*(\gamma_i, s)\Big).$$

Proof. Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)\gamma_i\mathrm{SL}_2(\mathbb{Z})_\infty, 1 \leq i \leq n$. By Theorem 39

$$q^{-\{|P_{m,r}(t)|p(\gamma,r)+p^*(\gamma,s)\}}(c\tau + d)^{-(\frac{1}{2}|P_{m,r}(t)|w(r)+\frac{1}{2}w(s))} F(s, r, m, t)(\gamma\tau)$$

is a Laurent series in powers of $q^{1/k}$ with 0 principal part, for some $k \in \mathbb{N}^*$. Noting $\frac{1}{2}|P_{m,r}(t)|w(r) + \frac{1}{2}w(s) = 0$ because of (34), the desired estimate follows. □

.

Given $m,\ M,\ N \in \mathbb{N}^*$, $t \in \{0, \dots, m-1\}$ and $r = (r_\delta) \in R(M)$, we want to decide if there exists $s = (s_\delta) \in R(N)$ such that

$$F(s, r, m, t) \in \langle E(N) \rangle_{\mathbb{Q}}. \tag{43}$$

We solve this problem by splitting in two cases.

First case: $(m, M, N, t, (r_\delta)) \in \Delta^*$*:*

Note that by [8, Thm. 1.64] $\langle E(N) \rangle_{\mathbb{Q}} \subseteq K(N)$. By Theorem 45 we know that there exists an $s = (s_\delta)_{\delta|N} \in R(N)$ such that $F(s, r, m, t) \in K(N)$ iff the conditions (34)-(37) are satisfied. By Lemma 20 there exists a modular function $\mu(\tau) := \prod_{\delta|N} \eta^{\mu_\delta}(\delta\tau)$ such that $\mathrm{ord}_\gamma^N(\mu) > 0$ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z}) \setminus \Gamma_0(N)$. Then by Theorem 47 we have that

$$\mathrm{ord}_\gamma^N(F(s, r, m, t)) \geq \frac{N}{\gcd(c^2, N)}(|P_{m,r}(t)|p(\gamma, r) + p^*(\gamma, s)),$$

and, consequently, for $k \in \mathbb{N}$,

$$\mathrm{ord}_\gamma^N(\mu^k F(s, r, m, t)) \geq k\,\mathrm{ord}_\gamma^N(\mu) + \frac{N}{\gcd(c^2, N)}(|P_{m,r}(t)|p(\gamma, r) + p^*(\gamma, s)), \tag{44}$$

for all $\gamma \in \mathrm{SL}_2(\mathbb{Z}) \setminus \Gamma_0(N)$.

Next we find a complete set of double coset representatives $R := \{\gamma_0, \gamma_1, \dots, \gamma_n\}$ for the double cosets $\Gamma_0(N)\backslash\mathrm{SL}_2(\mathbb{Z})/\mathrm{SL}_2(\mathbb{Z})_\infty$. We choose the k such that for each $\gamma \in R$, $\gamma \notin \Gamma_0(N)$, the right hand side of the expression (44) is positive. In this way

$$\mu^k F(s, r, m, t) \in K^\infty(N).$$

Next we use Lemma 25 to compute a finite set of generators $m_1, \dots, m_s$ for $E^\infty(N)$. That is

$$E^\infty(N) = \{m_1^{\alpha_1} m_s^{\alpha_2} \cdots m_s^{\alpha_s} : (\alpha_1, \dots, \alpha_s) \in \mathbb{N}^s\}.$$

Next we input $m_1, \dots, m_s$ to Algorithm AB to obtain a t and $z_1, \dots, z_r$ such that $\langle z_1, \dots, z_r \rangle_{\mathbb{Q}[t]} = \langle E^\infty(N) \rangle_{\mathbb{Q}}$. By Lemma 28 there exists a k' (which is constructed as in the proof of Lemma 28) such that $\mu^{k'} \langle E(N) \rangle_{\mathbb{Q}} \subseteq \langle E^\infty(N) \rangle_{\mathbb{Q}}$. Next we apply the Algorithm MC in order to see if $\mu^{k+k'} F(s, r, m, t) \in \langle z_1, \dots, z_r \rangle_{\mathbb{Q}[t]}$. In case this is true we apply the Algorithm MW to compute $c_1(t), \dots, c_r(t) \in \mathbb{Q}[t]$ such that

$$\mu^{k+k'} F(s, r, m, t) = c_1(t) z_1 + c_2(t) z_2 + \cdots + c_r(t) z_r.$$

This implies that

$$F(s, r, m, t) = \mu^{-(k+k')}\{c_1(t) z_1 + c_2(t) z_2 + \cdots + c_r(t) z_r\} \in \langle E(N) \rangle_{\mathbb{Q}}.$$

In this case the algorithm succeeds. In case the Algorithm MC returns "False", then $F(s, r, m, t) \notin \langle E(N) \rangle_{\mathbb{Q}}$.

Second case: $(m, M, N, t, (r_\delta)) \notin \Delta^*$*:* We decide the existence of a suitable $s \in R(N)$ by checking 24 cases: For each fixed $j \in \{0, \dots, 23\}$ we examine whether there exists $s = (s_\delta)_{\delta|N} \in R(N)$ with $\sigma_0(s) \equiv j \pmod{24}$ such that

$$F(s, r, m, t) \in \langle E(N) \rangle_{\mathbb{Q}} \subseteq K(N). \tag{45}$$

Let N' be minimal such that $(m, M, N', t, (r_\delta)) \in \Delta^*$. (Note that such N' always exists.) Given $x \in R(N)$, we define $\overline{x} = (\overline{x}_\delta) \in R(\mathrm{lcm}(N, N'))$ by

$$\overline{x}_\delta := \begin{cases} x_\delta, & \text{if } \delta | N \\ 0, & \text{otherwise} \end{cases}$$

In particular, if (45), then $F(\overline{s}, r, m, t) \in K(\mathrm{lcm}(N, N'))$. By Theorem 45, $(\overline{s}_\delta)_{\delta|N}$ satisfies (34)-(37). By solving (34)-(37), for $\overline{s} \in R(\mathrm{lcm}(N, N'))$ with $\sigma_0(s) \equiv j \pmod{24}$ and $\overline{s}_\delta = 0$ if $\delta \nmid N$, we determine $\overline{s}$ uniquely up to addition by an element $\overline{u}$, where $u = (u_\delta) \in R(N, 0, 0, 0, 1)$. But we know that $\prod_{\delta|N} \eta^{u_\delta}(\delta\tau) \in E(N)$. So in order to prove the existence of $s = (s_\delta)$ such that $F(s, r, m, t) \in K(N)$ it is sufficient to determine it up to addition by an element in $R(N, 0, 0, 0, 1)$.

Following the steps in the previous case we obtain by Theorem 47,

$$\begin{aligned} &\mathrm{ord}_\gamma^{\mathrm{lcm}(N,N')}(F(s, r, m, t)) \\ &\qquad \geq \frac{\mathrm{lcm}(N, N')}{\gcd(c^2, \mathrm{lcm}(N, N'))}\Big(|P_{m,r}(t)|p(\gamma, r) + p^*(\gamma, s)\Big). \end{aligned}$$

Let $\mu \in K^\infty(N)$ be as in Lemma 20. Then

$$\begin{aligned} &\mathrm{ord}_\gamma^{\mathrm{lcm}(N,N')}(\mu^k F(s, r, m, t)) \\ \geq &k \,\mathrm{ord}_\gamma^{\mathrm{lcm}(N,N')}(\mu) + \frac{\mathrm{lcm}(N, N')}{\gcd(c^2, \mathrm{lcm}(N, N'))}\Big(|P_{m,r}(t)|p(\gamma, r) + p^*(\gamma, s)\Big). \end{aligned} \tag{46}$$

Next we find a complete set of double coset representatives $R := \{\gamma_1, \ldots, \gamma_n\}$ for the double cosets

$$\Gamma_0(\mathrm{lcm}(N, N'))\backslash \mathrm{SL}_2(\mathbb{Z})/\mathrm{SL}_2(\mathbb{Z})_\infty.$$

We choose the k in (46) such that for all $\gamma \in R \cap (\mathrm{SL}_2(\mathbb{Z})\backslash\Gamma_0(N))$: $\mathrm{ord}_\gamma^{\mathrm{lcm}(N,N')}(\mu^k F(s, r, m, t))$ is positive. Then

$$\mu^k F(s, r, m, t) \in K^\infty(N).$$

Finally one needs to check if

$$\mu^k F(s, r, m, t) \in \langle E^\infty(N)\rangle_{\mathbb{Q}}, \tag{47}$$

which can be verified analogously to Case 1. If (47) is false for each $j \in \{0, \ldots, 23\}$, then $F(s, r, m, t) \notin \langle E(N)\rangle_{\mathbb{Q}}$.

3.2. *A Necessary Condition Regarding the Existence of Certain Ramanujan-Kolberg Identities*

Given $M \in \mathbb{N}^*$, $r \in R(M)$ and integers $m > t \geq 0$, we show that there exists $N \in \mathbb{N}^*$ and $s = (s_\delta) \in R(N)$ such that

$$A := q^{\alpha(m,t,r,s)} \prod_{\delta|N} \prod_{n=1}^{\infty} (1 - q^{\delta n})^{s_\delta} \prod_{t' \in P_{m,r}(t)} \sum_{n=0}^{\infty} a_r(mn + t)q^n \in K(N),$$

where

$$\alpha(m, t, r, s) := \frac{\sigma_\infty(s)}{24} + \sum_{t' \in P_{m,r}(t)} \frac{24t' + \sigma_\infty(r)}{24m}.$$

The existence of such (s, N) is a necessary condition for the existence of $B \in \langle E(N)\rangle_{\mathbb{Q}}$ such that (A, B) is a Ramanujan-Kolberg identity.

Lemma 48. *Given $x_1, x_2, x_3, d \in \mathbb{N}^*$. Let $p_1, \ldots, p_k$ be the odd primes dividing d. Let $N \in \mathbb{N}^*$ be such that $N \equiv 0 \pmod{24 p_1 \cdots p_k}$. Then there exists $(s_\delta) \in R(N, -x_1, -x_2, -x_3, d)$.*

Proof. Noting that $s_1 = -x_1 - \sum_{\substack{\delta|N \\ \delta\neq 1}} s_\delta$ we need to find $(s_\delta)_{\delta|N, \delta\neq 1}$ such that $d \prod_{\delta|N} \delta^{s_\delta}$ is a rational square and

$$\begin{aligned} x_2 - x_1 + \sum_{\substack{\delta|N \\ \delta\neq 1}} (\delta - 1) s_\delta &\equiv 0 \pmod{24}, \\ x_3 - N x_1 + \sum_{\substack{\delta|N \\ \delta\neq 1}} (\frac{N}{\delta} - N) s_\delta &\equiv 0 \pmod{24} \end{aligned}$$

which because of $N \equiv 0 \pmod{24}$ may be rewritten as:

$$x_2 - x_1 + \sum_{\substack{\delta|N \\ \delta\neq 1}} (\delta - 1) s_\delta \equiv 0 \pmod{24}, \tag{48}$$

$$x_3 + \sum_{\substack{\delta|N \\ \delta\neq 1}} \frac{N}{\delta} s_\delta \equiv 0 \pmod{24}. \tag{49}$$

By the Chinese remainder theorem it is sufficient to define $(s_\delta)_{\delta|N, \delta\neq 1}$ modulo 8 and modulo 3. Assume that $d = 2^{v_0} p_1^{v_1} \cdots p_k^{v_k}$ and $2^\phi | N$, $2^{\phi+1} \nmid N$. Then we choose $s_\delta \equiv 0 \pmod 8$ if $\delta \notin \{2, 4, 2^\phi, p_1, \ldots, p_k\}$. We choose $s_{p_j} \equiv v_j \pmod 8$ for $j \in \{1, \ldots, k\}$. Then $d \prod_{\delta|N} \delta^{s_\delta}$ is a square if

$$s_2 + \phi s_{2^\phi} \equiv 0 \pmod 8. \tag{50}$$

We see that s_2, s_4 and a_{2^ϕ} must satisfy

$$y_1 + s_2 + 3 s_4 + (2^\phi - 1) a_{2^\phi} \equiv 0 \pmod 8, \tag{51}$$

$$y_2 + \frac{N}{2} s_2 + \frac{N}{4} s_4 + \frac{N}{2^\phi} a_{2^\phi} \equiv 0 \pmod 8 \tag{52}$$

where

$$\begin{aligned} y_1 &:= x_2 - x_1 + \sum_{\substack{\delta|N \\ \delta\notin\{1,2,4,2^\phi\}}} (\delta - 1) s_\delta, \\ y_2 &:= x_3 + \sum_{\substack{\delta|N \\ \delta\notin\{1,2,4,2^\phi\}}} \frac{N}{\delta} s_\delta. \end{aligned}$$

There exists a solution (s_2, s_4, s_{2^ϕ}) to the equations (50)-(52) because of

$$\begin{vmatrix} 1 & 0 & \phi \\ 1 & 3 & 2^\phi - 1 \\ \frac{N}{2} & \frac{N}{4} & \frac{N}{2^\phi} \end{vmatrix} \equiv \begin{vmatrix} 1 & 0 & \phi \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} \equiv 1 \pmod 2.$$

Next we need to specify $(s_\delta)_{\delta|N,\delta\neq 1}$ modulo 3 such that (48)-(49) are satisfied. We choose $s_\delta \equiv 0 \pmod 3$ if $\delta \neq 2, N$. Then we have

$$\begin{aligned} x_2 - x_1 + s_2 - s_N &\equiv_3 0, \\ x_3 + \frac{N}{2} s_2 + s_N &\equiv_3 0. \end{aligned}$$

Since $\frac{N}{2} \equiv_3 0$ because of $24|N$ we see that this system has the solution $(s_2, s_N) = (x_1 - x_2 - x_3, -x_3)$ modulo 3. □

Corollary 49. *For all $(m, M, N, t, r = (r_\delta)) \in \Delta^*$ with $24|N$ there exists $s = (s_\delta)_{\delta|N} \in R(N)$ such that $F(s, r, m, t) \in K^\infty(N)$.*

Proof. Lemma 48 implies that if $24|N$ we may always find a solution $s \in R(N)$ to (34)-(37). The function that corresponds to this solution is then multiplied by a high enough power of a $\mu \in K^\infty(N)$ as in Lemma 20 such that the result will be in $K^\infty(N)$. □

Remark 50. We note that Corollary 49 implies that if $24|N$ and $(m, M, N, (r_\delta), t) \in \Delta^*$, then we can find a Ramanujan-Kolberg identity, whenever $\langle E^\infty(N)\rangle_{\mathbb{Q}} = K^\infty(N)$. That this is the case was conjectured by Newman [6, 7]. Assuming Newman's conjecture we thus could always find Ramanujan-Kolberg identities. But this conjecture is not true in general, as shown in the next subsection.

3.3. A Counterexample to a Conjecture by Newman

We want to find generators of $z_1, z_2, \ldots, z_n$ and $t \in E^\infty(49)$ such that every element $x \in \langle E^\infty(49)\rangle_{\mathbb{Q}}$ is expressible as

$$x = c_0(t) + c_1(t)z_1 + \cdots + c_n(t)z_n, \quad c_i(x) \in \mathbb{Q}[x].$$

We know from Example 27 that $E^\infty(49)$ is generated by

$$\begin{aligned} X_1 &:= \frac{\eta(\tau)}{\eta(49\tau)} \\ X_2 &:= \left(\frac{\eta(7\tau)}{\eta(49\tau)}\right)^4 \\ X_3 &:= \frac{\eta^8(7\tau)}{\eta(\tau)\eta^7(49\tau)}. \end{aligned}$$

We apply the Algorithm AB to the input $\{X_1, X_2, X_3\}$. Following the algorithm we start with $L := \{\}$ and $M := \{X_1, X_2, X_3\}$. Since $L \neq M$ we apply the Algorithm MB to M. The output is then $z_1 := X_2$ and $t := X_1$ because of

$$\begin{aligned} X_3 =& X_1^6 + 7X_1^5 + 21X_1^4 + 49X_1^3 \\ &+ (147 + 7X_2)X_1^2 + (343 + 35X_2)X_1 + 49X_2 + 343. \end{aligned} \tag{53}$$

Set $L := \{z_1\}$, $M := L$. At the next step we have to check if $z_1^2 = X_2^2 \in \langle z_1\rangle_{\mathbb{Q}[t]}$. We find that

$$\begin{aligned} X_2^2 =& X_1^7 + 7X_1^6 + 21X_1^5 + 49X_1^4 \\ &+ (147 + 7X_2)X_1^3 + (343 + 35X_2)X_1^2 + (49X_2 + 343)X_1. \end{aligned} \tag{54}$$

Consequently, $z_1^2 = X_2^2 \in \langle z_1 \rangle_{\mathbb{Q}[t]}$ and the algorithm stops returning $\{z_1\} = \{X_2\}$ and $t = X_1$.

Using (54) one finds

$$(X_2 - 49X_1 - 49)^2 = (X_1^2 + 7X_1 + 7)^2(X_1^3 - 7X_1^2 + 56X_1 - 49) \\ + (X_1^2 + 7X_1 + 7)(7X_1 - 14)(X_2 - 49X_1 - 49).$$

Dividing the above equation by $(X_1^2+7X_1+7)^2$ and setting $Z := \frac{X_2-49X_1-49}{X_1^2+7X_1+7}$ we obtain: $Z^2 = Z(7X_1 - 14) + X_1^3 - 7X_1^2 + 56X_1 - 49$. This last equation implies that Z has only poles at infinity because it is integral over $\mathbb{Q}[X_1]$ and X_1 has only poles at infinity. Hence $Z \in K^\infty(49)$. However using Algorithm MC we find that $Z \notin \langle E^\infty(49)\rangle_{\mathbb{Q}}$. This contradicts Newman's conjecture above.

4. Examples of Identities

4.1. A Ramanujan Identity Involving $\sum p(11n+6)q^n$

Given $m = 11$, $t = 6$, $M = 1$, $N = 22$ and $(r_1) = (-1) \in R(1)$ we want to decide if there exists $(s_\delta) \in R(22)$ such that (43). We are in the "First case" of the algorithm because $(m, M, N, t, (r_\delta)) \in \Delta^*$. We have to solve (34)-(37) for $s = (s_\delta)_{\delta|N} \in R(N)$:

$$\begin{aligned} -1 + w(s) &= 0; \\ -11 + \sigma_\infty(s) &\equiv 0 \pmod{24}; \\ -242 + \sigma_0^{22}(s) &\equiv 0 \pmod{24}; \\ \sqrt{11\Pi(s)} &\in \mathbb{Z}. \end{aligned}$$

Next we need a function $\mu(\tau) = \prod_{\delta|22} \eta^{\mu_\delta}(\delta\tau)$ such that $\mathrm{ord}_\gamma^{22}(\mu) > 0$ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. Note that if (s_δ) satisfies (34)-(37) above then so does $(s_\delta + k\mu_\delta)$. Furthermore, we have $\mu^k F(s, r, m, t) = F(s + k\mu, r, m, t)$ so that instead of constructing μ we solve directly (44):

$$p(\gamma, r) + p^*(\gamma, s) \geq 0, \tag{55}$$

for all $\gamma \in \mathrm{SL}_2(\mathbb{Z}) \setminus \Gamma_0(22)$. We can prove that in general

$$p(\gamma, r) = \min_{\substack{d|m \\ \gcd(d,c)=1}} \frac{1}{24} \sum_{\delta|M} r_\delta \frac{d\gcd^2(\delta, \frac{m}{d}c)}{\delta \frac{m}{d}}. \tag{56}$$

Since $\gamma \in \Gamma_0(22)$ iff $22|c$, we find that (55) is equivalent to

$$\Big(\min_{\substack{d|m \\ \gcd(d,c)=1}} \frac{1}{24} \sum_{\delta|M} r_\delta \frac{\gcd^2(\delta d, mc)}{\delta m} \Big) + \frac{1}{24} \sum_{\delta|N} s_\delta \frac{\gcd^2(\delta, c)}{\delta} \geq 0 \tag{57}$$

for all $c \in \mathbb{Z}$ such that $22 \nmid c$. From this new expression (56) of $p(\gamma, r)$ we see that the value of the expression is the same for $c = c_1$ and $c = c_2$ if $\gcd(c_1, 22) = \gcd(c_2, 22)$. Hence we only need to verify (57) for $c \in \{1, 2, 11\}$. In particular we find that

$$(s_\delta) = (s_1, s_2, s_{11}, s_{22}) = (10, 2, 11, -22)$$

satisfies (57) and (34)-(37). Consequently,

$$F(\tau) := q^{\frac{13}{24}} \frac{\eta^{10}(\tau)\eta^2(2\tau)\eta^{11}(11\tau)}{\eta^{22}(22\tau)} \sum_{n=0}^{\infty} p(11n+6)q^n \in K^{\infty}(22).$$

In the next step we need to multiply the above expression by an appropriate power of μ. In general, one can compute a suitable μ by solving a corresponding system of Diophantine inequalities. But for reasons of space, we will try directly to see if the above expression lies in $\langle E^{\infty}(22)\rangle_{\mathbb{Q}}$. Using the proof in Lemma 25 and removing the generators that can be written in terms of other generators, we find that a set of generators of $E^{\infty}(22)$ is given by:

$$\frac{\eta^8(2\tau)\eta^4(11\tau)}{\eta^4(\tau)\eta^8(22\tau)}, \frac{\eta(2\tau)\eta^{11}(11\tau)}{\eta(\tau)\eta^{11}(22\tau)}, \frac{\eta^7(\tau)\eta^3(11\tau)}{\eta^3(2\tau)\eta^7(22\tau)}, \frac{\eta^6(2\tau)\eta^6(11\tau)}{\eta^2(\tau)\eta^{10}(22\tau)},$$
$$\frac{\eta^4(2\tau)\eta^8(11\tau)}{\eta^{12}(22\tau)}, \frac{\eta^2(\tau)\eta^2(2\tau)\eta^{10}(11\tau)}{\eta^{14}(22\tau)}, \frac{\eta^4(\tau)\eta^{12}(11\tau)}{\eta^{16}(22\tau)}.$$

Applying Algorithm AB with this input we obtain t and $\{z_1, z_2\}$ where

$$t := \frac{3}{88}\frac{\eta^7(\tau)\eta^3(11\tau)}{\eta^3(2\tau)\eta^7(22\tau)} + \frac{1}{11}\frac{\eta^8(2\tau)\eta^4(11\tau)}{\eta^4(\tau)\eta^8(22\tau)} - \frac{1}{8}\frac{\eta(2\tau)\eta^{11}(11\tau)}{\eta(\tau)\eta^{11}(22\tau)},$$

$$z_1 := -\frac{5}{88}\frac{\eta^7(\tau)\eta^3(11\tau)}{\eta^3(2\tau)\eta^7(22\tau)} + \frac{2}{11}\frac{\eta^8(2\tau)\eta^4(11\tau)}{\eta^4(\tau)\eta^8(22\tau)} - \frac{1}{8}\frac{\eta(2\tau)\eta^{11}(11\tau)}{\eta(\tau)\eta^{11}(22\tau)} - 3,$$

$$z_2 := \frac{1}{44}\frac{\eta^7(\tau)\eta^3(11\tau)}{\eta^3(2\tau)\eta^7(22\tau)} - \frac{3}{11}\frac{\eta^8(2\tau)\eta^4(11\tau)}{\eta^4(\tau)\eta^8(22\tau)} + \frac{5}{4}\frac{\eta(2\tau)\eta^{11}(11\tau)}{\eta(\tau)\eta^{11}(22\tau)}.$$

Using the Algorithm MW described we find that $F(\tau) \in \langle E^{\infty}(22)\rangle_{\mathbb{Q}}$:

$$\begin{aligned} & q^{\frac{13}{24}} \frac{\eta^{10}(\tau)\eta^2(2\tau)\eta^{11}(11\tau)}{\eta^{22}(22\tau)} \sum_{n=0}^{\infty} p(11n+6)q^n \\ =& 1078t^4 + 13893t^3 + 31647t^2 + 11209t - 21967 \\ & + z_1(187t^3 + 5390t^2 + 594t - 9581) \\ & + z_2(11t^3 + 2761t^2 + 5368t - 6754). \end{aligned} \tag{58}$$

Note that

$$88t = 3\frac{\eta^7(\tau)\eta^3(11\tau)}{\eta^3(2\tau)\eta^7(22\tau)} + 8\frac{\eta^8(2\tau)\eta^4(11\tau)}{\eta^4(\tau)\eta^8(22\tau)} - 11\frac{\eta(2\tau)\eta^{11}(11\tau)}{\eta(\tau)\eta^{11}(22\tau)}.$$

Hence

$$\begin{aligned} 88t \equiv& 3\frac{\eta^7(\tau)\eta^3(11\tau)}{\eta^3(2\tau)\eta^7(22\tau)} + 8\frac{\eta^8(2\tau)\eta^4(11\tau)}{\eta^4(\tau)\eta^8(22\tau)} \\ \equiv& 3\frac{\eta^{40}(\tau)}{\eta^{80}(2\tau)} + 8\frac{\eta^{40}(\tau)}{\eta^{80}(2\tau)} \equiv 0 \pmod{11}. \end{aligned}$$

Similarly, using $\frac{\eta^8(\tau)}{\eta^4(2\tau)} \equiv 1 \pmod 8$ we find

$$\begin{aligned} 88t \equiv & 3\frac{\eta^7(\tau)\eta^3(11\tau)}{\eta^3(2\tau)\eta^7(22\tau)} - 11\frac{\eta(2\tau)\eta^{11}(11\tau)}{\eta(\tau)\eta^{11}(22\tau)} \\ \equiv & 3\frac{\eta^7(\tau)\eta^3(11\tau)}{\eta^3(2\tau)\eta^7(22\tau)} - 11\frac{\eta^7(\tau)\eta^3(11\tau)}{\eta^3(2\tau)\eta^7(22\tau)} \equiv 0 \pmod 8. \end{aligned}$$

This implies that the q-expansion of t has integer coefficients. Analogously, one can prove the same for z_1 and z_2. Consequently, (58) implies $p(11n+6) \equiv 0 \pmod{11}$ because each coefficient on the right side of (58) is an integer divisible by 11.

4.2. A Ramanujan-Kolberg Identity Involving Broken 2-Diamonds

In [1] George Andrews and Peter Paule introduce a new kind of partitions called broken k-diamond partitions which they denote by $\Delta_k(n)$. In their paper they conjecture that

$$\Delta_2(25n+14) \equiv 0 \pmod 5.$$

Chan [3] proved this conjecture and found and also proved the related congruence

$$\Delta_2(25n+24) \equiv 0 \pmod 5.$$

We will give a different proof of these two congruences here. From [1] we know that

$$\sum_{n=0}^{\infty} \Delta_2(n) q^n = \prod_{n=1}^{\infty} \frac{(1-q^{2n})(1-q^{5n})}{(1-q^n)^3(1-q^{10n})}.$$

Let

$$\sum_{n=0}^{\infty} a(n) q^n := \prod_{n=1}^{\infty} \frac{(1-q^n)^2(1-q^{2n})}{(1-q^{10n})}.$$

Because of $(1-q^n)^5 \equiv 1-q^{5n} \pmod 5$ it follows that $a(n) \equiv \Delta_2(n) \pmod 5$. Consequently, it is sufficient to prove

$$a(25n+14) \equiv a(25n+25) \equiv 0 \pmod 5. \tag{59}$$

Given $m = 25$, $t = 14$, $M = 10$, $N = 10$ and $(r_\delta) = (2, 1, 0, -1) \in R(10)$ we want to decide if there exists $(s_\delta) \in R(10)$ such that (43). We are again in the "First case" of the algorithm. Analogously to the previous example we find that

$$q^{3/2}\frac{\eta^{12}(2\tau)\eta^{10}(5\tau)}{\eta^6(\tau)\eta^{20}(10\tau)}\Big(\sum_{n=0}^{\infty} a(25n+14)q^n\Big)\Big(\sum_{n=0}^{\infty} a(25n+24)q^n\Big) \in K^{\infty}(10).$$

The generators of $E^{\infty}(10)$ are given by:

$$\frac{\eta^3(\tau)\eta(5\tau)}{\eta(2\tau)\eta^3(10\tau)}, \frac{\eta^4(2\tau)\eta^2(5\tau)}{\eta^2(\tau)\eta^4(10\tau)}, \frac{\eta(2\tau)\eta^5(5\tau)}{\eta(\tau)\eta^5(10\tau)}.$$

We apply the Algorithm AB with this input and obtain

$$t = \frac{\eta^3(\tau)\eta(5\tau)}{\eta(2\tau)\eta^3(10\tau)}$$

and empty set {} for the generators $z_1,\ldots,z_r$. Consequently $\langle E^\infty(10)\rangle_{\mathbb{Q}} = \mathbb{Q}[t]$. By using the algorithm in Note 1 we find:

$$\begin{aligned} q^{3/2}\frac{\eta^{12}(2\tau)\eta^{10}(5\tau)}{\eta^{6}(\tau)\eta^{20}(10\tau)}\Big(\sum_{n=0}^{\infty} a(25n+14)q^n\Big)\Big(\sum_{n=0}^{\infty} a(25n+24)q^n\Big) \\ = 25(2t^4+28t^3+155t^2+400t+400). \end{aligned} \tag{60}$$

One can prove that if A and B are power series such that $AB \equiv 0 \pmod{25}$, then only three cases may occur: Case 1 where $A \equiv 0 \pmod{25}$, Case 2 with $B \equiv 0 \pmod{25}$, and Case 3 where $A \equiv B \equiv 0 \pmod{5}$. Since $a(14) = 5$ and $a(24) = 10$, it follows that we are in Case 3. Consequently, we have proved (59).

Remark 51. For $\Delta_2(n)$, instead of $a(n)$, we could also find an identity similar to (60). However, this identity turns out to be much bigger.

5. The Ideal of Relations Among Certain Eta Products

We note that given arbitrary eta quotients $m_1,\ldots,m_s \in K(N)$ we can find generators for the ideal generated by all possible relations among $m_1,\ldots,m_s$ in the following way. First we multiply $m_1,\ldots,m_s$ by an eta quotient $\mu \in K^\infty(N)$ as in Lemma 20 such that $m_1\mu,\ldots,m_s\mu \in K^\infty(N)$. We define $m_i' := \mu m_i$, $m_{s+1}' := \mu$. Note that every polynomial relation $f(m_1,\ldots,m_s) = 0$ can be rewritten multiplying μ^k with a sufficiently large k such that

$$\mu^k f(m_1,\ldots,m_s) = F(m_1',\ldots,m_s',m_{s+1}')$$

for some multivariate polynomial F.

Next we input $m_1',\ldots,m_s',m_{s+1}'$ to Algorithm AB.

Assume that the output is t and $b_1,\ldots,b_r$. In particular $b_ib_j \in \langle b_1,\ldots,b_r\rangle_{\mathbb{Q}[t]}$. Consequently, for every $i,j \in \{1,\ldots,r\}$ we have relations of the form

$$b_ib_j = c_{1ij}(t)b_1 + \cdots + c_{rij}(t)b_r \tag{61}$$

for some $c_{1ij}(t),\ldots,c_{rij}(t) \in \mathbb{Q}[t]$. These relations generate all relations among $t,b_1,\ldots,b_r$ because every other relation can be reduced using these relations to a relation of the form $c_1(t)b_1+\cdots+c_r(t)b_r = 0$. Since $\operatorname{pord}(t^{j_1}b_{i_1}) \neq \operatorname{pord}(t^{j_2}b_{i_2})$ whenever $(i_1,j_1) \neq (i_2,j_2)$, we find that such a relation can only exist iff $c_1(t) = \cdots = c_r(t) = 0$.

Next we recall that there exist multivariate polynomials f_i such that

$$m_i' = f_i(t,b_1,\ldots,b_r) \tag{62}$$

for $i = 1,\ldots,s+1$. Computing a Gröbner basis S_1 of the ideal generated by the relations (61) and (62) with the order $b_s > \cdots > b_1 > t > m_{s+1}' > m_s' > \cdots > m_1'$ we obtain that $S_2 := S_1 \cap C[m_1',\ldots,m_s',m_{s+1}']$ generate all the relations among $m_1',\ldots,m_s',m_{s+1}'$. We then substitute back $m_i' = m_i\mu$ and $m_{s+1}' = \mu$ and view $\langle S_2\rangle$ as an ideal in $\mathbb{Q}[m_1,\ldots,m_s,\mu]$. The ideal of relations among $m_1,\ldots,m_s$ is then given by

$$I := \{f(m_1,\ldots,m_s) : \exists_{k\in\mathbb{N}}\mu^k f(m_1,\ldots,m_s) \in \langle S_2\rangle\}.$$

The generators of the ideal I are found by computing a Gröbner basis S_3 of the ideal $\langle S_2 \cup \{1-\mu z\}\rangle$ with the order $z > \mu > m_s > \cdots > m_1$. Then $I = \langle S_3 \cap \mathbb{Q}[m_1,\ldots,m_s]\rangle$.

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