

3-loop Massive $O(T_F^2)$ Contributions to the DIS Operator Matrix Element A_{gg} J. Ablinger^b, J. Blümlein^a, A. De Freitas^a, A. Hasselhuhn^{a,b}, A. von Manteuffel^c, M. Round^{a,b,*}, C. Schneider^b^aDeutsches Elektronen–Synchrotron, DESY, Platanenallee 6, D-15738 Zeuthen, Germany^bInstitute for Symbolic Computation (RISC), Johannes Kepler University, Altenbergerstraße 69, A-4040 Linz, Austria^cPRISMA Cluster of Excellence, Institute of Physics, J. Gutenberg University Mainz, D-55099 Mainz, Germany**Abstract**

Contributions to heavy flavour transition matrix elements in the variable flavour number scheme are considered at 3-loop order. In particular a calculation of the diagrams with two equal masses that contribute to the massive operator matrix element $A_{gg,Q}^{(3)}$ is performed. In the Mellin space result one finds finite nested binomial sums. In x -space these sums correspond to iterated integrals over an alphabet containing also square-root valued letters.

Keywords: deep-inelastic scattering, heavy quark

1. Introduction

Perturbative QCD contributions to the deep-inelastic structure functions play an important role in understanding the structure of the proton; to measure $\alpha_s(M_Z^2)$ [1], the parton distribution functions [2], and the mass of the charm quark m_c [3] at high precision. This applies to the data analyses at HERA [4] and future measurements at planned ep -facilities like the EIC [5] and LHeC [6]. At the present accuracy the QCD corrections both for the light and heavy flavours are requested to be known to 3-loop order. Feynman diagrams with two massive lines begin contributing to the structure functions at 3-loops. Both diagrams with two equal and two unequal masses contribute. Although this work focuses only on the equal mass diagrams there has also been progress on the unequal mass diagrams [7]. One may work in the asymptotic regime of large virtuality $Q^2 \gg m^2$. Here Q^2 denotes the virtuality of the exchanged gauge boson and m is the heavy quark mass. For the structure function $F_2(x, Q^2)$ it is known that the asymptotic regime holds to percent-level accuracy at NLO and for scales $Q^2/m^2 \gtrsim 10$ [8].

The heavy flavour Wilson coefficients are known to factorise into light flavour coefficients $C_{i,(2,L)}$ and process independent operator matrix elements (OMEs) A_{ij} , see [8, 9] for the corresponding relations. The light flavour Wilson coefficients have been calculated to 3-loop order in Ref. [10].

For the equal mass diagrams that contribute to $F_2(x, Q^2)$ a set of fixed Mellin moments has been calculated [11] in the asymptotic regime. Meanwhile the complete set of logarithmic contributions to $F_2(x, Q^2)$ is known [12] and for a significant list of operator matrix elements (OMEs) and Wilson coefficients; $L_{q,2}^{(3),PS}$, $L_{g,2}^{(3)}$ [12, 13], $L_{q,2}^{(3),NS}$ [14], $H_{q,2}^{(3),PS}$ [15], $A_{qg,Q}^{(3)}$, $A_{qq,Q}^{(3),PS}$ [13], $A_{gq,Q}^{(3)}$ [16], $A_{qq,Q}^{(3),NS}$ [14], and $A_{Qq}^{(3),PS}$ [15] the result at general values of the Mellin variable N has been computed in complete form.¹

In various calculations one may want to treat heavy quarks as light flavours at large enough energies. This is particularly the case for a series of reactions at high energy hadron colliders. The corresponding transition is described in the variable flavour number scheme (VFNS), decoupling one heavy quark at the time [9, 11, 17]². Since the masses of the charm and bottom quarks form no strong hierarchy because $m_c^2/m_b^2 \sim 1/10$, one rather has to decouple them together. This is needed in particular for processes, where m_c and m_b contribute in

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¹For the notation see [11].²The value of the decoupling scale in the VFNS is process dependent and is usually *not* the heavy quark’s mass [18].

the same Feynman diagram. Here a generalization of the VFNS is needed [7].

In this proceedings recent results on contributions to OMEs containing two massive fermion lines of equal quark masses are given. In Section 2 we discuss the contributions of $O(\alpha_s^3 T_F^2 C_F)$ and $O(\alpha_s^3 T_F^2 C_A)$ to the OME $A_{gg,Q}$ with the restriction of equal fermion masses. Section 3 contains the conclusions.

2. The $O(\alpha_s^3 T_F^2 C_{F,A})$ Contributions to $A_{gg,Q}$

The contributions to the OME $A_{gg,Q}$ proportional to $\alpha_s^3 T_F^2 C_{F,A}$ were given in [19]. Following the conventions of [11, 20] $A_{gg,Q}$ is defined as the expectation value,

$$A_{gg,Q} = \langle g | O_g | g \rangle, \quad (1)$$

$$O_g = 2i^{N-2} \text{SSp}(F_{\mu_1, \alpha} D_{\mu_2} \cdots D_{\mu_{N-1}} F_{\mu_N}^\alpha) - \text{trace terms}, \quad (2)$$

with the external gluon lines on-shell. \mathbf{S} represents the symmetrisation with respect to the Lorentz indices and Sp is the trace over colour indices. For further details see [11, 20] and references therein. For completeness the colour factors are, $C_F = (N_c^2 - 1)/2N_c$, $C_A = N_c$ and $T_F = 1/2$ in an $SU(N_c)$ gauge theory and $N_c = 3$ in QCD.

In total 12 unique diagrams contribute to the $\alpha_s^3 T_F^2 C_{F,A}$ contribution of interest. Most of the diagrams were computed directly. The momentum integration was performed by introducing Feynman parameters. Then a Mellin-Barnes representation was applied to the structures that arise. These representations requested specific ways to close the contour depending on other variables involved. Calculating each Mellin-Barnes integral by the residue theorem led to a large number of nested sums. These sums were handled with the summation technologies encoded in the package `Sigma` [21, 22], which uses advanced symbolic summation algorithms in the setting of difference fields [23, 24, 25, 26, 27, 28, 29, 30, 31]. In addition the packages `EvaluateMultiSums`, `SumProduction` [32, 33, 34], and `rhoSUM` [35] which are all based on `Sigma`, were used.

The remaining diagrams were solved using the integration-by-parts routines in `Reduze2` [36, 37]³ and calculating the corresponding master integrals using differential equations and also applying Mellin-Barnes techniques.

Of the main results in [19], here the constant part of the unrenormalised OME is given. If the unrenormalised OME corresponding to $A_{gg,Q}$ is denoted $\hat{A}_{gg,Q}$

then one may expand in the coupling,

$$\hat{A}_{gg,Q} = \frac{1}{2} [1 + (-1)^N] \left\{ 1 + \sum_{k=1}^{\infty} a_s^k \hat{A}_{gg,Q}^{(k)} \right\}, \quad (3)$$

for $a_s = \alpha_s/4\pi$. The contribution at 3-loop order reads,

$$\hat{A}_{gg,Q}^{(3)} = \frac{1}{\varepsilon^3} a_{gg,Q}^{(3,0)} + \frac{1}{\varepsilon^2} a_{gg,Q}^{(3,1)} + \frac{1}{\varepsilon} a_{gg,Q}^{(3,2)} + a_{gg,Q}^{(3)}. \quad (4)$$

Here $\varepsilon = D - 4$ denotes the dimensional parameter. One of our new results is the coefficient of T_F^2 in $a_{gg,Q}^{(3)}$. Expressed in Mellin space it is,

$$\begin{aligned} a_{gg,Q:T_F^2}^{(3)}(N) = & C_F T_F^2 \left\{ \frac{16}{27} F S_1^3 + \frac{16 Q_4}{27(N-1)N^3(N+1)^3(N+2)} S_1^2 \right. \\ & + \left[-\frac{32 Q_{10}}{81(N-1)N^4(N+1)^4(N+2)(2N-3)(2N-1)} \right. \\ & \left. - \frac{16}{3} F S_2 \right] S_1 - \frac{16 Q_4}{9(N-1)N^3(N+1)^3(N+2)} S_2 \\ & - \frac{2 Q_{13}}{243(N-1)N^5(N+1)^5(N+2)(2N-3)(2N-1)} \\ & - F \left[\frac{352}{27} S_3 - \frac{64}{3} S_{2,1} \right] \\ & + \left[\frac{16}{3} F S_1 - \frac{8 Q_8}{9(N-1)N^3(N+1)^3(N+2)} \right] \zeta_2 \\ & + \frac{Q_3}{9(N-1)N^2(N+1)^2(N+2)} \zeta_3 \\ & - \binom{2N}{N} \times \left(\sum_{i=1}^N \frac{4^i S_1(i-1)}{i^2 \binom{2i}{i}} - 7 \zeta_3 \right) \\ & \left. \frac{16 Q_5}{3(N-1)N(N+1)^2(N+2)(2N-3)(2N-1)} \frac{1}{4^N} \right\} \\ & + C_A T_F^2 \left\{ -\frac{4 Q_2}{135(N-1)N^2(N+1)^2(N+2)} S_1^2 \right. \\ & + \frac{16(4N^3 + 4N^2 - 7N + 1)}{15(N-1)N(N+1)} [S_{2,1} - S_3] \\ & + \frac{Q_{12}}{3645(N-1)N^4(N+1)^4(N+2)(2N-3)(2N-1)} \\ & - \frac{8 Q_{11}}{3645(N-1)N^3(N+1)^3(N+2)(2N-3)(2N-1)} S_1 \\ & + \frac{4 Q_7}{135(N-1)N^2(N+1)^2(N+2)} S_2 \\ & - \binom{2N}{N} \frac{1}{4^N} \left(\sum_{i=1}^N \frac{4^i S_1(i-1)}{i^2 \binom{2i}{i}} - 7 \zeta_3 \right) \\ & \times \frac{4 Q_9}{45(N-1)N(N+1)^2(N+2)(2N-3)(2N-1)} \\ & \left. + \left[\frac{4 Q_6}{27(N-1)N^2(N+1)^2(N+2)} - \frac{560}{27} S_1 \right] \zeta_2 \right\} \end{aligned}$$

³The package `Reduze2` uses `Fermat` [38] and `GiNac` [39].

$$+ \left[-\frac{7Q_1}{270(N-1)N(N+1)(N+2)} - \frac{1120}{27}S_1 \right] \zeta_3 \Big\}, \quad (5)$$

which uses,

$$F(N) = \frac{(2+N+N^2)^2}{(N-1)N^2(N+1)^2(N+2)} \equiv F. \quad (6)$$

Secondly, the Q_i denote polynomials in N , see [19] for expressions. Thirdly, the T_F^2 contribution is described using harmonic sums [40, 41] $S_{\vec{a}} \equiv S_{\vec{a}}(N)$,

$$S_{b,\vec{a}}(N) = \sum_{k=1}^N \frac{(\text{sign}(b))^k}{k^{|b|}} S_{\vec{a}}(k), \quad (7)$$

$$S_{\emptyset} = 1, \quad b, a_i \in \mathbb{Z} \setminus \{0\}.$$

In addition to the harmonic sums there is an inverse binomial type-sum in the T_F^2 contribution [42]⁴,

$$\frac{1}{4^N} \binom{2N}{N} \sum_{i=1}^N \frac{4^i S_1(i-1)}{i^{2\binom{2i}{i}}}. \quad (8)$$

Harmonic sums are related linearly to harmonic polylogarithms $H_{\vec{a}}(x)$ [44] in x -space by the Mellin transform. They can be represented as the following iterated integrals,

$$H(b, \vec{a}; x) = \int_0^x dt \frac{H(\vec{a}; t)}{t-b}, \quad (9)$$

$$H(\emptyset; x) = 1, \quad a, b \in \{-1, 0, 1\}.$$

To include (8) as the transform of an iterated integral, one must extend to generalised multiple polylogarithms [42, 45] including root-valued letters. For our purposes the following letter is needed,

$$w_9 = \frac{1}{(1-t)\sqrt{t}}, \quad (10)$$

see [42] for further details.

If the Mellin transform is denoted by,

$$M[f(x)](N) = \int_0^1 dx x^N f(x), \quad (11)$$

then in x -space (8) can be built by applying the Mellin convolution to,

$$\frac{1}{4^N} \binom{2N}{N} = \frac{1}{\pi} M \left[\frac{1}{\sqrt{x(1-x)}} \right], \quad (12)$$

⁴For recent surveys on the mathematical structures occurring in zero- and single scale higher order calculations see [43].

$$\sum_{i=1}^N \frac{4^i S_1(i-1)}{i^{2\binom{2i}{i}}} = M \left[\left(\frac{H(w_9, 0; 1-x)}{x-1} \right)_+ \right]$$

$$+ M \left[\left(\frac{H(w_9, 1; 1-x)}{x-1} \right)_+ \right]$$

$$+ 2 \log(2) M \left[\left(\frac{H(w_9; 1-x)}{x-1} \right)_+ \right], \quad (13)$$

using the +-distribution. Thus (8) is an iterated integral over the usual harmonic polylogarithm alphabet extended by square-root valued letters. This is the first time such root valued letters have been found in 3-loop Wilson coefficients for deep-inelastic scattering.

Further results, including the renormalised T_F^2 contribution, can be found in [19].

3. Conclusions

The contributions proportional to $T_F^2 C_{F,A}$ for the gluonic massive operator matrix element at 3-loop order were calculated. These are the diagrams with two massive lines of equal mass.

To calculate the diagrams Mellin-Barnes integrals were used leading to nested sums. These sums were expressed in terms of harmonic sums and their generalisations using automated computer algebra techniques. Both in intermediary and final results there are nested finite binomial sums, weighted by harmonic sums. Moving to x -space the nested finite binomial sums become generalised multiple polylogarithms with square-root valued letters.

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