DESINGULARIZATION OF ORE OPERATORS

SHAOSHI CHEN, MANUEL KAUERS, AND MICHAEL F. SINGER

ABSTRACT. We show that Ore operators can be desingularized by calculating a least common left multiple with a random operator of appropriate order. Our result generalizes a classical result about apparent singularities of linear differential equations, and it gives rise to a surprisingly simple desingularization algorithm.

1. INTRODUCTION

Consider a linear ordinary differential equation, like for example

$$x(1-x)f'(x) - f(x) = 0.$$

The leading coefficient polynomial $x(1-x)$ of the equation is of special interest because every point ξ which is a singularity of some solution of the differential equation is also a root of this polynomial. However, the converse is in general not true. In the example above, the root $\xi = 1$ indicates the singularity of the solution $x/(1-x)$, but there is no solution which has a singularity at the other root $\xi = 0$. To see this, observe that after differentiating the equation, we can cancel ("remove") the factor x from it. The result is the higher order equation

$$(1-x)f''(x) - 2f'(x) = 0,$$

whose solution space contains the solution space of the original equation. Such a calculation is called *desingularization.* The factor x is said to be *removable.*

Given a differential equation, it is of interest to decide which factors of its leading coefficient polynomial are removable, and to construct a higher order equation in which all the removable factors are removed. A classical algorithm, which is known since the end of the 19th century [14, 11], proceeds by taking the least common left multiple of the given differential operator with a suitably constructed auxiliary operator. This algorithm is summarized in Section 2 below. At the end of the 20th century, the corresponding problem for linear recurrence equations was studied and algorithms for identifying removable factors have been found and their relations to "singularities" of solutions have been investigated [3, 4, 1]. Also some steps towards a unified theory for desingularization of Ore operators have been made [10, 9]. Possible connections to Ore closures of an operator ideal have been noted in [10] and within the context of order-degree curves [9, 7, 8]. These will be further developed in a future paper.

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Our contribution in the present article is a three-fold generalization of the classical desingularization algorithm for differential equations. Our main result (Theorem 6 below) says that (a) instead of the particular auxiliary operator traditionally used, almost every other operator of appropriate order also does the job, (b) also the case when a multiple root of the leading coefficient can't be removed completely but only its multiplicity can be reduced is covered, and (c) the technique works not only for differential operators but for every Ore algebra.

For every removable factor p there is a smallest $n \in \mathbb{N}$ such that removing p from the operator requires increasing the order of the operator by at least n. Classical desingularization algorithms compute for each factor p an upper bound for this n, and then determine whether or not it is possible to remove p at the cost of increasing the order of the operator by at most n. In the present paper, we do not address the question of finding bounds on n but only discuss the second part: assuming some $n \in \mathbb{N}$ is given as part of the input, we consider the task of removing as many factors as possible without increasing the order of the operator by more than n. Of course, for Ore algebras where it is known how to obtain bounds on n, these bounds can be combined with our result.

Recall the notion of Ore algebras [13]. Let K be a field of characteristic zero. Let $\sigma\colon K[x] \to K[x]$ be a ring automorphism that leaves the elements of K fixed, and let $\delta\colon K[x] \to K[x]$ be a K-linear map satisfying the law $\delta(uv) = \delta(u)v + \sigma(u)\delta(v)$ for all $u, v \in K[x]$. The algebra $K[x][\partial]$ consists of all polynomials in ∂ with coefficients in $K[x]$ together with the usual addition and the unique (in general noncommutative) multiplication satisfying $\partial u = \sigma(u)\partial + \delta(u)$ for all $u \in K[x]$ is called an *Ore algebra.* The field K is called the *constant field* of the algebra. Every nonzero element L of an Ore algebra $K[x][\partial]$ can be written uniquely in the form

$$L = \ell_0 + \ell_1\partial + \cdots + \ell_r\partial^r$$

with $\ell_0, \ldots, \ell_r \in K[x]$ and $\ell_r \neq 0$. We call $\deg_\partial(L) := r$ the *order* of L and $\operatorname{lc}_\partial(L) := \ell_r$ the *leading coefficient* of L. Roots of the leading coefficient ℓ_r are called singularities of L. Prominent examples of Ore algebras are the algebra of linear differential operators (with $\sigma = \mathrm{id}$ and $\delta = \frac{d}{dx}$; we will write D instead of ∂ in this case) and the algebra of linear recurrence operators (with $\sigma(x) = x + 1$ and $\delta = 0$; we will write S instead of ∂ in this case).

We shall suppose that the reader is familiar with these definitions and facts, and will make free use of well-known facts about Ore algebras, as explained, for instance, in [13, 6, 2]. In particular, we will make use of the notion of least common left multiples (lclm) of elements of Ore algebras: $L \in K(x)[\partial]$ is a *common left multiple* of $P, Q \in K(x)[\partial]$ if we have $L = UP = VQ$ for some $U, V \in K(x)[\partial]$, it is called a *least common left multiple* if there is no common left multiple of lower order. Least common left multiples are unique up to left-multiplication by nonzero elements of $K(x)$. By $\operatorname{lclm}(P, Q)$ we denote a least common left multiple whose coefficients belong to $K[x]$ and share no common divisors in $K[x]$. Note that $\operatorname{lclm}(P, Q)$ is unique up to (left-)multiplication by nonzero elements of K. Efficient algorithms for computing least common left multiples are available [5].

2. The Differential Case

In order to motivate our result, we begin by recalling the classical results concerning the desingularization of linear differential operators. See the appendix of [1] for further details on this case.

Let $L = \ell_0 + \ell_1 D + \cdots + \ell_r D^r \in K[x][D]$ be a differential operator of order r. Consider the power series solutions of L. It can be shown that $x \nmid \ell_r$ if and only if L admits r power series solutions of the form $x^\alpha + \cdots$, for $\alpha = 0, \ldots, r-1$. Therefore, if $x \mid \ell_r$, then this factor is removable if and only if there exists some left multiple M of L, say with $\deg_\partial(M) = s$, such that M admits a power series solution with minimal exponent α for every $\alpha = 0, \ldots, s-1$. This is the case if and only if L has r linearly independent power series solutions with integer exponents $0 \le \alpha_1 < \alpha_2 < \cdots < \alpha_r$, because in this case (and only in this case) we can construct a left multiple M of L with power series solutions $x^\alpha + \cdots$ for each $\alpha = 0, \ldots, \max\{\alpha_1, \ldots, \alpha_r\} - 1$, by adding power series of the missing orders to the solution space of L.

These observations suggest the following desingularization algorithm for operators $L \in K[x][\partial]$ with $x \mid \mathrm{lc}_\partial(L)$. First find the set $\{\alpha_1, \ldots, \alpha_\ell\} \subseteq \mathbb{N}$ of all exponents α_i for which there exist power series solutions $x^{\alpha_i} + \cdots$. If $\ell < r$, return "not desingularizable" and stop. Otherwise, let $m = \max\{\alpha_1, \ldots, \alpha_\ell\}$ and let $e_1, e_2, \ldots, e_{m-\ell}$ be those nonnegative integers which are at most m but not among the α_i. Return the operator

$$M = \mathrm{lclm}(L, A),$$

where

$$A := \mathrm{lclm}(xD - e_1,\ xD - e_2,\ \ldots,\ xD - e_{m-\ell}).$$

Note that among the solutions of A there are the monomials $x^{e_1}, x^{e_2}, \ldots, x^{e_{m-\ell}}$, and that the solutions of M are linear combinations of solutions of A and solutions of L. Therefore, by the choice of the e_j and the remarks made above, M is desingularized.

Example 1. *Consider the operator*

$$L = (x-1)(x^2-3x+3)xD^2 - (x^2-3)(x^2-2x+2)D + (x-2)(2x^2-3x+3) \in K[x][D].$$

This operator has power series solutions with minimal exponents $\alpha = 0$ and $\alpha = 3$. Their first terms are

$$\begin{aligned}&1 + x + \tfrac{1}{2}x^2 - \tfrac{1}{8}x^4 - \tfrac{19}{120}x^5 - \tfrac{119}{720}x^6 + \cdots,\\ &x^3 + x^4 + x^5 + x^6 + \cdots.\end{aligned}$$

The missing exponents are $e_1 = 1$ and $e_2 = 2$. Therefore we take

$$A := \mathrm{lclm}(xD - 1, xD - 2) = x^2D^2 - 2xD + 2$$

and calculate

$$\begin{aligned}M = \mathrm{lclm}(L, A) &= (x^5 - 2x^4 + 4x^3 - 9x^2 + 12x - 6)D^4\\ &\quad - (x^5 - 2x^4 + x^3 - 12x^2 + 24x - 24)D^3\\ &\quad - (3x^3 + 9x^2)D^2 + (6x^2 + 18x)D - (6x + 18).\end{aligned}$$

Note that we have $x \nmid \mathrm{lc}_\partial(M)$, as predicted.

In the form sketched above, the algorithm applies only to the singularity 0. In order to get rid of a different singularity, move this singularity to 0 by a suitable change of variables, then proceed as described above, and after that undo the change of variables. Note that by removing the singularity 0 we will in general introduce new singularities at other points.

3. Removable Factors

We now turn from the algebra of linear differential operators to arbitrary Ore algebras. In the general case, removability of a factor of the leading coefficient is defined as follows.

Definition 2. *Let $L \in K[x][\partial]$ and let $p \in K[x]$ be such that $p \mid \operatorname{lc}_\partial(L) \in K[x]$. We say that p is* removable *from L at order n if there exists some $P \in K(x)[\partial]$ with $\deg_\partial(P) = n$ and some $v, w \in K[x]$ with $\gcd(p, w) = 1$ such that $PL \in K[x][\partial]$ and $\sigma^{-n}(\operatorname{lc}_\partial(PL)) = \frac{w}{vp}\operatorname{lc}_\partial(L)$. We then call P a* p-removing *operator for L, and PL the corresponding* p-removed *operator. p is simply called* removable *from L if it is removable at order n for some $n \in \mathbb{N}$.*

Example 3.

(1) *In the example from the introduction, we have $L = x(1-x)D - 1 \in K[x][D]$. An x-removing operator is $P = \frac{1}{x}D$: we have $PL = (1-x)D^2 - 2D$. Because of $\deg_\partial(P) = 1$ we say that x is removable at order 1.*

If P is a p-removing operator then so is QP, for every $Q \in K[x][\partial]$ with $\gcd(\operatorname{lc}_\partial(Q), \sigma^{\deg_\partial(P)+\deg_\partial(Q)}(p)) = 1$. In particular, note that the definition permits to introduce some new factors w into the leading coefficient while p is being removed. For instance, in our example also $\frac{2-3x}{x}D$ is an x-removing operator for L.

(2) *The definition does not imply that the leading coefficient of a p-removed operator is coprime with (a shifted copy of) p. In general, it only requires that the multiplicity is reduced. As an example, consider the operator*

$$L = x^2(x-2)(x-1)D^2 + 2x(x^2-3x+1)D - 2 \in K[x][D]$$

and $p = x$. The operator $P = \frac{x^4-x^3-4x^2+2x-2}{(x-2)x}D - (x^2+5x+3) \in K(x)[D]$ is a p-removing operator because the leading coefficient of

$$\begin{aligned} PL = x(x-1)(x^4-x^3-4x^2+2x-2)D^3 \\ &- (x^6-4x^5-x^4+22x^3-18x^2+18x-6)D^2 \\ &- 2(x^5-x^4-8x^3+8x^2-3x+6)D \\ &+ 2(x^2+5x+3) \end{aligned}$$

contains only one copy of p while there are two of them in L. This is called partial desingularization. *Observe that the definition permits to remove some factors v from the leading coefficient in addition to p.*

(3) *In the shift case, or more generally, in an Ore algebra where σ is not the identity, the leading coefficient changes when an operator is multiplied by a power of ∂ from the left. The application of σ^{-n} in the definition compensates this change. As an example, consider the operator*

$$\begin{aligned} L = x(x+1)(5x-2)S^2 - 2x(5x^2-2x-9)S \\ + (x-4)(x+2)(5x+3) \in K[x][S] \end{aligned}$$

and $p = x + 1$. *The operator* $P = \frac{5x^3+13x^2-18x-24}{(x+2)(5x+3)}S - \frac{2(5x^3+28x^2+23x-24)}{(x+2)(5x+3)}$ *is a* p*-removing operator because the leading coefficient of*

$$\begin{aligned} PL = {} & (x+1)(5x^3+13x^2-18x-24)S^3 \\ & - 2(x+1)(10x^3+21x^2-58x+24)S^2 \\ & + (25x^4+60x^3-217x^2-84x+288)S \\ & - 2(x-4)(5x^3+28x^2+23x-24) \end{aligned}$$

does not contain $\sigma(p) = x + 2$. *It is irrelevant that it contains* $x + 1$.

As indicated in the examples, when removing a factor p from an operator L, Def. 2 allows that we introduce other factors w, coprime to p. We are also always allowed to remove additional factors v besides p. The freedom for having v and w is convenient but not really necessary. In fact, whenever there exists an operator $P \in K(x)[\partial]$ of order n such that $\sigma^{-n}(\mathrm{lc}_\partial(PL)) = \frac{w}{vp}\,\mathrm{lc}_\partial(L)$, then there also exists an operator $Q \in K(x)[\partial]$ of order n such that $\sigma^{-n}(\mathrm{lc}_\partial(QL)) = \frac{1}{p}\,\mathrm{lc}_\partial(L)$. To see this, note that by the extended Euclidean algorithm there exist $s, t \in K[x]$ such that $sw + tp = 1$. Set $Q = \sigma^n(sv)P + \sigma^{-n}(t)\partial^n$. Then

$$\begin{aligned} \sigma^{-n}(\mathrm{lc}_\partial(QL)) &= sv\,\sigma^{-n}(\mathrm{lc}_\partial(PL)) + t\,\mathrm{lc}_\partial(\partial^n L) \\ &= sv\frac{w}{vp}\,\mathrm{lc}_\partial(L) + \frac{tp}{p}\,\mathrm{lc}_\partial(L) = \frac{1}{p}\,\mathrm{lc}_\partial(L), \end{aligned}$$

as desired. This argument is borrowed from [1]. The same argument can also be used to show the existence of operators that remove all the removable factors in one stroke:

Lemma 4. *Let* $L \in K[x][\partial]$, *let* $n \in \mathbb{N}$, *and let* $\mathrm{lc}_\partial(L) = p_1^{e_1} p_2^{e_2} \cdots p_m^{e_m}$ *be a factorization of the leading coefficient into irreducible polynomials. For each* $i = 1, \ldots, m$, *let* $k_i \le e_i$ *be maximal such that* p_i *is removable from* L *at order* n. *Then there exists an operator* $P \in K(x)[\partial]$ *of order* n *such that* $\sigma^{-n}(\mathrm{lc}_\partial(PL)) = \frac{1}{p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m}}\,\mathrm{lc}_\partial(L)$.

Proof. By the remark preceding the lemma, we may assume that for every i there exists an operator $P_i \in K(x)[\partial]$ of order n with $P_i L \in K[x][\partial]$ and $\sigma^{-n}(\mathrm{lc}_\partial(P_i L)) = p_i^{-k_i}\,\mathrm{lc}_\partial(L)$ (i.e., $w = v = 1$).

Next, observe that when p and q are two coprime factors of $\mathrm{lc}_\partial(L)$ which both are removable at order n, then also their product pq is removable at order n. Indeed, if $P, Q \in K(x)[\partial]$ are such that $\deg_\partial(P) = \deg_\partial(Q) = n$, $PL, QL \in K[x][\partial]$, $\sigma^{-n}(\mathrm{lc}_\partial(PL)) = \frac{1}{p}\,\mathrm{lc}_\partial(L)$, and $\sigma^{-n}(\mathrm{lc}_\partial(QL)) = \frac{1}{q}\,\mathrm{lc}_\partial(L)$, and if $s, t \in K[x]$ are such that $sq + tp = 1$, then for $R := \sigma^{-n}(s)P + \sigma^{-n}(t)Q$ we have $\sigma^{-n}(\mathrm{lc}_\partial(RL)) = \frac{1}{pq}\,\mathrm{lc}_\partial(L)$, as desired.

The claim of the lemma now follows by induction on m, taking $p = p_1^{e_1} \cdots p_{m-1}^{e_m}$ and $q = p_m^{e_m}$. □

4. DESINGULARIZATION BY TAKING LEAST COMMON LEFT MULTIPLES

As outlined in Section 2, the classical algorithm for desingularizing differential operators relies on taking the lclm of the operator to be desingularized with a suitably chosen auxiliary operator. Our contribution consists in a three-fold generalization of this approach: first, we show that it works in every Ore algebra and not just for differential operators, second, we show that almost every operator qualifies as an auxiliary operator in the lclm and not just the particular operator used traditionally, and third, we show that the approach also covers partial desingularization. From the second fact it follows directly that taking the lclm with a random operator of appropriate order removes, with high probability, *all* the removable singularities of the operator under consideration and not just a given one.

Consider an operator $L \in K[x][\partial]$ in an arbitrary Ore algebra, and let $p \mid \operatorname{lc}_\partial(L)$ be a factor of its leading coefficient. Assume that this factor is removable at order n. Our goal is to show that for almost all operators $A \in K[\partial]$ of order n with constant coefficients the operator $\operatorname{lclm}(L, A)$ is p-removed.

One way of computing the least common left multiple of two operators $L, A \in K[x][\partial]$ with $\deg_\partial(L) = r$ and $\deg_\partial(A) = n$ is as follows. Make an ansatz with undetermined coefficients $u_0, \dots, u_n$, $v_0, \dots, v_r$ and compare coefficients of ∂^i ($i = 0, \dots, n + r$) in the equation

$$(u_0 + \dots + u_{n-1}\partial^{n-1} + u_n\partial^n)L = (v_0 + \dots + v_{r-1}\partial^{r-1} + v_r\partial^r)A.$$

This leads to a system of homogeneous linear equations over $K(x)$ for the undetermined coefficients, which has more variables than equations and therefore must have a nontrivial solution. For each solution, the operator on either side of the equation is a common left multiple of L and A.

For most choices of A the solution space will have dimension 1, and in this case, for every nontrivial solution we have $u_n \neq 0$. In particular the least common left multiple $M = \operatorname{lclm}(L, A)$ has then order $r + n$. The singularities of M are then the roots of $\sigma^n(\operatorname{lc}_\partial(L))$ plus the roots u_n minus the common roots of $u_0, \dots, u_n$, which are cancelled out by convention. It is not obvious at this point why removable factors should appear among the common factors of $u_0, \dots, u_n$. To see that they systematically do, consider a p-removing operator $P \in K(x)[\partial]$ of order n, and observe that the operators $1, \partial, \dots, \partial^{n-1}, \partial^n$ generate the same $K(x)$-vector space as $1, \partial, \dots, \partial^{n-1}, P$. If we use the latter basis in the ansatz for the lclm, i.e., do coefficient comparison in

$$(u_0 + \dots + u_{n-1}\partial^{n-1} + u_nP)L = (v_0 + \dots + v_{r-1}\partial^{r-1} + v_r\partial^r)A,$$

then every nontrivial solution vector $(u_0, \dots, u_n, v_0, \dots, v_r)$ of the resulting linear system gives rise to a common left multiple of L and A in $K[x][\partial]$ whose singularities are the roots of $\operatorname{lc}_\partial(PL) = \sigma^n(\frac{1}{p}\operatorname{lc}_\partial(L))$ plus the roots of u_n minus the common roots of $u_0, \dots, u_n$. This argument shows that the removable factor p will have disappeared in the lclm *unless* it is reintroduced by u_n. The main technical difficulty to be addressed in the following is to show that this can happen only for very special choices of A. For the proof of this result we need the following lemma.

Lemma 5. *Let $n, m \in \mathbb{N}$, let $v_1, \dots, v_n \in K^{n+m}$ be linearly independent over K, and let $w_1, \dots, w_m \in K[x_1, \dots, x_n]^{n+m}$ be defined by*

$$\begin{aligned} w_1 &= (x_1, \dots, x_n, 1, 0, \dots, 0)\\ w_2 &= (0, x_1, \dots, x_n, 1, 0, \dots, 0)\\ &\vdots\\ w_m &= (0, \dots, 0, x_1, \dots, x_n, 1). \end{aligned}$$

Then $\Delta := \det(w_1, \dots, w_m, v_1, \dots, v_n)$ is a nonzero polynomial in $K[x_1, \dots, x_n]$.

Proof. Simultaneous induction on n and m: We show that the lemma holds for (n, m) if it holds for $(n-1, m)$ and for $(n, m-1)$.

As induction basis, observe that the lemma holds for $n = 1$, m arbitrary, and also for n arbitrary, $m = 1$.

Now let $(n, m) \in \mathbb{N}^2$ with $n \geq 2, m \geq 2$ be given. Let $v_1, \dots, v_n \in K^{n+m}$ be linearly independent. Write $v_i = (v_{1,i}, \dots, v_{n+m,i})$ for the coefficients.

Case 1. $v_{1,1} = v_{1,2} = \cdots = v_{1,n} = 0$. In this case, the vectors $\bar{v}_i \in K^{n+(m-1)}$ obtained from the v_i by chopping the first coordinate must be linearly independent. By expanding along the first row, we have

$$\Delta = x_1 \det(\bar{w}_2, \dots, \bar{w}_m, \bar{v}_1, \dots, \bar{v}_n).$$

The determinant on the right is nonzero by applying the lemma with n and $m-1$. Therefore the determinant on the left is also nonzero.

Case 2. If at least one of the $v_{1,j}$ is nonzero, then we may assume without loss of generality that $v_{1,1} = 1$ and $v_{1,2} = v_{1,3} = \cdots = v_{1,n} = 0$, by performing suitable column operations on $(v_1, \dots, v_n) \in K^{(n+m)\times n}$. Then the vectors $\bar{v}_2, \dots, \bar{v}_n \in K^{(n-1)+m}$ obtained from the v_i by chopping the first coordinate are linearly independent. Expanding along the first row, we now have

$$\Delta = x_1\,[[\text{poly}]] + v_{1,1} \begin{vmatrix} x_2 & x_1 & 0 & \cdots & 0 & v_{2,2} & \cdots & v_{2,n} \\ x_3 & x_2 & \ddots & \ddots & \vdots & \vdots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 & \vdots & & \vdots \\ \vdots & & \ddots & \ddots & x_1 & \vdots & & \vdots \\ x_n & & & \ddots & x_2 & \vdots & & \vdots \\ 1 & \ddots & & & x_3 & \vdots & & \vdots \\ 0 & \ddots & \ddots & & \vdots & \vdots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & & \vdots \\ \vdots & & \ddots & \ddots & x_n & \vdots & & \vdots \\ 0 & \cdots & \cdots & 0 & 1 & v_{n+m,2} & \cdots & v_{n+m,n} \end{vmatrix}.$$

By setting $x_1 = 0$, the first term on the right hand side disappears, and so do the entries x_1 in the determinant of the second term. By induction hypothesis, the determinant on the right with x_1 set to zero is a nonzero polynomial in $x_2, \dots, x_n$. Since also $v_{1,1} \neq 0$, the whole right hand side is nonzero for $x_1 = 0$. Consequently, when x_1 is not set to zero, it cannot be the zero polynomial. □

Theorem 6 (Main result). *Let $K[x][\partial]$ be an Ore algebra, let $L \in K[x][\partial]$ be an operator of order r, and let $n \in \mathbb{N}$. Let $p \in K[x][\partial]$ be an irreducible polynomial which appears with multiplicity e in $\operatorname{lc}_\partial(L)$ and let $k \leq e$ be maximal such that p^k is removable from L at order n. Let $A = a_0 + a_1\partial + \cdots + a_{n-1}\partial^{n-1} + \partial^n$ in $K[a_0, \dots, a_{n-1}][\partial]$, where $a_0, \dots, a_{n-1}$ are new constants, algebraically independent over K. Then the multiplicity of $\sigma^n(p)$ in $\operatorname{lc}_\partial(\operatorname{lclm}(L, A))$ is $e - k$.*

Proof. Let $P_0, \dots, P_n \in K(x)[\partial]$ be such that each P_i has order i and removes from L all the factors of $\operatorname{lc}_\partial(L)$ that can possibly be removed by an operator of order i.

Such operators exist by Lemma 4. Consider an ansatz

$$u_0P_0L + u_1P_1L + \cdots + u_nP_nL = v_0A + v_1\partial A + \cdots + v_r\partial^r A$$

with unknown $u_i, v_j \in K[a_0, \ldots, a_{n-1}][x]$. Compare coefficients with respect to powers of ∂ on both sides and solve the resulting linear system. This gives a polynomial solution vector with

$$u_n = \det\bigl([P_0L], [P_1L], \cdots [P_{n-1}L], [A], [\partial A], \cdots, [\partial^{r-1}A]\bigr),$$

where the notation $[U]$ refers to the coefficient vector of the operator U (padded with zeros, if necessary, to dimension $r + n$).

If $\sigma^n(p) \mid u_n$, then the columns of the determinant are linearly dependent when viewed as elements of $F[a_0, \ldots, a_{n-1}]$ with $F = K[x]/\langle\sigma^n(p)\rangle$. Then Lemma 5 with F in place of K implies that already $[P_0L], \ldots, [P_{n-1}L]$ are linearly dependent modulo $\sigma^n(p)$. In other words, there are polynomials $u_0, \ldots, u_{n-1} \in K[x]$ of degree $< \deg(p)$, not all zero, such that the linear combination $u_0P_0L + \cdots + u_{n-1}P_{n-1}L$ has content $\sigma^n(p)$. If d is maximal such that $u_d \neq 0$, then this means that $\frac{1}{\sigma^d(p)}(u_0P_0 + \cdots + u_dP_d)$ is an operator of order d which removes from L one factor $\sigma^{n-d}(p)$ more than P_d does, in contradiction to the assumption that P_d removes as much as possible. □

The theorem continues to hold when the indeterminates $a_0, \ldots, a_{n-1}$ are replaced by values in K which do not form a point on the zero set of the determinant polynomial $u_n \bmod \sigma^n(p)$, as discussed in the proof. As this is not the zero polynomial and we assume throughout that K has characteristic zero, it follows that almost all choices of $A \in K[\partial]$ will successfully remove all the factors of $\operatorname{lc}_\partial(L)$ that are removable at order $\deg_\partial(A)$.

The theorem thus gives rise to the following very simple probabilistic algorithm for removing, with high probability, as many factors as possible from a given operator $L \in K[x][\partial]$ at a given order n:

- Pick an operator $A \in K[\partial]$ of order n at random.
- Return $\operatorname{lclm}(L, A)$.

This is a Monte Carlo algorithm: it always terminates but with low probability may return an incorrect answer. For a Las Vegas algorithm (low probability of not terminating but every answer is guaranteed to be correct), inspect the multiplier u_n which appears during the construction of the lclm: if it is coprime with $\sigma^n(\operatorname{lc}_\partial(L))$, then no removed singularities get mistakenly re-introduced and the result is therefore correct. Otherwise, try again. For a deterministic algorithm, don't take the operators A at random but from some enumeration of $K[\partial]$ which is chosen in such a way that the Zariski closure of the set of the corresponding coefficient vectors is all of K^n.

The Monte Carlo version of the algorithm is included in the new `ore_algebra` package for Sage [12], and works very efficiently thanks to the efficient implementation of least common left multiples also available in this package. This package has been used for the calculations in the following concluding examples. The computation time for all these examples is negligible.

Example 7. (1) *For* $L \in \mathbb{Q}[x][D]$ *from Example 1 and the "randomly chosen" operator* $A = D^2 + D + 1$ *we have*

$$\begin{aligned}\operatorname{lclm}(L, A) = {} & (x^7 - 4x^6 + 6x^5 - 4x^4 + x^3 + 6x - 6)D^4 \\ & - (2x^6 - 9x^5 + 15x^4 - 11x^3 + 3x^2 - 24)D^3 \\ & - (x^7 - 4x^6 + 6x^5 - 4x^4 + x^3 + 6x - 6)D \\ & + (2x^6 - 9x^5 + 15x^4 - 11x^3 + 3x^2 - 24).\end{aligned}$$

This is not the same result as in Example 1, but it does have the required property $x \nmid \operatorname{lc}_\partial(\operatorname{lclm}(L, A))$.

(2) *This is an example for the recurrence case. Let*

$$\begin{aligned}L = {} & 2(x+3)^2(59x+94)S^3 - (2301x^3 + 15171x^2 + 32696x + 22876)S^2 \\ & - 5(59x^3 + 330x^2 + 600x + 359)S - (59x+153)(x+1)^2.\end{aligned}$$

Among the factors of $(x+3)$ *and* $(59x+94)$ *of the leading coefficient, the latter is removable at order 1 and the former is not removable. Accordingly, for the "randomly chosen" operator* $A = S - 2$ *we have*

$$\begin{aligned}\operatorname{lclm}(L, A) = {} & 2(x+4)^2(8909x^3 + 57087x^2 + 119629x + 81711)S^4 \\ & + (\cdots)S^3 + (\cdots)S^2 + (\cdots)S + (\cdots),\end{aligned}$$

where $(\cdots)$ *stands for some other polynomials. Note that the leading coefficient is coprime to* $\sigma(59x+94) = 59x+153$.

(3) *As an example for partial desingularization, consider the operator* $L = x^3D^3 - 3x^2D^2 - 2xD + 10 \in \mathbb{Q}[x][D]$. *Of the three copies of* x *in the leading coefficient, one is removable at order 2, another one at order 4, and the third is not removable. In perfect accordance, we find for example*

$$\begin{aligned}&\operatorname{lc}_\partial(\operatorname{lclm}(L, D+2)) = x^3(4x^3 + 6x^2 - 2x - 5), \\ &\operatorname{lc}_\partial(\operatorname{lclm}(L, D^2+1)) = x^2(x^6 + 10x^4 + 40x^2 + 80), \\ &\operatorname{lc}_\partial(\operatorname{lclm}(L, D^3 + 3D^2 - 1)) = x^2(x^8 - 30x^6 + \cdots + 2160x + 1920), \\ &\operatorname{lc}_\partial(\operatorname{lclm}(L, D^4 - D^2 + 1)) = x(x^{10} - 10x^8 + 120x^6 - 720x^4 - 3200), \\ &\operatorname{lc}_\partial(\operatorname{lclm}(L, D^5 + D - 1)) = x(x^{12} - 3x^{11} + \cdots + 25600x - 22400).\end{aligned}$$

(4) *There are unlucky choices for* A. *For example, consider*

$$\begin{aligned}L = {} & (x-7)(x^2 - 2x - 12)S^2 - (3x^3 - 23x^2 - 23x + 291)S \\ & + 2(x-6)(x^2 - 13) \in \mathbb{Q}[x][S].\end{aligned}$$

The factor $x - 7$ *is removable, as can be seen, for example, from the fact that* $\operatorname{lc}_\partial(\operatorname{lclm}(L, S-1)) = 2x^2 - x - 51$ *is coprime to* $\sigma(x-7) = x - 6$. *However, if we take* $A = S - \frac{9}{4}$, *then*

$$\begin{aligned}\operatorname{lclm}(L, A) = {} & 4(x-7)(x-6)(5x-28)S^3 \\ & - (x-7)(3092 - 1138x + 105x^2)S^2 \\ & + (x-5)(6081 - 2080x + 175x^2)S \\ & - 18(x-6)(x-5)(5x-23),\end{aligned}$$

which has $x - 6$ *in the leading coefficient. (It is irrelevant that also* $x - 7$ *appears as a factor.)*

(5) *Finally, as an example with an unusual Ore algebra, consider* $\mathbb{Q}[x][\partial]$ *with* $\sigma\colon \mathbb{Q}[x] \to \mathbb{Q}[x]$ *defined by* $\sigma(x) = x^2$ *and* $\delta\colon \mathbb{Q}[x] \to \mathbb{Q}[x]$ *defined by* $\delta(x) = 1 - x$*. Let*

$$L = (2x+1)\partial^2 + (x^2+3x-1)\partial - (2x^4+2x^3+x^2+1).$$

The factor $2x+1$ *is removable at order* 1*. For example, for* $A = \partial - 1$ *we find that* $\operatorname{lclm}(L, A)$ *equals*

$$\begin{aligned}&(2x^3+4x^2+4x-1)\partial^3 - (2x^6-x^4-4x^3-3x^2+x+5)\partial^2\\&\quad - (2x^9+4x^8+6x^7+4x^6+2x^5+3x^4+2x^3+3x^2+3x-2)\partial\\&\quad + (2x^9+4x^8+6x^7+6x^6+2x^5+2x^4-4x^3-4x^2+4).\end{aligned}$$

As expected, the leading coefficient does not contain the factor $\sigma(\mathrm{lc}_\partial(L)) = 2x^2+1$*.*

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Shaoshi Chen, KLMM, AMSS, Chinese Academy of Sciences, 100190 Beijing, China
E-mail address: `schen@amss.ac.cn`

Manuel Kauers, Research Institute for Symbolic Computation, J. Kepler University Linz, Austria
E-mail address: `mkauers@risc.uni-linz.ac.at`

Michael F. Singer, Department of Mathematics, North Carolina State University, Raleigh, NC, USA
E-mail address: `singer@math.ncsu.edu`