Improved Abramov–Petkovšek's Reduction for Hypergeometric Terms

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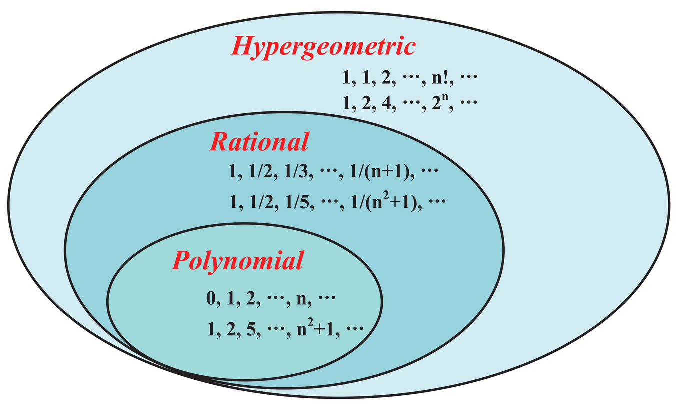
Hypergeometric Sequences

A sequence $T : \mathbb{N} \to \mathbb{C}$ is said to be *hypergeometric* if

$$\exists\, R \in \mathbb{C}(n) \text{ s.t. } T(n+1) = R(n)T(n) \text{ for all } n \gg 0.$$



Notation.

– For a sequence $A(n)$, $\Delta(A) = A(n+1) - A(n)$.

– For two hypergeometric sequences $T(n), S(n)$, write

$$T \equiv S \text{ if } T - S = \Delta(H) \text{ for some hypergeometric sequence } H(n).$$

Definition 1. A hypergeometric sequence $T(n)$ is *summable* if $T \equiv 0$.

Summability. Given a hypergeometric sequence $T(n)$, decide whether $T \equiv 0$.

Gosper algorithm [4] solves the summability problem.

Abramov–Petkovšek's Reduction

Additive Decomposition. Given a hypergeometric sequence $T(n)$, compute two hypergeometric sequences $T_1(n)$ and $T_2(n)$ s.t.

$$T = \Delta(T_1) + T_2 \quad \text{and} \quad T \equiv 0 \iff T_2 = 0. \tag{1}$$

Abramov–Petkovšek's reduction (AP reduction) [1, 2] computes $T_1(n)$ and $T_2(n)$ in (1).

The AP reduction solves the summability problem as well.

Definition 2.

– $p \in \mathbb{C}[n]$ is *shift-free* if $\gcd(p(n), p(n+i)) = 1$ for all $i \in \mathbb{Z} \setminus \{0\}$.

– $r \in \mathbb{C}(n)$ with $r = u/v$ is *shift-reduced* if $\gcd(u(n), v(n+i)) = 1$ for all $i \in \mathbb{Z}$.

– Let $r = u/v$ be a shift-reduced rational function in $\mathbb{C}(n)$. A polynomial $f \in \mathbb{C}[n]$ is *strongly prime* with r if either $f \in \mathbb{C}$, or, for every irreducible factor p of f,

$$p \nmid uv, \;\; p(n+i) \nmid u \text{ and } p(n-i) \nmid v \text{ for all } i \in \mathbb{Z}^+.$$

AP reduction. Given a hypergeometric sequence $T(n)$, compute two hypergeometric sequences $T_1(n)$ and $T_2(n)$ such that the two conditions in (1) hold.

1. Compute three polynomials $a, b, w \in \mathbb{C}[n]$ such that

$$T \equiv \left(\frac{a}{b} + \frac{w}{v}\right) H \tag{2}$$

where $H(n+1)/H(n) = u/v$ is shift-reduced, $\deg(a) < \deg(b)$, $\gcd(a, b) = 1$ and b is shift-free and strongly prime with u/v. Moreover, the degree of the numerator w is bounded.

2. Consider the equation

$$u(n)y(n+1) - v(n)y(n) = w(n). \tag{3}$$

Summable case	Non-summable case
$b \in \mathbb{C}^*$ and $w = 0$ or (3) has a polynomial solution $\Downarrow$ $T_2 = 0$	$b \notin \mathbb{C}^*$ or (3) has no polynomial solution $\Downarrow$ $T_2 = (a/b + w/v)\,H$

The sequence $T_1(n)$ can be constructed as the product of a rational function $r(n)$ and the sequence $H(n)$ incrementally.

Improved AP Reduction

Idea. Not only bound the degree of the numerator w in (2) as in [1, 2], but also reduce the number of its terms as in [3].

Definition 3. Let $K = u/v$ be shift-reduced. Define

$$\begin{aligned} \phi_K : \mathbb{C}[n] &\longrightarrow \mathbb{C}[n] \\ f(n) &\mapsto u(n)f(n+1) - v(n)f(n). \end{aligned}$$

Let $\mathcal{N}_K = \text{span}_{\mathbb{C}}\left\{n^\ell \mid \ell \in \mathbb{N} \text{ and } \ell \neq \deg(g) \text{ for all } g \in \text{im}(\phi_K)\right\}$.

Key Lemma. The $\mathbb{C}$-linear map ϕ_K is injective and $\mathbb{C}[n] = \text{im}(\phi_K) \oplus \mathcal{N}_K$.

Improved AP reduction. Given a hypergeometric sequence $T(n)$, compute two hypergeometric sequences $T_1(n)$ and $T_2(n)$ s.t. the two conditions in (1) hold.

1. Compute (2) as in step 1 of AP reduction.
2. Compute the projection p of w in $\mathcal{N}_K$. Set

$$T_2 := \left(\frac{a}{b} + \frac{p}{v}\right) H.$$

The sequence $T_1(n)$ can be constructed as the product of a rational function $r(n)$ and the sequence $H(n)$ incrementally.

The improved AP reduction avoids computing a polynomial solution of any auxiliary OΔE.

Experiments

We compare

– G: the Maple function Gosper in SumTools[Hypergeometric].

– S: a procedure that solves the summability problem based on improved AP, in which T_1 is not normalized if T is not summable.

– AP: the Maple function SumDecomposition in SumTools[Hypergeometric].

– IAP: the reduction algorithm of improved AP-reduction, in which T_1 is always normalized.

Test suite:

$$T(n) := \frac{f(n)}{g(n)\cdot g(n+\lambda)\cdot g(n+\mu)\cdot h(n)\cdot h(n+\lambda)\cdot h(n+\mu)} \cdot \prod_{k=n_0}^{n} \frac{u(k)}{v(k)},$$

where $f, g, h \in \mathbb{Z}[n]$ of respective degrees 20, 10 and 10, $u(n), v(n)$ are the product of two linear polynomials in $\mathbb{Z}[n]$, and $\lambda, \mu \in \mathbb{N}$ with $\lambda \leq \mu$.

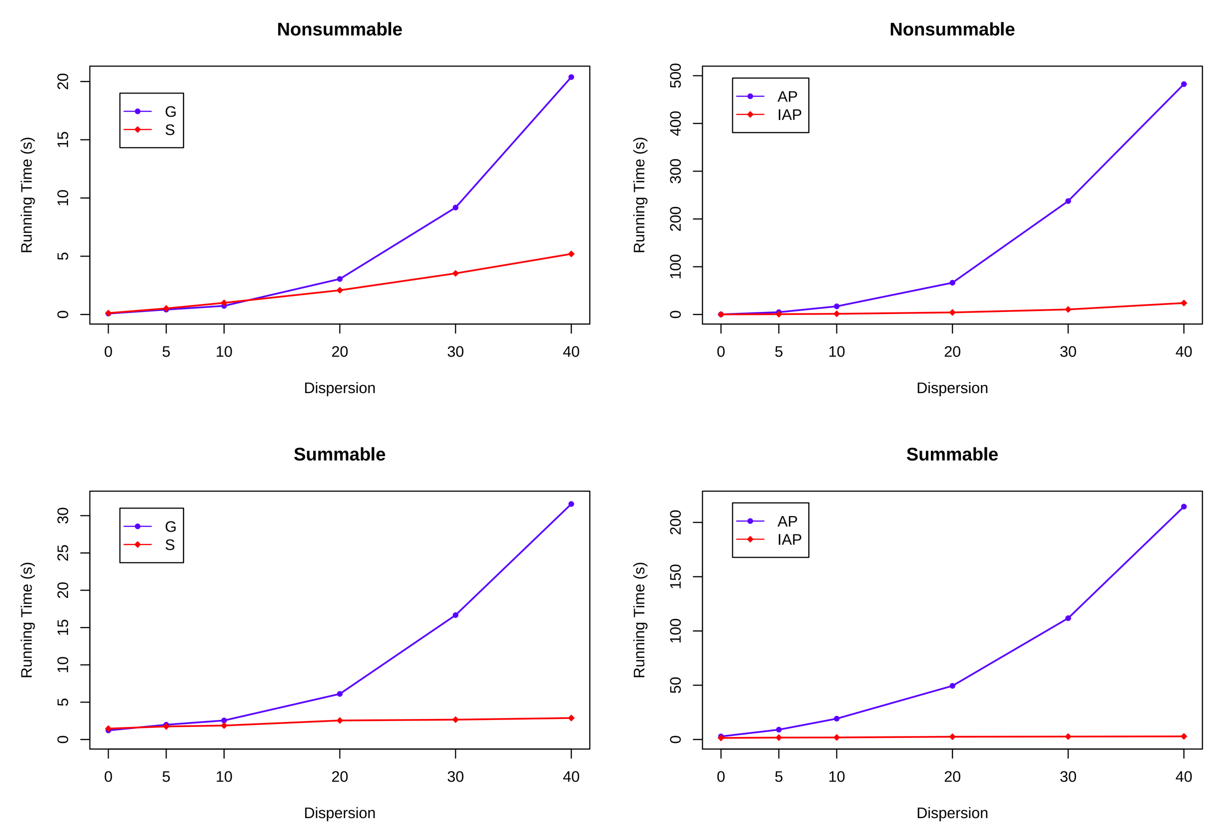
Input: $T(n)$

(λ, μ)	G	S	AP	IAP
[0, 0]	0.08	0.12	0.19	0.12
[5, 5]	0.42	0.52	4.80	0.64
[10, 10]	0.74	1.00	17.06	1.42
[10, 20]	3.05	2.08	66.50	4.30
[10, 30]	9.18	3.53	237.50	10.54
[10, 40]	20.38	5.20	482.34	24.02

Input: $T(n+1) - T(n)$

(λ, μ)	G	S	AP	IAP
[0, 0]	1.22	1.46	2.83	1.44
[5, 5]	1.98	1.75	9.06	1.76
[10, 10]	2.55	1.87	19.21	1.89
[10, 20]	6.11	2.55	49.43	2.55
[10, 30]	16.27	2.66	111.77	2.70
[10, 40]	31.56	2.88	214.57	2.89

Timings (in sec.) measured on a Mac computer, 4GB RAM, 3.06 GHz Core 2 Duo processor.



Experiments illustrate that the improved AP reduction is more efficient than both Gosper algorithm and AP-reduction.

A Potential Application

Can one compute the minimal telescoper for a bivariate hypergeometric term by the improved AP reduction, following the idea in [3]?

Advantage. Such an algorithm would separate the computation for telescopers from that for certificates so as to improve efficiency.

Difficulty. The least common multiple of shift-free polynomials is *not* necessarily shift-free.

References

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