A solution method for autonomous first-order algebraic partial differential equations

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In this paper we present a procedure for solving first-order autonomous algebraic partial differential equations. The method uses rational parametrizations of algebraic surfaces and generalizes a similar procedure for first-order autonomous ordinary differential equations. In particular we are interested in rational solutions and present certain classes in which such solutions exist. However, the method can also be used for finding non-rational solutions.

1 Introduction

Recently algebraic-geometric solution methods for algebraic ordinary differential equations (AODEs) were investigated. First results on solving first order AODEs can be found in [12] where Gröbner bases are used and [4] where a degree bound is computed which might be used for making an ansatz. The starting point for algebraic-geometric methods was an algorithm by Feng and Gao [5, 6] which decides whether or not an

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autonomous AODE, $F(y, y') = 0$ has a rational solution and in the affirmative case computes it. This result was then generalized by Ngô and Winkler [16, 18, 17] to the non-autonomous case $F(x, y, y') = 0$. First results on higher order AODEs can be found in [9, 10, 11]. Ngô, Sendra and Winkler [15] also classified AODEs in terms of rational solvability by considering affine linear transformations. A generalization to birational transformations can be found in [14]. In [7, 8] a solution method for autonomous AODEs is presented which generalizes the method of Feng and Gao to finding radical and also non-radical solutions. In this paper we present a generalization of the procedure to algebraic partial differential equations (APDEs). For the moment we restrict to first-order autonomous APDEs in two variables.

In Section 2 we will recall and introduce the necessary definitions and concepts. Then we will present the general procedure for solving APDEs in Section 3. In Section 4 we will consider the case of rational solutions. The section is divided into two parts. The first part proves some properties of rational solutions which can be found by the procedure. The second part presents a class of APDEs which has rational solutions. Finally in Section 5 we show that the procedure is not restricted to rational solutions.

2 Preliminaries

We consider the field of rational functions $\mathbb{K}(x, y)$. By $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ we denote the usual derivative by x and y respectively. Sometimes we might use the abbreviations $u_x = \frac{\partial u}{\partial x}$ and $u_y = \frac{\partial u}{\partial y}$. The ring of differential polynomials is denoted as $\mathbb{K}(x, y)\{u\}$. It consists of all polynomials in u and its derivatives, i. e.

$$\mathbb{K}(x, y)\{u\} = \mathbb{K}(x, y)[u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}, \ldots].$$

An algebraic partial differential equation (APDE) is defined by a differential polynomial $F \in \mathbb{K}(x, y)\{u\}$ which is also a polynomial in x and y. We write

$$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}, \ldots) = 0$$

for the considered APDE. In this paper we restrict to the first-order autonomous case, i. e. $F(u, u_x, u_y) = 0$.

An algebraic surface $\mathcal{S}$ is a two-dimensional algebraic variety, i. e. if we restrict to three-dimensional space this is a zero set of a squarefree non-constant polynomial $f \in \mathbb{K}[x, y, z]$, $\mathcal{S} = \{(a, b, c) \in \mathbb{A}^3 \mid f(a, b, c) = 0\}$. We call the polynomial f the defining polynomial. An important aspect of algebraic surfaces is their rational parametrizability. We consider an algebraic surface defined by an irreducible polynomial f. A triple of rational functions $\mathcal{P}(s, t) = (p_1(s, t), p_2(s, t), p_3(s, t))$ is called a rational parametrization of the surface if $f(p_1(s, t), p_2(s, t), p_3(s, t)) = 0$ for all s and t and the jacobian of $\mathcal{P}$ has generic rank 2. We observe that this condition is fundamental since, otherwise, we are parametrizing a point (if the rank is 0) or a curve on the surface (if the rank is 1). A parametrization can be considered as a dominant map $\mathcal{P}(s, t) : \mathbb{A}^2 \to \mathcal{S}$. By abuse of notation we also call this map a parametrization. We call a parametrization $\mathcal{P}(s,t)$ proper if it is a birational map or in other words if for almost every point (a,b,c) on the curve we find exactly one pair (s,t) such that $\mathcal{P}(s,t) = (a,b,c)$ or equivalently if $\mathbb{K}(\mathcal{P}(s,t)) = \mathbb{K}(s,t)$.

Above we have considered rational parametrizations of a surface. However, we might want to deal with more general parametrizations. If so, we will say that a triple of differentiable functions $\mathcal{Q}(s,t) = (q_1(s,t), q_2(s,t), q_3(s,t))$ is a parametrization of the surface if $f(\mathcal{Q}(s,t))$ is identically zero and the jacobian of $\mathcal{Q}(s,t)$ has generic rank 2.

Let $F(u, u_x, u_y) = 0$ be an autonomous APDE. We consider the corresponding algebraic surface by replacing the derivatives by independent transcendental variables, $F(z,p,q) = 0$. Given any differentiable function $u(x,y)$ with $F(u,u_x,u_y) = 0$, then $(u(s,t), u_x(s,t), u_y(s,t))$ is a parametrization. We call this parametrization the *corresponding parametrization of the solution*. We observe that the corresponding parametrization of a solution is not necessarily a parametrization of the associated surface. For instance, let us consider the APDE $u_x = 0$. A solution would be of the form $u(x,y) = g(y)$, with g differentiable. However, this solution generates $(g(t), 0, g'(t))$ that is a curve in the surface; namely the plane $p = 0$. Now, consider the APDE $u_x = \lambda$, with λ a nonzero constant. Hence, the solutions are of the form $u(x,y) = \lambda x + g(y)$. Then, $u(x,y) = \lambda x + y$ generates the line $(\lambda s + t, \lambda, 1)$ while $u(x,y) = \lambda x + y^2$ generates the parametrization $(\lambda s + t^2, \lambda, 2t)$ of the associated plane $p = \lambda$. These examples motivate the following definition. Clearly a solution of an APDE is a function $u(x,y)$ such that $F(u,u_x,u_y) = 0$.

Definition 2.1.
We say that a solution of an APDE is rational *if $u(x,y)$ is a rational function over an algebraic extension of $\mathbb{K}$.*
We say that a solution of an APDE is proper *if the corresponding parametrization is proper.*

In the case of autonomous ordinary differential equations, every non-constant solution induces a proper parametrization of the associated curve (see [5]). However, this is not true in general for autonomous APDEs. For instance, the solution $x + y^3$ of $u_x = 1$, induces the parametrization $(s + t^3, 1, 3t^2)$ which is, although its jacobian has rank 2, not proper.

Remark 2.2.
The jacobian of a proper parametrization $\mathcal{P}$ of a surface has generic rank 2 as we will see in the following. Since $\mathcal{P}$ is proper we know that $\mathbb{K}(s,t) = \mathbb{K}(\mathcal{P}(s,t))$. Hence, there is a rational function $R(a,b,c) = (R_1(a,b,c), R_2(a,b,c)) \in \mathbb{K}(a,b,c)^2$ such that $R(\mathcal{P}(s,t)) = (s,t)$. Thus, $\mathcal{J}_{\mathrm{id}} = \mathcal{J}_{R\circ\mathcal{P}} = \mathcal{J}_R(\mathcal{P}) \cdot \mathcal{J}_{\mathcal{P}}$. Taking into account, that the rank of a product of two matrices is smaller equal the minimal rank of the two matrices, we get that $\operatorname{rank}(\mathcal{J}_{\mathcal{P}}) = 2$.

We observe that, in the rational case, the condition on the rank of jacobian (see Definition 2.1) is equivalent to ask that the implicitization ideal of the parametrization is generated by F; compare with the notion of complete solution of suitable dimension in Definition 2.3. We denote by (F) the ideal generated by F.

Definition 2.3.
Let $F(u, u_x, u_y) = 0$ be an APDE. Assume we have a rational solution u depending on two constants c_1, c_2. Let $\mathcal{L} = (p_1, p_2, p_3)$ be the parametrization induced by the solution, i. e. $p_1 = u, p_2 = u_s, p_3 = u_t$. Assume $p_i = \frac{N_i}{D_i}$ with $\gcd(N_i, D_i) = 1$. We say that $u(s,t)$ is a complete *solution if $(F) = I \cap \mathbb{C}[s,t,z,p,q]$ where I is the ideal generated by $\{D_1 z - N_1, D_2 p - N_2, D_3 q - N_3, \operatorname{lcm}(D_1, D_2, D_3)w - 1\}$ over $\mathbb{C}[c_1, c_2, w, s, t, z, p, q]$.*

We call a solution complete of suitable dimension *if it is complete and $(F) = I \cap \mathbb{C}[c_1, c_2, z, p, q]$.*

Intuitively speaking, the notion of complete solution is requiring that the corresponding parametrization of the solution parametrizes an algebraic set on the surface, independently of the constants c_1 and c_2. The suitable dimension ensures that it parametrizes, indeed, the surface.

Note that the notion of complete also appears elsewhere in the theory of PDEs. In [13] a definition of completeness can be found which has a somehow similar purpose but in fact is not the same as the one we use here.

In the following example we will see complete and non-complete solutions of APDEs.

Example 2.4.
We consider the APDE $u_x = 0$, $F(z,p,q) = p$, as well as the solution $u(x,y) = y + c_1 + c_2$. The corresponding parametrization is $\mathcal{L} = (t + c_1 + c_2, 0, 1)$. Moreover, a Gröbner basis of I w.r.t. the lexicographic order with $c_1 > c_2 > w > u > s > t > p > q$ is $\{q-1, p, w-1, -u+t+c_1+c_2\}$. Thus, $I \cap \mathbb{C}[u,s,t,p,q]$ is generated by $\{q-1, p\}$, that is the line parametrized by $\mathcal{L}$, and hence $u(x,y)$ is not complete. However, if we take $u(x,y) = c_1 y + c_2$, the Gröbner basis is $\{p, w-1, qt + c_2 - u, -q + c_1\}$. So, $I \cap \mathbb{C}[u,s,t,p,q] = (p)$, and u is complete. However, it is not of suitable dimension because $I \cap \mathbb{C}[c_1, c_2, u, p, q] = (p, -q + c_1)$.

Now, if we take the APDE, $u_x = 1$. In Table 1 we see solutions and their properties. Note that the solution $s + c_1 + t^2 + c_2$ is not complete and hence, not complete of suitable dimension. However, the other property of suitable dimension is fulfilled.

solution	complete	suitable dim	proper	rank($\mathcal{J}$)
$s + c_1$	F	F	F	1
$s + t + c_1 + c_2$	F	F	F	1
$s + c_1 + c_2 t$	T	F	F	1
$s + c_1 + t^2 + c_2$	F	F	T	2
$s + c_1 + c_2 t^2$	T	T	T	2
$s + c_1 + (t + c_2)^2$	T	T	T	2
$s + c_1 + (t + c_2)^3$	T	T	F	2

Table 1: Properties of the solutions of $u_x = 1$ where T means true, F false

3 A method for solving first-order autonomous APDEs

Let $F(u, u_x, u_y) = 0$ be an algebraic partial differential equation. We consider the surface $F(z, p, q) = 0$ and assume it admits a proper (rational) surface parametrization

$$\mathcal{Q}(s,t) = (q_1(s,t), q_2(s,t), q_3(s,t)).$$

An algorithm for computing a proper rational parametrization of a surface can be found for instance in [19]. Here, we will stick to rational parametrizations, but the procedure which we present will work as well with other kinds of parametrizations, for instance radical ones. First results on radical parametrizations of surfaces can be found in [20]. Assume that $\mathcal{L}(s,t) = (p_1, p_2, p_3)$ corresponds to a solution of the APDE. Furthermore we assume that the parametrization $\mathcal{Q}$ can be expressed as

$$\mathcal{Q}(s,t) = \mathcal{L}(g(s,t))$$

for some invertible function $g(s,t) = (g_1(s,t), g_2(s,t))$. This assumption is motivated by the fact that in case of rational algebraic curves every non-constant rational solution of an AODE yields a proper rational parametrization of the associated algebraic curve and each proper rational parametrization can be obtained from any other proper one by a rational transformation. However, in the case of APDEs, not all rational solutions provide a proper parametrization, as mentioned in the remark after Definition 2.1. Now, using the assumption, if we can compute g^{-1} we have a solution $\mathcal{Q}(g^{-1}(s,t))$.

Let $\mathcal{J}$ be the jacobian matrix. Then we have

$$\mathcal{J}_{\mathcal{Q}}(s,t) = \mathcal{J}_{\mathcal{L}}(g(s,t)) \cdot \mathcal{J}_g(s,t).$$

Taking a look at the rows we get that

$$\begin{aligned} \frac{\partial q_1}{\partial s} &= \frac{\partial p_1}{\partial s}(g)\frac{\partial g_1}{\partial s} + \frac{\partial p_1}{\partial t}(g)\frac{\partial g_2}{\partial s} = q_2(s,t)\frac{\partial g_1}{\partial s} + q_3(s,t)\frac{\partial g_2}{\partial s} \\ \frac{\partial q_1}{\partial t} &= \frac{\partial p_1}{\partial s}(g)\frac{\partial g_1}{\partial t} + \frac{\partial p_1}{\partial t}(g)\frac{\partial g_2}{\partial t} = q_2(s,t)\frac{\partial g_1}{\partial t} + q_3(s,t)\frac{\partial g_2}{\partial t} \end{aligned} \tag{1}$$

This is a system of quasilinear equations in the unknown functions g_1 and g_2. In case q_2 or q_3 is zero the problem reduces to ordinary differential equations. Hence, from now on we assume that $q_2 \neq 0$ and $q_3 \neq 0$. First we divide by q_2:

$$\begin{aligned} a_1 &= \frac{\partial g_1}{\partial s} + b\frac{\partial g_2}{\partial s} \\ a_2 &= \frac{\partial g_1}{\partial t} + b\frac{\partial g_2}{\partial t} \end{aligned} \tag{2}$$

with

$$a_1 = \frac{\frac{\partial q_1}{\partial s}}{q_2}, \qquad a_2 = \frac{\frac{\partial q_1}{\partial t}}{q_2}, \qquad b = \frac{q_3}{q_2}. \tag{3}$$

By taking derivatives we get

$$\begin{aligned}\frac{\partial a_1}{\partial t} &= \frac{\partial^2 g_1}{\partial s \partial t} + \frac{\partial b}{\partial t}\frac{\partial g_2}{\partial s} + b\frac{\partial^2 g_2}{\partial s \partial t} \\ \frac{\partial a_2}{\partial s} &= \frac{\partial^2 g_1}{\partial t \partial s} + \frac{\partial b}{\partial s}\frac{\partial g_2}{\partial t} + b\frac{\partial^2 g_2}{\partial t \partial s}\end{aligned} \tag{4}$$

Subtraction of the two equations yields

$$\frac{\partial b}{\partial t}\frac{\partial g_2}{\partial s} - \frac{\partial b}{\partial s}\frac{\partial g_2}{\partial t} = \frac{\partial a_1}{\partial t} - \frac{\partial a_2}{\partial s} \tag{5}$$

This is a single quasilinear differential equation which can be solved by the method of characteristics (see for instance [22]). In case $\frac{\partial b}{\partial t} = 0$ or $\frac{\partial b}{\partial s} = 0$ equation (5) reduces to a simple ordinary differential equation.

Remark 3.1.
Remark that if both derivatives of b are zero then b is a constant. Then the left hand side of (5) is zero. In case the right hand side is non-zero we get a contradiction, and hence there is no solution. In case the right hand side is zero as well we get from (5) that

$$\begin{aligned}0 = \frac{\partial a_1}{\partial t} - \frac{\partial a_2}{\partial s} &= \frac{\partial}{\partial t}\left(\frac{\frac{\partial q_1}{\partial s}}{q_2}\right) - \frac{\partial}{\partial s}\left(\frac{\frac{\partial q_1}{\partial t}}{q_2}\right) \\ &= \frac{\frac{\partial q_1}{\partial t \partial s} q_2 - \frac{\partial q_1}{\partial s}\frac{\partial q_2}{\partial t}}{q_2^2} - \frac{\frac{\partial q_1}{\partial s \partial t} q_2 - \frac{\partial q_1}{\partial t}\frac{\partial q_2}{\partial s}}{q_2^2} \\ &= -\frac{\frac{\partial q_1}{\partial s}\frac{\partial q_2}{\partial t} - \frac{\partial q_1}{\partial t}\frac{\partial q_2}{\partial s}}{q_2^2}\end{aligned}$$

hence,

$$0 = \frac{\partial q_1}{\partial s}\frac{\partial q_2}{\partial t} - \frac{\partial q_1}{\partial t}\frac{\partial q_2}{\partial s}.$$

Moreover, since b is constant, $q_2 = kq_3$ for some constant k. But this means that the rank of the jacobian of $\mathcal{Q}$ is 1, a contradiction to $\mathcal{Q}$ being proper.

Therefore we assume from now on, that the derivatives of b are non-zero. According to the method of characteristics, we need to solve the following system of first-order ordinary differential equations

$$\begin{aligned}\frac{ds(t)}{dt} &= -\frac{\frac{\partial b}{\partial t}(s(t),t)}{\frac{\partial b}{\partial s}(s(t),t)}, \\ \frac{dv(t)}{dt} &= \frac{\frac{\partial a_1}{\partial t}(s(t),t) - \frac{\partial a_2}{\partial s}(s(t),t)}{-\frac{\partial b}{\partial s}(s(t),t)}.\end{aligned}$$

The second equation is linear and separable but depends on the solution of the first. The first ODE can be solved independently. Its solution $s(t) = \eta(t,k)$ will depend on an arbitrary constant k. Hence, also the solutions of the second ODE depends on k. Finally, the function g_2 we are looking for is $g_2(s,t) = v(t,\mu(s,t)) + \nu(\mu(s,t))$ where μ is computed such that $s = \eta(t,\mu(s,t))$ and ν is some function in k. In case we are only looking for rational solutions we can use the algorithm of Ngô and Winkler [16, 18, 17] for solving these ODEs.

Knowing g_2 we can compute g_1 by using equation (1) which now reduces to a separable ODE in g_1. The remaining task is to compute h_1 and h_2 such that $g(h_1(s,t), h_2(s,t)) = (s,t)$. Then $q_1(h_1,h_2)$ is a solution of the original PDE.

Finally the method reads as

Procedure 1.
Given an autonomous APDE, $F(u,u_x,u_y) = 0$, where F is irreducible.

1. *Compute a proper rational parametrization $\mathcal{Q} = (q_1,q_2,q_3)$ of $F(z,p,q) = 0$.*
2. *Compute the coefficients b and a_i as in (3).*
3. *If $\frac{\partial b}{\partial s} = 0$ and $\frac{\partial b}{\partial t} \neq 0$ compute $g_2 = \int \frac{\frac{\partial a_1}{\partial t} - \frac{\partial a_2}{\partial s}}{\frac{\partial b}{\partial t}} ds + \kappa(t)$ and go to step 7 otherwise continue.*
   If $\frac{\partial b}{\partial s} = \frac{\partial b}{\partial t} = 0$ return "No proper solution".
4. *Solve the ODE $\frac{ds(t)}{dt} = -\frac{\frac{\partial b}{\partial t}(s(t),t)}{\frac{\partial b}{\partial s}(s(t),t)}$ for $s(t) = \eta(t,k)$ with arbitrary constant k.*
5. *Solve the ODE $\frac{dv(t)}{dt} = \frac{\frac{\partial a_1}{\partial t}(\eta(t,k),t) - \frac{\partial a_2}{\partial s}(\eta(t,k),t)}{-\frac{\partial b}{\partial s}(\eta(t,k),t)}$*
   by $v(t) = v(t,k) = \int \frac{\frac{\partial a_1}{\partial t}(\eta(t,k),t) - \frac{\partial a_2}{\partial s}(\eta(t,k),t)}{-\frac{\partial b}{\partial s}(\eta(t,k),t)} dt + \nu(k)$.
6. *Compute μ such that $s = \eta(t,\mu(s,t))$ and then $g_2(s,t) = v(t,\mu(s,t))$.*
7. *Use the second equation of (2) to compute $g_1(s,t) = m(s) + \int a_2 - b\frac{\partial g_2}{\partial t} dt$.*
8. *Determine $m(s)$ by using the first equation of (2).*
9. *Compute h_1, h_2 such that $g(h_1(s,t), h_2(s,t)) = (s,t)$.*
10. *Return the solution $q_1(h_1,h_2)$.*

In general ν will depend on a constant c_2 and m on a constant c_1. As a special case of the procedure we will fix $\nu = c_2$. This choice is done for simplicity reasons but we may sometimes refer to cases with other choices which are subject of further research.

Furthermore, the procedure can be considered symmetrically in step 3 for the case that $\frac{\partial b}{\partial t} = 0$ and $\frac{\partial b}{\partial s} \neq 0$. In such a case the rest of the procedure has to be changed symmetrically as well. We will not go into further details.

Theorem 3.2.
Let $F(u,u_x,u_y) = 0$ be an autonomous APDE. If Procedure 1 returns a function $v(x,y)$ for input F, then v is a solution of F.

Proof. By the procedure we know that $v(x,y) = q_1(h_1(x,y), h_2(x,y))$ with h_i such that $g(h_1(s,t), h_2(s,t)) = (s,t)$. The function g fulfills the assumption that $u(g_1, g_2) = q_1$ for a solution u since it is a solution of the system (1). Hence, v is a solution. We have seen a more detailed description at the beginning of this section. □

Now, we will show that the result does not change if we postpone the introduction of c_1 and c_2 to the end of the procedure. It is easy to show that if $u(x,y)$ is a solution of an autonomous APDE then so is $u(x+c, y+d)$ for any constants c and d. From the procedure we see that in the computation of g_1 we use the derivative of g_2 only (and hence c_2 disappears). Therefore, we have that

$$g_2 = \bar{g}_2 + c_2, \qquad\qquad g_1 = \bar{g}_1 + c_1,$$

for some functions $\bar{g}_1, \bar{g}_2$. Let $g = (g_1, g_2)$ and $\bar{g} = (\bar{g}_1, \bar{g}_2)$. In the step 9 we are looking for a function h such that $g \circ h = \mathrm{id}$. Now $g \circ h = \bar{g} \circ h + (c_1, c_2)$. Take $\bar{h}$ such that $\bar{g} \circ \bar{h} = \mathrm{id}$. Then $g \circ \bar{h}(s - c_1, t - c_2)) = \mathrm{id}$. Hence, we can introduce the constants at the end.

In case the original APDE is in fact an AODE, the ODE in step 5 turns out to be trivial and the integral in step 8 is exactly the one which appears in the procedure for AODEs [7, 8]. Of course then g is univariate and so is its inverse. In this sense, this new procedure generalizes the procedure in [7, 8]. We do not specify Procedure 1 to handle this case.

In the following we see a simple example with a rational solution computed by the procedure.

Example 3.3.
We consider the autonomous APDE

$$F(u, u_x, u_y) = uu_x^2 - uu_xu_y + 7u_y^2 = 0.$$

Since F is of degree one in each of the derivatives, it is easy to compute a parametrization $\mathcal{Q} = \left(-\frac{7t^2}{s(s-t)}, s, t\right)$. We compute the coefficients

$$a_1 = \frac{7(2s-t)t^2}{s^3(s-t)^2}, \qquad a_2 = \frac{7t(-2s+t)}{s^2(s-t)^2}, \qquad b = \frac{t}{s}.$$

In step 4 we find $s(t) = tk$ and in step 5 we compute $v(t) = -\frac{7-14k}{(-1+k)^2kt} + \nu(k)$. Then $\mu(s,t) = \frac{s}{t}$ and hence (with $\nu = c_2$),

$$g_2 = \frac{7(2s-t)t}{s(s-t)^2} + c_2,$$
$$g_1 = \frac{7t^2(-2s+t)}{s^2(s-t)^2} + m(s).$$

Using step 8 we find out that $m(s) = c_1$. Computing the inverse of g we find

$$h_1 = -\frac{7\,(s-c_1)\,(s+2t-c_1-2c_2)}{(t-c_2)\,(s+t-c_1-c_2)^2},$$
$$h_2 = \frac{7\,(s-c_1)^2\,(s+2t-c_1-2c_2)}{(t-c_2)^2\,(s+t-c_1-c_2)^2}.$$

Finally, we get the solution $-\frac{7(x-c_1)^2}{(y-c_2)(x+y-c_1-c_2)}$.

4 Rational Solutions

For first-order autonomous AODE the algorithm of Feng and Gao [5] gives an answer on whether or not a rational solution exists. As Procedure 1 is a generalization of the the procedure for ODEs in [7, 8], it also generalizes this algorithm. However, as in [7, 8], the procedure gives a correct answer when everything is computable, but otherwise does not tell us whether a solution might exist. In the following we describe properties of rational solutions found by Procedure 1 and we give a class of APDEs that has a rational solution which can be found by the procedure.

4.1 Properties of Rational Solutions

In the following we will discuss the properties of rational solutions computed by our procedure. We will show that these solutions are proper and complete of suitable dimension.

Lemma 4.1.
If Procedure 1 yields a rational solution, then the solution is proper.

Proof. Let $\mathcal{L} = (p_1, p_2, p_3)$ be the corresponding parametrization of the output solution. In the procedure we start with a proper parametrization $\mathcal{Q}$ of the associated surface. When the procedure is successful we know that $\mathcal{L}(g) = \mathcal{Q}$ and the inverse h of g exists. Hence, $\mathcal{L} = \mathcal{Q}(h)$ is proper as well. □

Recall Remark 2.2 which proves that the jacobian of the corresponding parametrization of a proper solution computed by the procedure has generic rank 2.

Theorem 4.2.
If Procedure 1 yields a rational solution, then the solution is complete.

Proof. Let $\mathcal{L} = (p_1, p_2, p_3)$ be the parametrization corresponding to the solution. Let $\mathcal{L}^*$ be the parametrization without the constants c_1, c_2 (i. e. $\mathcal{L}(s, t, c_1, c_2) = \mathcal{L}^*(s+c_1, t+c_2)$).

Let $U(s,t,c_1,c_2) = \frac{N_1}{D_1}$ be the solution and $U_s = \frac{N_2}{D_2}$ and $U_t = \frac{N_3}{D_3}$ its derivatives w. r. t. s and t respectively. We consider the polynomials:

$$H_1 = D_1 z - N_1, \qquad H_2 = D_2 p - N_2,$$
$$H_3 = D_3 - qN_3, \qquad H_4 = W \operatorname{lcm}(D_1, D_2, D_3) - 1$$

Note that $H_1, \ldots, H_4 \in \mathbb{C}[s,t,c_1,c_2,z,p,q,W]$. Let $J = \langle H_1, \ldots, H_4 \rangle$ be the ideal generated by $\{H_1, \ldots, H_4\}$ over $\mathbb{C}[s,t,c_1,c_2,z,p,q,W]$. We want to prove that $J \cap \mathbb{C}[s,t,z,p,q] = \langle F \rangle$, where the ideal $\langle F \rangle$ is over $\mathbb{C}[s,t,z,p,q]$.

"$\subset$": Let $f \in J \cap \mathbb{C}[s,t,z,p,q]$. Then, f can be written as

$$f(s,t,z,p,q) = \sum_{i=1}^{4} A_i(s,t,c_1,c_2,z,p,q,W) H_i(s,t,c_1,c_2,z,p,q,W).$$

Let us denote

$$\Lambda = (s,t,U(s,t,c_1,c_2), U_s(s,t,c_1,c_2), U_t(s,t,c_1,c_2)) = (s,t,\mathcal{L}^*(s+c_1, t+c_2)).$$

We consider

$$f(\Lambda) = \sum_{i=1}^{4} A_i(\Lambda, W) H_i(\Lambda, W) = A_4(\Lambda, W) H_4(\Lambda, W)$$
$$= A_4(\Lambda, W)(W \cdot \operatorname{lcm}(D_1, D_2, D_3)(s,t,c_1,c_2) - 1).$$

Since $\operatorname{lcm}(D_1, D_2, D_3)(s,t,c_1,c_2)$ is not zero, because the corresponding rational functions $U(s,t,c_1,c_2), U_s(s,t,c_1,c_2), U_t(s,t,c_1,c_2)$ are well defined, and since $f(\Lambda)$ does not depend on W, we have that $A_4(\Lambda, W)$ is identically zero. Therefore,

$$f(\Lambda) = 0.$$

This means that for every particular value of the pair $(s_0, t_0) \in \mathbb{C}^2$, the polynomial $f(s_0, t_0, z, p, q)$ vanishes at the (jacobian-rank 2) parametrization $\mathcal{L}(s_0, t_0)$ of the surface $F(z,p,q) = 0$. Therefore, for every particular value of $(s_0, t_0) \in \mathbb{C}^2$, $F(z,p,q)$ divides $f(s_0, t_0, z, p, q)$. Let us see that this implies that F divides $f(s,t,z,p,q)$. Indeed, if we assume that F does not divide f, since F is irreducible, then $\gcd(F, f) = 1$. So, the resultant $R = \operatorname{res}_z(f, F)$ is not zero (if z does not appear take p or q). Now we take a value (s_0, t_0) that does not vanish R. Since the leading coefficient of F w. r. t. z does not vanish at (s_0, t_0), the resultant specializes properly (see for instance [21, Lemma 4.3.1]). So, $\gcd(f(s_0, t_0, z, p, q), F) = 1$ and hence F does not divide $f(s_0, t_0, z, p, q)$.

"$\supset$": Let us consider the polynomials

$$H_i^*(s,t,z,p,q,W) = H_i(s - c_1, t - c_2, c_1, c_2, z, p, q, W) \in \mathbb{C}[s,t,z,p,q,W]$$

and the corresponding ideal J^*. J^* is the implicitization ideal of $\mathcal{L}^*(s,t)$. Therefore, $J^* \cap \mathbb{C}[z,p,q] \supseteq \langle F \rangle$ where now $\langle F \rangle$ is over $\mathbb{C}[z,p,q]$. We write $\langle F \rangle_{z,p,q}$ and

$\langle F\rangle_{s,t,z,p,q}$ to distinguish between the two ideals. In any case, $F \in J^*$. So F can be expressed as

$$F = \sum_{i=1}^{4} A_i^*(s,t,z,p,q,W)H_i^*(s,t,z,p,q,W).$$

Since, F does not depend on s and t we get the following

$$\begin{aligned} F(z,p,q) &= \sum_{i=1}^{4} A_i^*(s+c_1,t+c_2,z,p,q,W)H_i^*(s+c_1,t+c_2,z,p,q,W) \\ &= \sum_{i=1}^{4} A_i^*(s+c_1,t+c_2,z,p,q,W)H_i(s,t,c_1,c_2,z,p,q,W). \end{aligned}$$

Now, take $g \in \langle F\rangle_{s,t,z,p,q}$. Then,

$$\begin{aligned} g &= M(s,t,z,p,q)F(z,p,q) \\ &= M(s,t,z,p,q)\sum_{i=1}^{4} A_i^*(s+c_1,t+c_2,z,p,q,W)H_i(s,t,c_1,c_2,z,p,q,W). \end{aligned}$$

Thus, $g \in J$ and by assumption $g \in \mathbb{C}[s,t,z,p,q]$.

□

4.2 APDEs with Rational Solutions

We are interested in whether or not a given APDE has rational solutions. We will not give a full answer but show classes of APDEs which have a rational solution that can be found by the procedure. The following lemma shows rational solvability for a certain class of APDEs.

Lemma 4.3.
Assume we have an APDE, $F(u,u_x,u_y)=0$, with a parametrization of the form $\mathcal{Q} = (s^{n+1}B(t), ts^nB(t), s^nB(t))$ where $B(t) = \frac{N(t)}{D(t)} \notin \mathbb{K}$ with $N(t), D(t) \in \mathbb{K}[t]$, $\gcd(N,D) = 1$ and $n \in \mathbb{Z}$.
Then F has an algebraic solution. Moreover, there is a rational solution if the equation

$$D(\alpha)N(\alpha)s(n+1) + (N'(\alpha)D(\alpha) - N(\alpha)D'(\alpha))(s\alpha + t) = 0$$

has a linear factor in α which also depends on s or t.

Proof. Let $\mathcal{Q} = (s^{n+1}B(t), ts^nB(t), s^nB(t))$. We observe that $\mathcal{Q}$ is proper (its inverse is $(z/q, p/q)$), and hence we can take $\mathcal{Q}$ in the first step of the procedure. Following the procedure, we get:

$$b = \frac{1}{t}, \qquad a_1 = \frac{n+1}{t}, \qquad a_2 = \frac{s}{t}\frac{B'}{B},$$

and hence, $\frac{\partial b}{\partial s} = 0$ but $\frac{\partial b}{\partial t} \neq 0$. Therefore

$$g_2 = \int \frac{\frac{\partial a_1}{\partial t} - \frac{\partial a_2}{\partial s}}{\frac{\partial b}{\partial t}} ds = s(n + 1 + t\frac{B'}{B}),$$

$$g_1 = \int a_2 - b\frac{\partial g_2}{\partial t} dt + m(s) = -s\frac{B'}{B} + m(s).$$

Now we need to find $m(s)$. We do so by using the equation as in the procedure

$$\begin{aligned}
\frac{\partial q_1}{\partial s} &= q_2\frac{\partial g_1}{\partial s} + q_3\frac{\partial g_2}{\partial s} \\
(n+1)s^n B &= s^n tB(-\frac{B'}{B} + m'(s)) + s^n B(n + 1 + t\frac{B'}{B}) \\
(n+1)s^n B &= -s^n tB' + s^n tBm' + s^n(n+1)B + s^n tB' \\
0 &= s^n tBm' \\
0 &= m'.
\end{aligned}$$

Hence, m is a constant and we choose $m = 0$. Finally we need to find h_1, h_2 fulfilling

$$\begin{aligned}
g_1(h_1, h_2) &= s, & g_2(h_1, h_2) &= t, \\
-h_1\frac{B'(h_2)}{B(h_2)} &= s, & h_1(n + 1 + h_2\frac{B'(h_2)}{B(h_2)}) &= t.
\end{aligned}$$

This means

$$\begin{aligned}
-s\frac{B(h_2)}{B'(h_2)} &= t(n + 1 + h_2\frac{B'(h_2)}{B(h_2)})^{-1} \\
-sB(h_2)(n + 1 + h_2\frac{B'(h_2)}{B(h_2)}) &= tB'(h_2) \\
B(h_2)s(n+1) + B'(h_2)(sh_2 + t) &= 0.
\end{aligned}$$

Hence, after clearing denominators, we have an algebraic equation for h_2 and therefore also for h_1. Thus, we get an algebraic solution. Furthermore we get a rational solution if the last equation has a factor with degree 1 in h_2 which also depends on s or t. □

Corollary 4.4.
Let the APDE be of the form

$$F(u, u_x, u_y) = \lambda u^m + \gamma_{m-1}(u_x, u_y) = 0$$

where $m \in \mathbb{N}, \lambda \in \mathbb{C} \setminus \{0\}$ and $\gamma_{m-1}(p, q)$ be a form of degree $m - 1$. Then F has an algebraic solution.

Proof. Observe that $F(z, p, q)$ is irreducible, and can be parametrized as

$$\mathcal{Q}(s, t) = \left(-s\frac{\gamma_{m-1}(t, 1)}{\lambda s^m}, -t\frac{\gamma_{m-1}(t, 1)}{\lambda s^m}, -\frac{\gamma_{m-1}(t, 1)}{\lambda s^m}\right),$$

that corresponds to the parametrization form in Lemma 4.3 with $n = -m$ and $B(t) = -\gamma_{m-1}(t, 1)/\lambda$. Hence, there is an algebraic solution. □

Note, that the same is applicable to APDEs $F(u, u_x, u_y) = \lambda u_x^m + \gamma_{m-1}(u, u_y) = 0$ and $F(u, u_x, u_y) = \lambda u_y^m + \gamma_{m-1}(u, u_x) = 0$. The following example is of the form required in the corollary and yields a rational solution.

Example 4.5.
We consider the APDE

$$F(u, u_x, u_y) = 6u^4 + 5u_x^3 + 5u_x^2 u_y = 0.$$

This example fulfills the requirements of Lemma 4.3. We compute a parametrization

$$\mathcal{Q} = \left(-\frac{5t^2 + 5t^3}{6s^3}, -\frac{t\,(5t^2 + 5t^3)}{6s^4}, -\frac{5t^2 + 5t^3}{6s^4} \right).$$

With the notation of Lemma 4.3 we have $B(t) = -\frac{5t^2+5t^3}{6}$ *and* $n = -4$*. Hence, we have to solve the following equation for* h_2

$$\begin{aligned} B(h_2)s(n+1) + B'(h_2)(sh_2 + t) &= 0 \\ -3(5h_2^2 + 5h_2^3)s + (10h_2 + 15h_2^2)(sh_2 + t) &= 0 \\ h_2\left(-3(5h_2 + 5h_2^2)s + (10 + 15h_2)(sh_2 + t)\right) &= 0 \\ 5h_2\left(-h_2 s + 3h_2 t + 2t\right) &= 0. \end{aligned}$$

Doing so we get

$$h_2 = -\frac{2t}{-s + 3t}.$$

Now using $-h_1 \frac{B'(h_2)}{B(h_2)} = s$ *we compute*

$$h_1 = -\frac{t(-s + t)}{-s + 3t}.$$

Finally, we get the solution $u(x, y) = \frac{10}{3(x-y)^2 y}$ *and hence* $u(x + c_1, y + c_2)$ *is a solution for any constants* c_1 *and* c_2*.*

5 Other Solutions

The procedure presented in this paper is, however, not restricted to rational solutions nor to rational parametrizations as we will see in the following examples. In this section we will show examples with non-rational solutions which can be computed by the procedure. We start with an example which has a radical solution.

Example 5.1.
We consider the APDE

$$F(u, u_x, u_y) = 5u^3 u_x - 7u_x^5 + 5u^3 u_y - u_x^4 u_y = 0.$$

This example fulfills the requirements of Lemma 4.3. We compute a parametrization

$$\mathcal{Q} = \left(-\frac{s\left(5s^3+5s^3t\right)}{-t^4-7t^5}, -\frac{t\left(5s^3+5s^3t\right)}{-t^4-7t^5}, -\frac{5s^3+5s^3t}{-t^4-7t^5}\right).$$

With the notation of Lemma 4.3 we have $B(t) = -\frac{5+5t}{-t^4-7t^5}$ *and* $n = 3$*. Hence, we have to solve the following equation for* h_2

$$B(h_2)s(n+1) + B'(h_2)(sh_2+t) = 0.$$

Doing so we get

$$h_2 = \frac{-19t - \sqrt{361t^2 - 8t(3s+14t)}}{2(3s+14t)}.$$

Now using $-h_1\frac{B'(h_2)}{B(h_2)} = s$ *we compute*

$$h_1 = \frac{1}{4}\left(t - \frac{19st}{2(3s+14t)} - \frac{s\sqrt{361t^2-8t(3s+14t)}}{2(3s+14t)}\right).$$

Finally, we get the solution

$$u(x,y) = \frac{5\left(-6x-9y+\sqrt{3}\sqrt{y(-8x+83y)}\right)\left(13xy-28y^2+\sqrt{3}x\sqrt{y(-8x+83y)}\right)^4}{256\left(19y+\sqrt{3}\sqrt{y(-8x+83y)}\right)^4\left(-6x+105y+7\sqrt{3}\sqrt{y(-8x+83y)}\right)}.$$

Furthermore, $u(x+c_1, y+c_2)$ *is a solution for any constants* c_1 *and* c_2*.*

In a further example we compute an exponential solution of an APDE.

Example 5.2.
We consider the APDE

$$F(u, u_x, u_y) = 4u^4 - 8u_x^3 + 8u^3u_y = 0.$$

We compute a parametrization $\mathcal{Q} = \left(\frac{8st^3}{8s^3+4s^4}, \frac{8t^4}{8s^3+4s^4}, \frac{8t^3}{8s^3+4s^4}\right)$*. We compute the coefficients*

$$a_1 = -\frac{4+3s}{2t+st}, \qquad a_2 = \frac{3s}{t^2}, \qquad b = \frac{1}{t}.$$

Solving the ODEs we get

$$g_2 = \log(2+s), \qquad g_1 = -\frac{3s}{t}.$$

Computing the inverse of g *we find*

$$h_1 = -2 + e^{t/2}, \qquad h_2 = \frac{3\left(2-e^{t/2}\right)}{s}.$$

Finally, we get the solution $-\frac{54e^{-y/2}\left(-2+e^{y/2}\right)}{x^3}$*.*

6 Conclusion

We have introduced a procedure which, in case all steps are computable, yields a solution of the input APDE. In case one step of the procedure is not computable (in a certain class of functions) we cannot give an answer. Furthermore, in the case of rational solutions we have shown that the output of the procedure is proper and complete. We have also shown classes of APDEs which have rational solutions. The investigation of rational solutions will be subject of further research. The procedure finds solutions of the following well known PDEs.

Burgers (inviscid) [22, p. 174]
$F(u, u_x, u_y) = uu_x + u_y = 0$ with $\mathcal{Q} = (-\frac{t}{s}, s, t)$ yields the solution $\frac{s-c_1}{t-c_2}$.

Traffic [3, p. 151]
$F(u, u_x, u_y) = u_y + u_x \left(-\frac{uv_m}{r_m} + \left(1 - \frac{u}{r_m}\right) v_m\right) = 0$ with $\mathcal{Q} = (\frac{r_m(t+sv_m)}{2sv_m}, s, t)$ yields the solution $\frac{r_m(-s+tv_m+c_1-v_m c_2)}{2v_m(t-c_2)}$.

Eikonal [2, p. 2]
$F(u, u_x, u_y) = u_x^2 + u_y^2 - 1 = 0$ with $\mathcal{Q} = (s, \frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2})$ yields the solution $\pm\sqrt{s^2 + t^2 - 2sc_1 + c_1^2 - 2tc_2 + c_2^2}$.

Convection-Reaction [1, p. 7]
$F(u, u_x, u_y) = u_x + cu_y - du = 0$ with parametrization $\mathcal{Q} = (\frac{s+ct}{d}, s, t)$ yields the solution $\frac{\mathrm{e}^{d(s-c_1)} + c\mathrm{e}^{\frac{d(t-c_2)}{c}}}{d}$.

Generalized Burgers (special case) [22, p. 176]
$F(u, u_x, u_y) = u_y + u^n u_x + \left(\frac{j}{2y} + \alpha\right) u + \left(\beta + \frac{\gamma}{x}\right) u^{n+1} - \frac{\delta}{2} u_{xx} = 0$ with $j = \gamma = \delta = 0$ and $n = 1$ has the parametrization $\mathcal{Q} = (-\frac{s(1+s\alpha)}{st+s^2\beta}, -\frac{t(1+s\alpha)}{st+s^2\beta}, -\frac{1+s\alpha}{st+s^2\beta})$ which yields the solution $\frac{\mathrm{e}^{-s\beta}\left(-\mathrm{e}^{s\beta}+\mathrm{e}^{\beta c_1}\right)\alpha}{\left(1+\mathrm{e}^{\alpha(t-c_2)}\right)\beta}$.

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