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Symmetry and Prime Divisibility Properties of Partitions of *n* into Exactly *m* Parts

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Abstract. Let $p(n, m)$ denote the number of partitions of n into exactly m parts. In this paper we uncover new congruences for the function $p(n, m)$ and give an alternate proof to a known theorem in addition to extending it. The methods of proof rely on identifying generating functions to polynomials and then using the symmetric properties of those polynomials. The theorems proved here provide further motivation and description for a full characterisation of Ramanujan-like divisibility statements about the partition numbers $p(n, m)$.

Keywords: partition, generating function, congruence, Ramanujan

1. Introduction

A partition of a positive integer n is a finite non-increasing sequence of positive integers $\{\lambda_1, \lambda_2, \ldots, \lambda_r\}$which sum to n. The λ_i are called the parts of the partition. The function $p(n)$ denotes the number of partitions of n and $p(n, m)$ is the number of partitions of n into exactly m parts. The relationship between $p(n)$ and $p(n, m)$ is clear:

$$p(n) = p(n, 1) + p(n, 2) + \cdots + p(n, n). \tag{1.1}$$

Divisibility and periodicity properties of the sequence $\{p(n, m) \pmod{M}\}_{n=0}$ have been studied on several occasions [2–7]. Gupta gives an extensive history and tables of values of $p(n, m)$ in [1]. The purpose of this paper is to study prime divisibility and symmetry properties of the restricted partition function $p(n, m)$ and in doing so, report several new theorems. Moreover, our methods allow us to extend the results of a known theorem in addition to providing an alternate proof for it.

2. Main Theorem

For ℓ an odd prime and $0 \le m \le \frac{\ell-3}{2}$ we define

$$x_\ell(m) = \frac{\ell+1-|\ell-4m-3|}{4}.$$

We further define $lcm(k)$ to be the least common multiple of the numbers from 1 to k.

Let $p(n)$ denote the number of partitions of n and $p(n,m)$ denote the number of partitions of n into exactly m parts. We note that $p(n) = 0$ when $n < 0$ and $p(n,m) = 0$ for $n < m$. The purpose of this paper is to prove the following theorem:

Theorem 2.1. *For ℓ an odd prime, $-\ell \cdot x_\ell(m) \le n \le \ell - 1$ and $k \ge 0$;*

$$\begin{aligned} &p\big(lcm(\ell)k - \ell m - [\ell x_\ell(m)+n], \ell\big) + p\big(lcm(\ell)k - \ell m + [\ell x_\ell(m)+n], \ell\big) \\ &\equiv p(n) \pmod{\ell}, \end{aligned} \tag{2.1}$$

unless $m = \frac{\ell-3}{4}$ is an integer, then

$$\begin{aligned} &p\big(lcm(\ell)k - \ell m - [\ell x_\ell(m)+n], \ell\big) + p\big(lcm(\ell)k - \ell m + [\ell x_\ell(m)+n], \ell\big) \\ &\equiv 2p(n) \pmod{\ell}. \end{aligned} \tag{2.2}$$

For every m for which $x_\ell(m)$ is defined, Theorem 2.1 gives a list of $\ell x_\ell(m) + \ell$ partition congruences. Theorem 2.1 extends a previous theorem of the first author [2] which we state here as a corollary to Theorem 2.1.

Corollary 2.2. ([2]) *For ℓ an odd prime, $k \ge 0$, and $0 \le m \le \frac{\ell-3}{2}$,*

$$p(lcm(\ell)k - \ell m, \ell) \equiv 0 \pmod{\ell}. \tag{2.3}$$

Corollary 2.2 is proved in Section 8 of this paper.

3. Background Material

Proposition 3.1. $p(n) = p(m+n, m)$ *for $n \le m$.*

Proposition 3.1 is stated without proof.

Definition 3.2. ([2]) *A polynomial $P(q) = b_0 + b_1 q + \cdots + b_n q^n$ of degree n is called anti-reciprocal if for each i,*

$$b_i = -b_{n-i}. \tag{3.1}$$

Definition 3.3. *If a polynomial $P(q)$ of degree n is antireciprocal then for any integer k we will call $q^k P(q)$ a termwise antireciprocal polynomial of degree $k+n$.*

When $q^k P(q)$ is a termwise antireciprocal polynomial (termwise reciprocal polynomial, respectively) we will call the coefficients b_{i+k} and b_{n-i+k} termwise antireciprocal partners.

Theorem 3.4. *The generating function for the difference between the number of partitions of n into exactly $\ell-1$ parts and the number of partitions of $n-lcm(\ell)$ into exactly $\ell-1$ parts is congruent modulo ℓ to a termwise periodic antireciprocal polynomial. We will call this polynomial $A(q;\ell-1)$. The coefficients of $A(q;\ell-1)$ are* $p(n,\ell-1) \pmod{\ell}$.

$$\begin{aligned}\sum_{n=0}^{\infty}[p(n,\ell-1)-p(n-lcm(\ell),\ell-1)]q^n &\equiv \frac{q^{\ell-1}\left(1-q^{lcm(\ell-1)}\right)^{\ell}}{(q;q)_{\ell-1}} \\ &\equiv A(q;\ell-1) \pmod{\ell}.\end{aligned} \tag{3.2}$$

Depending on $\ell \pmod 4$ *and for* $0\le j\le lcm(\ell)$, $A(q;\ell-1)$ *has the following properties*:

- *If* $\ell\equiv 1 \pmod 4$, *then* $A(q;\ell-1)$ *has periodic degree* $lcm(\ell)-\frac{\ell^2-5\ell+4}{4}$ *with* $lcm(\ell)+1$ *terms and the coefficients have the following antireciprocal property*:

$$\begin{aligned}&p\left(-\frac{\ell^2-5\ell+4}{4}+j,\ell-1\right) \\ &\equiv -p\left(lcm(\ell)-\frac{\ell^2-5\ell+4}{4}-j,\ell-1\right) \pmod{\ell}.\end{aligned} \tag{3.3}$$

- *If* $\ell\equiv 3 \pmod 4$, *then* $A(q;\ell-1)$ *has periodic degree* $lcm(\ell)-\frac{(\ell-1)^2-3(\ell-1)+2}{4}$ *with* $lcm(\ell)$ *terms and and the coefficients have the following antireciprocal property*:

$$p\left(-\frac{\ell^2-5\ell+2}{4}+j,\ell-1\right)\equiv -p\left(lcm(\ell)-\frac{\ell^2-5\ell+6}{4}-j,\ell-1\right) \pmod{\ell}. \tag{3.4}$$

Our results will come from an examination of the symmetric properties of the coefficients of $A(q;\ell-1)$.

We now prove Theorem 3.4.

Proof of Theorem 3.4. In [2], it was shown that the rational function $\left(1-q^{lcm(\ell)}\right)/(q;q)_{\ell-1}$ can be congruentially identified to an antireciprocal polynomial of degree $lcm(\ell)-\frac{\ell^2-\ell}{2}$. Call this polynomial $L(q;\ell-1)$.

$$\begin{aligned}\frac{1-q^{lcm(\ell)}}{(q;q)_{\ell-1}} &\equiv \frac{\left(1-q^{lcm(\ell-1)}\right)^{\ell}}{(q;q)_{\ell-1}} \\ &\equiv 1+a_1q+a_2q^2+\cdots+a_{lcm(\ell)-\frac{\ell^2-\ell}{2}}q^{lcm(\ell)-\frac{\ell^2-\ell}{2}} \\ &\equiv L(q;\ell-1) \pmod{\ell}.\end{aligned} \tag{3.5}$$

From Definition 3.3 we see that $q^{\ell-1}L(q;\ell-1)$ is a termwise antireciprocal polynomial of degree $lcm(\ell)-\frac{\ell^2-3\ell+2}{2}$ with $lcm(\ell)-\frac{\ell^2-\ell-2}{2}$ terms. The number of terms is somewhat less than what we would expect given that the sequence $\{p(n,\ell-1) \pmod{\ell}\}_{n\geq 0}$ has period $lcm(\ell)$. So that the period of the sequence and the number of terms of the polynomial correspond we embellish $q^{\ell-1}L(q;\ell-1)$ by including additional terms, each of which has coefficient zero. We will call these additional terms *periodic terms*, we will call this embellished polynomial a *periodic polynomial,* and we will take the periodic terms into consideration when determining the degree of the periodic polynomial. We call the embellished polynomial $A(q;\ell-1)$.

Though $lcm(\ell)$ is always even, $A(q;\ell-1)$ may have an even or odd number of terms depending on $\ell \pmod 4$. Our goal is simultaneously to have the number of terms of $A(q;\ell-1)$ reflect the period of $\{p(n,\ell-1) \pmod{\ell}\}_{n\geq 0}$, and to make efficient use of the antireciprocal properties of $A(q;\ell-1)$.

Case 1. If $\ell\equiv 1 \pmod 4$ then $q^{\ell-1}L(q;\ell-1)$ has exactly $lcm(\ell)-\frac{\ell^2-\ell-2}{2}$ terms, where $lcm(\ell)-\frac{\ell^2-\ell-2}{2}$ is an odd number. We locate $\frac{\ell^2-\ell}{4}$ of the periodic terms before the first non-zero term $q^{\ell-1}$ of $q^{\ell-1}L(q;\ell-1)$. Namely, $\left\{0q^{-\frac{\ell^2-5\ell+4}{4}}, 0q^{\frac{-\ell^2-5\ell}{4}},\ldots, 0q^{\ell-3}, 0q^{\ell-2}\right\}$. We position the remaining $\frac{\ell^2-\ell}{4}$ periodic terms following the last non-zero term $q^{lcm(\ell)-\frac{\ell^2-3\ell+2}{2}}$ of $q^{\ell-1}L(q;\ell-1)$. Namely, $\left\{0q^{lcm(\ell)-\frac{\ell^2-3\ell}{2}}, 0q^{lcm(\ell)-\frac{\ell^2-3\ell-2}{2}}, \ldots, 0q^{lcm(\ell)-\frac{\ell^2-5\ell+8}{4}}, 0q^{lcm(\ell)-\frac{\ell^2-5\ell+4}{4}}\right\}$. In other words, for $-\frac{\ell^2-5\ell+4}{4}\leq j\leq \ell-2$ and $lcm(\ell)-\frac{\ell^2-3\ell}{2}\leq j\leq lcm(\ell)-\frac{\ell^2-5\ell+4}{4}$, the coefficent $a_j=0$. This makes for a total of $lcm(\ell)-\frac{\ell^2-\ell-2}{2}+\frac{\ell^2-\ell}{2}=lcm(\ell)+1$ terms for $A(q;\ell-1)$ and for $\ell\equiv 1 \pmod 4$,

$$A(q;\ell-1)=0q^{-\frac{\ell^2-5\ell+4}{4}}+\cdots+0q^{\ell-2}+\left[q^{\ell-1}L(q;\ell-1)\right] + 0q^{lcm(\ell)-\frac{\ell^2-3\ell}{2}}+\cdots+0q^{lcm(\ell)-\frac{\ell^2-5\ell+4}{4}}.$$

Lastly, it is clear that $A(q;\ell-1)$ is termwise periodic antireciprocal and so we have (3.3). Hence,

$$p\left(-\frac{\ell^2-5\ell+4}{4}+j,\ell-1\right)\equiv p\left(lcm(\ell)-\frac{\ell^2-5\ell+4}{4}-j,\ell-1\right) \pmod{\ell}.$$

Case 2. If $\ell\equiv 3 \pmod 4$ then $q^{\ell-1}L(q;\ell-1)$ has exactly $lcm(\ell)-\frac{\ell^2-\ell-2}{2}$ terms, where $lcm(\ell)-\frac{\ell^2-\ell-2}{2}$ is an even number. We locate $\frac{\ell^2-\ell-2}{4}$ of the periodic terms before the first non-zero term $q^{\ell-1}$ of $q^{\ell-1}L(q;\ell-1)$. Namely, $\left\{0q^{-\frac{\ell^2-5\ell+2}{4}}, 0q^{-\frac{\ell^2-5\ell-2}{4}}, \ldots, 0q^{\ell-3}, 0q^{\ell-2}\right\}$. We position the remaining $\frac{\ell^2-\ell+2}{4}$ periodic terms following the last non-zero term $q^{lcm(\ell)-\frac{\ell^2-3\ell+2}{2}}$ of $q^{\ell-1}L(q;\ell-1)$. Namely, $\left\{0q^{lcm(\ell)-\frac{\ell^2-3\ell}{2}}, 0q^{lcm(\ell)-\frac{\ell^2-3\ell-2}{2}},\ldots, 0q^{lcm(\ell)-\frac{\ell^2-5\ell+10}{4}}, 0q^{lcm(\ell)-\frac{\ell^2-5\ell+6}{4}}\right\}$. In other words, for $-\frac{\ell^2-5\ell+2}{4}\leq j\leq \ell-2$ and $lcm(\ell)-\frac{\ell^2-3\ell}{2}\leq j\leq lcm(\ell)-\frac{\ell^2-5\ell+6}{4}$, the coefficent $a_j=0$. This makes for a total of $lcm(\ell)-\frac{\ell^2-\ell-2}{2}+\frac{\ell^2-\ell+2}{2}=lcm(\ell)$ terms for $A(q;\ell-1)$ and for

$\ell \equiv 3 \pmod 4$,

$$A(q;\ell-1) = 0q^{-\frac{\ell^2-5\ell+2}{4}} + \cdots + 0q^{\ell-2} + \left[q^{\ell-1}L(q;\ell-1)\right] + 0q^{lcm(\ell)-\frac{\ell^2-3\ell}{2}} + \cdots + 0q^{lcm(\ell)-\frac{\ell^2-5\ell+6}{4}}. \tag{3.6}$$

Lastly, it is clear that $A(q;\ell-1)$ is termwise periodic antireciprocal and so we have (3.4). Hence,

$$p\left(-\frac{\ell^2-5\ell+2}{4}+j,\ell-1\right) \equiv -p\left(lcm(\ell)-\frac{\ell^2-5\ell+6}{4}-j,\ell-1\right) \pmod{\ell}.$$

∎

As an immediate result of Theorem 3.4, we highlight the following statement by labelling it as a corollary.

Corollary 3.5. *The coefficients of the generating function*

$$\sum_{n=0}^{\infty} p(n,\ell-1)q^n = \frac{q^{\ell-1}}{(q;q)_{\ell-1}} \tag{3.7}$$

are periodic modulo ℓ with period $lcm(\ell)$. Equivalently, the sequence

$$\{p(n,\ell-1) \pmod{\ell}\}_{n\geq 0} \tag{3.8}$$

is periodic with period $lcm(\ell)$.

Corollary 3.5 allows us to congruentially identify partition numbers $p(n,\ell-1)$ to the coefficients of the polynomial $A(q;\ell-1)$. Corollary 3.5 is a special case of a more general theorem of Wilf and Nijenhuis [7] which was later refined by Kwong in [6].

Proposition 3.6. *Depending on $\ell \pmod 4$ we consider the following two cases*:

- *For $\ell \equiv 1 \pmod 4$ and $r \equiv \rho \pmod{lcm(\ell)}$ with $-\frac{\ell^2-5\ell+4}{4} \leq \rho \leq lcm(\ell) - \frac{\ell^2-5\ell+4}{4}$,*

$$p(lcm(\ell)k+r,\ell-1) \equiv p(r,\ell-1) \equiv p(\rho,\ell-1) \pmod{\ell}. \tag{3.9}$$

- *For $\ell \equiv 3 \pmod 4$ and $r \equiv \rho \pmod{lcm(\ell)}$ with $-\frac{\ell^2-5\ell+2}{4} \leq \rho \leq lcm(\ell) - \frac{\ell^2-5\ell+6}{4}$,*

$$p(lcm(\ell)k+r,\ell-1) \equiv p(r,\ell-1) \equiv p(\rho,\ell-1) \pmod{\ell}, \tag{3.10}$$

  where $p(\rho,\ell-1) \pmod{\ell}$ is the coefficient on the ρ^{th} term of $A(q,\ell-1)$.

For example, $p(60k+7,4) \equiv p(67,4) \equiv p(7,4) \pmod 5$ where $p(7,4) \pmod 5$ is the coefficient on q^7 in the polynomial $A(q;4)$. In the case $\ell \equiv 1 \pmod 4$ and $r \equiv -\frac{\ell^2-5\ell+4}{4}$ or $lcm(\ell) - \frac{\ell^2-5\ell+4}{4} \pmod{lcm(\ell)}$ we consider $r \equiv -\frac{\ell^2-5\ell+4}{4}$. For example, $p(60k+59,4) \equiv p(59,4) \equiv p(-1,4) \pmod 4$.

4. A Short Example of Theorem 3.4

We illustrate some of these ideas by an examination of the sequence $\{p(n,2)\}$. From Corollary 3.5 the sequence $\{p(n,2) \pmod 3\}_{n\geq 0}$ is periodic with period $lcm(3)=6$. Using Theorem 3.4 and Proposition 3.6 we identify the partition numbers $\{p(n,2) \pmod 3\}_{n\geq 0}$ to the coefficients $p(\rho,2) \pmod 3$ of $A(q;2)$.

$$\sum_{n=0}^{\infty}[p(n,2)-p(n-6,2)]q^n = \frac{q^2(1-q^6)}{(q;q)_2} \equiv q^2L(q;2) \pmod 3, \tag{4.1}$$

where

$$q^2L(q;2) = \frac{q^2\left(1-q^2\right)^3}{(1-q)\left(1-q^2\right)} = q^2+q^3-q^4-q^5 \tag{4.2}$$

is a termwise antireciprocal polynomial of degree 5 with 4 terms. By including two periodic terms, namely, $0q$ and $0q^6$, we arrive at $A(q;2)$, our termwise antireciprocal periodic polynomial of degree 6 with 6 terms:

$$0q+q^2L(q;2)-0q^6 = A(q;2) = 0q+q^2+q^3-q^4-q^5-0q^6. \tag{4.3}$$

We display the antireciprocal properties of the coefficients of $A(q;2)$ below;

$$p(1,2)\equiv 0 \pmod 3 \leftrightarrow -p(6,2)\equiv 0 \pmod 3,$$
$$p(2,2)\equiv 1 \pmod 3 \leftrightarrow -p(5,2)\equiv 1 \pmod 3,$$
$$p(3,2)\equiv 1 \pmod 3 \leftrightarrow -p(4,2)\equiv 1 \pmod 3.$$

5. Further Background Material

Proposition 5.1.

$$p(n,m) = \sum_{i=0}^{\lfloor n/m\rfloor-1} p(n-1-mi, m-1). \tag{5.1}$$

Proposition 5.1 comes from iterating the following partition identity due to Euler,

$$p(n,m) = p(n-1,m-1)+p(n-m,m). \tag{5.2}$$

We note that for $n<m,\ p(n,m)=0$.

Proposition 5.2. *For $\rho,\ \tau$ as in Proposition 3.6 and the coefficients of $A(q;\ell-1)$, $p(\rho,\ell-1),\ p(\tau,\ell-1)$ we have*

$$p(lcm(\ell)k+t,\ell) \equiv \sum_{i=0}^{lcm(\ell-1)-1} p(\rho-\ell i,\ell-1) + \sum_{i=0}^{\lfloor\frac{t-1}{\ell}\rfloor} p(\tau-\ell i,\ell-1) \pmod \ell. \tag{5.3}$$

Proof.

$$p(lcm(\ell)k+t,\ell) = \sum_{j=1}^{k}\sum_{i=0}^{lcm(\ell-1)-1} p(lcm(\ell)j+t-1-\ell i,\ell-1) + \sum_{i=0}^{\lfloor\frac{t-1}{\ell}\rfloor} p(t-1-\ell i,\ell-1) \tag{5.4}$$

$$\equiv \sum_{i=0}^{lcm(\ell-1)-1} p(\rho-\ell i,\ell-1) + \sum_{i=0}^{\lfloor\frac{t-1}{\ell}\rfloor} p(\tau-\ell i,\ell-1) \pmod{\ell}. \tag{5.5}$$

The periodicity of $\{p(n,\ell-1) \pmod{\ell}\}_{n\geq 0}$ from Corollary 3.5, (3.8) along with Proposition 5.1 allows us to write (5.4) while (5.5) follows from Proposition 3.6. ∎

The second sum of (5.5) may have redundant summands. For example,

Example 5.3.

$$p(60k+7,5) = \sum_{j=1}^{k}\sum_{i=0}^{11} p(60j+6-5i,4) + \sum_{i=0}^{\lfloor\frac{6}{5}\rfloor} p(66-5i,4),$$

where

$$\sum_{i=0}^{1} p(6-5i,4) \equiv p(6,4)+p(1,4) \pmod{5}.$$

6. Proof of Main Theorem

We now prove Theorem 2.1.
Proof. We treat both summands in the left side of (2.2) separately at first.

$$p(lcm(\ell)k-\ell m-[\ell x_\ell(m)+n],\ell)$$

$$= \sum_{j=1}^{k}\sum_{i=0}^{lcm(\ell-1)-1} p(lcm(\ell)j-\ell m-[\ell x_\ell(m)+n]-1-\ell i,\ell-1) + \sum_{i=0}^{lcm(\ell-1)-m-x_\ell(m)-\lfloor\frac{n+1}{\ell}\rfloor} p(lcm(\ell)-\ell m-[\ell x_\ell(m)+n]-1-\ell i,\ell-1) \tag{6.1}$$

$$\equiv \sum_{i=0}^{lcm(\ell-1)-1} p(lcm(\ell)-\rho-\ell i,\ell-1) + \sum_{i=0}^{lcm(\ell-1)-m-x_\ell(m)-\lfloor\frac{n+1}{\ell}\rfloor} p(lcm(\ell)-\tau-\ell i,\ell-1) \pmod{\ell}, \tag{6.2}$$

where (6.2) is due to Proposition 5.2.

Again by Proposition 5.2 and for some η and ξ, we treat the second sum in the right hand side of (2.2);

$$\begin{aligned} & p(lcm(\ell)k-\ell m+[\ell x_\ell(m)+n],\ell) \\ & \equiv \sum_{i=0}^{lcm(\ell-1)-1} p(\eta+\ell i,\ell-1) \\ & \quad + \sum_{i=0}^{lcm(\ell-1)-m+x_\ell(m)+\lfloor\frac{n-1}{\ell}\rfloor} p(\xi+\ell i,\ell-1) \pmod{\ell}. \end{aligned} \tag{6.3}$$

We now consider the sum of (6.2) and (6.3):

$$\begin{aligned} & p(lcm(\ell)k-\ell m-[\ell x_\ell(m)+n],\ell)+p(lcm(\ell)k-\ell m+[\ell x_\ell(m)+n],\ell) \\ & \equiv \sum_{i=0}^{lcm(\ell-1)-1} [p(\eta+\ell i,\ell-1)+p(lcm(\ell)-\rho-\ell i,\ell-1)] \\ & \quad + \sum_{i=0}^{lcm(\ell-1)-m-x_\ell(m)-\lfloor\frac{n+1}{\ell}\rfloor} [p(\xi+\ell i,\ell-1)+p(lcm(\ell)-\eta-\ell i,\ell-1)] \\ & \quad + p(\alpha,\ell-1)+\cdots+p(\omega,\ell-1) \pmod{\ell}. \end{aligned} \tag{6.4}$$

By Theorem 3.4 (lines (3.3) or (3.4)), both sums in the left side of (6.4) vanish leaving only $p(\alpha,\ell-1)+\cdots+p(\omega,\ell-1) \pmod{\ell}$ to consider. Almost all of these terms are zero because they are periodic terms. If for some β, it is the case that $\ell-1\le\beta\le 2(\ell-1)$, then $p(\beta,\ell-1)$ is not a periodic term and $p(\beta,\ell-1) = p(\ell-1+n,\ell-1)\equiv p(n) \pmod{\ell}$.

In the special case that $\ell\equiv 3 \pmod 4$ and $m=\frac{\ell-3}{4}$ we have $2p(\beta,\ell-1) = 2p(\ell-1+n,\ell-1)\equiv 2p(n) \pmod{\ell}$. ∎

Example 6.1. We use Therorem 2.1, line (2.2) to show that

$$p(60k+47,5)+p(60k+63,5)\equiv p(3)\equiv 3 \pmod 5 \tag{6.5}$$

Let $\ell=5$, $k\ge 0$, $n=3$, and $m=1$ so that $x_\ell(1)=1$. By (2.2) we have

$$\begin{aligned} & p(60k-5\cdot 1-[5\cdot 1+3],5)+p(60k-5\cdot 1+[5\cdot 1+3],5) \\ & = p(60k+47,5)+p(60k+63,5) \\ & = \sum_{j=1}^{k}\sum_{i=0}^{11} p(60j+46-5i,4)+\sum_{i=0}^{\lfloor 46/5\rfloor} p(46-5i,4) \\ & \quad + \sum_{j=1}^{k-1}\sum_{i=0}^{11} p(60j+62-5i,4)+\sum_{i=0}^{\lfloor 62/5\rfloor} p(62-5i,4) \end{aligned}$$

$$\begin{aligned}
&\equiv \left[\sum_{i=0}^{11} p(46-5i,4)+\sum_{i=0}^{9} p(46-5i,4)\right] \\
&\quad + \left[\sum_{i=0}^{11} p(62-5i,4)+\sum_{i=0}^{12} p(62-5i,4)\right] \\
&\equiv \left[\sum_{i=0}^{11} p(56-5i,4)+\sum_{i=0}^{9} p(46-5i,4)\right] \\
&\quad + \left[\sum_{i=0}^{11} p(2+5i,4)+\sum_{i=0}^{11} p(2+5i,4)+p(2,4)\right] \\
&\equiv \sum_{i=0}^{11} [p(56-5i,4)+p(2+5i,4)] \\
&\quad + \sum_{i=0}^{9} [p(46-5i,4)+p(12+5i,4)] \\
&\quad + p(2,4)+p(7,4)+p(2,4) \\
&\equiv p(8,5) \\
&\equiv p(3) \\
&\equiv 3 \pmod 5.
\end{aligned}$$

By (3.3) both $\sum_{i=0}^{11}[p(56-5i,4)+p(2+5i,4)] \equiv 0 \pmod 5$ and $\sum_{i=0}^{9}[p(46-5i,4)+p(12+5i,4)] \equiv 0 \pmod 5$. Since $p(2,4)=0$ because it is a periodic term we are left with just $p(7,4)$ and

$$p(7,4) \equiv p(8,5) \equiv p(3) \equiv 3 \pmod 5.$$

7. A Short Corollary to Theorem 3.4

The following Corollary has the apperance of a Ramanujan-like partition congruence and is an immediate result of Theorem 3.4, Case 1. Moreover, this result will be useful in the proof of Corollary 2.2.

Corollary 7.1. *For $k \geq 0$ and $\ell \equiv 1 \pmod 4$,*

$$p\left(lcm(\ell)k+\frac{lcm(\ell)}{2}-\frac{\ell^2-5\ell+4}{4},\ell-1\right) \equiv 0 \pmod{\ell}. \tag{7.1}$$

Proof. Since $A(q;\ell-1)$ is a termwise periodic antireciprocal polynomial then setting $i = lcm(\ell)/2$ in (3.3) we have

$$p\left(\frac{lcm(\ell)}{2}-\frac{\ell^2-5\ell+4}{4},\ell-1\right) \equiv -p\left(\frac{lcm(\ell)}{2}-\frac{\ell^2-5\ell+4}{4},\ell-1\right) \pmod{\ell}. \tag{7.2}$$

Since (7.2) is true if and only if $p\big(\frac{lcm(\ell)}{2}-\frac{\ell^2-5\ell+4}{4},\ell-1\big)\equiv 0 \pmod{\ell}$, the proof is complete. ∎

In other words, when $\ell\equiv 1 \pmod 4$, the coefficient on the "central" term of $A(q,\ell-1)$ is zero. For example, $p(60k+29,4)\equiv p(29,4)\equiv 185\equiv 0 \pmod 5$.

8. Proof of Corollary 2.2

Proof. We prove the case $\ell\equiv 1 \pmod 4$. Our proof begins by setting $n=-x_\ell(m)$ with $m=\frac{\ell-3}{2}$ in line (2.2) of Theorem 2.1.

$$\begin{aligned}&p(lcm(\ell)k-\ell m-[\ell x_\ell(m)+n],\ell)+p(lcm(\ell)k-\ell m+[\ell x_\ell(m)+n],\ell)\\&=p\left(lcm(\ell)k-\frac{\ell^2-3\ell}{2},\ell\right)+p\left(lcm(\ell)k-\frac{\ell^2-3\ell}{2},\ell\right).\end{aligned}\tag{8.1}$$

Since our claim is true if and only if $p\big(lcm(\ell)k-\frac{\ell^2-3\ell}{2},\ell\big)\equiv 0 \pmod{\ell}$, we consider just this one summand.

$$\begin{aligned}&p\left(lcm(\ell)k-\frac{\ell^2-3\ell}{2},\ell\right)\\&=\sum_{j=1}^{k-1}\sum_{i=0}^{lcm(\ell-1)-1}p\left(lcm(\ell)j+lcm(\ell)-\frac{\ell^2-3\ell+2}{2}-\ell i,\ell-1\right)\\&\quad+\sum_{i=0}^{lcm(\ell-1)-\frac{\ell-1}{2}}p\left(lcm(\ell)-\frac{\ell^2-3\ell+2}{2}-\ell i,\ell-1\right).\end{aligned}\tag{8.2}$$

We consider the far right sum in (8.2) modulo the prime ℓ.

$$\begin{aligned}&\equiv\sum_{i=0}^{\frac{lcm(\ell-1)}{2}-\frac{\ell+3}{4}}p\left(lcm(\ell)-\frac{\ell^2-3\ell+2}{2}-\ell i,\ell-1\right)\\&\quad+p\left(\frac{lcm(\ell)}{2}-\frac{\ell^2-5\ell+4}{4},\ell-1\right)+\sum_{i=0}^{\frac{lcm(\ell-1)}{2}-\frac{\ell+3}{4}}p(\ell-1+\ell i,\ell-1)\end{aligned}\tag{8.3}$$

$$\begin{aligned}&\equiv\sum_{i=0}^{\frac{lcm(\ell-1)}{2}-\frac{\ell+3}{4}}\left[p(\ell-1+\ell i,\ell-1)+p\left(lcm(\ell)-\frac{\ell^2-3\ell+2}{2}-\ell i,\ell-1\right)\right]\\&\equiv 0 \pmod{\ell}.\end{aligned}\tag{8.4}$$

The coefficient $p\big(\frac{lcm(\ell)}{2}-\frac{\ell^2-5\ell+4}{4},\ell-1\big)=0$ in (8.3) by Corollary 7.1 and the summand in (8.4) is identically zero by setting $j=\frac{\ell^2-\ell}{4}+\ell i$ in (3.3). Similarly, the double sum in (8.2) vanishes by periodicity. For any other m, with $0 \leq m \leq \frac{\ell-3}{2}$, we gain additional coefficients $p(\rho, \ell-1)$ each of which is necessarily a periodic term and therefore equal to zero. Hence the sum in (8.4) remains zero and thus Corollary 2.2 is proved. ∎

9. Concluding Remarks and Conjectures

The results in this paper are focused on prime symmetry and divisibility properties of those partition numbers that occur near the end of the period of their generating function. However, for small primes, there are similar if not nearly identical results that have been observed for partition numbers that occur elsewhere within the period of their generating function. These observations are not yet generalised. In this sense, the results we offer in Theorem 2.1 are incomplete.

In Example 9.1 below, with $\ell = 5$, there are in fact four collections of partition congruences and not just two indicated by Theorem 2.1, namely the collections about $p(60k+60,5)$ and $p(60k+55,5)$. The unexpected third and fourth collections of congruences are centered about $p(60k+40,5)$ and $p(60k+15,5)$, respectively. The extent of predictability of these congruences comes from [4] it is established that $p(60k+40,5)$ (9.21) is partnered with $p(60k+15,5)$ (9.40), just as $p(60k+60,5)$ (9.1) is partnered with $p(60k+55,5)$ (9.11). Example 9.1 gives a characterisation of congruences for partitions into exactly five parts. Lines (9.1) and (9.11) are known from [2], and all of lines (9.1) through (9.20) are due to Theorem 2.1. Lines (9.21) through (9.40) are known and proved through other methods [5].

Example 9.1.

$$p(60k+60,5) \equiv p(-5) \equiv 0 \pmod 5, \tag{9.1}$$

$$p(60k+59,5)+p(60k+61,5) \equiv p(-4) \equiv 0 \pmod 5, \tag{9.2}$$

$$p(60k+58,5)+p(60k+62,5) \equiv p(-3) \equiv 0 \pmod 5, \tag{9.3}$$

$$p(60k+57,5)+p(60k+63,5) \equiv p(-2) \equiv 0 \pmod 5, \tag{9.4}$$

$$p(60k+56,5)+p(60k+64,5) \equiv p(-1) \equiv 0 \pmod 5, \tag{9.5}$$

$$p(60k+55,5)+p(60k+65,5) \equiv p(0) \equiv 1 \pmod 5, \tag{9.6}$$

$$p(60k+54,5)+p(60k+66,5) \equiv p(1) \equiv 1 \pmod 5, \tag{9.7}$$

$$p(60k+53,5)+p(60k+67,5) \equiv p(2) \equiv 2 \pmod 5, \tag{9.8}$$

$$p(60k+52,5)+p(60k+68,5) \equiv p(3) \equiv 3 \pmod 5, \tag{9.9}$$

$$p(60k+51,5)+p(60k+69,5) \equiv p(4) \equiv 0 \pmod 5, \tag{9.10}$$

$$p(60k+55,5) \equiv p(-5) \equiv 0 \pmod 5, \tag{9.11}$$

$$p(60k+54,5)+p(60k+56,5) \equiv p(-4) \equiv 0 \pmod 5, \tag{9.12}$$

$$p(60k+53,5)+p(60k+57,5) \equiv p(-3) \equiv 0 \pmod 5, \tag{9.13}$$

$$p(60k+52,5)+p(60k+58,5) \equiv p(-2) \equiv 0 \pmod 5, \tag{9.14}$$

$$p(60k+51,5)+p(60k+59,5) \equiv p(-1) \equiv 0 \pmod 5, \tag{9.15}$$

$$p(60k+50,5)+p(60k+60,5) \equiv p(0) \equiv 1 \pmod 5, \tag{9.16}$$

$$p(60k+49,5)+p(60k+61,5) \equiv p(1) \equiv 1 \pmod 5, \tag{9.17}$$

$$p(60k+48,5)+p(60k+62,5) \equiv p(2) \equiv 2 \pmod 5, \tag{9.18}$$

$$p(60k+47,5)+p(60k+63,5) \equiv p(3) \equiv 3 \pmod 5, \tag{9.19}$$

$$p(60k+46,5)+p(60k+64,5) \equiv p(4) \equiv 0 \pmod 5, \tag{9.20}$$

$$p(60k+40,5) \equiv p(-5) \equiv 0 \pmod 5, \tag{9.21}$$

$$p(60k+39,5)+p(60k+41,5) \equiv p(-4) \equiv 0 \pmod 5, \tag{9.22}$$

$$p(60k+38,5)+p(60k+42,5) \equiv p(-3) \equiv 0 \pmod 5, \tag{9.23}$$

$$p(60k+37,5)+p(60k+43,5) \equiv p(-2) \equiv 0 \pmod 5, \tag{9.24}$$

$$p(60k+36,5)+p(60k+44,5) \equiv p(-1) \equiv 0 \pmod 5, \tag{9.25}$$

$$p(60k+35,5)+p(60k+45,5) \equiv p(0) \equiv 1 \pmod 5, \tag{9.26}$$

$$p(60k+34,5)+p(60k+46,5) \equiv p(1) \equiv 1 \pmod 5, \tag{9.27}$$

$$p(60k+33,5)+p(60k+47,5) \equiv p(2) \equiv 2 \pmod 5, \tag{9.28}$$

$$p(60k+32,5)+p(60k+48,5) \equiv p(3) \equiv 3 \pmod 5, \tag{9.29}$$

$$p(60k+31,5)+p(60k+49,5) \equiv p(4) \equiv 0 \pmod 5, \tag{9.30}$$

$$p(60k+15,5) \equiv p(-5) \equiv 0 \pmod 5, \tag{9.31}$$

$$p(60k+14,5)+p(60k+16,5) \equiv p(-4) \equiv 0 \pmod 5, \tag{9.32}$$

$$p(60k+13,5)+p(60k+17,5) \equiv p(-3) \equiv 0 \pmod 5, \tag{9.33}$$

$$p(60k+12,5)+p(60k+18,5) \equiv p(-2) \equiv 0 \pmod 5, \tag{9.34}$$

$$p(60k+11,5)+p(60k+19,5) \equiv p(-1) \equiv 0 \pmod 5, \tag{9.35}$$

$$p(60k+10,5)+p(60k+20,5) \equiv p(0) \equiv 1 \pmod 5, \tag{9.36}$$

$$p(60k+9,5)+p(60k+21,5) \equiv p(1) \equiv 1 \pmod 5, \tag{9.37}$$

$$p(60k+8,5)+p(60k+22,5) \equiv p(2) \equiv 2 \pmod 5, \tag{9.38}$$

$$p(60k+7,5)+p(60k+23,5) \equiv p(3) \equiv 3 \pmod 5, \tag{9.39}$$

$$p(60k+6,5)+p(60k+24,5) \equiv p(4) \equiv 0 \pmod 5. \tag{9.40}$$

With the goal of a more complete characterisation of Ramanujan-like congruences for $p(n,\ell)$, the second author makes the following conjectures.

Conjecture 9.2. For $k \geq 0$ and a prime $\ell > 3$ with $\ell \equiv 3 \pmod 4$

$$p\left(\operatorname{lcm}(\ell)k - \frac{\operatorname{lcm}(\ell)}{2}, \ell\right) \equiv p\left(\operatorname{lcm}(\ell)(k+1) - \frac{\operatorname{lcm}(\ell)+\ell^2-3\ell}{2}, \ell\right) \equiv 0 \pmod{\ell}. \tag{9.41}$$

Conjecture 9.3. For $k \geq 0$ and a prime $\ell > 3$ with $\ell \equiv 3 \pmod 4$ and $-\ell \leq n \leq \ell - 1$,

$$p\left(\operatorname{lcm}(\ell)k - \frac{\operatorname{lcm}(\ell)}{2} - [\ell+n], \ell\right) + p\left(\operatorname{lcm}(\ell)k - \frac{\operatorname{lcm}(\ell)}{2} + [\ell+n], \ell\right) \equiv 0 \pmod{\ell}. \tag{9.42}$$

Moreover,

$$p\left(\operatorname{lcm}(\ell)k - \frac{\operatorname{lcm}(\ell)+\ell^2-3\ell}{2} - [\ell+n], \ell\right) + p\left(\operatorname{lcm}(\ell)k - \frac{\operatorname{lcm}(\ell)+\ell^2-3\ell}{2} + [\ell+n], \ell\right) \equiv 0 \pmod{\ell}. \tag{9.43}$$

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