

# A NOTE ON UNIFORM APPROXIMATION OF FUNCTIONS HAVING A DOUBLE POLE

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ABSTRACT. We consider the classical problem of finding the best uniform approximation by polynomials of  $1/(x-a)^2$ , where  $a > 1$  is given, on the interval  $[-1, 1]$ . First, using symbolic computation tools we derive the explicit expressions of the polynomials of best approximation of low degrees and then give a parametric solution of the problem in terms of elliptic functions. Symbolic computation is invoked then once more to derive a recurrence relation for the coefficients of the polynomials of best uniform approximation based on a Pell-type equation satisfied by the solutions.

**Keywords:** polynomial approximation, uniform norm, elliptic functions, holonomic functions, symbolic computation.

## 1. INTRODUCTION

The aim of this paper is to show how symbolic computation tools can be integrated in the study of classical problems in uniform approximation theory. To exemplify this, we consider the problem of finding the polynomial of best uniform approximation of the function  $1/(x-a)^2$ , where  $a > 1$  is given, on the interval  $[-1, 1]$ ; more precisely, we are looking for  $p_n^* \in \Pi_n$ , where  $\Pi_n := \{p_n : p_n(x) = \sum_{i=0}^n c_i x^i, c_i \in \mathbb{R}\}$ , such that

$$(1) \quad \begin{aligned} \left\| \frac{1}{(x-a)^2} - p_n^* \right\| &:= \max_{x \in [-1, 1]} \left| \frac{1}{(x-a)^2} - p_n^*(x) \right| \\ &= \min_{p_n \in \Pi_n} \max_{x \in [-1, 1]} \left| \frac{1}{(x-a)^2} - p_n(x) \right|. \end{aligned}$$

We call  $p_n^*$  the polynomial of best approximation, respectively

$$y_n(x) := \frac{1}{(x-a)^2} - p_n^*(x) \quad \text{and} \quad L_n := \left\| \frac{1}{(x-a)^2} - p_n^*(x) \right\|$$

error function of best approximation and minimum deviation.

An asymptotic expression for the minimum deviation is well-known, e.g. [5, p. 122],

$$L_n \sim \frac{n}{(a^2 - 1)^{(n+1)/2}} \frac{1}{(a + \sqrt{a^2 - 1})^n},$$

however explicit expressions for the error function of best approximation and the minimum deviation do not seem to appear in the classical textbooks on approximation theory, we mention here [1, 2, 5]. The reason behind this omission is most probably the fact that not all the parameters involved in the description of the solution, which

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can be done in terms of elliptic functions, can be given explicitly. As we will also see here, a pair of parameters is defined as the solution of a system of equations rather involved, but which can be easily solved numerically in any of the available computer algebra systems.

The case of a function having a simple pole, namely  $1/(x - a)$ , where  $a > 1$  is given, is significantly simpler, explicit expressions for the polynomial of best approximation as well as for the minimum deviation in terms of elementary functions being well-known, see e.g. [5, p. 120]. These polynomials were recently used in [14] to provide simple convergence analysis for the algebraic multilevel methods, a three term recurrence relation for the polynomials being derived for this analysis. Naturally it would be interesting whether the polynomials of best approximation corresponding to the problem (1) also satisfy such a recurrence relation, however we will not dwell on this here, but rather study recurrence relations for their coefficients.

The paper is organized as follows: in Section 2 we give the explicit solution of the problem (1) in terms of elliptic functions as well as explicit expressions for the polynomials of lower degrees. Section 3 is devoted to the proof of the explicit solution and some additional remarks. In the concluding Section 4 we comment on how to derive a recurrence relation for the coefficients of the polynomials of best approximation.

## 2. MAIN RESULTS

By the Chebyshev Alternation Theorem, see e.g. [1, p. 55], the error function of best approximation  $y_n(x)$  must have at least  $n + 2$  alternation points in the interval  $[-1, 1]$ . It is easy to show that the derivative of a function of the form  $1/(x - a)^2 + p_n(x)$ ,  $p_n \in \Pi_n$ , has at most  $n$  real zeros. Hence  $y'_n(x)$  has precisely  $n$  distinct real zeros  $\xi_1, \dots, \xi_n \in (-1, 1)$  and two complex conjugate zeros  $z_1, z_2$ , and thus the alternation points of  $y_n(x)$  are  $-1 < \xi_1 < \dots < \xi_n < 1$ . In addition,  $y_n(x)$  decreases from  $+\infty$  to  $-\infty$  on the interval  $(a, \infty)$ .

With this clear view of how the graph of the error function of best approximation looks like, we determined with the help of Mathematica the explicit expressions of  $p_n^*(x)$  and  $L_n$  for  $n = 1$  and  $n = 2$  and arbitrary  $a > 1$ , see also Figure 2. More precisely,

for  $n = 1$  :

$$\text{In}[1]:= p_1^*[x_-, a_-] := (1 - a^2 + 3(a(-1 + a^2))^{2/3})/(2(a^2 - 1)^2) + 2a/(a^2 - 1)^2 x$$

$$\text{In}[2]:= L_1[a_-] := (1 + 3a^2 - 3(a(a^2 - 1))^{2/3})/(2(a^2 - 1)^2)$$

for  $n = 2$  :

$$\text{In}[3]:= r[a_-] := \text{Root}[-16 + 28a + 45a^2 - 90a^3 - 35a^4 + 84a^5 + 16a^6 - 32a^7 + (-60 + 70a + 200a^2 - 230a^3 - 168a^4 + 172a^5 + 80a^6 - 64a^7)\#1 + (-99 + 42a + 383a^2 - 196a^3 - 350a^4 + 44a^5 + 176a^6)\#1^2 + (-88 - 64a + 352a^2 + 84a^3 - 176a^4 - 172a^5)\#1^3 + (-38 - 100a + 99a^2 + 94a^3 + 73a^4)\#1^4 + (-4 - 38a - 8a^2 - 14a^3)\#1^5 + (1 - 2a + a^2)\#1^6 \&, 1]$$

$$\text{In}[4]:= c_0[a_-] := ((a - 1)^{-2} + (a + 1)^{-2} + 2(2 - 3a + r[a])/((a - 1)^2(a - r[a])^3))/2$$

$$\text{In}[5]:= c_1[a_-] := 2(1 - 2a + a^2 + 2r[a] - 3ar[a] + r[a]^2)/((a - 1)^2(a - r[a])^3)$$

$$\text{In}[6]:= c_2[a_-] := -(2 - 3a + r[a])/((a - 1)^2(a - r[a])^3)$$

$$\text{In}[7]:= p_2^*[x_-, a_-] := c_0[a] + c_1[a]x + c_2[a]x^2$$

$$\text{In}[8]:= L_2[a_-] := -2(1 + r[a])^2(1 - 2a^2 + ar[a])/((a^2 - 1)^2(a - r[a])^3)$$

For the computation of these explicit formulas Gröbner bases [7, 17, 9] were used, a well established tool in symbolic computation for solving systems of polynomial equations. Gröbner bases computations may be computationally very expensive depending exponentially on the number of variables in the input and also badly on the polynomial degrees. Finding the linear polynomial of best approximation is an easy task that can be solved on any computer algebra system. However starting from degree 2 the computations become very involved and we used Singular [10] via a Mathematica interface [11].

In order to determine the polynomial of best approximation, we use a generic ansatz for the polynomial  $p_n^*(x) = \sum_{k=0}^n c_k x^k$ . Using that the  $n + 2$  alternation points are  $-1, \xi_1, \dots, \xi_n, 1$  (sorted in increasing order) and that the interior alternation points  $\xi_j$  are extreme points, hence their first derivative vanishes, we obtain a system of  $2n + 1$  equations in the  $2n + 1$  unknowns  $c_0, \dots, c_n, \xi_1, \dots, \xi_n$ :

$$(2) \quad \begin{aligned} y_n(-1) &= (-1)^{n+1} y_n(1), \quad \text{and} \quad y_n(\xi_j) = (-1)^{n+j} y_n(1), \quad 1 \leq j \leq n \\ y_n'(\xi_j) &= 0, \quad 1 \leq j \leq n. \end{aligned}$$

Even though ultimately we only need to determine the coefficients  $c_k$ , currently we do not see how to set up a system without using the alternation points. Bringing everything to common denominator and equating the numerators of these equations to zero yields the polynomial system. The coefficients  $c_k$  appear only linearly and the  $\xi_k$  appear in monomials up to degree  $n + 2$ . This increase in numbers of variables and degrees explains why we are reaching a limit in our computations quickly. For  $n = 2$  it is still possible to obtain the coefficients comparably fast for symbolic  $a$ . Also for degree 3 it is still possible to obtain the coefficients in reasonable time (less than 10 minutes) for specific choices of  $a$ . Note that these choices are not floating point numbers and the result is still exact and can be evaluated to arbitrary precision. Starting from degree 4 we could not find closed form solutions, not even for specific choices of  $a$ .

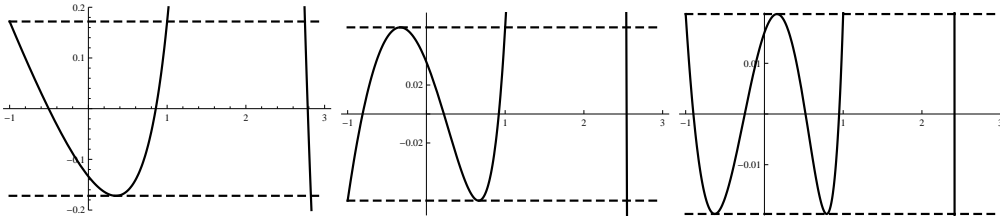


FIGURE 1. Graphs of  $y_1(x), y_2(x)$  and  $y_3(x)$  for  $a = 2$ .

For  $n = 3$  and  $a = 2$  the coefficients of the polynomial of best approximation are algebraic numbers that are approximately given by

$$c_0 = 0.2339547125, \quad c_1 = 0.2062172304, \quad c_2 = 0.3018857394, \quad c_3 = 0.2382272140.$$

The minimal polynomial for, e.g.,  $c_3$  is given by:

$$\begin{aligned} \chi(\alpha) = & 215892499727278669824\alpha^{11} - 5459947971497905618944\alpha^{10} \\ & + 1699378437698250434795520\alpha^9 + 4985576127353627620088832\alpha^8 \\ & - 1487598201477303010010019\alpha^7 + 49652231802046011126432\alpha^6 \\ & - 128259012822418301760\alpha^5 - 39565695530318777856\alpha^4 \\ & + 1028117822899995648\alpha^3 - 23136252868952064\alpha^2 \\ & + 277724396519424\alpha - 1099511627776. \end{aligned}$$

A Gröbner basis is a basis for the polynomial ideal spanned by the given set of polynomials that is unique once a monomial ordering on the variables has been fixed, the basis is normalized to be monic and it is auto-reduced. If the monomial ordering is lexicographic, then a Gröbner basis has the elimination property, i.e., if we are computing in the polynomial ring  $\mathbb{K}[x_1, \dots, x_n]$  (for some field  $\mathbb{K}$ ) with a lexicographic ordering such that  $x_1 < \dots < x_n$  then (with  $k < n$ )

$$I \cap \mathbb{K}[x_1, \dots, x_k] = \langle G \cap \mathbb{K}[x_1, \dots, x_k] \rangle,$$

for  $I$  being the ideal spanned by the input,  $G$  its Gröbner basis and  $\langle S \rangle$  denotes the polynomial ideal spanned by the set  $S$ . This property allows to use Gröbner bases for solving polynomial systems of equations.

In our computations in the first step we eliminate the variables  $c_k$  from the given system. Certainly it would be desirable to eliminate the  $\xi_k$  and solve the remaining equations for the coefficients, however this is computationally too expensive. In the second step, we determine the solutions  $\xi_k$  of the remaining equations. From this solution set we pick the solution satisfying  $-1 < \xi_1 < \dots < \xi_n < 1$ . Plugging into the original system gives the coefficients  $c_k$  we are after.

The following theorem gives the parametric solution to the problem (1) for arbitrary  $n \in \mathbb{N}$ . As usual  $K := K(k)$  is the complete integral of the first kind of modulus  $k$ ,  $0 < k < 1$ ,  $K' := K'(k) := K(k')$  where  $k' := \sqrt{1 - k^2}$ ,  $\text{sn}(u) := \text{sn}(u; k)$  is the Jacobi elliptic function and  $\Theta(u) := \Theta(u; k)$ ,  $\text{H}(u) := \text{H}(u; k)$ ,  $\text{H}_1(u) := \text{H}_1(u; k)$  and  $\Theta_1(u) := \Theta_1(u; k)$  are the four Jacobi theta functions.

**Theorem 2.1.** *Let  $k$ ,  $0 < k < 1$ , and  $\rho$ ,  $0 < \rho < K$ , be defined by the system:*

$$(3) \quad \begin{cases} \left( n(a-1) + \frac{2(a-\alpha)(a-\beta) - (a^2-1)(2a-\alpha-\beta)}{\sqrt{(a^2-1)(a-\alpha)(a-\beta)}} \right) \frac{\text{cn } \rho \text{ dn } \rho}{\text{sn } \rho} \\ \quad - 2n \frac{\Theta'(\rho)}{\Theta(\rho)} - 4 \frac{\Theta'(\hat{\rho} + iK')}{\Theta(\hat{\rho} + iK')} - \frac{2\pi i}{K} = 0 \\ \text{sn}(\hat{\rho} + iK') - \sqrt{\frac{a+1}{a-1}} \text{sn } \rho = 0, \end{cases}$$

where

$$(4) \quad \alpha = \frac{1 + k^2 \text{sn}^2 \rho}{1 - k^2 \text{sn}^2 \rho}, \quad \beta = \frac{1 + \text{sn}^2 \rho}{1 - \text{sn}^2 \rho} \quad \text{and} \quad \hat{\rho} = \frac{K}{2} - \frac{n}{2} \rho,$$

and let

$$(5) \quad x = \frac{\operatorname{sn}^2 u + \operatorname{sn}^2 \rho}{\operatorname{sn}^2 u - \operatorname{sn}^2 \rho}.$$

Then the solution of the approximation problem is

$$(6) \quad y_n(x) = (-1)^{n-1} \frac{L_n}{2} \left\{ \left[ \frac{\mathbf{H}(\rho + u)}{\mathbf{H}(\rho - u)} \right]^n \left[ \frac{\Theta(\hat{\rho} + u)}{\Theta(\hat{\rho} - u)} \right]^2 + \left[ \frac{\mathbf{H}(\rho - u)}{\mathbf{H}(\rho + u)} \right]^n \left[ \frac{\Theta(\hat{\rho} - u)}{\Theta(\hat{\rho} + u)} \right]^2 \right\},$$

$$(7) \quad L_n = \frac{-e^{-\frac{\pi}{2} \frac{K'}{K}} [\Theta'(-iK')]^2 [\Theta(\rho - \hat{\rho})]^{n+4}}{2 k'^2 [\Theta(\rho + \hat{\rho})]^{n-4}} \frac{\Theta^6(0)}{\Theta_1^2(0) \mathbf{H}_1^2(0) \Theta^4(\rho) \mathbf{H}^4(\rho) \mathbf{H}^4(2\hat{\rho})}.$$

The proof of the theorem given in the next section is based on the functional equation (9) satisfied by the error function of best approximation  $y_n(x)$ , recalling Pell's equation from number theory. This equation is the key tool in deriving a recurrence relation for the coefficients of  $p_n^*(x)$ , see Section 4.

Pell-type equations are satisfied by many polynomials appearing in approximation theory, like Chebyshev polynomials, Zolotarev polynomials, and Achieser polynomials, see e.g. [1]. We refer to the survey paper [15] for further details on this topic. Therefore, the approach we used here to derive a recurrence relation for the coefficients of the polynomials of best approximation to  $1/(x-a)^2$ ,  $a > 1$ , can be carried over also to the above mentioned cases.

### 3. PROOF OF THEOREM 2.1

The key tool in obtaining the solution of the approximation problem (1) we are considering here is a Pell-type equation satisfied by the error function of best approximation  $y_n(x)$ . To write down this equation, let  $a < \alpha < \beta$  be such that  $y_n(\alpha) = -y_n(\beta) = L_n$ . Based on the explanation from the beginning of the previous section and denoting  $p_n^*(x) = c_n x^n + c_{n-1} x^{n-1} + \dots + c_0$ , we have:

$$(8) \quad \begin{aligned} [y_n(x)]^2 - L_n^2 &= c_n^2 \frac{(x - \xi_1)^2 \dots (x - \xi_n)^2 (x^2 - 1)(x - \alpha)(x - \beta)}{(x - a)^4} \\ y_n'(x) &= -n c_n \frac{(x - \xi_1) \dots (x - \xi_n)(x - z_1)(x - z_2)}{(x - a)^3}. \end{aligned}$$

Combining the two relations, we conclude that  $y_n(x)$  satisfies the following Pell-type equation:

$$(9) \quad \frac{[y_n'(x)]^2}{[y_n(x)]^2 - L_n^2} = n^2 \frac{(x^2 - \gamma_1 x + \gamma_2)^2}{(x - a)^2} \frac{1}{(x^2 - 1)(x - \alpha)(x - \beta)}$$

where  $\gamma_1 := z_1 + z_2, \gamma_2 := z_1 z_2$ . The formulas of  $\gamma_1$  and  $\gamma_2$  can be easily obtained using the explicit form of  $y_n(x) = 1/(x-a)^2 - (c_n x^n + \dots c_0)$  and of  $y_n'(x)$  derived from this one, in combination with equations (8), namely:

$$(10) \quad \begin{aligned} \gamma_1 &= 2a - \frac{2a(a - \alpha)(a - \beta) + (a^2 - 1)(2a - (\alpha + \beta))}{n\sqrt{(a^2 - 1)(a - \alpha)(a - \beta)}} \\ \gamma_2 &= a^2 - \frac{2(a - \alpha)(a - \beta) + a(a^2 - 1)(2a - (\alpha + \beta))}{n\sqrt{(a^2 - 1)(a - \alpha)(a - \beta)}}. \end{aligned}$$

Equation (9) implies that

$$(11) \quad \log \frac{y_n(x) + \sqrt{y_n^2(x) - L_n^2}}{\pm L_n} = \int_{-1}^x \frac{n(x^2 - \gamma_1 x + \gamma_2)}{(x-a)\sqrt{(x^2-1)(x-\alpha)(x-\beta)}} dx,$$

where on the right hand-side we have an elliptic integral, which we solve by making a suitable substitution. For this, let  $x(u)$  be the conformal mapping from the rectangle with vertices at the points  $u = 0, K, K + iK', iK'$ , with  $K$  and  $K'$  of modulus  $k$ ,  $0 < k < 1$ , onto the upper half-plane with the following normalization:

$$x(0) = -1, \quad x(iK') = 1, \quad x(K + iK') = \alpha, \quad x(K) = \beta.$$

The conformal mapping is given by

$$(12) \quad x(u) = \frac{\operatorname{sn}^2 u + \operatorname{sn}^2 \rho}{\operatorname{sn}^2 u - \operatorname{sn}^2 \rho}$$

where  $k, 0 < k < 1$ , and  $\rho, 0 < \rho < K$ , are defined by

$$k = \sqrt{\frac{\alpha - 1}{\alpha + 1} \frac{\beta + 1}{\beta - 1}} \quad \text{and} \quad \operatorname{sn}^2 \rho = \frac{\beta - 1}{\beta + 1}.$$

The expression (12) was found as  $x = g \circ f$ , where  $f$  is the conformal mapping from the rectangle with vertices at the points  $u = 0, K, K + iK', iK'$ , onto the upper half-plane with the normalization  $f(0) = 0, f(K) = 1, f(K + iK') = 1/k^2, f(iK') = \infty$ , and  $g$  is the conformal mapping of the upper half-plane onto the upper half-plane with the normalization  $g(0) = -1, g(1) = \beta, g(\infty) = 1, g(1/k^2) = \alpha$ . We have  $f(u) = \operatorname{sn}^2 u$ , by writing first the expression of the inverse map with the help of the Schwarz-Christoffel formula, and  $g(u) = \frac{(\beta+1)u + (\beta-1)}{(\beta+1)u - (\beta-1)}$  obtained as a linear fractional transform satisfying the first three normalizing conditions and then obtaining  $k$  from the last one. The substitution we use in the computation of the elliptic integral in (11) is then

$$x = \frac{z^2 + \lambda^2}{z^2 - \lambda^2} \quad \text{with} \quad \lambda^2 = \frac{\beta - 1}{\beta + 1}, \quad k^2 = \frac{\alpha - 1}{\alpha + 1} \frac{\beta + 1}{\beta - 1}$$

which implies

$$(13) \quad \alpha = \frac{1 + k^2 \operatorname{sn}^2 \rho}{1 + k^2 \operatorname{sn}^2 \rho}, \quad \beta = \frac{1 + \operatorname{sn}^2 \rho}{1 + \operatorname{sn}^2 \rho}.$$

Therefore

$$\begin{aligned} & \int_{-1}^x \frac{x^2 - \gamma_1 x + \gamma_2}{(x-a)\sqrt{(x^2-1)(x-\alpha)(x-\beta)}} dx \\ &= \frac{1}{a-1} \frac{\sqrt{(1-\lambda^2)(1-k^2\lambda^2)}}{\lambda} \int_0^z \frac{(1-\gamma_1+\gamma_2)z^4 + 2(1-\gamma_2)\lambda^2 z^2 + (1+\gamma_1+\gamma_2)\lambda^4}{(z^2-\lambda^2)(z^2-\frac{a+1}{a-1}\lambda^2)\sqrt{(1-z^2)(1-k^2 z^2)}} dz, \end{aligned}$$

and, after partial fraction decomposition, we arrive at

$$\begin{aligned} & \int_{-1}^x \frac{x^2 - \gamma_1 x + \gamma_2}{(x-a)\sqrt{(x^2-1)(x-\alpha)(x-\beta)}} dx \\ &= \frac{1-\gamma_1+\gamma_2}{a-1} \frac{\sqrt{(1-\lambda^2)(1-k^2\lambda^2)}}{\lambda} \int_0^z \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}} \\ & \quad - 2\lambda\sqrt{(1-\lambda^2)(1-k^2\lambda^2)} \int_0^z \frac{dz}{(z^2-\lambda^2)\sqrt{(1-z^2)(1-k^2z^2)}} \\ & \quad + \frac{2(a^2-a\gamma_1+\gamma_2)}{(a-1)^2} \lambda\sqrt{(1-\lambda^2)(1-k^2\lambda^2)} \int_0^z \frac{dz}{(z^2-\frac{a+1}{a-1}\lambda^2)\sqrt{(1-z^2)(1-k^2z^2)}}. \end{aligned}$$

For the second and third integral in the last equality above, let  $\rho$  and  $\tilde{\rho}$  be defined by  $\operatorname{sn} \rho = \lambda$  and  $\operatorname{sn} \tilde{\rho} = \sqrt{\frac{a+1}{a-1}}\lambda$ , respectively. We note here that the two conditions imply that

$$(14) \quad a = \frac{\operatorname{sn}^2 \tilde{\rho} + \operatorname{sn}^2 \rho}{\operatorname{sn}^2 \tilde{\rho} - \operatorname{sn}^2 \rho},$$

hence  $\tilde{\rho}$  is the pre-image of  $a$  through the conformal mapping (12) and thus it is of the form  $\tilde{\rho} = \hat{\rho} + iK'$ ,  $\hat{\rho} \in (0, K)$ . With the above notation for  $\rho$  and using [18, p. 523], it can be easily shown that

$$\int_0^z \frac{dz}{(z^2-\lambda^2)\sqrt{(1-z^2)(1-k^2z^2)}} = \frac{1}{\operatorname{sn} \rho \operatorname{cn} \rho \operatorname{dn} \rho} \left[ \frac{1}{2} \log \frac{\operatorname{H}(\rho-u)}{\operatorname{H}(\rho+u)} + \frac{\Theta'(\rho)}{\Theta(\rho)} u \right],$$

with a similar expression holding for the third integral. Therefore, using the fact that

$$\frac{n(a^2-a\gamma_1+\gamma_2)}{(a-1)^2} \frac{\operatorname{sn} \rho \operatorname{cn} \rho \operatorname{dn} \rho}{\operatorname{sn} \tilde{\rho} \operatorname{cn} \tilde{\rho} \operatorname{dn} \tilde{\rho}} = -2,$$

we arrive at

$$\begin{aligned} & \int_{-1}^x n \frac{x^2 - \gamma_1 x + \gamma_2}{(x-a)\sqrt{(x^2-1)(x-\alpha)(x-\beta)}} dx \\ &= \left[ \frac{n(1-\gamma_1+\gamma_2)}{a-1} \frac{\operatorname{cn} \rho \operatorname{dn} \rho}{\operatorname{sn} \rho} - 2n \frac{\Theta'(\rho)}{\Theta(\rho)} - 4 \frac{\Theta'(\tilde{\rho})}{\Theta(\tilde{\rho})} \right] u + n \log \frac{\operatorname{H}(\rho+u)}{\operatorname{H}(\rho-u)} + 2 \log \frac{\operatorname{H}(\tilde{\rho}+u)}{\operatorname{H}(\tilde{\rho}-u)}. \end{aligned}$$

Taking into account also that  $\tilde{\rho} = \hat{\rho} + iK'$  we have thus by (11) and the reduction formulas for Jacobi theta functions:

$$(15) \quad \begin{aligned} & \log \frac{y_n(x) + \sqrt{y_n^2(x) - L_n^2}}{\pm L_n} \\ &= \left[ \frac{n(1-\gamma_1+\gamma_2)}{a-1} \frac{\operatorname{cn} \rho \operatorname{dn} \rho}{\operatorname{sn} \rho} - 2n \frac{\Theta'(\rho)}{\Theta(\rho)} - 4 \frac{\Theta'(\hat{\rho} + iK')}{\Theta(\hat{\rho} + iK')} - \frac{2\pi i}{K} \right] u \\ & \quad + \log \left[ \frac{\operatorname{H}(\rho+u)}{\operatorname{H}(\rho-u)} \right]^n + \log \left[ \frac{\Theta(\hat{\rho}+u)}{\Theta(\hat{\rho}-u)} \right]^2. \end{aligned}$$

From the condition that  $y_n(x)$  has  $n+2$  alternation points in  $[-1, 1]$  we must have

$$\frac{n(1-\gamma_1+\gamma_2)}{a-1} \frac{\operatorname{cn} \rho \operatorname{dn} \rho}{\operatorname{sn} \rho} - 2n \frac{\Theta'(\rho)}{\Theta(\rho)} - 4 \frac{\Theta'(\hat{\rho} + iK')}{\Theta(\hat{\rho} + iK')} - \frac{2\pi i}{K} = \left( n \frac{\rho}{K} + 2 \frac{\hat{\rho}}{K} - 1 \right) \frac{\pi}{K'}$$

and from the condition that  $y_n(x)$  strictly decreases from  $L_n$  to  $-L_n$  in  $[\alpha, \beta]$  that

$$n \frac{\rho}{K} + 2 \frac{\hat{\rho}}{K} - 1 = 0.$$

Combining these two we conclude from (15) that

$$y_n(x) = (-1)^{n-1} \frac{L_n}{2} \left\{ \left[ \frac{H(\rho + u)}{H(\rho - u)} \right]^n \left[ \frac{\Theta(\hat{\rho} + u)}{\Theta(\hat{\rho} - u)} \right]^2 + \left[ \frac{H(\rho - u)}{H(\rho + u)} \right]^n \left[ \frac{\Theta(\hat{\rho} - u)}{\Theta(\hat{\rho} + u)} \right]^2 \right\}$$

with  $\rho$  and  $k$  defined by (3) and  $\alpha, \beta$  and  $\hat{\rho}$  by (4).

The value of  $L_n$  follows now from the condition that  $\lim_{x \rightarrow a} (x - a)^2 y_n(x) = 1$ . By (12) and (14) we get that  $x - a$  is equal to

$$(16) \quad g(u) := \frac{2 \operatorname{sn}^2 \rho}{\operatorname{sn}^2 \rho - \operatorname{sn}^2(\hat{\rho} + iK')} \frac{\operatorname{sn}^2 u - \operatorname{sn}^2(\hat{\rho} + iK')}{\operatorname{sn}^2 u - \operatorname{sn}^2 \rho},$$

and therefore

$$(17) \quad L_n = \frac{1}{\lim_{u \rightarrow \hat{\rho} + iK'} f(u) g(u)^2}$$

which is easily shown to be equal to

$$L_n = \frac{(-1)^{n-1} H^n(\rho - \hat{\rho} - iK')}{8 H^n(\rho + \hat{\rho} + iK')} \frac{[\Theta'(-iK')]^2 (\operatorname{sn}^2 \rho - \operatorname{sn}^2(\hat{\rho} + iK'))^4}{\Theta^2(2\hat{\rho} + iK') \operatorname{sn}^4 \rho [\operatorname{sn}(\hat{\rho} + iK') \operatorname{cn}(\hat{\rho} + iK') \operatorname{dn}(\hat{\rho} + iK')]^2}$$

and, after some transformations, to (7).

The proof of the theorem is complete up to the fact that the system (3) has a unique solution which satisfies  $0 < k < 1$  and, since  $\rho, \hat{\rho} \in (0, K)$ , that  $0 < \rho < K/n$ . We do not consider this matter here, but rather give the numerical solutions of the system for  $a = 2$  and  $n = 1, 2, \dots, 8$ , which were found with the help of the command FindRoot in Mathematica:

$n$	$k$	$\rho$
1	0.988912134768707	0.846885238283369
2	0.998506984080587	0.793161228023789
3	0.999760160023856	0.764276888897998
4	0.999957225505635	0.745744327757575
5	0.999991825727615	0.732764470377829
6	0.999998359426700	0.723151786732571
7	0.999999658473260	0.715743316327916
8	0.999999926858805	0.709858166486558

We note that we have chosen to prove the theorem by computing the elliptic integral on the right-hand side of (11), rather than introducing Riemann surfaces and considering rational functions on them, in the hope for availability of algorithms for dealing with such integrals. Even though many integrals over special functions nowadays can be dealt with by symbolic computation [6, 3], these particular ones are still out of scope. We consider this paper also as a challenge to derive new methods that extend to this class of integrals.



We conclude this section with some remarks concerning the best uniform approximation on the interval  $[-1, 1]$  of the function  $\frac{1}{(x-a)^2} + \frac{B}{x-a}$ , where  $a > 1$  and  $B \in \mathbb{R}$ , by polynomials of degree  $n \in \mathbb{N}$ .

In the case when  $B = 0$ , the expression of the error function of best approximation was given in Theorem 2.1 in terms of elliptic functions. As we will briefly explain in what follows, with a different choice of the parameters  $\rho$  and  $k$ , this expression describes the solution also when  $B$  is in some small neighborhood of 0. More precisely, a function defined by (6), with  $L_n$  and  $\hat{\rho}$  given by (7) and (4), respectively, and with  $k \in (0, 1)$  and  $\rho \in (0, K/n)$  arbitrary, is a rational function of the form  $\tilde{y}_n(x) = \frac{1}{(x-a)^2} + \frac{B}{x-a} + p_n(x)$ , where  $p_n \in \Pi_n$  and

$$(18) \quad B = \frac{e^{\frac{\pi i}{4K}(2K+4\hat{\rho}+3iK')}}{8(1-k^2)} \frac{[\Theta'(-iK')]^2 \Theta^6(0) \Theta^2(\rho - \hat{\rho}) \Theta^2(\rho + \hat{\rho})}{\mathrm{H}'(0) \Theta^2(\rho) \mathrm{H}^2(\rho) \Theta^2(\hat{\rho}) \Theta_1^2(\hat{\rho}) \mathrm{H}^2(\hat{\rho}) \mathrm{H}_1^2(\hat{\rho})} \left[ 4\Theta'(2\hat{\rho} + iK') \right. \\ \left. + \left( (n+2) \frac{\mathrm{H}'(\rho - \hat{\rho} - iK')}{\mathrm{H}(\rho - \hat{\rho} - iK')} + (n-2) \frac{\mathrm{H}'(\rho + \hat{\rho} + iK')}{\mathrm{H}(\rho + \hat{\rho} + iK')} \right) \Theta(2\hat{\rho} + iK') \right].$$

It is easy to show that indeed also in this general case  $y_n(x)$  is a rational function with a single pole  $x = a$  of order two, while as we already stated, the definition of  $L_n$  is such that the coefficient of  $1/(x-a)^2$  is equal to 1, see (17) above. The expression (18) was then found as  $B = \lim_{x \rightarrow a} (y_n(x)(x-a)^2 - 1)/(x-a)$ , making use of (6) and (16).

Now when  $u$  runs in the segment from 0 to  $iK'$ , then  $|\mathrm{H}(\rho + u)/\mathrm{H}(\rho - u)| = 1$  and  $|\Theta(\hat{\rho} + u)/\Theta(\hat{\rho} - u)| = 1$ , hence  $|y_n(x)| \leq L_n$  on  $[-1, 1]$ . In addition, by a simple application of the Argument Principle,  $y_n(x)$  alternates at least  $n + 2$  times between  $\pm L_n$  on  $[-1, 1]$ . Hence the Chebyshev Alternation Theorem can be applied, and thus for those values of  $B$  in some neighborhood of 0 for which the system in the unknowns  $k$  and  $\rho$  formed by (18) and the second equation in (3) has a solution, we obtain the best approximation to  $\frac{1}{(x-a)^2} + \frac{B}{x-a}$ . That is, plugging in the solution values for  $k$  and  $\rho$ , then  $y_n(x)$  defined by (6), with  $L_n$  given by (7) and  $\hat{\rho}$  by (4), is the corresponding error function of best approximation.

On the other hand, in the case when  $B = \frac{2a+n\sqrt{a^2-1}}{a^2-1}$ , the expression of the error function of best approximation  $y_n(x)$  of  $\frac{1}{(x-a)^2} + \frac{B}{x-a}$  by polynomials of degree  $n$  can be given in terms of elementary functions; namely with  $x = \frac{1}{2}(z + \frac{1}{z})$ ,  $|z| = 1$ , then

$$y_n(x) = \frac{L_n}{2} \left\{ z^n \left( \frac{\lambda - z}{1 - \lambda z} \right)^2 + z^{-n} \left( \frac{1 - \lambda z}{\lambda - z} \right)^2 \right\}, \quad L_n = \frac{8\lambda^{n+4}}{(1 - \lambda^2)^4},$$

where  $\lambda := a - \sqrt{a^2 - 1}$ . This follows immediately by Chebyshev Alternation Theorem, using again the Argument Principle.

#### 4. RECURRENCE RELATION FOR THE POLYNOMIALS OF BEST APPROXIMATION

With  $k$  and  $\rho$  defined by the system (3) from Theorem 2.1, we have  $\alpha$  and  $\beta$  defined by (4),  $\gamma_1$  and  $\gamma_2$  defined by (10) and the Pell-type equation (9) for  $y_n(x)$ . By differentiating relation (9) we obtain the following linear differential equation for  $y_n(x)$  with polynomial coefficients

$$(19) \quad 2f_0(x)f_1(x)y_n''(x) + (f_0'(x)f_1(x) - f_0(x)f_1'(x))y_n'(x) - 2f_1^2(x)y_n(x) = 0,$$

where

$$f_0(x) = (x^2 - 1)(x - a)^2(x - \alpha)(x - \beta) \quad \text{and} \quad f_1(x) = n^2(x^2 - \gamma_1x + \gamma_2)^2.$$

It is well known that this type of differential equations can be turned into a recurrence relation for the coefficients in the series expansion of  $y_n(x)$  around  $x_0 = 0$ . Also the given function  $\frac{1}{(x-a)^2}$  satisfies a linear differential equation with polynomial coefficients. Hence the difference  $p_n^*(x) = 1/(x-a)^2 - y_n(x)$  satisfies such a differential equation and a recurrence for its coefficients in the monomial expansion can be computed. This transfer from differential to recurrence equation can be done entirely automatically using symbolic computation. The framework for this are holonomic functions, i.e., functions (discrete and/or continuous) satisfying systems of linear difference/differential relations with polynomial coefficients. Algorithms for executing the transfer between differential equation for the function and recurrence relation for the coefficients or for executing closure properties, e.g., given recurrences for two sequences return the recurrence for the termwise sum of these sequences, have been implemented in different computer algebra systems [8, 12]. Here we use Koutschan's Mathematica implementation [13] "HolonomicFunctions"<sup>1</sup>.

The algorithms implemented in HolonomicFunctions deal with difference/differential equations in operator form over some appropriate algebra. For the underlying computations again Gröbner bases are involved, this time also in non-commutative rings. We use this tool to derive a recurrence relation for the coefficients  $b_k$  of the power series expansion  $y_n(x) = \sum_{k \geq 0} b_k x^k$ . After loading the package in the first step we translate the given differential equation (19) into operator notation:

```
In[9]:= annY = ToOrePolynomial[ode, y[x]];
```

Then in the second step we pass from the differential equation of  $y_n$  to the recurrence relation for the coefficients  $b_k$  using the "DFiniteDE2RE" command (differential equation to recurrence equation). The output is again in operator notation, but we can use the "ApplyOreOperator" command to write it in traditional form.

```
In[10]:= annB = DFiniteDE2RE[annY, {x}, {k}];
In[11]:= recB = ApplyOreOperator[annB, b[k]];
Out[11]= {2(a-1)a^2(a+1)(k+7)(k+8){αβ(a-α)(a-β)(a^6n - a^5n(α+β) + a^4(n(αβ-1) - 2R)
+ a^3(n+R)(α+β) - a^2nαβ + aR(α+β) - 2Rαβ)b[k+8]
...
- 2(a-1)^2(a+1)^2n(a-α)^2(a-β)^2(k-n)(k+n)b[k]}
```

Here we omit the full output because of size and abbreviate the root expression  $R = \sqrt{(a^2 - 1)(a - \alpha)(a - \beta)}$  for better readability. The resulting recurrence is of order eight, because the polynomial coefficients of the given ordinary differential equation are up to degree eight. The polynomial coefficients in the recurrence are of degree 2 in the variable  $k$  each.

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<sup>1</sup>available at <http://www.risc.jku.at/research/combinat/software/>

The Taylor series expansion of  $1/(x-a)^2$  around  $x_0 = 0$  is well known and thus we have

$$y_n(x) = \sum_{k \geq 0} b_k x^k = \frac{1}{(x-a)^2} - p_n^*(x) = \sum_{k \geq 0} (k+1) \left(\frac{1}{a}\right)^{k+2} x^k - \sum_{k=0}^n c_k x^k.$$

In other words,  $b_k = (k+1)a^{-k-2}$  for  $k > n$ , which is also reflected by the factor  $n-k$  in the trailing coefficient of the recurrence above. Given a recurrence for the coefficients of  $y_n(x)$  and a recurrence for  $(k+1)a^{-k-2}$  (that is easily calculated), by holonomic closure properties a recurrence for the coefficients  $c_k$  of the polynomial of best uniform approximation can be computed:

```
In[12]:= annF = Annihilator[a^{-k-2}(k+1), {S[k]}]
Out[12]= {a(k+1)S_k - (k+2)}
In[13]:= annC = DFinitePlus[annB, annF];
In[14]:= Support[annC]
Out[14]= {{S_k^8, S_k^7, S_k^6, S_k^5, S_k^4, S_k^3, S_k^2, S_k, 1}}
In[15]:= rec = ApplyOreOperator[annC, c[k]]
Out[15]= {-2a^2(a^2-1)\alpha\beta(k^2+15k+56)(a-\alpha)(a-\beta)(a^6n-a^5n(\alpha+\beta)+a^4(n(\alpha\beta-1)-2R)+
a^3(\alpha+\beta)(n+R)-\alpha a^2\beta n+aR(\alpha+\beta)-2\alpha\beta R)c[k+8]
...
+2(a^2-1)^2n(a-\alpha)^2(a-\beta)^2(k^2-n^2)c[k]}
```

As can be seen from the Mathematica output above, the resulting recurrence is also of order eight, where we do not display the full output<sup>2</sup> and reuse the abbreviation  $R = \sqrt{(a^2-1)(a-\alpha)(a-\beta)}$ . For the recursive evaluation of the coefficients we follow the approach in [16]. First of all, we need the numerical values from the table of the previous section in order to initialize the parameters  $\alpha$  and  $\beta$ . The recurrence is unwinded starting from the coefficient  $c_n$  down to  $c_0$ . As initial value first we normalize  $c_n = 1$  and, since we know that  $p_n^*(x)$  is polynomial, all  $c_{n+m}$  for  $m = 1, \dots, 7$  are set to zero. In the second step the coefficients are adjusted to match  $L_n$  as the minimum deviation. Using the precomputed values from the previous section,  $\alpha, \beta$  and  $L_n$  can be computed and using the recurrence relation we obtain the approximations

$$\begin{aligned} \tilde{p}_7^*(x) &= 0.0378802x^7 + 0.056484x^6 + 0.0243862x^5 + 0.0569877x^4 \\ &\quad + 0.13294x^3 + 0.192055x^2 + 0.249238x + 0.249851, \end{aligned}$$

and

$$\begin{aligned} \tilde{p}_8^*(x) &= 0.0227427x^8 + 0.0345011x^7 + 0.0110035x^6 + 0.0303003x^5 + 0.0854089x^4 \\ &\quad + 0.129983x^3 + 0.186372x^2 + 0.249608x + 0.250028, \end{aligned}$$

for the polynomials of best uniform approximation for degree  $n = 7$  and  $n = 8$ , respectively, and  $a = 2$ . Figure 4 below displays the error functions of best approximation in these cases. It is clearly visible that the approximation is a rather accurate one, showing also the required number of alternation points, at least approximately.

<sup>2</sup>for the complete recurrence see the Mathematica notebook available for download at <http://www.risc.jku.at/people/vpillwei/Results.nb>

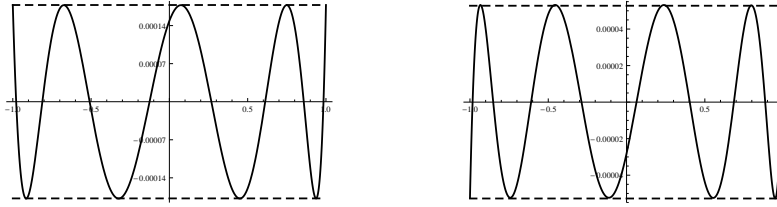


FIGURE 2. Error function for  $n = 7, 8$  for the approximate polynomials  $\tilde{p}_n^*(x)$

In [16] also a recursive evaluation for the Zolotarev polynomials in Chebyshev expansion is given. We note that such a recurrence for the coefficients of  $y_n(x)$  can be obtained similarly entirely automatic using the results of Benoit and Salvy [4].

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