

RICAM, Special Semester; November 28, 2013

Workshop 3: Computer algebra and polynomials

Fast Algorithms for Refined Parameterized Telescoping in Difference Fields

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Der Wissenschaftsfonds

Simplify

$$\sum_{k=1}^m H_k$$

with $H_k = \sum_{i=1}^k \frac{1}{i}$

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Telescoping

Find $g(k)$ s.t.,

$$H_k = g(k+1) - g(k)$$

for all $1 \leq k \leq m$ and $m \geq 0$

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$$H_k = g(k+1) - g(k)$$

for all $1 \leq k \leq m$ and $m \geq 0$

We compute

$$g(k) = (H_k - 1)k.$$

Simplify

$$\sum_{k=1}^m H_k$$

with $H_k = \sum_{i=1}^k \frac{1}{i}$

Telescoping

Find $g(k)$ s.t,

$$H_k = g(k+1) - g(k)$$

for all $1 \leq k \leq m$ and $m \geq 0$

Summing this equation over k from 1 to m gives

$$\begin{aligned} \sum_{k=1}^m H_k &= g(m+1) - g(1) \\ &= (H_{m+1} - 1)(m+1). \end{aligned}$$

Telescoping in the given difference field

FIND a closed form for

$$\sum_{k=1}^m H_k.$$

A difference field for the **summand**

Consider the rational function field

$$\mathbb{F} := \mathbb{Q}$$

with the automorphism $\sigma : \mathbb{F} \rightarrow \mathbb{F}$ defined by

$$\sigma(c) = c \quad \forall c \in \mathbb{Q},$$

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$$\sigma(k) = k + 1,$$

$$\mathcal{S} k = k + 1,$$

Telescoping in the given difference field

FIND a closed form for

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$$\mathbb{F} := \mathbb{Q}(k)(h)$$

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Telescoping in the given difference field

FIND $g \in \mathbb{F}$:

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with

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This gives

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with

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Hence,

$$(H_{m+1} - 1)(m + 1) = \sum_{k=1}^m H_k.$$

CONSTRUCT a difference field (\mathbb{F}, σ) :

- ▶ a rational function field

$$\mathbb{F} := \mathbb{K}$$

- ▶ with an automorphism

$$\sigma(c) = c \quad \forall c \in \mathbb{K}$$

CONSTRUCT a difference field (\mathbb{F}, σ) :

- ▶ a rational function field

$$\mathbb{F} := \mathbb{K}(t_1)$$

- ▶ with an automorphism

$$\sigma(c) = c \quad \forall c \in \mathbb{K}$$

$$\sigma(t_1) = a_1 t_1 + f_1, \quad a_1 \in \mathbb{K}^*, \quad f_1 \in \mathbb{K}$$

CONSTRUCT a difference field (\mathbb{F}, σ) :

- ▶ a rational function field

$$\mathbb{F} := \mathbb{K}(t_1)(t_2)$$

- ▶ with an automorphism

$$\sigma(c) = c \quad \forall c \in \mathbb{K}$$

$$\sigma(t_1) = a_1 t_1 + f_1, \quad a_1 \in \mathbb{K}^*, \quad f_1 \in \mathbb{K}$$

$$\sigma(t_2) = a_2 t_2 + f_2, \quad a_2 \in \mathbb{K}(t_1)^*, \quad f_2 \in \mathbb{K}(t_1)$$

CONSTRUCT a difference field (\mathbb{F}, σ) :

- ▶ a rational function field

$$\mathbb{F} := \mathbb{K}(t_1)(t_2) \dots (t_e)$$

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$$\sigma(t_1) = a_1 t_1 + f_1, \quad a_1 \in \mathbb{K}^*, \quad f_1 \in \mathbb{K}$$

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$$\vdots$$

$$\sigma(t_e) = a_e t_e + f_e, \quad a_e \in \mathbb{K}(t_1, \dots, t_{e-1})^*, \quad f_e \in \mathbb{K}(t_1, \dots, t_{e-1})$$

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GIVEN $f \in \mathbb{F}$;

FIND $g \in \mathbb{F}$ such that

$$\sigma(g) - g = f.$$

CONSTRUCT a $\Pi\Sigma^*$ -field (\mathbb{F}, σ) :

- ▶ a rational function field

$$\mathbb{F} := \mathbb{K}(t_1)(t_2) \dots (t_e)$$

- ▶ with an automorphism

$$\sigma(c) = c \quad \forall c \in \mathbb{K}$$

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such that

$$\text{const}_\sigma \mathbb{F} = \{c \in \mathbb{K}(t_1)(t_2) \dots (t_e) \mid \sigma(c) = c\} = \mathbb{K}.$$

GIVEN $f \in \mathbb{F}$;

FIND $g \in \mathbb{F}$ such that

Karr's algorithm (1981)

$$\sigma(g) - g = f.$$

Simplified telescoping algorithm

FIND a closed form for

$$\sum_{k=1}^m H_k.$$

A $\Pi\Sigma^*$ -field for the summand

$$\text{const}_\sigma \mathbb{F} = \mathbb{Q}$$

Consider the rational function field

$$\mathbb{F} := \mathbb{Q}(k)(h)$$

with the automorphism $\sigma : \mathbb{F} \rightarrow \mathbb{F}$ defined by

$$\sigma(c) = c \quad \forall c \in \mathbb{Q},$$

$$\sigma(k) = k + 1,$$

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FIND $g \in \mathbb{Q}(k)(h)$:

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Denominator bound: COMPUTE a polynomial $d \in \mathbb{Q}(k)[h]^*$:

$$\forall g \in \mathbb{Q}(k)(h) : \quad \sigma(g) - g = h \quad \Rightarrow \quad gd \in \mathbb{Q}(k)[h].$$

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FIND $g' \in \mathbb{Q}(k)[h]$ with

$$\sigma\left(\frac{g'}{d}\right) - \frac{g'}{d} = h.$$

FIND $g \in \mathbb{Q}(k)(h)$:

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FIND $g' \in \mathbb{Q}(k)[h]$ with

$$\frac{1}{\sigma(d)}\sigma(g') - \frac{1}{d}g' = \sigma\left(\frac{g'}{d}\right) - \frac{g'}{d} = h.$$

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Degree bound: COMPUTE $m \geq 0$:

$$\forall g \in \mathbb{Q}(k)[h] \quad \sigma(g) - g = h \Rightarrow \deg(g) \leq m.$$

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Degree bound: COMPUTE $m \geq 0$:

$$m = 2$$

$$\forall g \in \mathbb{Q}(k)[h] \quad \sigma(g) - g = h \Rightarrow \deg(g) \leq m.$$

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Polynomial Solution: FIND

$$g = g_2 h^2 + g_1 h + g_0 \in \mathbb{Q}(k)[h].$$

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$$g = hk - k$$

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ANSATZ $g = g_2 h^2 + g_1 h + g_0 \in \mathbb{Q}(k)[h]$

$$\sigma(g) - g = h$$



ANSATZ $g = g_2 h^2 + g_1 h + g_0 \in \mathbb{Q}(k)[h]$

$$\left[\sigma(g_2) \left(h + \frac{1}{k+1} \right)^2 + \sigma(g_1 h + g_0) \right] - [g_2 h^2 + g_1 h + g_0] = h$$



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coeff. comp.

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$$\sigma(g_2) - g_2 = 0$$

$$g_2 = c \in \mathbb{Q}$$

$$\sigma(g_1 h + g_0) - (g_1 h + g_0) = h - c \left[\frac{2h(k+1)+1}{(k+1)^2} \right]$$

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$$\sigma(g_1) - g_1 = 1 - c \frac{2}{k+1}$$

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$$\sigma(g_1) - g_1 = 1 - c \frac{2}{k+1}$$

$$\sigma(g_0) - g_0 = -1 - d \frac{1}{k+1}$$

$$c = 0, \quad g_1 = k + d \\ d \in \mathbb{Q}$$

ANSATZ $g = g_2 h^2 + g_1 h + g_0 \in \mathbb{Q}(k)[h]$

$$[\sigma(g_2)(h + \frac{1}{k+1})^2 + \sigma(g_1 h + g_0)] - [g_2 h^2 + g_1 h + g_0] = h$$

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$$g = hk - k$$

$$\sigma(g_2) - g_2 = 0$$

$$g_2 = c \in \mathbb{Q}$$

$$\sigma(g_1 h + g_0) - (g_1 h + g_0) = h - c \left[\frac{2h(k+1)+1}{(k+1)^2} \right]$$

coeff. comp.

$$\sigma(g_1) - g_1 = 1 - c \frac{2}{k+1}$$

$$\begin{aligned} g_0 &= -k \\ d &= 0 \end{aligned}$$

$$\leftarrow \sigma(g_0) - g_0 = -1 - d \frac{1}{k+1}$$

$$\leftarrow c = 0, \quad g_1 = k + d \\ d \in \mathbb{Q}$$

First order difference equations in difference fields

Let (\mathbb{F}, σ) be a $\Pi\Sigma$ -field with constant field \mathbb{K}

Telescoping

- ▶ Given $f \in \mathbb{F}$
- ▶ Find $g \in \mathbb{F}$:

$$\sigma(g) - g = f$$

First order difference equations in difference fields

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↓

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Parameterized Telescoping

- ▶ Given $\mathbf{f} = (f_1, \dots, f_n) \in \mathbb{F}^n$
- ▶ Find $\left\{ (c_1, \dots, c_n, g) \in \mathbb{K}^n \times \mathbb{F} \mid \right.$

$$\left. \sigma(g) - g = c_1 f_1 + \dots + c_n f_n \right\}$$

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Parameterized first order linear difference equation

- ▶ Given $\mathbf{f} = (f_1, \dots, f_n) \in \mathbb{F}^n$, $(0, 0) \neq (a_0, a_1) \in \mathbb{F}^2$
- ▶ Find $\left\{ (c_1, \dots, c_n, g) \in \mathbb{K}^n \times \mathbb{F} \mid \right.$

$$\left. a_1 \sigma(g) + a_0 g = c_1 f_1 + \dots + c_n f_n \right\}$$

First order difference equations in difference fields

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$$\left. a_1 \sigma(g) + a_0 g = c_1 f_1 + \dots + c_n f_n \right\} =: V(\mathbf{a}, \mathbf{f}, \mathbb{F})$$

First order difference equations in difference fields

Let (\mathbb{F}, σ) be a $\Pi\Sigma$ -field with constant field \mathbb{K}

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Note:

- ▶ $V(\mathbf{a}, \mathbf{f}, \mathbb{F})$ is a subspace of $\mathbb{K}^n \times \mathbb{F}$ over \mathbb{K}

First order difference equations in difference fields

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Note:

- ▶ $V(\mathbf{a}, \mathbf{f}, \mathbb{F})$ is a subspace of $\mathbb{K}^n \times \mathbb{F}$ over \mathbb{K}
- ▶ $\dim_{\mathbb{K}} V(\mathbf{a}, \mathbf{f}, \mathbb{F}) \leq n + 1$

First order difference equations in difference fields

Let (\mathbb{F}, σ) be a $\Pi\Sigma$ -field with constant field \mathbb{K}

Telescoping

- ▶ Given $f \in \mathbb{F}$
- ▶ Find $g \in \mathbb{F}$:

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Parameterized first order linear difference equation

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Note:

- ▶ $V(\mathbf{a}, \mathbf{f}, \mathbb{F})$ is a subspace of $\mathbb{K}^n \times \mathbb{F}$ over \mathbb{K}
- ▶ $\dim_{\mathbb{K}} V(\mathbf{a}, \mathbf{f}, \mathbb{F}) \leq n + 1$
- ▶ Task: Compute a basis of $V(\mathbf{a}, \mathbf{f}, \mathbb{F})$

Find a basis of

$$V(\mathbf{a}, \mathbf{f}, \mathbb{G}(t))$$

$(\mathbb{G}(t), \sigma)$ $\Pi\Sigma^*$ -field

Find a basis of

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Compute a denominator bound
and search for polynomial solutions

$(\mathbb{G}(t), \sigma)$ $\Pi\Sigma^*$ -field

Find a basis of

$$V(\mathbf{a}, \mathbf{f}, \mathbb{G}(t))$$



Compute a denominator bound
and search for polynomial solutions



Compute a degree bound m

$(\mathbb{G}(t), \sigma)$ $\Pi\Sigma^*$ -field

Find a basis of

$$V(\mathbf{a}, \mathbf{f}, \mathbb{G}(t))$$



Compute a denominator bound
and search for polynomial solutions



Compute a degree bound m



Compute a basis of

$$V(\mathbf{a}_m, \mathbf{f}_m, \mathbb{G}[t]_m)$$

$(\mathbb{G}(t), \sigma)$ $\Pi\Sigma^*$ -field

$$\mathbb{G}[t]_m := \{p \in \mathbb{G}[t] \mid \deg(p) \leq m\}$$

Find a basis of

$$V(\mathbf{a}, \mathbf{f}, \mathbb{G}(t))$$

$(\mathbb{G}(t), \sigma)$ $\Pi\Sigma^*$ -field



Compute a denominator bound
and search for polynomial solutions



Compute a degree bound m



Compute a basis of

$$V(\mathbf{a}_m, \mathbf{f}_m, \mathbb{G}[t]_m)$$

coeff. comparison

→ Compute a basis of

$$V(\tilde{\mathbf{a}}_m, \tilde{\mathbf{f}}_m, \mathbb{G})$$

Find a basis of

$$V(\mathbf{a}, \mathbf{f}, \mathbb{G}(t))$$

$(\mathbb{G}(t), \sigma)$ $\Pi\Sigma^*$ -field



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Plugin highest term

Compute a basis of

$$V(\mathbf{a}_{m-1}, \mathbf{f}_{m-1}, \mathbb{G}[t]_{m-1})$$

Find a basis of

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Compute a basis of

Combine solution

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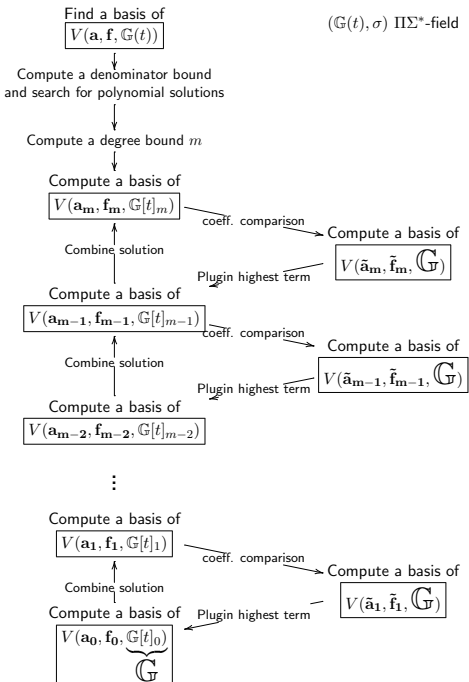
$$V(\tilde{\mathbf{a}}_{m-1}, \tilde{\mathbf{f}}_{m-1}, \mathbb{G})$$

Combine solution

Plugin highest term

Compute a basis of

$$V(\mathbf{a}_{m-2}, \mathbf{f}_{m-2}, \mathbb{G}[t]_{m-2})$$

Simplified version of
Karr's algorithm

degree reduction
by solving problems in \mathbb{G}

Example 2

FIND $g(k)$:

$$g(k+1) - g(k) = \frac{H_k}{k}$$

for all $1 \leq k \leq m$ and $m \geq 0$.

Find $g \in \mathbb{Q}(k)(h)$ s.t.

$$\sigma(g) - g = \frac{h}{k}$$

Find $g \in \mathbb{Q}(k)[h]$ s.t.

$$\sigma(g) - g = \frac{h}{k}$$

Find $g = g_2h^2 + g_1h + g_0 \in \mathbb{Q}(k)[h]_2$ s.t.

$$\sigma(g) - g = \frac{h}{k}$$

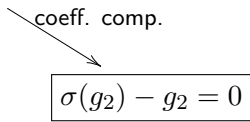
Find $g = g_2 h^2 + g_1 h + g_0 \in \mathbb{Q}(k)[h]_2$ s.t.

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$$\begin{aligned} & [\sigma(g_2) \left(h + \frac{1}{k+1}\right)^2 + \sigma(g_1 h + g_0)] \\ & - [g_2 h^2 + g_1 h + g_0] = \frac{h}{k} \end{aligned}$$

Find $g = g_2h^2 + g_1h + g_0 \in \mathbb{Q}(k)[h]_2$ s.t.
$$\sigma(g) - g = \frac{h}{k}$$

$$\begin{aligned} & [\sigma(g_2)(h + \frac{1}{k+1})^2 + \sigma(g_1h + g_0)] \\ & - [g_2h^2 + g_1h + g_0] = \frac{h}{k} \end{aligned}$$

coeff. comp.


$$\sigma(g_2) - g_2 = 0$$

Find $g = g_2h^2 + g_1h + g_0 \in \mathbb{Q}(k)[h]_2$ s.t.

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$$\sigma(g_2) - g_2 = 0$$

$g_2 = c \in \mathbb{Q}$

$$\sigma(g_1 h + g_0) - (g_1 h + g_0) = \frac{h}{k} + c \left[\frac{-2h(k+1) - 1}{(k+1)^2} \right]$$

Find $g = g_2 h^2 + g_1 h + g_0 \in \mathbb{Q}(k)[h]_2$ s.t.

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$$\sigma(g_0) - g_0 = \frac{2}{(k+1)^2} + d \frac{-1}{k+1}$$

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coeff. comp.

$$\sigma(g_1) - g_1 = \frac{1}{k} + c \frac{-2}{k+1}$$

no

solution

$$\leftarrow \sigma(g_0) - g_0 = \frac{2}{(k+1)^2} + d \frac{-1}{k+1}$$

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FIND $g(k)$:

$$g(k+1) - g(k) = \frac{H_k}{k}$$

for all $1 \leq k \leq m$ and $m \geq 0$.

No solution 😞

Refined telescoping

FIND $g(k), \phi(k)$:

$$g(k+1) - g(k) + \phi(k) = \frac{H_k}{k}$$

for all $1 \leq k \leq m$ and $m \geq 0$.

Find $g = g_2 h^2 + g_1 h + g_0 \in \mathbb{Q}(k)[h]_2$ s.t.

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$$\left[\sigma(g_2) \left(h + \frac{1}{k+1} \right)^2 + \sigma(g_1 h + g_0) \right] - [g_2 h^2 + g_1 h + g_0] = \frac{h}{k}$$

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$$c = \frac{1}{2}, \quad g_1 = -\frac{1}{k} + d$$

$$\sigma(0) - 0 + \frac{2}{(k+1)^2} = \frac{2}{(k+1)^2} + 0 \frac{-1}{k+1}$$

Find $g = g_2 h^2 + g_1 h + g_0 \in \mathbb{Q}(k)[h]_2$ s.t.

$$\sigma(g) - g + \frac{1}{2(k+1)^2} = \frac{h}{k}$$

$$g = \frac{1}{2}h + \frac{1}{k}$$

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Refined telescoping

FIND $g(k), \phi(k)$:

$$g(k+1) - g(k) + \phi(k) = \frac{H_k}{k}$$

for all $1 \leq k \leq m$ and $m \geq 0$.

We compute

$$g(k) = \frac{1}{2}H_k + \frac{1}{k}$$

$$\phi(k) = \frac{1}{2(k+1)^2}$$

Refined telescoping

FIND $g(k), \phi(k)$:

$$g(k+1) - g(k) + \phi(k) = \frac{H_k}{k}$$

for all $1 \leq k \leq m$ and $m \geq 0$.

Summing this equation over k from 1 to m gives

$$\sum_{k=1}^m \frac{H_k}{k} = g(m+1) - g(1) + \sum_{k=1}^m \phi(k)$$
$$= \frac{1}{2} \left(H_m^2 + \sum_{k=1}^m \frac{1}{k^2} \right).$$

Parameterized telescoping

Given $\Pi\Sigma^*$ -field $(\mathbb{K}(t_1) \dots (t_e), \sigma)$.

Define $\mathbb{F}_i = \mathbb{K}(t_1) \dots (t_i)$, i.e.,

$$\mathbb{F}_0 < \mathbb{F}_1 < \dots < \mathbb{F}_i < \dots < \mathbb{F}_e$$

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Given $f_1, \dots, f_n \in \mathbb{F}_e$.

Find $c_1, \dots, c_n \in \mathbb{K}$, $g \in \mathbb{F}_e$:

$$\sigma(g) - g = c_1 f_1 + \dots + c_n f_n$$

Special cases are

- ▶ telescoping ($n = 1$)
- ▶ creative telescoping (take $f_i = F(r - 1 + i, k)$)

First entry telescoping

Given $\Pi\Sigma^*$ -field $(\mathbb{K}(t_1) \dots (t_e), \sigma)$.

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Note: Gives rise to faster algorithms

Find $g = g_2 h^2 + g_1 h + g_0 \in \mathbb{Q}(k)[h]_2$ s.t.

$$\sigma(g) - g = \frac{h}{k}$$

$$\left[\sigma(g_2) \left(h + \frac{1}{k+1} \right)^2 + \sigma(g_1 h + g_0) \right] - [g_2 h^2 + g_1 h + g_0] = \frac{h}{k}$$

coeff. comp.

$$\sigma(g_2) - g_2 = 0$$

$$g_2 = c \in \mathbb{Q}$$

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coeff. comp.

$$\sigma(g_1) - g_1 = \frac{1}{k} + c \frac{-2}{k+1}$$

no

solution

$$\leftarrow \sigma(g_0) - g_0 = \frac{2}{(k+1)^2} + d \frac{-1}{k+1}$$

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Find $c_1, \dots, c_n \in \mathbb{K}$, $g \in \mathbb{F}_e$

$\phi \in \mathbb{F}_i$ with i minimal :

$$\sigma(g) - g + \boxed{\phi} = \underbrace{c_1}_{\neq 0} f_1 + \dots + c_n f_n$$

Find $c_j \in \mathbb{K}$, $g \in \mathbb{F}_e$, $\phi \in \mathbb{F}_e$ s.t.

$$\sigma(g) - g + \phi = \underbrace{c_1}_{\neq 0} f_1 + \cdots + c_n f_n.$$

$$V((-1, 1), (f_1, \dots, f_n), \mathbb{F}_e)$$



$$V((-1, 1), (f_1^{(e-1)}, \dots, f_{n_{e-1}}^{(e-1)}), \mathbb{F}_{e-1})$$

Find $g = g_2h^2 + g_1h + g_0 \in \mathbb{Q}(k)[h]_2$ s.t.

$$\sigma(g) - g + \frac{1}{2(k+1)^2} = \frac{h}{k}$$

$$g = \frac{1}{2}h + \frac{1}{k}$$

$$V\left(\left(-1, 1\right), \left(\frac{h}{k}\right), \mathbb{Q}(k)(h)\right)$$

coeff. comp.

$$\sigma(g_2) - g_2 = 0$$

$$g_2 = c \in \mathbb{Q}$$

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$$\sigma(g_1) - g_1 = \frac{1}{k} + c \frac{-2}{k+1}$$

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$$\vdots$$


$$V((-1, 1), (f_1^{(i)}, \dots, f_{n_i}^{(i)}), \mathbb{F}_i)$$

Find $c_j \in \mathbb{K}$, $g \in \mathbb{F}_e$, $\phi \in \mathbb{F}_e$ s.t.

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⋮



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Take $\bar{c}_j \in \mathbb{K}$, $\bar{g} \in \mathbb{F}_e$, $\bar{\phi} \in \mathbb{F}_{i-1}$ s.t.

$$\sigma(\bar{g}) - \bar{g} + \bar{\phi} = \underbrace{\bar{c}_1}_{\neq 0} f_1 + \cdots + \bar{c}_n f_n.$$

Find $c_j \in \mathbb{K}$, $g \in \mathbb{F}_e$, $\phi \in \mathbb{F}_e$ s.t.

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$$V((-1, 1), (f_1, \dots, f_n), \mathbb{F}_e)$$



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⋮



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$$V((-1, 1), (f_1^{(i)}, \dots, f_{n_i}^{(i)}, \bar{\phi}), \mathbb{F}_i)$$

Find $c_j \in \mathbb{K}$, $g \in \mathbb{F}_e$, $\phi \in \mathbb{F}_e$ s.t.

$$\sigma(g) - g + \phi = \underbrace{c_1}_{\neq 0} f_1 + \cdots + c_n f_n.$$

$$V((-1, 1), (f_1, \dots, f_n), \mathbb{F}_e)$$



$$V((-1, 1), (f_1^{(e-1)}, \dots, f_{n_{e-1}}^{(e-1)}), \mathbb{F}_{e-1})$$



⋮



$$V((-1, 1), (f_1^{(i)}, \dots, f_{n_i}^{(i)}), \mathbb{F}_i)$$

Take $\bar{c}_j \in \mathbb{K}$, $\bar{g} \in \mathbb{F}_e$, $\bar{\phi} \in \mathbb{F}_{i-1}$ s.t.

$$\sigma(\bar{g}) - \bar{g} = \underbrace{\bar{c}_1}_{\neq 0} f_1 + \cdots + \bar{c}_n f_n - \bar{\phi}.$$

$$V((-1, 1), (f_1, \dots, f_n, \bar{\phi}), \mathbb{F}_e)$$



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$$\vdots$$


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Refined telescoping

Given $\Pi\Sigma^*$ -field $(\mathbb{K}(t_1) \dots (t_e), \sigma)$.

Define $\mathbb{F}_i = \mathbb{K}(t_1) \dots (t_i)$, i.e.,

$$\underbrace{\mathbb{F}_{-1}}_{=\{0\}} \subset \mathbb{F}_0 < \mathbb{F}_1 < \dots < \mathbb{F}_i < \dots < \mathbb{F}_e$$

Given $f_1, \dots, f_n \in \mathbb{F}_e$.

Find $c_1, \dots, c_n \in \mathbb{K}$, $g \in \mathbb{F}_e$

$\phi \in \mathbb{F}_i$ with i minimal :

$$\sigma(g) - g + \boxed{\phi} = \underbrace{c_1}_{\neq 0} f_1 + \dots + c_n f_n$$

Refined telescoping

Given $\Pi\Sigma^*$ -extension $(\mathbb{G}(t_1) \dots (t_e), \sigma)$ of a difference field (\mathbb{G}, σ) .

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Requirements: an algorithm to solve first order linear difference equations in (\mathbb{G}, σ) plus technical conditions.

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$\phi \in \mathbb{F}_i$ with i minimal :

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Example 1: $\mathbb{G} = \mathbb{K}$

Examples

$$\sum_{k=1}^m (k^2 + 1) k! H_k^2 = \sum_{k=1}^m \frac{(k+1)!}{k^2} + m(m+1)! H_m^2 - 2m! H_m.$$

$$\sum_{k=1}^m k \sum_{j=1}^k \frac{H_j}{j^2} = \frac{1}{2} \left(\sum_{k=1}^m \frac{1}{k^2} + m(m+1) \sum_{i=1}^m \frac{H_i}{i^2} + H_m^2 - (m+1)H_m + m \right)$$

$$\begin{aligned} \sum_{k=1}^m \left(\sum_{j=1}^k \binom{r}{j} \right)^2 &= - \frac{1}{2} r \sum_{k=1}^m \binom{r}{k}^2 + \frac{1}{2} (2m - r + 2) \left(\sum_{i=1}^m \binom{r}{i} \right)^2 \\ &\quad - (m - r) \binom{r}{m} \sum_{i=1}^m \binom{r}{i} \end{aligned}$$

Refined telescoping

Given $\Pi\Sigma^*$ -extension $(\mathbb{G}(t_1) \dots (t_e), \sigma)$ of a difference field (\mathbb{G}, σ) .

Define $\mathbb{F}_i = \mathbb{G}(t_1) \dots (t_i)$, i.e.,

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Find $c_1, \dots, c_n \in \mathbb{K}$, $g \in \mathbb{F}_e$

$\phi \in \mathbb{F}_i$ with i minimal :

$$\sigma(g) - g + \boxed{\phi} = \underbrace{c_1}_{\neq 0} f_1 + \dots + c_n f_n$$

Example 2 (free difference field): $\mathbb{G} = \mathbb{K}(\dots, x_{-1}, x_0, x_1, \dots)$
with $\sigma(x_i) = x_{i+1}$.

Unspecified sequences (M. Kauers, CS; 2006)

$$\sum_{k=1}^n a^k \sum_{j=1}^k X_j = \frac{a^{n+1} \sum_{k=1}^n X_k - \sum_{k=1}^n a^k X_k}{a - 1}, \quad a \neq 1$$

$$\begin{aligned} \sum_{k=0}^n k^2 \sum_{i=0}^k X_i &= \frac{1}{6} \left(n(n+1)(2n+1) \sum_{k=0}^n X_k - \sum_{k=0}^n k X_k \right. \\ &\quad \left. + 3 \sum_{k=0}^n k^2 X_k - 2 \sum_{k=0}^n k^3 X_k \right) \end{aligned}$$

Refined telescoping

Given $\Pi\Sigma^*$ -extension $(\mathbb{G}(t_1) \dots (t_e), \sigma)$ of a difference field (\mathbb{G}, σ) .

Define $\mathbb{F}_i = \mathbb{G}(t_1) \dots (t_i)$, i.e.,

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Find $c_1, \dots, c_n \in \mathbb{K}, g \in \mathbb{F}_e$

$\phi \in \mathbb{F}_i$ with i minimal :

$$\sigma(g) - g + \boxed{\phi} = \underbrace{c_1}_{\neq 0} f_1 + \dots + c_n f_n$$

Example 3 (radicals of order $d > 0$): $\mathbb{G} = \mathbb{K}(k)(\dots, x_{-1}, x_0, x_1, \dots)$ with $\sigma(k) = k + 1$ and $\sigma(x_i) = x_{i+1}$ subject to the relation $x_i^d = k$.

Radical expressions (M. Kauers, CS; 2007)

$$\sum_{k=0}^n \frac{1}{\sqrt{k+1} + \sqrt{k}} = \sqrt{n+1}$$

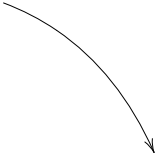
$$\begin{aligned} \sum_{k=2}^n H_k \left(\frac{(\sqrt{k})^3}{k-1} + \sum_{i=1}^k \frac{\sqrt{i}}{i + \sqrt{i}} \right) &= \frac{1}{2} \left(-3 - 5n + (5n + 3)H_n \right. \\ &\quad \left. - (2n + 1)H_n^2 + H_n^{(2)} \right) + \sum_{k=1}^n \sqrt{k} \\ &\quad + \left((n + 1)H_n - (n - 1) \right) \sum_{k=2}^n \frac{\sqrt{k}}{k-1} \end{aligned}$$

Conclusion

simplified version of
Karr's algorithm

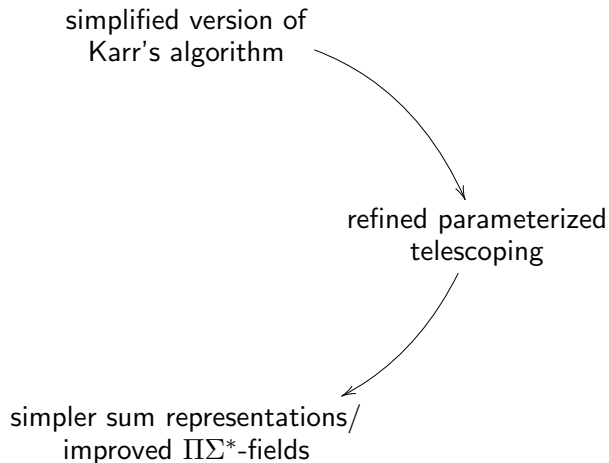
Conclusion

simplified version of
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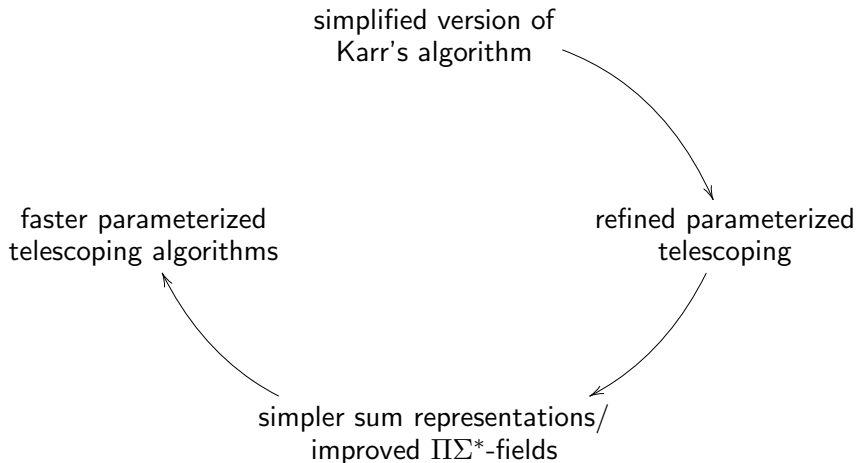


refined parameterized
telescoping

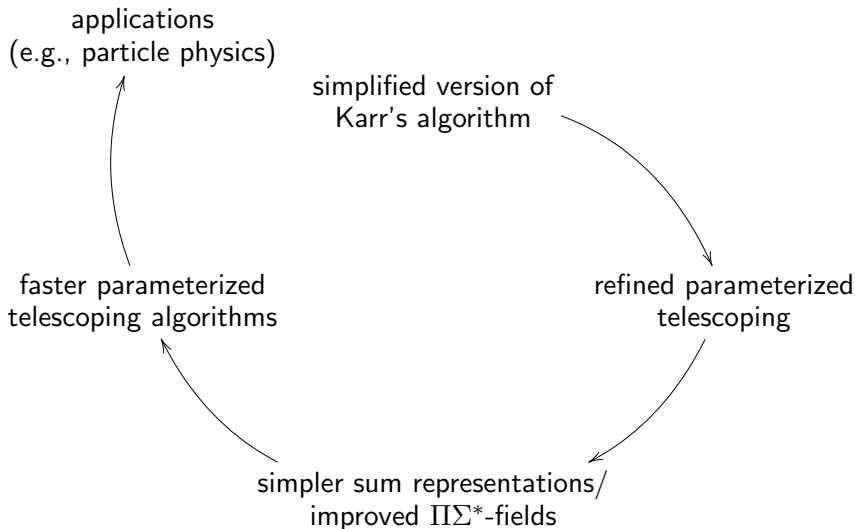
Conclusion



Conclusion



Conclusion



Further details: Proceedings of this workshop (LNCS series) [arXiv:1307.7887]