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Institute of High Energy Physics (IHEP) of the Chinese Academy of Sciences

Solution of Loop Integrals in Quantum Field Theory using Modern Summation Methods

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Outline of talk

1. Symbolic summation techniques
2. Application to massive 3-loop integrals

A warm up example

$$\begin{aligned}
 \text{GIVEN } F(n) &= \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{e^{-\varepsilon\gamma}}{\Gamma(\varepsilon + 1)} \times \\
 &\times \left(\frac{\Gamma(k+1)}{\Gamma(k+2+n)} \frac{\Gamma(\frac{\varepsilon}{2})\Gamma(1-\frac{\varepsilon}{2})\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+1+\frac{\varepsilon}{2})\Gamma(k+j+1+n)}{\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+2+n)\Gamma(k+j+2)} \right. \\
 &\left. + \frac{\Gamma(k+1)}{\Gamma(k+2+n)} \frac{\Gamma(-\frac{\varepsilon}{2})\Gamma(1+\frac{\varepsilon}{2})\Gamma(j+1+\varepsilon)\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(k+j+1+\frac{\varepsilon}{2}+n)}{\Gamma(j+1)\Gamma(j+2+\frac{\varepsilon}{2}+n)\Gamma(k+j+2+\frac{\varepsilon}{2})} \right) \\
 &\qquad \qquad \qquad \underbrace{\hspace{15em}}_{f(n, k, j)}.
 \end{aligned}$$

Arose in the context of

I. Bierenbaum, J. Blümlein, and S. Klein, *Evaluating two-loop massive operator matrix elements with Mellin-Barnes integrals*. 2006

A warm up example

$$\begin{aligned} \text{GIVEN } F(n) &= \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{e^{-\epsilon\gamma}}{\Gamma(\epsilon + 1)} \times \\ &\times \left(\frac{\Gamma(k+1)}{\Gamma(k+2+n)} \frac{\Gamma(\frac{\epsilon}{2})\Gamma(1-\frac{\epsilon}{2})\Gamma(j+1-\frac{\epsilon}{2})\Gamma(j+1+\frac{\epsilon}{2})\Gamma(k+j+1+n)}{\Gamma(j+1-\frac{\epsilon}{2})\Gamma(j+2+n)\Gamma(k+j+2)} \right. \\ &\left. + \frac{\Gamma(k+1)}{\Gamma(k+2+n)} \frac{\Gamma(-\frac{\epsilon}{2})\Gamma(1+\frac{\epsilon}{2})\Gamma(j+1+\epsilon)\Gamma(j+1-\frac{\epsilon}{2})\Gamma(k+j+1+\frac{\epsilon}{2}+n)}{\Gamma(j+1)\Gamma(j+2+\frac{\epsilon}{2}+n)\Gamma(k+j+2+\frac{\epsilon}{2})} \right) \\ &\underbrace{\hspace{10em}}_{f(n, k, j)}. \end{aligned}$$

FIND the first coefficients of the ϵ -expansion

$$F(n) = F_0(n) + \epsilon F_1(n) + \dots$$

Arose in the context of

I. Bierenbaum, J. Blümlein, and S. Klein, *Evaluating two-loop massive operator matrix elements with Mellin-Barnes integrals*. 2006

A warm up example

$$\text{GIVEN } F(n) = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{e^{-\epsilon\gamma}}{\Gamma(\epsilon + 1)} \times$$

$$\times \left(\frac{\Gamma(k+1)}{\Gamma(k+2+n)} \frac{\Gamma(\frac{\epsilon}{2})\Gamma(1-\frac{\epsilon}{2})\Gamma(j+1-\frac{\epsilon}{2})\Gamma(j+1+\frac{\epsilon}{2})\Gamma(k+j+1+n)}{\Gamma(j+1-\frac{\epsilon}{2})\Gamma(j+2+n)\Gamma(k+j+2)} \right.$$

$$\left. + \frac{\Gamma(k+1)}{\Gamma(k+2+n)} \frac{\Gamma(-\frac{\epsilon}{2})\Gamma(1+\frac{\epsilon}{2})\Gamma(j+1+\epsilon)\Gamma(j+1-\frac{\epsilon}{2})\Gamma(k+j+1+\frac{\epsilon}{2}+n)}{\Gamma(j+1)\Gamma(j+2+\frac{\epsilon}{2}+n)\Gamma(k+j+2+\frac{\epsilon}{2})} \right).$$

$$f(n, k, j)$$

Step 1: Compute the first coefficients of the ϵ -expansion

$$f(n, k, j) = f_0(n, k, j) + \epsilon f_1(n, k, j) + \dots$$

Arose in the context of

I. Bierenbaum, J. Blümlein, and S. Klein, *Evaluating two-loop massive operator matrix elements with Mellin-Barnes integrals*. 2006

A warm up example

$$\text{GIVEN } F(n) = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{e^{-\varepsilon\gamma}}{\Gamma(\varepsilon + 1)} \times$$

$$\times \left(\frac{\Gamma(k+1)}{\Gamma(k+2+n)} \frac{\Gamma(\frac{\varepsilon}{2})\Gamma(1-\frac{\varepsilon}{2})\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+1+\frac{\varepsilon}{2})\Gamma(k+j+1+n)}{\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+2+n)\Gamma(k+j+2)} \right.$$

$$\left. + \frac{\Gamma(k+1)}{\Gamma(k+2+n)} \frac{\Gamma(-\frac{\varepsilon}{2})\Gamma(1+\frac{\varepsilon}{2})\Gamma(j+1+\varepsilon)\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(k+j+1+\frac{\varepsilon}{2}+n)}{\Gamma(j+1)\Gamma(j+2+\frac{\varepsilon}{2}+n)\Gamma(k+j+2+\frac{\varepsilon}{2})} \right).$$

$$f(n, k, j)$$

Step 2: **Simplify** the sums in

$$\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} f(n, k, j) = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} f_0(n, k, j) + \varepsilon \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} f_1(n, k, j) + \dots$$

Arose in the context of

I. Bierenbaum, J. Blümlein, and S. Klein, *Evaluating two-loop massive operator matrix elements with Mellin-Barnes integrals*. 2006

Simplify

$$\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \left(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} \right. \\ \left. + \frac{j!k!(j+k+n)!(-S_1(j) + S_1(j+k) + S_1(j+n) - S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!} \right) \\ \underbrace{\hspace{15em}}_{f(j)}$$

where

$$S_1(n) = \sum_{i=1}^n \frac{1}{i}$$

Simplify

$$\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \left(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} \right. \\ \left. + \frac{j!k!(j+k+n)!(-S_1(j) + S_1(j+k) + S_1(j+n) - S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!} \right) \\ \underbrace{\hspace{15em}}_{f(j)}$$

FIND $g(j)$:

$$f(j) = g(j+1) - g(j)$$

Simplify

$$\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \left(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} \right. \\ \left. + \frac{j!k!(j+k+n)!(-S_1(j) + S_1(j+k) + S_1(j+n) - S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!} \right) \\ \underbrace{\hspace{15em}}_{f(j)}$$

FIND $g(j)$:

$$\boxed{f(j) = g(j+1) - g(j)}$$

↑ summation package Sigma

$$g(j) = \frac{(j+k+1)(j+n+1)j!k!(j+k+n)!(S_1(j) - S_1(j+k) - S_1(j+n) + S_1(j+k+n))}{kn(j+k+1)!(j+n+1)!(k+n+1)!}$$

Simplify

$$\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \left(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} \right. \\ \left. + \frac{j!k!(j+k+n)!(-S_1(j) + S_1(j+k) + S_1(j+n) - S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!} \right) \\ \underbrace{\hspace{15em}}_{f(j)}$$

FIND $g(j)$:

$$\boxed{f(j) = g(j+1) - g(j)}$$

Summing the telescoping equation over j from 0 to a gives

$$\sum_{j=0}^a f(n, k, j) = g(a+1) - g(0)$$

Simplify

$$\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \left(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} \right. \\ \left. + \frac{j!k!(j+k+n)!(-S_1(j) + S_1(j+k) + S_1(j+n) - S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!} \right) \\ \underbrace{\hspace{15em}}_{f(j)}$$

FIND $g(j)$:

$$f(j) = g(j+1) - g(j)$$

Summing the telescoping equation over j from 0 to a gives

$$\sum_{j=0}^a f(n, k, j) = g(a+1) - g(0) \\ = \frac{(a+1)!(k-1)!(a+k+n+1)!(S_1(a) - S_1(a+k) - S_1(a+n) + S_1(a+k+n))}{n(a+k+1)!(a+n+1)!(k+n+1)!} \\ + \frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)n!} + \frac{(2a+k+n+2)a!k!(a+k+n)!}{(a+k+1)(a+n+1)(a+k+1)!(a+n+1)!(k+n+1)!} \\ \underbrace{\hspace{15em}}_{a \rightarrow \infty}$$

Simplify

$$\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \left(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} \right. \\ \left. + \frac{j!k!(j+k+n)!(-S_1(j) + S_1(j+k) + S_1(j+n) - S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!} \right) \\ \underbrace{\hspace{15em}}_{f(j)}$$

$$\sum_{j=0}^{\infty} f(n, k, j) = \frac{1}{n!} \frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}$$

Simplify

$$\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \left(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} \right. \\ \left. + \frac{j!k!(j+k+n)!(-S_1(j) + S_1(j+k) + S_1(j+n) - S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!} \right) \\ \underbrace{\hspace{15em}}_{f(j)}$$

$$\sum_{k=1}^{\infty} \sum_{j=0}^{\infty} f(n, k, j) = \frac{1}{n!} \sum_{k=1}^{\infty} \frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}$$

Telescoping

GIVEN

$$A(n) := \sum_{k=1}^a \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=: f(n, k)}.$$

FIND $g(n, k)$:

$$\boxed{g(n, k+1) - g(n, k)} = \boxed{f(n, k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.no solution 

Zeilberger's creative telescoping paradigm

GIVEN

$$A(n) := \sum_{k=1}^a \frac{S_1(k) + S_1(n) - S_1(k+n)}{\underbrace{kn(k+n+1)}_{=: f(n,k)}}.$$

FIND $g(n, k)$ and $c_0(n), c_1(n)$:

$$\boxed{g(n, k+1) - g(n, k)} = \boxed{c_0(n)f(n, k) + c_1(n)f(n+1, k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

Zeilberger's creative telescoping paradigm

GIVEN

$$A(n) := \sum_{k=1}^a \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=: f(n, k)}.$$

FIND $g(n, k)$ and $c_0(n), c_1(n)$:

$$\boxed{g(n, k+1) - g(n, k)} = \boxed{c_0(n)f(n, k) + c_1(n)f(n+1, k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

Sigma computes: $c_0(n) = -n$, $c_1(n) = (n+2)$ and

$$g(n, k) = \frac{kS_1(k) + (-n-1)S_1(n) - kS_1(k+n) - 2}{(k+n+1)(n+1)^2}$$

Zeilberger's creative telescoping paradigm

GIVEN

$$A(n) := \sum_{k=1}^a \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=: f(n, k)}.$$

FIND $g(n, k)$ and $c_0(n), c_1(n)$:

$$\boxed{g(n, k+1) - g(n, k)} = \boxed{c_0(n)f(n, k) + c_1(n)f(n+1, k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

Summing this equation over k from 1 to a gives:

$$\boxed{g(n, a+1) - g(n, 1)} = \boxed{\sum_{k=1}^a [c_0(n)f(n, k) + c_1(n)f(n+1, k)]}$$

Zeilberger's creative telescoping paradigm

GIVEN

$$A(n) := \sum_{k=1}^a \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=: f(n, k)}.$$

FIND $g(n, k)$ and $c_0(n), c_1(n)$:

$$\boxed{g(n, k+1) - g(n, k)} = \boxed{c_0(n)f(n, k) + c_1(n)f(n+1, k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

Summing this equation over k from 1 to a gives:

$$\boxed{g(n, a+1) - g(n, 1)} = \boxed{\sum_{k=1}^a c_0(n) f(n, k) + \sum_{k=1}^a c_1(n) f(n+1, k)}$$

Zeilberger's creative telescoping paradigm

GIVEN

$$A(n) := \sum_{k=1}^a \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=: f(n, k)}.$$

FIND $g(n, k)$ and $c_0(n), c_1(n)$:

$$\boxed{g(n, k+1) - g(n, k)} = \boxed{c_0(n)f(n, k) + c_1(n)f(n+1, k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

Summing this equation over k from 1 to a gives:

$$\boxed{g(n, a+1) - g(n, 1)} = \boxed{c_0(n) \sum_{k=1}^a f(n, k) + c_1(n) \sum_{k=1}^a f(n+1, k)}$$

Zeilberger's creative telescoping paradigm

GIVEN

$$A(n) := \sum_{k=1}^a \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=: f(n, k)}.$$

FIND $g(n, k)$ and $c_0(n), c_1(n)$:

$$\boxed{g(n, k+1) - g(n, k)} = \boxed{c_0(n)f(n, k) + c_1(n)f(n+1, k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

Summing this equation over k from 1 to a gives:

$$\boxed{g(n, a+1) - g(n, 1)} = \boxed{c_0(n)A(n) + c_1(n)A(n+1)}$$

Zeilberger's creative telescoping paradigm

GIVEN

$$A(n) := \sum_{k=1}^a \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=: f(n, k)}.$$

FIND $g(n, k)$ and $c_0(n), c_1(n)$:

$$\boxed{g(n, k+1) - g(n, k)} = \boxed{c_0(n)f(n, k) + c_1(n)f(n+1, k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

Summing this equation over k from 1 to a gives:

$$\begin{aligned} \boxed{g(n, a+1) - g(n, 1)} &= \boxed{c_0(n)A(n) + c_1(n)A(n+1)} \\ \parallel & \qquad \qquad \qquad \parallel \\ \frac{(a+1)(S_1(a)+S_1(n)-S_1(a+n))}{(n+1)^2(a+n+2)} & \qquad - nA(n) + (2+n)A(n+1) \\ + \frac{a(a+1)}{(n+1)^3(a+n+1)(a+n+2)} & \end{aligned}$$

$$(n+2)\mathbf{A}(n+1) - n\mathbf{A}(n) = \frac{(n+1)S_1(n) + 1}{(n+1)^3}$$

recurrence finder

$$A(n) = \sum_{k=1}^{\infty} \frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}$$

$$(n+2)\mathbf{A}(n+1) - n\mathbf{A}(n) = \frac{(n+1)S_1(n) + 1}{(n+1)^3}$$

recurrence solver

$$A(n) = \sum_{k=1}^{\infty} \frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}$$

\in

$$\left\{ c \times \frac{1}{n(n+1)} + \frac{S_1(n)^2 + S_2(n)}{2n(n+1)} \mid c \in \mathbb{R} \right\}$$

where

$$S_1(n) = \sum_{i=1}^n \frac{1}{i} \quad S_2(n) = \sum_{i=1}^n \frac{1}{i^2}$$

$$(n+2)\mathbf{A}(n+1) - n\mathbf{A}(n) = \frac{(n+1)S_1(n) + 1}{(n+1)^3}$$

Summation package Sigma

(based on difference field algorithms/theory)

see, e.g., Karr 1981, Bronstein 2000, Schneider 2001/2004/2005a-c/2007/2008/2010a-c)

$$A(n) = \sum_{k=1}^{\infty} \frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}$$

$$= 0 \times \frac{1}{n(n+1)} + \frac{S_1(n)^2 + S_2(n)}{2n(n+1)}$$

where

$$S_1(n) = \sum_{i=1}^n \frac{1}{i}$$

$$S_2(n) = \sum_{i=1}^n \frac{1}{i^2}$$

Simplify

$$\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \left(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} \right. \\ \left. + \frac{j!k!(j+k+n)!(-S_1(j) + S_1(j+k) + S_1(j+n) - S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!} \right) \\ \underbrace{\hspace{15em}}_{f(j)}$$

$$\sum_{k=1}^{\infty} \sum_{j=0}^{\infty} f(n, k, j) = \frac{1}{n!} \sum_{k=1}^{\infty} \frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)} \\ = \frac{1}{n!} \frac{S_1(n)^2 + S_2(n)}{2n(n+1)}$$

where

$$S_1(n) = \sum_{i=1}^n \frac{1}{i} \qquad S_2(n) = \sum_{i=1}^n \frac{1}{i^2}$$

GIVEN

$$\begin{aligned} & \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{e^{-\varepsilon\gamma}}{\Gamma(\varepsilon+1)} \left(\frac{\Gamma(k+1)}{\Gamma(k+2+n)} \frac{\Gamma(\frac{\varepsilon}{2})\Gamma(1-\frac{\varepsilon}{2})\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+1+\frac{\varepsilon}{2})\Gamma(k+j+1+n)}{\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+2+n)\Gamma(k+j+2)} \right. \\ & \left. + \frac{\Gamma(k+1)}{\Gamma(k+2+n)} \frac{\Gamma(-\frac{\varepsilon}{2})\Gamma(1+\frac{\varepsilon}{2})\Gamma(j+1+\varepsilon)\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(k+j+1+\frac{\varepsilon}{2}+n)}{\Gamma(j+1)\Gamma(j+2+\frac{\varepsilon}{2}+n)\Gamma(k+j+2+\frac{\varepsilon}{2})} \right). \\ & = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} f_0(n, k, j) + \end{aligned}$$

Sigma produces

$$\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} f_0(n, k, j) = \frac{S_1(n)^2 + 3S_2(n)}{2n(n+1)!}.$$

GIVEN

$$\begin{aligned} & \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{e^{-\varepsilon\gamma}}{\Gamma(\varepsilon+1)} \left(\frac{\Gamma(k+1)}{\Gamma(k+2+n)} \frac{\Gamma(\frac{\varepsilon}{2})\Gamma(1-\frac{\varepsilon}{2})\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+1+\frac{\varepsilon}{2})\Gamma(k+j+1+n)}{\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+2+n)\Gamma(k+j+2)} \right. \\ & \left. + \frac{\Gamma(k+1)}{\Gamma(k+2+n)} \frac{\Gamma(-\frac{\varepsilon}{2})\Gamma(1+\frac{\varepsilon}{2})\Gamma(j+1+\varepsilon)\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(k+j+1+\frac{\varepsilon}{2}+n)}{\Gamma(j+1)\Gamma(j+2+\frac{\varepsilon}{2}+n)\Gamma(k+j+2+\frac{\varepsilon}{2})} \right). \\ & = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} f_0(n, k, j) + \varepsilon \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} f_1(n, k, j) + \end{aligned}$$

Sigma produces

$$\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} f_0(n, k, j) = \frac{S_1(n)^2 + 3S_2(n)}{2n(n+1)!}.$$

$$\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} f_1(n, k, j) = \frac{-S_1(n)^3 - 3S_2(n)S_1(n) - 8S_3(n)}{6n(n+1)!}.$$

GIVEN

$$\begin{aligned}
& \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{e^{-\varepsilon\gamma}}{\Gamma(\varepsilon+1)} \left(\frac{\Gamma(k+1)}{\Gamma(k+2+n)} \frac{\Gamma(\frac{\varepsilon}{2})\Gamma(1-\frac{\varepsilon}{2})\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+1+\frac{\varepsilon}{2})\Gamma(k+j+1+n)}{\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+2+n)\Gamma(k+j+2)} \right. \\
& \left. + \frac{\Gamma(k+1)}{\Gamma(k+2+n)} \frac{\Gamma(-\frac{\varepsilon}{2})\Gamma(1+\frac{\varepsilon}{2})\Gamma(j+1+\varepsilon)\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(k+j+1+\frac{\varepsilon}{2}+n)}{\Gamma(j+1)\Gamma(j+2+\frac{\varepsilon}{2}+n)\Gamma(k+j+2+\frac{\varepsilon}{2})} \right). \\
& = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} f_0(n, k, j) + \varepsilon \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} f_1(n, k, j) + \varepsilon^2 \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} f_1(n, k, j) +
\end{aligned}$$

Sigma produces

$$\begin{aligned}
\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} f_2(n, k, j) &= \frac{1}{96n(n+1)} \left(S_1(n)^4 + (12\zeta_2 + 54S_2(n))S_1(n)^2 \right. \\
&+ 104S_3(n)S_1(n) - 48S_{2,1}(n)S_1(n) + 51S_2(n)^2 + 36\zeta_2 S_2(n) \\
&\left. + 126S_4(n) - 48S_{3,1}(n) - 96S_{1,1,2}(n) \right)
\end{aligned}$$

GIVEN

$$\begin{aligned}
& \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{e^{-\epsilon\gamma}}{\Gamma(\epsilon+1)} \left(\frac{\Gamma(k+1)}{\Gamma(k+2+n)} \frac{\Gamma(\frac{\epsilon}{2})\Gamma(1-\frac{\epsilon}{2})\Gamma(j+1-\frac{\epsilon}{2})\Gamma(j+1+\frac{\epsilon}{2})\Gamma(k+j+1+n)}{\Gamma(j+1-\frac{\epsilon}{2})\Gamma(j+2+n)\Gamma(k+j+2)} \right. \\
& \left. + \frac{\Gamma(k+1)}{\Gamma(k+2+n)} \frac{\Gamma(-\frac{\epsilon}{2})\Gamma(1+\frac{\epsilon}{2})\Gamma(j+1+\epsilon)\Gamma(j+1-\frac{\epsilon}{2})\Gamma(k+j+1+\frac{\epsilon}{2}+n)}{\Gamma(j+1)\Gamma(j+2+\frac{\epsilon}{2}+n)\Gamma(k+j+2+\frac{\epsilon}{2})} \right). \\
& = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} f_0(n, k, j) + \epsilon \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} f_1(n, k, j) + \epsilon^2 \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} f_1(n, k, j) + \epsilon^3 \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} f_1(n, k, j) + \dots
\end{aligned}$$

Sigma produces

$$\begin{aligned}
\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} f_3(n, k, j) = & \frac{1}{960n(n+1)} \left(S_1(n)^5 + (20\zeta_2 + 130S_2(n))S_1(n)^3 + \right. \\
& (40\zeta_3 + 380S_3(n))S_1(n)^2 + (135S_2(n)^2 + 60\zeta_2S_2(n) + 510S_4(n))S_1(n) \\
& - 240S_{3,1}(n)S_1(n) - 240S_{1,1,2}(n)S_1(n) + 160\zeta_2S_3(n) + S_2(n)(120\zeta_3 \\
& + 380S_3(n)) + 624S_5(n) + (-120S_1(n)^2 - 120S_2(n))S_{2,1}(n) \\
& \left. - 240S_{4,1}(n) - 240S_{1,1,3}(n) + 240S_{2,2,1}(n) \right)
\end{aligned}$$

Toolbox 1: Symbolic summation algorithms

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a **definite** sum

$$S(n) = \sum_{k=0}^n f(n, k);$$

$f(n, k)$: indefinite nested product-sum in k ;
 n : extra parameter

FIND a **recurrence** for $S(n)$

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2. Recurrence solving

GIVEN a recurrence

$$a_0(n), \dots, a_d(n), h(n):$$

indefinite nested product-sum expressions.

$$a_0(n)S(n) + \dots + a_d(n)S(n+d) = h(n);$$

FIND **all solutions** expressible by indefinite nested products/sums

(Abramov/Bronstein/Petkovšek/CS, in preparation)

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FIND **all solutions** expressible by indefinite nested products/sums
 (Abramov/Bronstein/Petkovšek/CS, in preparation)

3. Find a "closed form"

$S(n)$ = combined solutions in terms of **indefinite nested** sums.

In[1]:= << Sigma.m

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= mySum = $\sum_{k=1}^A \frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}$;

In[1]:= << **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= **mySum** = $\sum_{k=1}^A \frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}$;

Compute a recurrence

In[3]:= **rec** = **GenerateRecurrence**[**mySum**, **n**][[1]]

Out[3]= $-n\text{SUM}[n] + (1+n)(2+n)\text{SUM}[n+1] == \frac{(a+1)(S_1(a)+S_1(n)-S_1(a+n))}{(n+1)^2(a+n+2)n!} + \frac{a(a+1)}{(n+1)^3(a+n+1)(a+n+2)n!}$

In[4]:= **rec** = **LimitRec**[**rec**, **SUM**[**n**], {**n**}, **A**]

Out[4]= $-n\text{SUM}[n] + (1+n)(2+n)\text{SUM}[n+1] == \frac{(n+1)S_1(n) + 1}{(n+1)^3}$

In[1]:= << Sigma.m

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= mySum =
$$\sum_{k=1}^A \frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)};$$

Compute a recurrence

In[3]:= rec = GenerateRecurrence[mySum, n][[1]]

Out[3]=
$$n\text{SUM}[n] + (1+n)(2+n)\text{SUM}[n+1] == \frac{(a+1)(S_1(a)+S_1(n)-S_1(a+n))}{(n+1)^2(a+n+2)n!} + \frac{a(a+1)}{(n+1)^3(a+n+1)(a+n+2)n!}$$

In[4]:= rec = LimitRec[rec, SUM[n], {n}, A]

Out[4]=
$$-n\text{SUM}[n] + (1+n)(2+n)\text{SUM}[n+1] == \frac{(n+1)S_1(n) + 1}{(n+1)^3}$$

Solve a recurrence

In[5]:= recSol = SolveRecurrence[rec, SUM[n]]

Out[5]=
$$\left\{ \left\{ 0, \frac{1}{n(n+1)} \right\}, \left\{ 1, \frac{S_1(n)^2 + \sum_{i=1}^n \frac{1}{i^2}}{2n(n+1)} \right\} \right\}$$

In[1]:= << Sigma.m

Sigma - A summation package by Carsten Schneider © RISC-Linz

$$\text{In[2]:= mySum} = \sum_{k=1}^A \frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)};$$

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In[4]:= rec = LimitRec[rec, SUM[n], {n}, A]

$$\text{Out[4]= } -n\text{SUM}[n] + (1+n)(2+n)\text{SUM}[n+1] == \frac{(n+1)S_1(n) + 1}{(n+1)^3}$$

Solve a recurrence

In[5]:= recSol = SolveRecurrence[rec, SUM[n]]

$$\text{Out[5]= } \left\{ \left\{ 0, \frac{1}{n(n+1)} \right\}, \left\{ 1, \frac{S_1(n)^2 + \sum_{i=1}^n \frac{1}{i^2}}{2n(n+1)} \right\} \right\}$$

Combine the solutions

In[6]:= FindLinearCombination[recSol, {1, {1/2}}, n, 2]

$$\text{Out[6]= } \frac{S_1(n)^2 + \sum_{i=1}^n \frac{1}{i^2}}{2n(n+1)}$$

Iterative application from inside to outside
transforms

definite sums



indefinite sums

$$\sum_{j=0}^{n-2} \sum_{r=0}^{j+1} \sum_{s=0}^{n-j+s-2} \frac{(-1)^{r+s} \binom{j+1}{r} \binom{-j+n+r-2}{s} (-j+n-2)! r!}{(n-s)(s+1)(-j+n+r)!}$$

$$\sum_{j=0}^{n-2} \sum_{r=0}^{j+1} \sum_{s=0}^{n-j+s-2} \frac{(-1)^{r+s} \binom{j+1}{r} \binom{-j+n+r-2}{s} (-j+n-2)! r!}{(n-s)(s+1)(-j+n+r)!}$$

||

$$\sum_{j=0}^{n-2} \sum_{r=0}^{j+1} \boxed{\sum_{s=0}^{n-j+s-2} \frac{(-1)^{r+s} \binom{j+1}{r} \binom{-j+n+r-2}{s} (-j+n-2)! r!}{(n-s)(s+1)(-j+n+r)!}}$$

$$\sum_{j=0}^{n-2} \sum_{r=0}^{j+1} \sum_{s=0}^{n-j+s-2} \frac{(-1)^{r+s} \binom{j+1}{r} \binom{-j+n+r-2}{s} (-j+n-2)! r!}{(n-s)(s+1)(-j+n+r)!}$$

||

$$\sum_{j=0}^{n-2} \sum_{r=0}^{j+1} \boxed{\sum_{s=0}^{n-j+s-2} \frac{(-1)^{r+s} \binom{j+1}{r} \binom{-j+n+r-2}{s} (-j+n-2)! r!}{(n-s)(s+1)(-j+n+r)!}}$$

||

$$\boxed{\binom{j+1}{r} \left(\frac{(-1)^r (-j+n-2)! r!}{(n+1)(-j+n+r-1)(-j+n+r)!} + \frac{(-1)^{n+r} (j+1)! (-j+n-2)! (-j+n-1)_r r!}{(n-1)n(n+1)(-j+n+r)! (-j-1)_r (2-n)_j} \right)}$$

$$\sum_{j=0}^{n-2} \sum_{r=0}^{j+1} \sum_{s=0}^{n-j+s-2} \frac{(-1)^{r+s} \binom{j+1}{r} \binom{-j+n+r-2}{s} (-j+n-2)! r!}{(n-s)(s+1)(-j+n+r)!}$$

$$\sum_{j=0}^{n-2} \left(\sum_{r=0}^{j+1} \binom{j+1}{r} \left(\frac{(-1)^r (-j+n-2)! r!}{(n+1)(-j+n+r-1)(-j+n+r)!} + \frac{(-1)^{n+r} (j+1)! (-j+n-2)! (-j+n-1)_r r!}{(n-1)n(n+1)(-j+n+r)! (-j-1)_r (2-n)_j} \right) \right)$$

$$\sum_{j=0}^{n-2} \sum_{r=0}^{j+1} \sum_{s=0}^{n-j+s-2} \frac{(-1)^{r+s} \binom{j+1}{r} \binom{-j+n+r-2}{s} (-j+n-2)! r!}{(n-s)(s+1)(-j+n+r)!}$$

$$\sum_{j=0}^{n-2} \left(\sum_{r=0}^{j+1} \binom{j+1}{r} \left(\frac{(-1)^r (-j+n-2)! r!}{(n+1)(-j+n+r-1)(-j+n+r)!} + \frac{(-1)^{n+r} (j+1)! (-j+n-2)! (-j+n-1)_r r!}{(n-1)n(n+1)(-j+n+r)! (-j-1)_r (2-n)_j} \right) \right)$$

||

$$\left(\frac{n^2 - n + 1}{(n-1)^2 n^2 (n+1) (2-n)_j} + \frac{\sum_{i=1}^j \frac{(2-n)_i}{(-i+n-1)^2 (i+1)!}}{(n+1) (2-n)_j} + \frac{(-1)^{j+n} (-j-2) (-j+n-2)!}{(j-n+1) (n+1)^2 n!} \right) (j+1)! - \frac{1}{(n+1)^2 (-j+n-1)}$$

$$\sum_{j=0}^{n-2} \sum_{r=0}^{j+1} \sum_{s=0}^{n-j+s-2} \frac{(-1)^{r+s} \binom{j+1}{r} \binom{-j+n+r-2}{s} (-j+n-2)! r!}{(n-s)(s+1)(-j+n+r)!}$$

||

$$\sum_{j=0}^{n-2} \left(\left(\frac{n^2 - n + 1}{(n-1)^2 n^2 (n+1)(2-n)_j} + \frac{\sum_{i=1}^j \frac{(2-n)_i}{(-i+n-1)^2 (i+1)!}}{(n+1)(2-n)_j} + \frac{(-1)^{j+n} (-j-2)(-j+n-2)!}{(j-n+1)(n+1)^2 n!} \right) (j+1)! - \frac{1}{(n+1)^2 (-j+n-1)} \right)$$

$$\sum_{j=0}^{n-2} \sum_{r=0}^{j+1} \sum_{s=0}^{n-j+s-2} \frac{(-1)^{r+s} \binom{j+1}{r} \binom{-j+n+r-2}{s} (-j+n-2)! r!}{(n-s)(s+1)(-j+n+r)!}$$

||

$$\sum_{j=0}^{n-2} \left(\left(\frac{n^2 - n + 1}{(n-1)^2 n^2 (n+1)(2-n)_j} + \frac{\sum_{i=1}^j \frac{(2-n)_i}{(-i+n-1)^2 (i+1)!}}{(n+1)(2-n)_j} + \frac{(-1)^{j+n} (-j-2)(-j+n-2)!}{(j-n+1)(n+1)^2 n!} \right) (j+1)! - \frac{1}{(n+1)^2 (-j+n-1)} \right)$$

||

$$\frac{-n^2 - n - 1}{n^2 (n+1)^3} + \frac{(-1)^n (n^2 + n + 1)}{n^2 (n+1)^3} - \frac{2S_{-2}(n)}{n+1} + \frac{S_1(n)}{(n+1)^2} + \frac{S_2(n)}{-n-1}$$

Note: $S_a(n) = \sum_{i=1}^n \frac{\text{sign}(a)^i}{i^{|a|}}$, $a \in \mathbb{Z} \setminus \{0\}$.

Toolbox 2: Special function algorithms

Computer algebra and special functions:

Harmonic sums (Vermaseren, Remiddi, Blümlein; Hoffman, Broadhurst, . . .)

$$S_{2,1}(n) = \sum_{i=1}^n \frac{1}{i^2} \sum_{j=1}^i \frac{1}{j}$$

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Harmonic sums (Vermaseren, Remiddi, Blümlein; Hoffman, Broadhurst, ...)

$$S_{2,1}(n) = \sum_{i=1}^n \frac{1}{i^2} \sum_{j=1}^i \frac{1}{j}$$

Integral representation:

$$= \int_0^1 \frac{x^n - 1}{1-x} \left(\int_0^x \frac{\int_0^y \frac{1}{1-z} dz}{y} dy - \zeta(2) \right) dx,$$

$$\zeta(z) := \sum_{i=1}^{\infty} 1/i^z$$

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Harmonic sums (Vermaseren, Remiddi, Blümlein; Hoffman, Broadhurst, ...)

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$$= \int_0^1 \frac{x^n - 1}{1-x} \left(\int_0^x \frac{\int_0^y \frac{1}{1-z} dz}{y} dy - \zeta(2) \right) dx, \quad \zeta(z) := \sum_{i=1}^{\infty} 1/i^z$$

Asymptotic expansion:

$$= \left(\frac{1}{30n^5} - \frac{1}{6n^3} + \frac{1}{2n^2} - \frac{1}{n} \right) \ln(n) - \frac{1}{100n^5} - \frac{1}{6n^4} + \frac{13}{36n^3} - \frac{1}{4n^2} - \frac{1}{n} + 2\zeta(3) + O\left(\frac{\ln(n)}{n^6}\right).$$

limit computations

numerical evaluation

Computer algebra and special functions:

Generalization to cyclotomic harmonic sums

$$\boxed{\sum_{k=1}^n \frac{(-1)^k}{2k+1}} =$$

Integral representation:

$$= -(-1)^n \int_0^1 \frac{x^{2n}}{x^2+1} dx + \frac{(-1)^n}{2n+1} - 1 + \frac{\pi}{4},$$

Asymptotic expansion:

$$= (-1)^n \left(-\frac{3}{64n^5} - \frac{1}{16n^4} + \frac{3}{16n^3} - \frac{1}{4n^2} + \frac{1}{4n} \right) + \frac{\pi}{4} - 1 + O\left(\frac{1}{n^6}\right)$$

limit computations

numerical evaluation

(J. Ablinger, J. Blümlein, CS; J. Math. Phys. 2011 [arXiv:1105.6063 [math-ph]])

$$\sum_{i=1}^n \frac{\sum_{j=1}^i \frac{1}{j^2}}{(2i+1)^2} = \text{asymptotic expansion?}$$

In[1]:= << HarmonicSums.m

HarmonicSums by Jakob Ablinger © RISC-Linz

In[2]:= **SExpansion[S[{{2, 1, 2}, {1, 0, 2}}, n], n, 10]**

$$\sum_{i=1}^n \frac{\sum_{j=1}^i \frac{1}{j^2}}{(2i+1)^2} = \text{asymptotic expansion?}$$

In[1]:= << HarmonicSums.m

HarmonicSums by Jakob Ablinger © RISC-Linz

In[2]:= SExpansion[S[{{2, 1, 2}, {1, 0, 2}}, n], n, 10]

$$\begin{aligned} \text{Out[2]} = & \left(-\frac{16\ln^2 2}{3} + \frac{3}{128n^{10}} - \frac{367}{5760n^9} + \frac{7}{96n^8} - \frac{221}{2016n^7} + \frac{5}{24n^6} - \frac{127}{360n^5} + \frac{1}{2n^4} - \frac{11}{18n^3} + \right. \\ & \left. \frac{2}{3n^2} - \frac{2}{3n} - \frac{1936}{15} \right) \frac{1}{4} (\pi - 4)^2 + \left(-\frac{32\ln^2 2}{3} + \frac{3}{64n^{10}} - \frac{367}{2880n^9} + \frac{7}{48n^8} - \frac{221}{1008n^7} + \right. \\ & \left. \frac{5}{12n^6} - \frac{127}{180n^5} + \frac{1}{n^4} - \frac{11}{9n^3} + \frac{4}{3n^2} - \frac{4}{3n} - \frac{3872}{45} \right) \frac{1}{4} (\pi - 4) - \frac{968}{45} \frac{1}{4} (\pi - 4)^4 - \\ & \frac{3872}{45} \frac{1}{4} (\pi - 4)^3 + 8\text{li4half} + \frac{\ln^2 4}{3} - \frac{16\ln^2 2}{3} + 7\ln 2 z^3 + \frac{125891}{1075200n^{10}} - \frac{10259}{80640n^9} + \\ & \frac{645120n^8}{92257} - \frac{5507}{20160n^7} + \frac{2837}{5760n^6} - \frac{509}{720n^5} + \frac{161}{192n^4} - \frac{31}{36n^3} + \frac{19}{24n^2} - \frac{2}{3n} - \frac{968}{45} \end{aligned}$$

More involved massive 3-loop diagrams

Emergence of new nested sums :

$$\sum_{i=1}^n \binom{2i}{i} (-2)^i \sum_{j=1}^i \frac{1}{j \binom{2j}{j}} S_{1,2} \left(\frac{1}{2}, -1; j \right)$$

More involved massive 3-loop diagrams

Emergence of new nested sums :

$$\begin{aligned} & \sum_{i=1}^n \binom{2i}{i} (-2)^i \sum_{j=1}^i \frac{1}{j \binom{2j}{j}} S_{1,2} \left(\frac{1}{2}, -1; j \right) \\ &= \int_0^1 dx \frac{x^n - 1}{x - 1} \sqrt{\frac{x}{8+x}} \left[H_{w_{17}, -1, 0}^*(x) - 2H_{w_{18}, -1, 0}^*(x) \right] \\ &+ \frac{\zeta_2}{2} \int_0^1 dx \frac{(-x)^n - 1}{x + 1} \sqrt{\frac{x}{8+x}} \left[H_{12}^*(x) - 2H_{13}^*(x) \right] + c_3 \int_0^1 dx \frac{(-8x)^n - 1}{x + \frac{1}{8}} \sqrt{\frac{x}{1-x}}, \end{aligned}$$

with the constant

$$c_3 = \sum_{j=0}^{\infty} S_{1,2} \left(\frac{1}{2}, -1; j \right) \frac{(j!)^2}{j(2j)! \pi}$$

and the letters

$$w_{12} = \frac{1}{\sqrt{x(8-x)}}, \quad w_{13} = \frac{1}{(2-x)\sqrt{x(8-x)}}, \quad w_{17} = \frac{1}{\sqrt{x(8+x)}}, \quad w_{18} = \frac{1}{(2+x)\sqrt{x(8+x)}}.$$

(J. Ablinger, J. Bümlein, J. Raab, C. Schneider 2013.)

For more details see: [Johannes Blümlein's talk \(Saturday, 16:10\)](#)

The full machinery:

In[1]:= << **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= << **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[3]:= << **EvaluateMultiSums.m**

EvaluateMultiSums by Carsten Schneider © RISC-Linz

In[4]:= **EvaluateMultiSum**[

$$\frac{e^{-\varepsilon\gamma}}{\Gamma(\varepsilon+1)} \left(\frac{\Gamma(k+1)\Gamma(\frac{\varepsilon}{2})\Gamma(1-\frac{\varepsilon}{2})\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+1+\frac{\varepsilon}{2})\Gamma(k+j+1+n)}{\Gamma(k+2+n)\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+2+n)\Gamma(k+j+2)} + \frac{\Gamma(k+1)\Gamma(-\frac{\varepsilon}{2})\Gamma(1+\frac{\varepsilon}{2})\Gamma(j+1+\varepsilon)\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(k+j+1+\frac{\varepsilon}{2}+n)}{\Gamma(k+2+n)\Gamma(j+1)\Gamma(j+2+\frac{\varepsilon}{2}+n)\Gamma(k+j+2+\frac{\varepsilon}{2})} \right)$$

{ {j, 0, ∞}, {k, 0, ∞} }, {n}, {1}, **ExpandIn** → {ε, 0, 2}

The full machinery:

In[1]:= << **Sigma.m**

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In[2]:= << **HarmonicSums.m**

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In[3]:= << **EvaluateMultiSums.m**

EvaluateMultiSums by Carsten Schneider © RISC-Linz

In[4]:= **EvaluateMultiSum**[

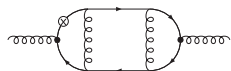
$$\frac{e^{-\varepsilon\gamma}}{\Gamma(\varepsilon+1)} \left(\frac{\Gamma(k+1)\Gamma(\frac{\varepsilon}{2})\Gamma(1-\frac{\varepsilon}{2})\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+1+\frac{\varepsilon}{2})\Gamma(k+j+1+n)}{\Gamma(k+2+n)\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+2+n)\Gamma(k+j+2)} + \frac{\Gamma(k+1)\Gamma(-\frac{\varepsilon}{2})\Gamma(1+\frac{\varepsilon}{2})\Gamma(j+1+\varepsilon)\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(k+j+1+\frac{\varepsilon}{2}+n)}{\Gamma(k+2+n)\Gamma(j+1)\Gamma(j+2+\frac{\varepsilon}{2}+n)\Gamma(k+j+2+\frac{\varepsilon}{2})} \right)$$

{ {j, 0, ∞}, {k, 0, ∞} }, {n}, {1}, ExpandIn → {ε, 0, 2}

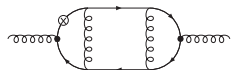
$$\text{Out[4]} = \left\{ \frac{S_1(n)^2 + 3S_2(n)}{2n(n+1)!}, \frac{-S_1(n)^3 - 3S_2(n)S_1(n) - 8S_3(n)}{6n(n+1)!}, \frac{1}{96n(n+1)} \left(S_1(n)^4 + (12\zeta_2 + 54S_2(n))S_1(n)^2 + 104S_3(n)S_1(n) - 48S_{2,1}(n)S_1(n) + 51S_2(n)^2 + 36\zeta_2 S_2(n) + 126S_4(n) - 48S_{3,1}(n) - 96S_{1,1,2}(n) \right) \right\}$$

Example 1: 3-Loop Ladder Graphs

Joint work with J. Ablinger (RISC), J. Blümlein (DESY),
A. Hasselhuhn (DESY), S. Klein (RWTH), F. Wissbrock (DESY)

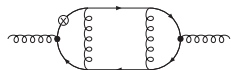


(massive 3-loop ladder graph with operator insertion)



J. Blümlein
A. Hasselhuhn
=

$$\frac{C_3}{(n+1)(n+2)} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{l=2}^{n+2} \binom{n+2}{l} \sum_{j=2}^l \binom{l}{j} \left\{ \sum_{k=1}^j \binom{j}{k} \sum_{r=0}^{l-k} \binom{l-k}{r} (-1)^{l+j+k+r} \frac{B(k, m+1+\frac{\epsilon}{2}) \Gamma(k+r+m+n+\frac{\epsilon}{2})}{\Gamma(m+1) \Gamma(n+1) \Gamma(k+r+\frac{\epsilon}{2})} \frac{B(r+l-1, n+1+\frac{\epsilon}{2})}{(n+3-j)} \frac{B(k+m-\frac{\epsilon}{2}, r+1+n-\frac{\epsilon}{2})}{(k+r+1+m+n-\epsilon)} \right. \\ \left. + \sum_{r=0}^{l-j} \binom{l-j}{r} (-1)^{l+j+r} \frac{B(r+l-1, n+1+\frac{\epsilon}{2}) \Gamma(j+r+m+n+\frac{\epsilon}{2})}{\Gamma(m+1) \Gamma(n+1) \Gamma(j+r+\frac{\epsilon}{2})} \frac{B(j, m+1+\frac{\epsilon}{2}) B(j+m-\frac{\epsilon}{2}, r+1+n-\frac{\epsilon}{2})}{(j+r+1+m+n-\epsilon)(n+3-j)} \right\}$$



J. Blümlein
A. Hasselhuhn
=

$$\frac{C_3}{(n+1)(n+2)} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{l=2}^{n+2} \binom{n+2}{l} \sum_{j=2}^l \binom{l}{j} \left\{$$

$$\sum_{k=1}^j \binom{j}{k} \sum_{r=0}^{l-k} \binom{l-k}{r} (-1)^{l+j+k+r} \frac{B(k, m+1+\frac{\epsilon}{2}) \Gamma(k+r+m+n+\frac{\epsilon}{2})}{\Gamma(m+1) \Gamma(n+1) \Gamma(k+r+\frac{\epsilon}{2})} \frac{B(r+l-1, n+1+\frac{\epsilon}{2})}{(n+3-j)} \frac{B(k+m-\frac{\epsilon}{2}, r+1+n-\frac{\epsilon}{2})}{(k+r+1+m+n-\epsilon)}$$

$$+ \sum_{r=0}^{l-j} \binom{l-j}{r} (-1)^{l+j+r} \frac{B(r+l-1, n+1+\frac{\epsilon}{2}) \Gamma(j+r+m+n+\frac{\epsilon}{2})}{\Gamma(m+1) \Gamma(n+1) \Gamma(j+r+\frac{\epsilon}{2})} \frac{B(j, m+1+\frac{\epsilon}{2}) B(j+m-\frac{\epsilon}{2}, r+1+n-\frac{\epsilon}{2})}{(j+r+1+m+n-\epsilon)(n+3-j)} \left. \right\}$$

|| EvaluateMultiSums

$$\frac{C_3}{(n+1)(n+2)(n+3)} \left\{ \frac{1}{6} S_1^3(n) + \frac{n^2+12n+16}{2(n+1)(n+2)} S_1(n)^2 + \frac{4(2n+3)}{(n+1)^2(n+2)} S_1(n) \right.$$

$$+ 2 \left[-2^{n+3} + 3 - (-1)^n \right] \zeta_3 + \left[\frac{3n^2+40n+56}{2(n+1)(n+2)} - \frac{1}{2} S_1(n) \right] S_2(n)$$

$$- (-1)^n S_{-3}(n) + \frac{8(2n+3)}{(n+1)^3(n+2)} - \frac{3n+17}{3} S_3(n) - 2(-1)^n S_{-2,1}(n) - (n+3) S_{2,1}(n)$$

$$+ 2^{n+4} S_{1,2} \left(\frac{1}{2}, 1; n \right) + 2^{n+3} \left. \left[S_{1,1,1} \left(\frac{1}{2}, 1, 1; n \right) \right] \right\} + O(\epsilon)$$

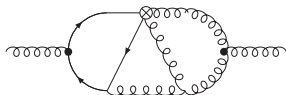
$$\boxed{S_{1,1,1} \left(\frac{1}{2}, 1, 1; n \right)} = \sum_{i=1}^n \frac{\left(\frac{1}{2}\right)^i \sum_{j=1}^i \frac{\sum_{k=1}^j \frac{1}{k}}{j}}{i}$$

$$\boxed{S_{1,1,1} \left(\frac{1}{2}, 1, 1; n \right)} = \sum_{i=1}^n \frac{\left(\frac{1}{2}\right)^i \sum_{j=1}^i \frac{\sum_{k=1}^j \frac{1}{k}}{j}}{i}$$

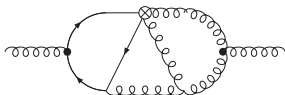
|| asymptotic expansion (HarmonicSums package)

$$\begin{aligned} & 2^{-n} \left(+\frac{541}{n^6} - \frac{75}{n^5} + \frac{13}{n^4} - \frac{3}{n^3} + \frac{1}{n^2} - \frac{1}{2n} \right) (\ln(n) + \gamma)^2 \\ & + 2^{-n-3} \left(-\frac{114686}{5n^6} + \frac{44099}{15n^5} - \frac{1372}{3n^4} + \frac{266}{3n^3} - \frac{20}{n^2} \right) (\ln(n) + \gamma) \\ & + 2^{-n} \left(+\frac{541}{n^6} - \frac{75}{n^5} + \frac{13}{n^4} - \frac{3}{n^3} + \frac{1}{n^2} - \frac{1}{2n} \right) \zeta(2) + \frac{3\zeta(3)}{4} \\ & + 2^{-n-9} \left(\frac{69280576}{45n^6} - \frac{1582096}{9n^5} + \frac{69184}{3n^4} - \frac{3264}{n^3} + \frac{256}{n^2} \right) + O\left(\frac{1}{2^n n^7}\right) \end{aligned}$$

(J. Ablinger, J. Blümlein, CS; r arXiv:1302.0378 [math-ph])



$$= F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + F_{-1}(n)\varepsilon^{-1} + \boxed{F_0(n)}$$



$$= F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + F_{-1}(n)\varepsilon^{-1} + \boxed{F_0(n)}$$

Simplify

$$\sum_{j=0}^{n-3} \sum_{k=0}^j \sum_{l=0}^k \sum_{q=0}^{-j+n-3} \sum_{s=1}^{-l+n-q-3} \sum_{r=0}^{-l+n-q-s-3} (-1)^{-j+k-l+n-q-3} \times$$

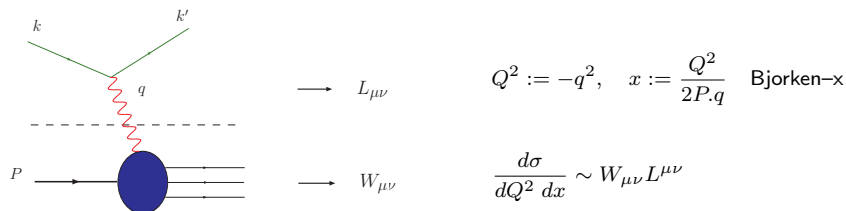
$$\times \frac{\binom{j+1}{k+1} \binom{k}{l} \binom{n-1}{j+2} \binom{-j+n-3}{q} \binom{-l+n-q-3}{s} \binom{-l+n-q-s-3}{r} r! (-l+n-q-r-s-3)! (s-1)!}{(-l+n-q-2)! (-j+n-1) (n-q-r-s-2) (q+s+1)}$$

$$\left[\begin{aligned} &4S_1(-j+n-1) - 4S_1(-j+n-2) - 2S_1(k) \\ &- (S_1(-l+n-q-2) + S_1(-l+n-q-r-s-3) - 2S_1(r+s)) \\ &+ 2S_1(s-1) - 2S_1(r+s) \end{aligned} \right] + \mathbf{3 \text{ further 6-fold sums}}$$

$$\begin{aligned}
\boxed{F_0(n)} = & \\
& \frac{7}{12} S_1(n)^4 + \frac{(17n+5)S_1(n)^3}{3n(n+1)} + \left(\frac{35n^2-2n-5}{2n^2(n+1)^2} + \frac{13S_2(n)}{2} + \frac{5(-1)^n}{2n^2} \right) S_1(n)^2 \\
& + \left(-\frac{4(13n+5)}{n^2(n+1)^2} + \left(\frac{4(-1)^n(2n+1)}{n(n+1)} - \frac{13}{n} \right) S_2(n) + \left(\frac{29}{3} - (-1)^n \right) S_3(n) \right. \\
& + \left. (2+2(-1)^n)S_{2,1}(n) - 28S_{-2,1}(n) + \frac{20(-1)^n}{n^2(n+1)} \right) S_1(n) + \left(\frac{3}{4} + (-1)^n \right) S_2(n)^2 \\
& - 2(-1)^n S_{-2}(n)^2 + S_{-3}(n) \left(\frac{2(3n-5)}{n(n+1)} + (26+4(-1)^n)S_1(n) + \frac{4(-1)^n}{n+1} \right) \\
& + \left(\frac{(-1)^n(5-3n)}{2n^2(n+1)} - \frac{5}{2n^2} \right) S_2(n) + S_{-2}(n) \left(10S_1(n)^2 + \frac{8(-1)^n(2n+1)}{n(n+1)} \right. \\
& + \left. \frac{4(3n-1)}{n(n+1)} \right) S_1(n) + \frac{8(-1)^n(3n+1)}{n(n+1)^2} + \left(-22+6(-1)^n \right) S_2(n) - \frac{16}{n(n+1)} \\
& + \left(\frac{(-1)^n(9n+5)}{n(n+1)} - \frac{29}{3n} \right) S_3(n) + \left(\frac{19}{2} - 2(-1)^n \right) S_4(n) + \left(-6+5(-1)^n \right) S_{-4}(n) \\
& + \left(-\frac{2(-1)^n(9n+5)}{n(n+1)} - \frac{2}{n} \right) S_{2,1}(n) + (20+2(-1)^n)S_{2,-2}(n) + (-17+13(-1)^n)S_{3,1}(n) \\
& - \frac{8(-1)^n(2n+1)+4(9n+1)}{n(n+1)} S_{-2,1}(n) - (24+4(-1)^n)S_{-3,1}(n) + (3-5(-1)^n)S_{2,1,1}(n) \\
& + 32S_{-2,1,1}(n) + \left(\frac{3}{2} S_1(n)^2 - \frac{3S_1(n)}{n} + \frac{3}{2} (-1)^n S_{-2}(n) \right) \zeta(2)
\end{aligned}$$

Example 2: Heavy Flavor Wilson Coefficients

Unpolarized Deep-Inelastic Scattering (DIS):



$$W_{\mu\nu}(q, P, s) = \frac{1}{2x} \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) F_L(x, Q^2) + \frac{2x}{Q^2} \left(P_\mu P_\nu + \frac{q_\mu P_\nu + q_\nu P_\mu}{2x} - \frac{Q^2}{4x^2} g_{\mu\nu} \right) F_2(x, Q^2).$$

Structure Functions: $F_{2,L}$ contain light and heavy quark contributions.

$$F_i(x, Q^2) = \sum_k C_{i,k} \left(\frac{Q^2}{\mu^2}, x \right) \otimes f_k \left(\frac{\mu^2}{\mu_0^2}, x \right)$$

$$C_{i,k} \left(\frac{Q^2}{\mu^2}, x \right) = C_{i,k}^{\text{light}} \left(\frac{Q^2}{\mu^2}, x \right) + C_{i,k}^{\text{heavy}} \left(\frac{Q^2}{\mu^2}, x \right)$$

Heavy Flavor Wilson Coefficients

Present team:

J. Ablinger^a, J. Blümlein^b, A. De Freitas^b, A. Hasselhuhn^{a,b}, A. von Manteuffel^c, C. Raab^b, C. Schneider^a, M. Round^a, F. Wißbrock^b

^a RISC, J. Kepler University, Linz, Austria

^b DESY, Zeuthen, Germany

^c Gutenberg-University, Mainz, Germany

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There are **eight** massive Wilson coefficients in the unpolarized case.

- ▶ The OMEs $A_{qq,Q}^{\text{PS}}$, $A_{qg,Q}$ were calculated.

J. Ablinger, J. Blümlein, S. Klein, C. Schneider, F. Wissbrock, 2010.

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- ▶ Essential technologies to calculate the further two Wilson coefficients have been developed.

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production by extension of QGRAF (P. Nogueira)

apply color algebra by Color (T.v. Ritbergen, A.N. Schellekens, J.A.M. Vermaseren)



diagrams with local operator insertions
for the respective Wilson coefficient

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apply all integration-by-parts relations, including those affecting
the operators; extension of REDUCER_2 (A.v. Manteuffel)

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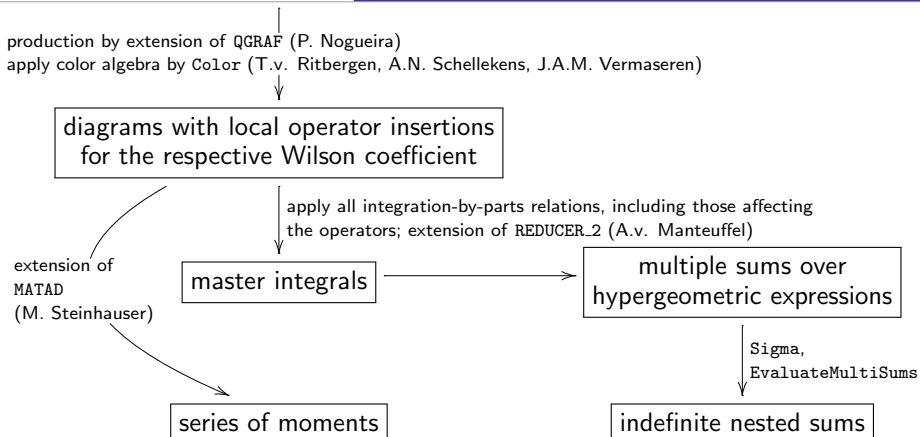
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Perform the renormalization (Mass, Charge, Operators, Factorization)
Assemble OMEs and Wilson coefficients in the \overline{MS} -scheme

production by extension of QGRAF (P. Nogueira)

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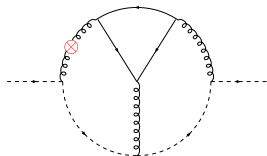
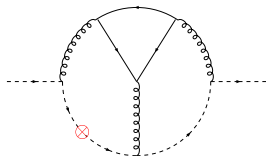
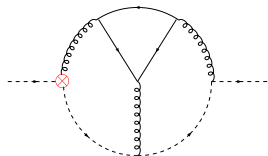
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Calculation of Benz-Diagrams

 $D_1(n)$

 $A_{gq,Q}$
 $D_2(n)$

 $A_{qq,Q}^{NS}$
 $D_3(n)$

 $A_{qq,Q}^{NS}$

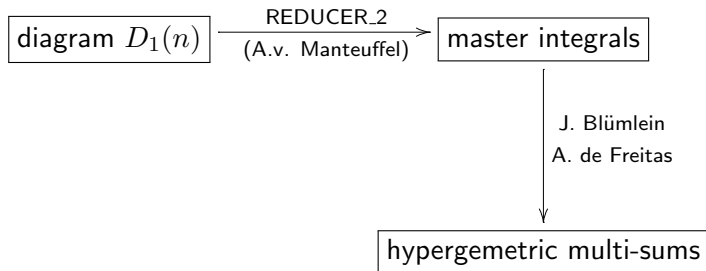
Only one scalar diagram needs to be calculated to obtain the two others :

$$D_2(n) = \sum_{k=0}^n \binom{n}{k} (-1)^k D_1(k), \quad \text{conjugation}$$

$$D_3(n) = \sum_{k=0}^n D_2(k)$$

[Use summation techniques.]

Symbolic summation



$$\sum_{j=0}^n \sum_{k=0}^{\infty} \sum_{i=0}^k (-1)^{i+1} e^{-\frac{3\varepsilon\gamma}{2}} \times$$

$$\times \frac{n! \Gamma\left(1 - \frac{\varepsilon}{2}\right) \Gamma\left(\frac{\varepsilon}{2}\right) \Gamma\left(i - \frac{\varepsilon}{2}\right) \Gamma(-\varepsilon + i + j + 1) \Gamma\left(k - \frac{3\varepsilon}{2}\right) \Gamma\left(\frac{\varepsilon}{2} + j + k + 1\right) \Gamma(n+2) \Gamma(-\varepsilon - j + n + 1)}{i! j! (k-i)! (n-j)! \Gamma\left(-\frac{\varepsilon}{2} + i + 1\right) \Gamma\left(-\frac{3\varepsilon}{2} + i + j + 2\right) \Gamma\left(\frac{\varepsilon}{2} + n + 2\right) \Gamma\left(-\frac{\varepsilon}{2} + k + n + 2\right)}$$

$$\sum_{j=0}^n \sum_{k=0}^{\infty} \sum_{i=0}^k (-1)^{i+1} e^{-\frac{3\varepsilon\gamma}{2}} \times$$

$$\times \frac{n! \Gamma\left(1 - \frac{\varepsilon}{2}\right) \Gamma\left(\frac{\varepsilon}{2}\right) \Gamma\left(i - \frac{\varepsilon}{2}\right) \Gamma(-\varepsilon + i + j + 1) \Gamma\left(k - \frac{3\varepsilon}{2}\right) \Gamma\left(\frac{\varepsilon}{2} + j + k + 1\right) \Gamma(n+2) \Gamma(-\varepsilon - j + n + 1)}{i! j! (k-i)! (n-j)! \Gamma\left(-\frac{\varepsilon}{2} + i + 1\right) \Gamma\left(-\frac{3\varepsilon}{2} + i + j + 2\right) \Gamma\left(\frac{\varepsilon}{2} + n + 2\right) \Gamma\left(-\frac{\varepsilon}{2} + k + n + 2\right)}$$

||Sigma, EvaluateMultiSums, HarmonicSums

$$-\frac{8}{3} \left[\frac{S_1(n)}{n+1} + \frac{1}{(n+1)^2} \right] \varepsilon^{-3}$$

$$+ \left[\frac{4S_1(n)^2}{3(n+1)} + \frac{8S_1(n)}{3(n+1)^2} - \frac{4S_2(n)}{3(n+1)} \right] \varepsilon^{-2}$$

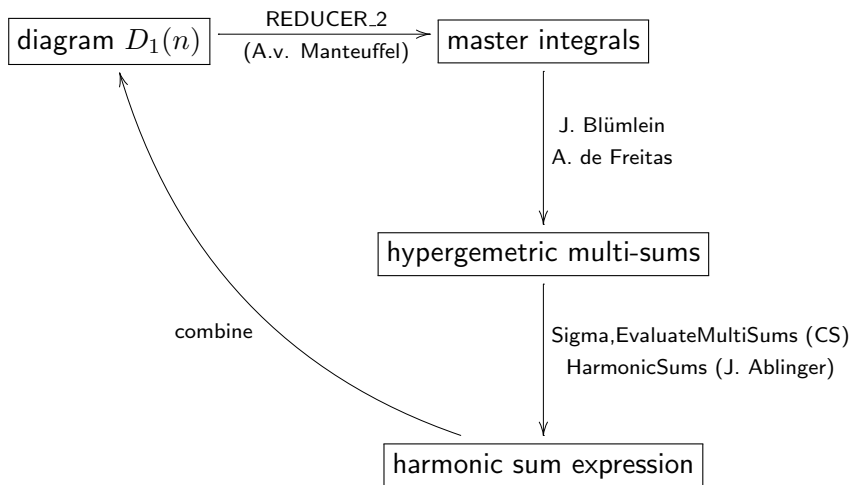
$$+ \left[\frac{4S_1(n)^3}{9(n+1)} - \frac{4S_{2,1}(n)}{3(n+1)} - \frac{4S_1(n)^2}{3(n+1)^2} - \frac{8S_1(n)}{3(n+1)^3} - \frac{8S_3(n)}{9(n+1)} - \frac{8}{3(n+1)^4} - \zeta_2 \frac{(n+1)S_1(n)+1}{(n+1)^2} \right] \varepsilon^{-1}$$

$$+ \left[S_1(n) \left(\frac{2S_{2,1}(n)}{n+1} + \frac{2S_3(n)}{9(n+1)} + \frac{8}{3(n+1)^4} \right) + \frac{2S_{2,1}(n)}{(n+1)^2} + \frac{2S_{3,1}(n)}{3(n+1)} - \frac{8S_{2,1,1}(n)}{3(n+1)} \right.$$

$$\left. + \zeta_2 \left(\frac{S_1(n)^2}{2(n+1)} + \frac{S_1(n)}{(n+1)^2} - \frac{S_2(n)}{2(n+1)} \right) + \zeta_3 \left(-\frac{5S_1(n)}{3(n+1)} - \frac{5}{3(n+1)^2} \right) \right.$$

$$\left. + \frac{S_1(n)^4}{9(n+1)} + \frac{4S_1(n)^3}{9(n+1)^2} + \frac{4S_1(n)^2}{3(n+1)^3} - \frac{4S_2(n)}{3(n+1)^3} + \frac{2S_3(n)}{9(n+1)^2} - \frac{S_4(n)}{3(n+1)} \right] \varepsilon^0 + O(\varepsilon)$$

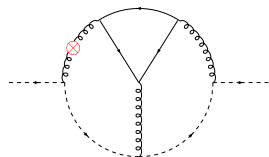
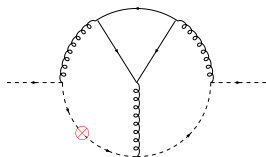
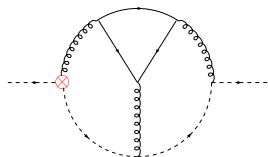
Symbolic summation



Constant term of

$$\begin{aligned}
 D_1(n) = & \left[\frac{S_2(n)}{96(n+1)} - \frac{52n^2 + 32n + 7}{432(n+1)^3} \right] S_1(n) - \zeta_2 \left[\frac{3n+4}{32(n+1)^2} - \frac{S_1(n)}{32(n+1)} \right] \\
 & - \frac{7S_1^3(n)}{288(n+1)} + \frac{(10n+3)S_1(n)^2}{96(n+1)^2} + \frac{(10n+11)S_2(n)}{96(n+1)^2} + \frac{5S_3(n)}{144(n+1)} \\
 & - \frac{204n^3 + 592n^2 + 527n + 130}{432(n+1)^4}
 \end{aligned}$$

Calculation of Benz-Diagrams

 $D_1(n)$

 $A_{gq,Q}$
 $D_2(n)$

 $A_{qq,Q}^{NS}$
 $D_3(n)$

 $A_{qq,Q}^{NS}$

Only one scalar diagram needs to be calculated to obtain the two others :

$$D_2(n) = \sum_{k=0}^n \binom{n}{k} (-1)^k D_1(k), \quad \text{conjugation}$$

$$D_3(n) = \sum_{k=0}^n D_2(k)$$

[Use summation techniques.]

Constant term of

$$D_1(n) = \left[\frac{S_2(n)}{96(n+1)} - \frac{52n^2 + 32n + 7}{432(n+1)^3} \right] S_1(n) - \zeta_2 \left[\frac{3n+4}{32(n+1)^2} - \frac{S_1(n)}{32(n+1)} \right] \\ - \frac{7S_1^3(n)}{288(n+1)} + \frac{(10n+3)S_1(n)^2}{96(n+1)^2} + \frac{(10n+11)S_2(n)}{96(n+1)^2} + \frac{5S_3(n)}{144(n+1)} \\ - \frac{204n^3 + 592n^2 + 527n + 130}{432(n+1)^4}$$

$$D_2(n) = \frac{(8n^2 + 13n + 9) S_1(n)}{48(n+1)^3} - \frac{S_1(n)^2}{32(n+1)} + \frac{(1-7n)S_2(n)}{96(n+1)^2} - \frac{(3n+4)\zeta_2}{32(n+1)^2} \\ - \frac{204n^3 + 592n^2 + 527n + 130}{432(n+1)^4}$$

$$D_3(n) = - \frac{32n^2 + 127n + 23}{864(n+1)^2} S_2(n) - \left[\frac{68n^3 + 180n^2 + 165n + 41}{144(n+1)^3} - \frac{7}{96} S_2(n) \right] S_1(n) \\ + \frac{1}{24} S_2(n)^2 - \frac{1}{32} \zeta_2 \left(3S_1(n) + S_2(n) - \frac{3n+4}{(n+1)^2} \right) - \frac{1}{96} S_1(n)^3 + \frac{(8n+5)}{96(n+1)} S_1(n)^2 \\ + \frac{7}{48} S_3(n) - \frac{5}{48} S_4(n) + \frac{1}{24} S_{2,1}(n) + \frac{1}{12} S_{3,1}(n) - \frac{204n^3 + 592n^2 + 527n + 130}{432(n+1)^4}$$

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- ▶ The (ongoing) development of computer algebra algorithms for special functions is crucial.
- ▶ Massive Wilson coefficients for DIS will be calculated with the present techniques very soon.
- ▶ Multi-leg calculations (e.g., for two loop diagrams) are in preparation (cooperation with J. Blümlein, J. Gluza, T. Riemann).