DESY, Zeuthen, Germany
Computer Algebra and Particle Physics - CAPP 2013

Symbolic summation and the Sigma package II

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Summation Paradigms:

Telescoping, creative telescoping, recurrence finding

Recall: Simplify

$$\sum_{k=1}^{n} S_1(k)$$

where $S_1(k) = \sum_{i=1}^{k} \frac{1}{i}$

Telescoping

GIVEN $f(k) = S_1(k)$.

FIND $g(k)$:

$$\boxed{f(k) = g(k+1) - g(k)}$$

for all $1 \leq k \leq n$ and $n \geq 0$.

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for all $1 \leq k \leq n$ and $n \geq 0$.

We compute

$$g(k) = (S_1(k) - 1)k.$$

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FIND $g(k)$:

$$\boxed{f(k) = g(k+1) - g(k)}$$

for all $1 \leq k \leq n$ and $n \geq 0$.

Summing this equation over k from 1 to n gives

$$\boxed{\sum_{k=1}^{n} S_1(k)} = \boxed{g(n+1) - g(1)}$$

$$= (S_1(n+1) - 1)(n+1).$$

Telescoping

FIND a closed form for

$$\sum_{k=1}^{n} S_1(k).$$

A $\Pi\Sigma^*$-field for the summand

$$\boxed{\text{const}_\sigma \mathbb{F} = \mathbb{Q}}$$

Consider the rational function field

$$\mathbb{F} := \mathbb{Q}$$

with the automorphism $\sigma : \mathbb{F} \to \mathbb{F}$ defined by

$$\sigma(c) = c \quad \forall c \in \mathbb{Q},$$

Telescoping

FIND a closed form for

$$\sum_{k=1}^{n} S_1(k).$$

A $\Pi\Sigma^*$-field for the summand

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Consider the rational function field

$$\mathbb{F} := \mathbb{Q}(k)$$

with the automorphism $\sigma : \mathbb{F} \to \mathbb{F}$ defined by

$$\sigma(c) = c \quad \forall c \in \mathbb{Q},$$

$$\sigma(k) = k + 1, \qquad\qquad \mathcal{S}\,k = k + 1,$$

Telescoping

FIND a closed form for

$$\sum_{k=1}^{n} S_1(k).$$

A $\Pi\Sigma^*$-field for the summand $\boxed{\text{const}_\sigma \mathbb{F} = \mathbb{Q}}$

Consider the rational function field

$$\mathbb{F} := \mathbb{Q}(k)(h)$$

with the automorphism $\sigma : \mathbb{F} \to \mathbb{F}$ defined by

$$\begin{aligned} \sigma(c) &= c \quad \forall c \in \mathbb{Q}, & & \\ \sigma(k) &= k+1, & \mathcal{S}\, k &= k+1, \\ \sigma(h) &= h + \frac{1}{k+1}, & \mathcal{S}\, S_1(k) &= S_1(k) + \frac{1}{k+1}. \end{aligned}$$

Telescoping in the given difference field

FIND $g \in \mathbb{F}$:

$$\boxed{\sigma(g) - g = h.}$$

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This gives

$$\boxed{g(k+1) - g(k) = S_1(k)}$$

with

$$g(k) = (S_1(k) - 1)k.$$

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with

$$g(k) = (S_1(k) - 1)k.$$

Hence,

$$(S_1(n+1) - 1)(n+1) = \sum_{k=1}^{n} S_1(k).$$

Example

Simplify

$$\sum_{k=1}^{n}\binom{n}{k}S_1(k)$$

where $S_1(k)=\sum_{i=1}^{k}\frac{1}{i}$

Simplify

$$A(n) := \sum_{k=1}^{n} \binom{n}{k} S_1(k).$$

A difference field for the summand

Consider the rational function field

$$\mathbb{F} := \mathbb{Q}(n)(k)(h)(b)$$

with the automorphism $\sigma : \mathbb{F} \to \mathbb{F}$ defined by

$$\sigma(c) = c \quad \forall c \in \mathbb{Q}(n),$$

$$\sigma(k) = k + 1, \qquad \mathcal{S}\, k = k + 1,$$

$$\sigma(h) = h + \frac{1}{k+1}, \qquad \mathcal{S}\, S_1(k) = S_1(k) + \frac{1}{k+1},$$

$$\sigma(b) = \frac{n-k}{k+1}\, b, \qquad \mathcal{S} \binom{n}{k} = \frac{n-k}{k+1} \binom{n}{k}.$$

Creative telescoping

REPRESENT $f(n,k)$ in $\mathbb{F}$:

$$f(n,k) = S_1(k)\binom{n}{k}$$

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REPRESENT $f(n,k)$ in $\mathbb{F}$:

$$f(n,k) = S_1(k)\binom{n}{k} \longleftrightarrow h\,b =: f_0$$

FIND $g \in \mathbb{F}$:

$$\boxed{\sigma(g) - g = f_0}$$

☹

Creative telescoping

REPRESENT $f(n+i,k)$ in $\mathbb{F}$:

$$f(n,k) = S_1(k)\binom{n}{k} \longleftrightarrow h\,b =: f_0$$

$$f(n+1,k) = \frac{(n+1)\,S_1(k)\,\binom{n}{k}}{n+1-k}$$

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FIND $g\in\mathbb{F}$ and $c_0,c_1\in\mathbb{Q}(n)$:

$$\boxed{\sigma(g)-g=c_0\,f_0+c_1\,f_1}$$

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$$f(n+2,k)=\frac{(n+1)\,(n+2)\,S_1(k)\,\binom{n}{k}}{(n+1-k)\,(n+2-k)} \longleftrightarrow \frac{(n+1)\,(n+2)\,h\,b}{(n+1-k)\,(n+2-k)} =: f_2.$$

FIND $g\in\mathbb{F}$ and $c_0,c_1,c_2\in\mathbb{Q}(n)$:

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$$\boxed{\sigma(g)-g=c_0f_0+c_1f_1+c_2f_2}$$

We compute

$$c_0:=4\,(1+n),\quad c_1:=-2\,(3+2\,n),\quad c_2:=2+n,$$

$$g:=\frac{(1+n)\,(-2+k-n+(2\,k-2\,k^2+k\,n)h)\,b)}{(1-k+n)\,(2-k+n)}.$$

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This gives

$$\boxed{g(n,k+1)-g(n,k)} = \boxed{c_0(n)f(n,k)+c_1(n)f(n+1,k)+c_2(n)f(n+2,k)}$$

with

$$c_0(n) := 4\,(1+n), \quad c_1(n) := -2\,(3+2\,n), \quad c_2(n) := 2+n,$$

$$g(n,k) := \frac{(1+n)\,(-2+k-n+(2\,k-2\,k^2+k\,n)S_1(k))\binom{n}{k})}{(1-k+n)\,(2-k+n)}.$$

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Summing over k from 0 to n gives

$$\boxed{g(n,n+1)-g(n,0)} = \boxed{\begin{array}{l} c_0(n)\,\mathsf{A}(n)+\\ c_1(n)\,\big[\mathsf{A}(n+1)-f(n+1,n+1)\big]\\ c_2(n)\,\big[\mathsf{A}(n+2)-f(n+2,n+1)-f(n+2,n+2)\big].\end{array}}$$

for $A(n)=\sum_{k=0}^{n}\binom{n}{k}S_1(k)$

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Summing over k from 0 to n gives

$$\boxed{1 = 4(1+n)A(n) - 2(3+2n)A(n+1) + (2+n)A(n+2)}$$

for $A(n) = \sum_{k=0}^{n}\binom{n}{k}S_1(k)$ Example

Strategies to find recurrences

- Search with the objects given by the input sum

In[1]:= $$\mathbf{GenerateRecurrence}[\sum_{k=0}^{n}\binom{n}{k}S_1(k), n, \mathbf{SimplifyByExt} \to \mathbf{None}]$$

Out[1]= $$\{-4(n+1)\mathsf{SUM}[n]+2(2n+3)\mathsf{SUM}[n+1]+(-n-2)\mathsf{SUM}[n+2]==-1\}$$

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- Allow sums with lower depth (default)

In[2]:= $\mathbf{GenerateRecurrence}[\sum_{k=0}^{n}\binom{n}{k}S_1(k), n, \mathbf{SimplifyByExt} \to \mathbf{MinDepth}]$

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- Allow sums with less objects

In[3]:= $\mathbf{GenerateRecurrence}[\sum_{k=0}^{n}\binom{n}{k}S_1(k), n,$
$\mathbf{SimplifyByExt} \to \mathbf{DepthNumber}]$

Out[3]= $\{2\mathsf{SUM}[n]-\mathsf{SUM}[n+1]==-\sum_{i_1=0}^{n}\frac{1}{1+n-i_1}\binom{n}{i_1}\}$

Example

Summation paradigms

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$A(n)=\sum_{k=0}^{n} f(n,k);$$

$f(n,k)$: nested product-sum expr. w.r.t. k;
n: extra parameter

FIND a recurrence for $A(n)$

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2. Recurrence solving

GIVEN a recurrence

$a_0(n),\dots,a_d(n),h(n)$:
nested product-sum expressions.

$$a_d(n)A(n+d)+\dots+a_0(n)A(n)=h(n);$$

FIND all solutions expressible by indefinite nested products and sums
(Abramov/Bronstein/Petkovšek/CS, in preparation)

Recurrence solving

Special case: homogeneous recurrences with $a_i(n) \in \mathbb{K}[n]$

$$a_d(n)A(n+d) + a_{d-1}(n)A(n+d-1) + \cdots + a_0(n)A(n) = 0$$

Recurrence solving

Special case: homogeneous recurrences with $a_i(n) \in \mathbb{K}[n]$

$$a_d(n)A(n+d) + a_{d-1}(n)A(n+d-1) + \cdots + a_0(n)A(n) = 0$$

$$||$$

$$\Big[a_d(n)S^d + a_{d-1}(n)S^{d-1} + \cdots + a_0(n)I\Big]A(n)$$

Recurrence solving

Special case: homogeneous recurrences with $a_i(n) \in \mathbb{K}[n]$

$$\begin{tikzcd}
a_d(n)A(n+d) + a_{d-1}(n)A(n+d-1) + \cdots + a_0(n)A(n) = 0 \arrow[d, equal] \\
\Big[a_d(n)S^d + a_{d-1}(n)S^{d-1} + \cdots + a_0(n)I\Big]A(n) \arrow[d, bend right, "\text{Hyper}"'] \\
\prod_{j=\lambda}^{n} b_1(j-1)
\end{tikzcd}$$

Recurrence solving

Special case: homogeneous recurrences with $a_i(n) \in \mathbb{K}[n]$

$$a_d(n)A(n+d) + a_{d-1}(n)A(n+d-1) + \dots + a_0(n)A(n) = 0$$

$$\|$$

$$\Big[\Big(\tilde{a}_{d-1}(n)S^{d-1} + \tilde{a}_{d-2}(n)S^{d-2} + \dots + \tilde{a}_0(n)I\Big)\Big(S - b_1(n)\Big)\Big]A(n)$$

$$\prod_{j=\lambda}^{n} b_1(j-1)$$

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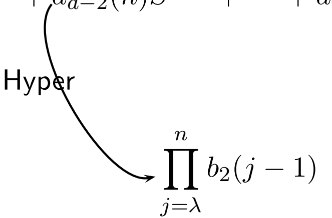

$$\prod_{j=\lambda}^{n} b_2(j-1)$$

Recurrence solving

Special case: homogeneous recurrences with $a_i(n) \in \mathbb{K}[n]$

$$a_d(n)A(n+d) + a_{d-1}(n)A(n+d-1) + \dots + a_0(n)A(n) = 0$$

$$\|$$

$$\Big[\Big(\tilde{a}_{d-2}(n)S^{d-2}+\tilde{a}_{d-3}(n)S^{d-3}+\dots+\tilde{a}_0(n)I\Big)\Big(S-b_2(n)\Big)\Big(S-b_1(n)\Big)\Big]A(n)$$

$$\prod_{j=\lambda}^{n} b_2(j-1)$$

Recurrence solving

Special case: homogeneous recurrences with $a_i(n) \in \mathbb{K}[n]$

$$a_d(n)A(n+d) + a_{d-1}(n)A(n+d-1) + \dots + a_0(n)A(n) = 0$$
$$\Vert$$
$$c(n)\Big(S - b_d(n)\Big)\dots\Big(S - b_2(n)\Big)\Big(S - b_1(n)\Big)A(n)$$

Recurrence solving

Special case: homogeneous recurrences with $a_i(n)\in\mathbb{K}[n]$

$$\begin{gathered}
a_d(n)A(n+d)+a_{d-1}(n)A(n+d-1)+\dots+a_0(n)A(n)=0\\
\Vert\\
c(n)\Big(S-b_d(n)\Big)\dots\Big(S-b_2(n)\Big)\Big(S-b_1(n)\Big)A(n)
\end{gathered}$$

$$L_1(n)=\prod_{j=\lambda}^{n}b_1(j-1)$$

Recurrence solving

Special case: homogeneous recurrences with $a_i(n) \in \mathbb{K}[n]$

$$a_d(n)A(n+d) + a_{d-1}(n)A(n+d-1) + \dots + a_0(n)A(n) = 0$$

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$$c(n)\Big(S - b_d(n)\Big)\dots\Big(S - b_2(n)\Big)\Big(S - b_1(n)\Big)A(n)$$

$$L_1(n) = \prod_{j=\lambda}^{n} b_1(j-1)$$

$$L_2(n) = \prod_{j=\lambda}^{n} b_1(j-1) \sum_{i_1=\lambda}^{n-1} \frac{\prod_{j=\lambda}^{i_1} b_2(j-1)}{\prod_{j=\lambda}^{i1+1} b_1(j-1)}$$

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$$L_1(n) = \prod_{j=\lambda}^{n} b_1(j-1)$$

$$\boxed{d \text{ linearly independent solutions}}$$

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$$\vdots$$

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Recurrence solving

Special case: homogeneous recurrences with $a_i(n)\in\mathbb{K}[n]$

$$a_d(n)A(n+d)+a_{d-1}(n)A(n+d-1)+\dots+a_0(n)A(n)=0$$
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$$c(n)\Big(S-b_d(n)\Big)\dots\Big(S-b_2(n)\Big)\Big(S-b_1(n)\Big)A(n)$$

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Example

$$\vdots$$

$$L_d(n)=\prod_{j=\lambda}^{n}b_1(j-1)\sum_{i_1=\lambda}^{n-1}\frac{\prod_{j=\lambda}^{i_1}b_2(j-1)}{\prod_{j=\lambda}^{i_1+1}b_1(j-1)}\dots\sum_{i_{d-1}=\lambda}^{i_{d-2}-1}\frac{\prod_{j=\lambda}^{i_{d-1}}b_d(j-1)}{\prod_{j=\lambda}^{i_{d-1}+1}b_{d-1}(j-1)}$$

Summation paradigms

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$A(n) = \sum_{k=0}^{n} f(n,k);$$

$f(n,k)$: nested product-sum expr. w.r.t. k;
n: extra parameter

FIND a recurrence for $A(n)$

2. Recurrence solving

GIVEN a recurrence

$a_0(n), \dots, a_d(n), h(n)$:
nested product-sum expressions.

$$a_d(n)A(n+d) + \dots + a_0(n)A(n) = h(n);$$

FIND all solutions expressible by indefinite nested products and sums

(Abramov/Bronstein/Petkovšek/CS, in preparation)

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NOTE: By construction, the solutions are highly nested.

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3. Indefinite summation for simplification

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$a_0(n), \dots, a_d(n), h(n)$:
nested product-sum expressions.

$$a_d(n)A(n+d) + \dots + a_0(n)A(n) = h(n);$$

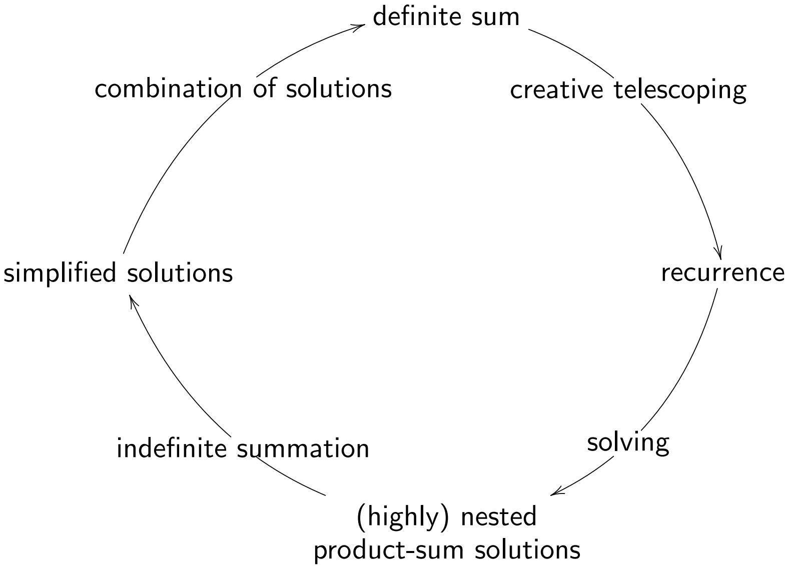
FIND all solutions expressible by indefinite nested products and sums
(Abramov/Bronstein/Petkovšek/CS, in preparation)

4. Find a “closed form”

A(n)=combined solutions.

The Sigma-summation spiral

Example



definite sum

creative telescoping

recurrence

solving

(highly) nested
product-sum solutions

indefinite summation

simplified solutions

combination of solutions

In[4]:= $\mathbf{mySum = SumName[SigmaBinomial[n, k]S[1, k], \{k, 0, n\}]}$

Out[4]= $\sum_{k=0}^{n} \binom{n}{k} S[1, k]$

Compute a recurrence

In[5]:= $\mathbf{rec = GenerateRecurrence[mySum, n][[1]]}$

Out[5]= $4(1+n)SUM[n] - 2(3+2n)SUM[n+1] + (2+n)SUM[n+2] == 1$

Solve a recurrence

In[6]:= $\mathbf{recSol = SolveRecurrence[rec, SUM[n]]}$

Out[6]= $\{\{0, -2^n\}, \{0, -2^n \sum_{i_1=1}^{n} \frac{1}{i_1}\}, \{1, -2^n \sum_{i_1=1}^{n} \frac{2^{-i_1}}{i_1}\}\}$

Combine the solutions

In[7]:= $\mathbf{FindLinearCombination[recSol, mySum, n, 2]}$

Out[7]= $2^n \sum_{i_1=1}^{n} \frac{1}{i_1} - 2^n \sum_{i_1=1}^{n} \frac{2^{-i_1}}{i_1}$

Example

Expansion of sums

Expansion of hypergeometric sum (Exercise)

GIVEN

$$F(n)=\sum_{k=0}^{n-1}\underbrace{\frac{(-2)^k(k+2)(3-\mathsf{ep})!\left(\frac{\varepsilon}{2}+2\right)!(n-1)!\left(-\frac{\varepsilon}{2}+k+1\right)!}{\left(1-\frac{\varepsilon}{2}\right)!(-\varepsilon+k+3)!\left(\frac{\varepsilon}{2}+k+2\right)!(-k+n-1)!}}_{=:f(n,k)}$$

Expansion of hypergeometric sum (Exercise)

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$$F(n)=\sum_{k=0}^{n-1}\underbrace{\frac{(-2)^k(k+2)(3-\mathsf{ep})!\big(\frac{\varepsilon}{2}+2\big)!(n-1)!\big(-\frac{\varepsilon}{2}+k+1\big)!}{\big(1-\frac{\varepsilon}{2}\big)!(-\varepsilon+k+3)!\big(\frac{\varepsilon}{2}+k+2\big)!(-k+n-1)!}}_{=:f(n,k)}$$

FIND the first coefficients of the ϵ-expansion

$$F(n)=F_0(n)+\varepsilon F_1(n)+\dots$$

Expansion of hypergeometric sum (Exercise)

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$$F(n)=\sum_{k=0}^{n-1}\underbrace{\frac{(-2)^k(k+2)(3-\mathsf{ep})!\big(\frac{\varepsilon}{2}+2\big)!(n-1)!\big(-\frac{\varepsilon}{2}+k+1\big)!}{\big(1-\frac{\varepsilon}{2}\big)!(-\varepsilon+k+3)!\big(\frac{\varepsilon}{2}+k+2\big)!(-k+n-1)!}}_{=:f(n,k)}$$

Step 1: Compute the first coefficients of the ϵ-expansion

$$f(n,k)=\qquad f_0(n,k)+\qquad \varepsilon f_1(n,k)+\dots$$

Expansion of hypergeometric sum (Exercise)

GIVEN

$$F(n)=\sum_{k=0}^{n-1}\underbrace{\frac{(-2)^k(k+2)(3-\text{ep})!\big(\frac{\varepsilon}{2}+2\big)!(n-1)!\big(-\frac{\varepsilon}{2}+k+1\big)!}{\big(1-\frac{\varepsilon}{2}\big)!(-\varepsilon+k+3)!\big(\frac{\varepsilon}{2}+k+2\big)!(-k+n-1)!}}_{=:f(n,k)}$$

Step 2: Simplify the sums in

$$\sum_{k=0}^{n-1}f(n,k)=\sum_{k=0}^{n-1}f_0(n,k)+\varepsilon\sum_{k=0}^{n-1}f_1(n,k)+\dots$$

Example

A more involved 2-loop example

GIVEN $F(n)=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{e^{-\varepsilon\gamma}}{\Gamma(\varepsilon+1)}\times$

$$\times\bigg(\underbrace{\frac{\Gamma(k+1)}{\Gamma(k+2+n)}\frac{\Gamma(\frac{\varepsilon}{2})\Gamma(1-\frac{\varepsilon}{2})\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+1+\frac{\varepsilon}{2})\Gamma(k+j+1+n)}{\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+2+n)\Gamma(k+j+2)} + \frac{\Gamma(k+1)}{\Gamma(k+2+n)}\frac{\Gamma(-\frac{\varepsilon}{2})\Gamma(1+\frac{\varepsilon}{2})\Gamma(j+1+\varepsilon)\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(k+j+1+\frac{\varepsilon}{2}+n)}{\Gamma(j+1)\Gamma(j+2+\frac{\varepsilon}{2}+n)\Gamma(k+j+2+\frac{\varepsilon}{2})}}_{f(n,k,j)}\bigg).$$

Arose in the context of
I. Bierenbaum, J. Blümlein, and S. Klein, *Evaluating two-loop massive operator matrix elements with Mellin-Barnes integrals*. 2006

A more involved 2-loop example

GIVEN $$F(n)=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{e^{-\varepsilon\gamma}}{\Gamma(\varepsilon+1)}\times$$

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FIND the first coefficients of the ϵ-expansion

$$F(n)=F_0(n)+\varepsilon F_1(n)+\dots$$

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A more involved 2-loop example

$$\boxed{\begin{aligned}&\text{GIVEN } F(n)=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{e^{-\varepsilon\gamma}}{\Gamma(\varepsilon+1)}\times\\ &\times\Bigg(\underbrace{\frac{\Gamma(k+1)}{\Gamma(k+2+n)}\frac{\Gamma(\frac{\varepsilon}{2})\Gamma(1-\frac{\varepsilon}{2})\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+1+\frac{\varepsilon}{2})\Gamma(k+j+1+n)}{\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+2+n)\Gamma(k+j+2)}+\frac{\Gamma(k+1)}{\Gamma(k+2+n)}\frac{\Gamma(-\frac{\varepsilon}{2})\Gamma(1+\frac{\varepsilon}{2})\Gamma(j+1+\varepsilon)\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(k+j+1+\frac{\varepsilon}{2}+n)}{\Gamma(j+1)\Gamma(j+2+\frac{\varepsilon}{2}+n)\Gamma(k+j+2+\frac{\varepsilon}{2})}}_{f(n,k,j)}\Bigg).\end{aligned}}$$

Step 1: Compute the first coefficients of the ϵ-expansion

$$f(n,k,j)=\qquad f_0(n,k,j)+\qquad \varepsilon f_1(n,k,j)+\dots$$

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A more involved 2-loop example

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Step 2: Simplify the sums in

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f(n,k,j)=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_0(n,k,j)+\varepsilon\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_1(n,k,j)+\dots$$

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I. Bierenbaum, J. Blümlein, and S. Klein, *Evaluating two-loop massive operator matrix elements with Mellin-Barnes integrals*. 2006

Simplify

$$\underbrace{\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}\Big)}_{f(j)}$$

where

$$S_1(n)=\sum_{i=1}^{n}\frac{1}{i}(=H_n)$$

Simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\underbrace{\Bigg(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!}+\frac{j!k!(j+k+n)!\left(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n)\right)}{(j+k+1)!(j+n+1)!(k+n+1)!}\Bigg)}_{f(j)}$$

FIND $g(j)$:

$$\boxed{f(j)=g(j+1)-g(j)}$$

Simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\underbrace{\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!}+\frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}\Big)}_{f(j)}$$

FIND $g(j)$:

$$\boxed{f(j)=g(j+1)-g(j)}$$

$$\uparrow$$

$$g(j)=\frac{(j+k+1)(j+n+1)j!k!(j+k+n)!\Big(S_1(j)-S_1(j+k)-S_1(j+n)+S_1(j+k+n)\Big)}{kn(j+k+1)!(j+n+1)!(k+n+1)!}$$

Simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\underbrace{\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!}+\frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}\Big)}_{f(j)}$$

FIND $g(j)$:

$$\boxed{f(j)=g(j+1)-g(j)}$$

Summing the telescoping equation over j from 0 to a gives

$$\sum_{j=0}^{a}f(n,k,j)=g(a+1)-g(0)$$

$$=\underbrace{\frac{(a+1)!(k-1)!(a+k+n+1)!(S_1(a)-S_1(a+k)-S_1(a+n)+S_1(a+k+n))}{n(a+k+1)!(a+n+1)!(k+n+1)!}+\frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)n!}+\frac{(2a+k+n+2)a!k!(a+k+n)!}{(a+k+1)(a+n+1)(a+k+1)!(a+n+1)!(k+n+1)!}}_{a\,\to\,\infty}$$

Simplify

$$\sum_{k=0}^{\infty}\underbrace{\sum_{j=0}^{\infty}\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!}+\frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}\Big)}_{f(j)}$$

$$\sum_{j=0}^{\infty}f(n,k,j)=\qquad\frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)n!}$$

Simplify

$$\sum_{k=0}^{\infty}\underbrace{\sum_{j=0}^{\infty}\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}\Big)}_{f(j)}$$

$$\sum_{k=1}^{\infty}\sum_{j=0}^{\infty}f(n,k,j)=\sum_{k=1}^{\infty}\frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)n!}$$

Telescoping

GIVEN

$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)n!}}_{=:\, f(n,k)}.$$

FIND $g(n,k)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{f(n,k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

no solution ☹

Zeilberger's creative telescoping paradigm

GIVEN

$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)n!}}_{=:\, f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n), c_1(n)$:

$$\boxed{g(n,k+1)-g(n,k)} = \boxed{c_0(n)f(n,k)+c_1(n)\,f(n+1,k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

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for all $0 \le k \le n$ and all $n \ge 0$.

Sigma *computes*: $c_0(n) = -n$, $c_1(n) = (n+1)(n+2)$ and

$$g(n,k) = \frac{kS_1(k)+(-n-1)S_1(n)-kS_1(k+n)-2}{(k+n+1)n!(n+1)^2}$$

Zeilberger's creative telescoping paradigm

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for all $0 \le k \le n$ and all $n \ge 0$.

Summing this equation over k from 0 to a gives:

$$\boxed{g(n,a+1)-g(n,0)} = \boxed{c_0(n)\,\mathsf{A}(n)+c_1(n)\,\mathsf{A}(n+1)}$$

Zeilberger's creative telescoping paradigm

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$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)n!}}_{=:\, f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n), c_1(n)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n) f(n,k) + c_1(n)\, f(n+1,k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

Summing this equation over k from 0 to a gives:

$$\boxed{g(n,a+1) - g(n,0)} = \boxed{c_0(n)\,\mathsf{A}(n) + c_1(n)\,\mathsf{A}(n+1)}$$

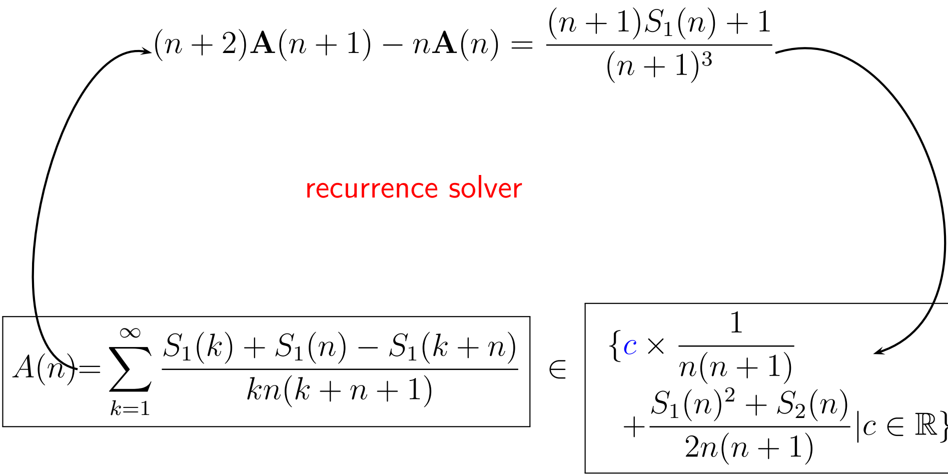
$$||\qquad\qquad\qquad\qquad ||$$

$$\frac{(a+1)(S_1(a)+S_1(n)-S_1(a+n))}{(n+1)^2(a+n+2)n!} + \frac{a(a+1)}{(n+1)^3(a+n+1)(a+n+2)n!} \qquad -n\mathsf{A}(n) + (1+n)(2+n)\mathsf{A}(n+1)$$

$$(n+2)\mathbf{A}(n+1) - n\mathbf{A}(n) = \frac{(n+1)S_1(n)+1}{(n+1)^3}$$

recurrence finder

$$\boxed{A(n) = \sum_{k=1}^{\infty} \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}}$$



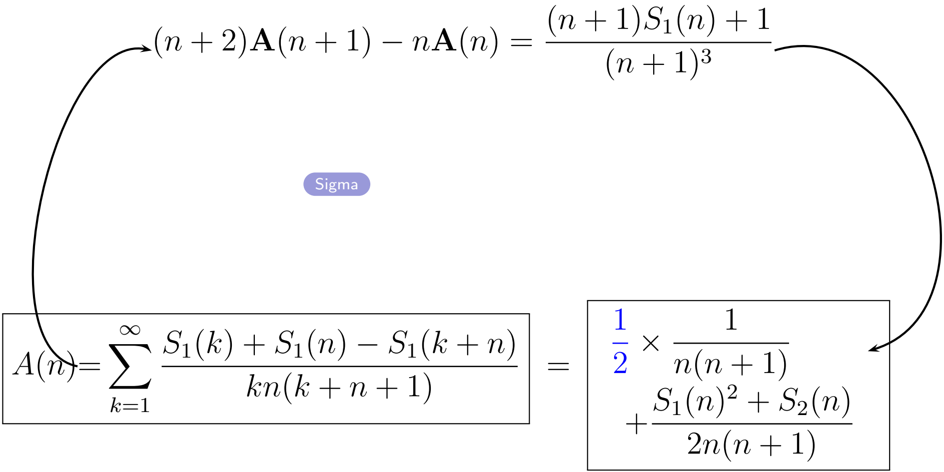
$$(n+2)\mathbf{A}(n+1) - n\mathbf{A}(n) = \frac{(n+1)S_1(n)+1}{(n+1)^3}$$

recurrence solver

$$\boxed{A(n) = \sum_{k=1}^{\infty} \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}} \in \boxed{\{c \times \frac{1}{n(n+1)} + \frac{S_1(n)^2+S_2(n)}{2n(n+1)} | c \in \mathbb{R}\}}$$

where

$$S_1(n) = \sum_{i=1}^{n} \frac{1}{i} \qquad S_2(n) = \sum_{i=1}^{n} \frac{1}{i^2}$$



$$(n+2)\mathbf{A}(n+1) - n\mathbf{A}(n) = \frac{(n+1)S_1(n)+1}{(n+1)^3}$$

$$\boxed{A(n) = \sum_{k=1}^{\infty} \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}} = \boxed{\begin{aligned}&\frac{1}{2}\times\frac{1}{n(n+1)}\\&+\frac{S_1(n)^2+S_2(n)}{2n(n+1)}\end{aligned}}$$

where

$$S_1(n) = \sum_{i=1}^{n} \frac{1}{i} \qquad S_2(n) = \sum_{i=1}^{n} \frac{1}{i^2}$$

Simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\underbrace{\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}\Big)}_{f(j)}$$

$$\sum_{k=1}^{\infty}\sum_{j=0}^{\infty} f(n,k,j) = \sum_{k=1}^{\infty}\frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)n!}$$

$$= \frac{S_1(n)^2+S_2(n)}{2n(n+1)!}$$

where

$$S_1(n)=\sum_{i=1}^{n}\frac{1}{i} \qquad S_2(n)=\sum_{i=1}^{n}\frac{1}{i^2}$$

GIVEN

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{e^{-e\gamma}}{\Gamma(\varepsilon+1)}\Bigg(\frac{\Gamma(k+1)}{\Gamma(k+2+n)}\,\frac{\Gamma(\frac{\varepsilon}{2})\Gamma(1-\frac{\varepsilon}{2})\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+1+\frac{\varepsilon}{2})\Gamma(k+j+1+n)}{\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+2+n)\Gamma(k+j+2)}$$

$$+\frac{\Gamma(k+1)}{\Gamma(k+2+n)}\,\frac{\Gamma(-\frac{\varepsilon}{2})\Gamma(1+\frac{\varepsilon}{2})\Gamma(j+1+\varepsilon)\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(k+j+1+\frac{\varepsilon}{2}+n)}{\Gamma(j+1)\Gamma(j+2+\frac{\varepsilon}{2}+n)\Gamma(k+j+2+\frac{\varepsilon}{2})}\Bigg).$$

$$=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_0(n,k,j)+$$

Sigma computes

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_0(n,k,j)=\frac{S_1(n)^2+3S_1(n)}{2n(n+1)!}.$$

GIVEN

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{e^{-e\gamma}}{\Gamma(\varepsilon+1)}\Bigg(\frac{\Gamma(k+1)}{\Gamma(k+2+n)}\,\frac{\Gamma(\frac{\varepsilon}{2})\Gamma(1-\frac{\varepsilon}{2})\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+1+\frac{\varepsilon}{2})\Gamma(k+j+1+n)}{\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+2+n)\Gamma(k+j+2)}$$

$$+\frac{\Gamma(k+1)}{\Gamma(k+2+n)}\,\frac{\Gamma(-\frac{\varepsilon}{2})\Gamma(1+\frac{\varepsilon}{2})\Gamma(j+1+\varepsilon)\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(k+j+1+\frac{\varepsilon}{2}+n)}{\Gamma(j+1)\Gamma(j+2+\frac{\varepsilon}{2}+n)\Gamma(k+j+2+\frac{\varepsilon}{2})}\Bigg).$$

$$=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_0(n,k,j)+\varepsilon\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_1(n,k,j)+$$

Sigma computes

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_0(n,k,j)=\frac{S_1(n)^2+3S_1(n)}{2n(n+1)!}.$$

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_1(n,k,j)=\frac{-S_1(n)^3-3S_2(n)S_1(n)-8S_3(n)}{6n(n+1)!}.$$

GIVEN

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{e^{-e\gamma}}{\Gamma(\varepsilon+1)}\Bigg(\frac{\Gamma(k+1)}{\Gamma(k+2+n)}\,\frac{\Gamma(\frac{\varepsilon}{2})\Gamma(1-\frac{\varepsilon}{2})\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+1+\frac{\varepsilon}{2})\Gamma(k+j+1+n)}{\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+2+n)\Gamma(k+j+2)}$$

$$+\frac{\Gamma(k+1)}{\Gamma(k+2+n)}\,\frac{\Gamma(-\frac{\varepsilon}{2})\Gamma(1+\frac{\varepsilon}{2})\Gamma(j+1+\varepsilon)\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(k+j+1+\frac{\varepsilon}{2}+n)}{\Gamma(j+1)\Gamma(j+2+\frac{\varepsilon}{2}+n)\Gamma(k+j+2+\frac{\varepsilon}{2})}\Bigg).$$

$$=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_0(n,k,j)+\varepsilon\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_1(n,k,j)+\varepsilon^2\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_1(n,k,j)+$$

Sigma computes

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_2(n,k,j)=\frac{1}{96n(n+1)}\Big(S_1(n)^4+(12\zeta_2+54S_2(n))S_1(n)^2$$

$$+104S_3(n)S_1(n)-48S_{2,1}(n)S_1(n)+51S_2(n)^2+36\zeta_2S_2(n)$$

$$+126S_4(n)-48S_{3,1}(n)-96S_{1,1,2}(n)\Big)$$

GIVEN

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{e^{-e\gamma}}{\Gamma(\varepsilon+1)}\Bigg(\frac{\Gamma(k+1)}{\Gamma(k+2+n)}\,\frac{\Gamma(\frac{\varepsilon}{2})\Gamma(1-\frac{\varepsilon}{2})\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+1+\frac{\varepsilon}{2})\Gamma(k+j+1+n)}{\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+2+n)\Gamma(k+j+2)}$$

$$+\frac{\Gamma(k+1)}{\Gamma(k+2+n)}\,\frac{\Gamma(-\frac{\varepsilon}{2})\Gamma(1+\frac{\varepsilon}{2})\Gamma(j+1+\varepsilon)\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(k+j+1+\frac{\varepsilon}{2}+n)}{\Gamma(j+1)\Gamma(j+2+\frac{\varepsilon}{2}+n)\Gamma(k+j+2+\frac{\varepsilon}{2})}\Bigg).$$

$$=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_0(n,k,j)+\varepsilon\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_1(n,k,j)+\varepsilon^2\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_1(n,k,j)+\varepsilon^3\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_1(n,k,j)+\dots$$

Sigma computes

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_3(n,k,j)=\frac{1}{960n(n+1)}\Big(S_1(n)^5+(20\zeta_2+130S_2(n))S_1(n)^3+$$

$$(40\zeta_3+380S_3(n))S_1(n)^2+\left(135S_2(n)^2+60\zeta_2S_2(n)+510S_4(n)\right)S_1(n)$$

$$-240S_{3,1}(n)S_1(n)-240S_{1,1,2}(n)S_1(n)+160\zeta_2S_3(n)+S_2(n)(120\zeta_3$$

$$+380S_3(n))+624S_5(n)+\left(-120S_1(n)^2-120S_2(n)\right)S_{2,1}(n)$$

$$-240S_{4,1}(n)-240S_{1,1,3}(n)+240S_{2,2,1}(n)\Big)$$

Guessing and Finding

(J. Blümlein, M. Kauers, S. Klein, CS; Comput. Phys. Comm. 180, pp. 2143-2165. 2009; arXiv 0902.4091)

In the non-singlet (3-loop) case ~ 360 diagrams contribute. The integrals are of the form:

$$F(n,\varepsilon)=\int_0^1 dx_1\dots\int_0^1 dx_7\sum_{i=1}^{K}\frac{p_i(x_1,x_2,\dots,x_7)^{n+\dots+r_i\varepsilon+\dots}}{q_i(x_1,x_2,\dots,x_7)^{\dots+s_i\varepsilon+\dots}}$$

where $K\in\mathbb{N}$, $r_i,s_i\in\mathbb{Q}$, and p_i,q_i are polynomials in $x_1,\dots,x_7$.

In the non-singlet (3-loop) case ~ 360 diagrams contribute. The integrals are of the form:

$$\begin{aligned} F(n,\varepsilon) &= \int_0^1 dx_1 \ldots \int_0^1 dx_7 \sum_{i=1}^{K} \frac{p_i(x_1,x_2,\ldots,x_7)^{n+\dots+r_i\varepsilon+\dots}}{q_i(x_1,x_2,\ldots,x_7)^{\dots+s_i\varepsilon+\dots}} \\ &= F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + F_{-1}(n)\varepsilon^{-1} + \boxed{F_0(n)}\,\varepsilon^0 + \ldots \end{aligned}$$

The 3-loop anomalous dimensions can be derived from the single pole part of $F(n,\varepsilon)$. The other poles are needed for the renormalization.

Vermaseren, Moch: 3-5 CPU years (2004)

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$$=F_{-3}(n)\varepsilon^{-3}+F_{-2}(n)\varepsilon^{-2}+F_{-1}(n)\varepsilon^{-1}+\boxed{F_0(n)}\varepsilon^0+\dots$$

$$\downarrow$$

Initial values $F_0(i)$, $i=1,\dots,5114$ (unsolved problem)

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$$F(n,\varepsilon)=\int_0^1 dx_1\ldots\int_0^1 dx_7\sum_{i=1}^{K}\frac{p_i(x_1,x_2,\ldots,x_7)^{n+\cdots+r_i\varepsilon+\ldots}}{q_i(x_1,x_2,\ldots,x_7)^{\cdots+s_i\varepsilon+\ldots}}$$

$$=F_{-3}(n)\varepsilon^{-3}+F_{-2}(n)\varepsilon^{-2}+F_{-1}(n)\varepsilon^{-1}+\boxed{F_0(n)}\varepsilon^0+\ldots$$

$$\downarrow$$

Initial values $F_0(i)$, $i=1,\ldots,5114$ (unsolved problem)

$\downarrow$ Recurrence finder (M. Kauers)

$$a_0(n)F_0(n)+a_1(n)F_0(n+1)+\cdots+a_{35}(n)F_0(n+35)=0$$

$$a_0(n)F_0(n)+a_1(n)F_0(n+1)+\dots+\boxed{a_{35}(n)}F_0(n+35)=0$$

$$a_0(n)F_0(n)+a_1(n)F_0(n+1)+\dots+\boxed{a_{35}(n)}\,F_0(n+35)=0$$

$$a_{35}(n)=\boxed{A_0}+A_1n+A_2n^2+\dots+A_{938}n^{983}\in\mathbb{Z}[n]$$

$$a_0(n)F_0(n) + a_1(n)F_0(n+1) + \dots + \boxed{a_{35}(n)}\, F_0(n+35) = 0$$

$$a_{35}(n) = \boxed{A_0} + A_1 n + A_2 n^2 + \dots + A_{938} n^{983} \in \mathbb{Z}[n]$$

$A_0 =$ 4640944309211313672503980223716264124200407085993854002412460315194
95765021269344971048446299722216293405285738333200767150194016391501666
27950213807356109710952045603966273388757782697588602201277983560532017
37487592671445911325765145271945214255462153147308420597210761595329365
51563452998613135384718911305253299053198893606401464021608911620974192
09001668029951620780182947258262939450801154511774527832503874341661898
89167522107378468797979810265385510643937043867557563467523740406094658
99100467933353731959645624977524424672990654427732309881685346483771128
69020837147452024401528169079406933665344476181260243344172097691636706
62803059675535809027169693064474147719610219849628486896079642312975136
20776876867741883488363846944854496482629372436829699055391369178850397
00381638011612302679580897488076647721311930634735316787779620757659951
5202809978299053753901432067359626151

(885 decimal digits)

In the non-singlet (3-loop) case ~ 360 diagrams contribute. The integrals are of the form:

$$\begin{aligned} F(n,\varepsilon) &= \int_0^1 dx_1 \dots \int_0^1 dx_7 \sum_{i=1}^{K} \frac{p_i(x_1,x_2,\dots,x_7)^{n+\dots+r_i\varepsilon+\dots}}{q_i(x_1,x_2,\dots,x_7)^{\dots+s_i\varepsilon+\dots}} \\ &= F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + F_{-1}(n)\varepsilon^{-1} + \boxed{F_0(n)}\varepsilon^0 + \dots \end{aligned}$$

$\downarrow$

Initial values $F_0(i)$, $i = 1, \dots, 5114$ (unsolved problem)

$\downarrow$ Recurrence finder (M. Kauers)

$$a_0(n)F_0(n) + a_1(n)F_0(n+1) + \dots + a_{35}(n)F_0(n+35) = 0$$

$\downarrow$ Sigma

CLOSED FORM

New Strategies

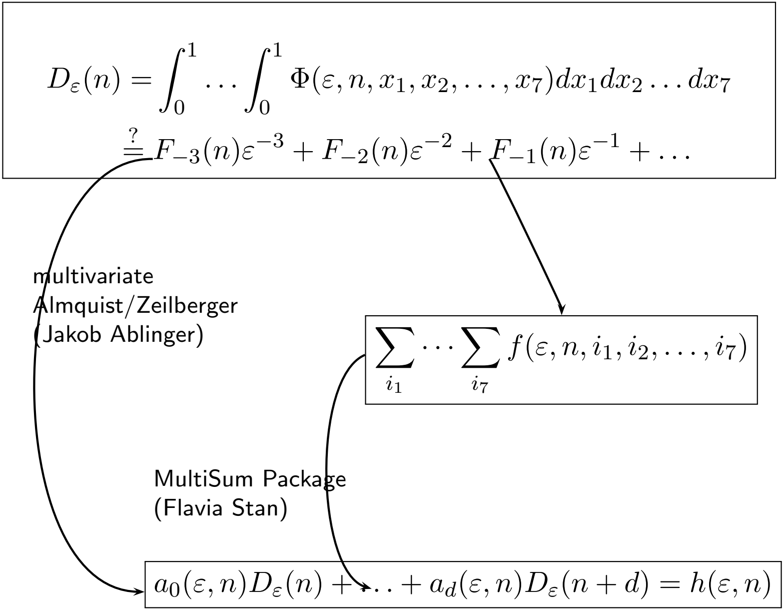
Find a recurrence for the integral/sum

$$\boxed{\begin{aligned} D_\varepsilon(n) &= \int_0^1 \dots \int_0^1 \Phi(\varepsilon, n, x_1, x_2, \dots, x_7) dx_1 dx_2 \dots dx_7 \\ &\overset{?}{=} F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + F_{-1}(n)\varepsilon^{-1} + \dots \end{aligned}}$$

multivariate
Almquist/Zeilberger
(Jakob Ablinger)

$$\boxed{a_0(\varepsilon, n) D_\varepsilon(n) + \dots + a_d(\varepsilon, n) D_\varepsilon(n+d) = h(\varepsilon, n)}$$

Find a recurrence for the integral/sum



$$D_\varepsilon(n) = \int_0^1 \dots \int_0^1 \Phi(\varepsilon, n, x_1, x_2, \dots, x_7) dx_1 dx_2 \dots dx_7$$
$$\overset{?}{=} F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + F_{-1}(n)\varepsilon^{-1} + \dots$$

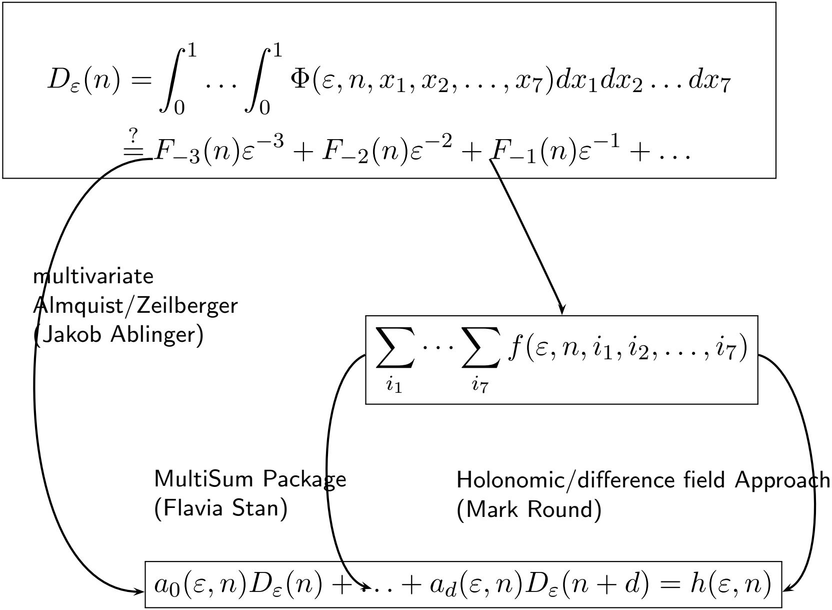
multivariate
Almquist/Zeilberger
(Jakob Ablinger)

$$\sum_{i_1} \dots \sum_{i_7} f(\varepsilon, n, i_1, i_2, \dots, i_7)$$

MultiSum Package
(Flavia Stan)

$$a_0(\varepsilon, n) D_\varepsilon(n) + \dots + a_d(\varepsilon, n) D_\varepsilon(n+d) = h(\varepsilon, n)$$

Find a recurrence for the integral/sum



$$D_\varepsilon(n) = \int_0^1 \dots \int_0^1 \Phi(\varepsilon, n, x_1, x_2, \dots, x_7) dx_1 dx_2 \dots dx_7$$

$$\overset{?}{=} F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + F_{-1}(n)\varepsilon^{-1} + \dots$$

multivariate
Almquist/Zeilberger
(Jakob Ablinger)

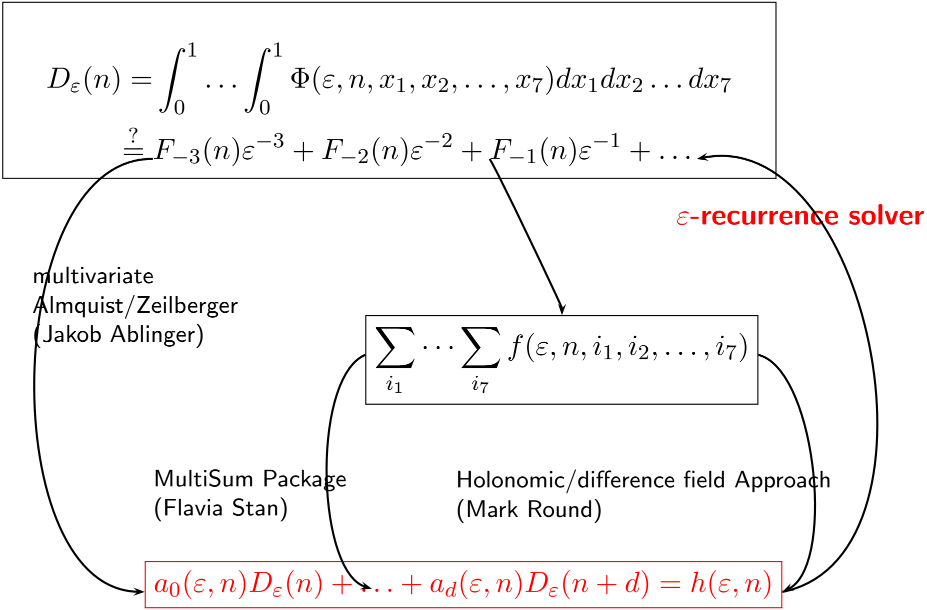
$$\sum_{i_1} \dots \sum_{i_7} f(\varepsilon, n, i_1, i_2, \dots, i_7)$$

MultiSum Package
(Flavia Stan)

Holonomic/difference field Approach
(Mark Round)

$$a_0(\varepsilon, n)D_\varepsilon(n) + \dots + a_d(\varepsilon, n)D_\varepsilon(n+d) = h(\varepsilon, n)$$

Find a recurrence for the integral/sum



$$D_\varepsilon(n)=\int_0^1\dots\int_0^1\Phi(\varepsilon,n,x_1,x_2,\dots,x_7)dx_1dx_2\dots dx_7$$
$$\overset{?}{=}F_{-3}(n)\varepsilon^{-3}+F_{-2}(n)\varepsilon^{-2}+F_{-1}(n)\varepsilon^{-1}+\dots$$

ε-recurrence solver

multivariate
Almquist/Zeilberger
(Jakob Ablinger)

$$\sum_{i_1}\dots\sum_{i_7}f(\varepsilon,n,i_1,i_2,\dots,i_7)$$

MultiSum Package
(Flavia Stan)

Holonomic/difference field Approach
(Mark Round)

$$a_0(\varepsilon,n)D_\varepsilon(n)+\dots+a_d(\varepsilon,n)D_\varepsilon(n+d)=h(\varepsilon,n)$$

Ansatz (for power series)

$$
\begin{aligned}
&a_0(\varepsilon,n)\Big[F_0(n)+F_1(n)\varepsilon+F_2(n)\varepsilon^2+\dots\Big]\\
+&a_1(\varepsilon,n)\Big[F_0(n+1)+F_1(n+1)\varepsilon+F_2(n+1)\varepsilon^2+\dots\Big]\\
+&\\
&\vdots\\
+&a_d(\varepsilon,n)\Big[F_0(n+d)+F_1(n+d)\varepsilon+F_2(n+d)\varepsilon^2+\dots\Big]\\
&\qquad\qquad\qquad\qquad\qquad=h_0(n)+h_1(n)\varepsilon+h_1(n)\varepsilon^2+\dots
\end{aligned}
$$

given

Ansatz (for power series)

$$
\begin{aligned}
&a_0(\varepsilon,n)\Big[F_0(n)+F_1(n)\varepsilon+F_2(n)\varepsilon^2+\dots\Big]\\
+&a_1(\varepsilon,n)\Big[F_0(n+1)+F_1(n+1)\varepsilon+F_2(n+1)\varepsilon^2+\dots\Big]\\
+&\\
&\vdots\\
+&a_d(\varepsilon,n)\Big[F_0(n+d)+F_1(n+d)\varepsilon+F_2(n+d)\varepsilon^2+\dots\Big]\\
&\qquad\qquad\qquad\qquad = h_0(n)+h_1(n)\varepsilon+h_1(n)\varepsilon^2+\dots
\end{aligned}
$$

$\Downarrow$ constant terms must agree

$$\boxed{a_0(0,n)F_0(n)+a_1(0,n)F_0(n+1)+\dots+a_d(0,n)F_0(n+d)=h_0(n)}$$

Ansatz (for power series)

$$\begin{aligned} &a_0(\varepsilon,n)\Big[F_0(n)+F_1(n)\varepsilon+F_2(n)\varepsilon^2+\dots\Big]\\ +&a_1(\varepsilon,n)\Big[F_0(n+1)+F_1(n+1)\varepsilon+F_2(n+1)\varepsilon^2+\dots\Big]\\ +&\\ &\vdots\\ +&a_d(\varepsilon,n)\Big[F_0(n+d)+F_1(n+d)\varepsilon+F_2(n+d)\varepsilon^2+\dots\Big]\\ &\qquad\qquad\qquad\qquad\qquad=h_0(n)+h_1(n)\varepsilon+h_1(n)\varepsilon^2+\dots\end{aligned}$$

$\Downarrow$ constant terms must agree

$$\boxed{a_0(0,n)F_0(n)+a_1(0,n)F_0(n+1)+\dots+a_d(0,n)F_0(n+d)=h_0(n)}$$

If $F_0(n)$ (with required initial values) is not expressible in terms of indefinite nested sums and products: $\boxed{\textbf{game over}}$

Ansatz (for power series)

$$
\begin{aligned}
&a_0(\varepsilon,n)\Big[F_0(n)+F_1(n)\varepsilon+F_2(n)\varepsilon^2+\dots\Big]\\
+&a_1(\varepsilon,n)\Big[F_0(n+1)+F_1(n+1)\varepsilon+F_2(n+1)\varepsilon^2+\dots\Big]\\
+&\\
&\vdots\\
+&a_d(\varepsilon,n)\Big[F_0(n+d)+F_1(n+d)\varepsilon+F_2(n+d)\varepsilon^2+\dots\Big]\\
&\qquad\qquad\qquad\qquad=h_0(n)+h_1(n)\varepsilon+h_1(n)\varepsilon^2+\dots
\end{aligned}
$$

$\Downarrow$ constant terms must agree

$$\boxed{a_0(0,n)F_0(n)+a_1(0,n)F_0(n+1)+\dots+a_d(0,n)F_0(n+d)=h_0(n)}$$

Ansatz (for power series)

$$
\begin{aligned}
&a_0(\varepsilon,n)\Big[F_0(n)+F_1(n)\varepsilon+F_2(n)\varepsilon^2+\dots\Big]\\
+&a_1(\varepsilon,n)\Big[F_0(n+1)+F_1(n+1)\varepsilon+F_2(n+1)\varepsilon^2+\dots\Big]\\
+&\\
&\vdots\\
+&a_d(\varepsilon,n)\Big[F_0(n+d)+F_1(n+d)\varepsilon+F_2(n+d)\varepsilon^2+\dots\Big]\\
&\qquad\qquad\qquad\qquad = h_0(n)+h_1(n)\varepsilon+h_1(n)\varepsilon^2+\dots
\end{aligned}
$$

$\Downarrow$ constant terms must agree

$$\boxed{a_0(0,n)F_0(n)+a_1(0,n)F_0(n+1)+\dots+a_d(0,n)F_0(n+d)=h_0(n)}$$

$$
\begin{aligned}
&\quad a_0(\varepsilon,n)\Big[F_1(n)\varepsilon+F_2(n)\varepsilon^2+\dots\Big]\\
&+a_1(\varepsilon,n)\Big[F_1(n+1)\varepsilon+F_2(n+1)\varepsilon^2+\dots\Big]\\
&+\\
&\ \vdots\\
&+a_d(\varepsilon,n)\Big[F_1(n+d)\varepsilon+F_2(n+d)\varepsilon^2+\dots\Big]\\
&\qquad\qquad\qquad\qquad\qquad\qquad=h'_0(n)+h'_1(n)\varepsilon+h'_2(n)\varepsilon^2+\dots
\end{aligned}
$$

$$
\begin{aligned}
&a_0(\varepsilon,n)\Big[F_1(n)\varepsilon+F_2(n)\varepsilon^2+\dots\Big]\\
&+a_1(\varepsilon,n)\Big[F_1(n+1)\varepsilon+F_2(n+1)\varepsilon^2+\dots\Big]\\
&+\\
&\ \vdots\\
&+a_d(\varepsilon,n)\Big[F_1(n+d)\varepsilon+F_2(n+d)\varepsilon^2+\dots\Big]\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad=\underbrace{h_0'(n)}_{\mathbf{=0}}+h_1'(n)\varepsilon+h_2'(n)\varepsilon^2+\dots
\end{aligned}
$$

Devide by ε

$$
\begin{aligned}
&a_0(\varepsilon,n)\Big[F_1(n)+F_2(n)\varepsilon+\dots\Big]\\
+&a_1(\varepsilon,n)\Big[F_1(n+1)+F_2(n+1)\varepsilon+\dots\Big]\\
+&\\
&\vdots\\
+&a_d(\varepsilon,n)\Big[F_1(n+d)+F_2(n+d)\varepsilon+\dots\Big]=h_1'(n)+h_2'(n)\varepsilon+\dots
\end{aligned}
$$

Now repeat for $F_1(n), F_2(n), \dots$ Example

Remark: Works the same for Laurent series.

(see J. Blümlein, S. Klein, CS, F. Stan. J. Symbolic Comput. 47, 2012; arXiv:1011.2656v2)