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ON CONGRUENCE PROPERTIES OF $p(n, m)$

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ABSTRACT. In the late 19th century, Sylvester and Cayley investigated the properties of the partition function $p(n, m)$. This function enumerates the partitions of a non-negative integer n into exactly m parts. Here we investigate the congruence properties of such functions and we obtain several infinite classes of Ramanujan-type congruences.

1. INTRODUCTION

Through the study of the theory of invariants, J.J. Sylvester and Arthur Cayley exploited and improved the methods of DeMorgan, Warburton and Herschel [1] to make tremendous contributions to the understanding of the restricted partition function $p(n, m)$, the function which enumerates the partitions of n into exactly m parts. They were able to establish explicit formulas for $p(n, m)$ for m up to 12.

It was the work of Ramanujan [5] that initiated great interest in partition congruences. He proved several divisibility properties of $p(n)$, the general partition function. The most basic of these are

$$p(5n + 4) \equiv 0 \pmod{5},$$
$$p(7n + 5) \equiv 0 \pmod{7},$$
$$p(11n + 6) \equiv 0 \pmod{11}.$$

The study of these and similar divisibility properties continues to be a very active area of mathematical research. After Ramanujan, Watson [6] and Atkin [2] made great contributions and most recently Ono [4] has proved that there are infinitely many congruences for $p(n)$.

The relationship betwen $p(n, m)$ and $p(n)$ is clear. The total number of of partitions of n is equal to the sum of the number of partitions of n into exactly one part, exactly two parts, and so forth until we conclude with the number of partitions of n into exactly n parts:

$$p(n) = p(n, 1) + p(n, 2) + p(n, 3) + ... + p(n, n - 1) + p(n, n). \tag{1}$$

Before we begin examining and proving the results of this paper, we will need one definition so that we may properly state our results.

Definition 1. For any natural number k, we define $lcm(k)$ to be the least common multiple of the numbers from 1 to k.

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For example, $\mathrm{lcm}(4)$ equals the least common multiple of 1, 2, 3, and 4 and so we have $\mathrm{lcm}(4) = 12$. Furthermore, we note that for any odd prime ℓ we have $\mathrm{lcm}(\ell-1)\cdot\ell = \mathrm{lcm}(\ell)$.

Although the discovery method for the results of this paper was combinatorial and done without congruence properties in mind, the proofs utilize generating functions and, as such, continue to underscore the power of this technique. The main object of this paper is to prove the following theorem and a striking corollary.

Theorem 1. *For ℓ an odd prime and $n \geq lcm(\ell-1) - \frac{\ell-3}{2}$,*

$$p(n\ell,\ell) - p\left(n\ell - lcm\left(\ell\right),\ell\right) \equiv 0 \pmod{\ell}.$$

Corollary 1. *For ℓ an odd prime, $n \geq 0$, and $0 \leq t \leq \frac{\ell-3}{2}$,*

$$p(n \cdot lcm(\ell) - t\ell, \ell) \equiv 0 \pmod{\ell}.$$

These general results give us intriguing specific examples such as

$$\begin{gathered} p(60k+35,5) - p(35,5) \equiv 0 \pmod{5}, \\ p(420k,7) \equiv 0 \pmod{7}, \\ p(420k+406,7) \equiv 0 \pmod{7}, \\ \text{and } p(420k+413,7) \equiv 0 \pmod{7}. \end{gathered}$$

2. Background

We begin this section with an additional definition which is crucial in the work below.

Definition 2. A polynomial $P(q)$ of degree d is called an anti-reciprocal polynomial if

$$q^d\left(P(1/q)\right) = -P(q). \tag{2}$$

Lemma 1. *The rational function $\frac{(1-q^{\mathrm{lcm}(m-1)})^m}{(1-q)(1-q^2)\cdots(1-q^m)}$ is an anti-reciprocal polynomial of degree $d = \mathrm{lcm}(m-1)\cdot m - \frac{m^2-m}{2}$.*

Proof. We begin with the rational function

$$K_m(q) = \frac{(1-q^{\mathrm{lcm}(m-1)})^m}{(1-q)(1-q^2)\cdots(1-q^{m-1})}, \tag{3}$$

which is a polynomial of degree $d = \mathrm{lcm}(m-1)\cdot m - \frac{m^2-m}{2}$ because

$$\begin{aligned} K_m(q) \quad = \quad & (1-q^{\mathrm{lcm}(m-1)})\cdot(1+q+q^2+\cdots+q^{\mathrm{lcm}(m-1)-1}) \\ & \cdot(1+q^2+q^4+\cdots+q^{\mathrm{lcm}(m-1)-2}) \\ & \cdots(1+q^{(m-1)}+q^{2(m-1)}+\cdots+q^{\mathrm{lcm}(m-1)-(m-1)}). \end{aligned} \tag{4}$$

We will now prove that $K_m(q)$ is an anti-reciprocal polynomial by showing that $q^d \cdot K_m(1/q) = -K_m(q)$. Multiplying $K_m(1/q)$ by q^d and simplifying yields

$$\begin{aligned}
q^d\left(K_m(1/q)\right) &= \frac{q^d \cdot \left(1-\frac{1}{q^{\operatorname{lcm}(m-1)}}\right)^m}{\left(1-\frac{1}{q}\right)\left(1-\frac{1}{q^2}\right)\cdots\left(1-\frac{1}{q^{m-1}}\right)}\\
&= \frac{q^d \cdot \left(\frac{q^{\operatorname{lcm}(m-1)}-1}{q^{\operatorname{lcm}(m-1)}}\right)^m}{\left(\frac{q-1}{q}\right)\left(\frac{q^2-1}{q^2}\right)\cdots\left(\frac{q^{m-1}-1}{q^{m-1}}\right)} = \frac{q^d\cdot\frac{(q^{\operatorname{lcm}(m-1)}-1)^m}{q^{\operatorname{lcm}(m)\cdot m}}}{\left(\frac{(q-1)(q^2-1)\cdots(q^{m-1}-1)}{q^{\sum_{n=0}^{m-1} n}}\right)}\\
&= \frac{q^d(q^{\operatorname{lcm}(m-1)}-1)^m}{q^{\operatorname{lcm}(m)\cdot m}} \cdot \frac{q^{\frac{m^2-m}{2}}}{(q-1)(q^2-1)\cdots(q^{m-1}-1)}\\
&= \frac{q^d}{q^{\operatorname{lcm}(m)\cdot m}\cdot q^{-\left(\frac{m^2-m}{2}\right)}} \cdot \frac{(q^{\operatorname{lcm}(m-1)}-1)^m}{(q-1)(q^2-1)\cdots(q^{m-1}-1)}\\
&= \frac{q^d}{q^d}\cdot\frac{(q^{\operatorname{lcm}(m-1)}-1)^m}{(q-1)(q^2-1)\cdots(q^{m-1}-1)} = \frac{(q^{\operatorname{lcm}(m-1)}-1)^m}{(q-1)(q^2-1)\cdots(q^{m-1}-1)} = -K_m(q).
\end{aligned}$$

Thus $K_m(q)$ is an anti-reciprocal polynomial. □

3. Proof of the main theorem

We now prove Theorem 1.

Proof. It is well known that the generating function for the number of partitions of n into exactly ℓ parts is given by

$$\sum_{n=0}^{\infty} p(n,\ell)\,q^n = \frac{q^\ell}{(1-q)(1-q^2)\cdots(1-q^\ell)}. \tag{5}$$

The generating formula for $p(n,\ell)-p(n-\operatorname{lcm}(\ell),\ell)$ is likewise given by

$$\sum_{n=0}^{\infty}\left(p(n,\ell)-p(n-\operatorname{lcm}(\ell),\ell)\right)q^n = \frac{q^\ell(1-q^{\operatorname{lcm}(\ell)})}{(1-q)(1-q^2)\cdots(1-q^\ell)}. \tag{6}$$

Let us now consider the right-hand side of our generating function, which we rewrite as

$$\frac{q^\ell}{1-q^\ell}\cdot\frac{\left(1-q^{\operatorname{lcm}(\ell)}\right)}{(1-q)(1-q^2)\cdots(1-q^{\ell-1})}. \tag{7}$$

In light of the fact that ℓ divides $\binom{\ell}{j}$ for $0<j<\ell$, we see that

$$\frac{\left(1-q^{\operatorname{lcm}(\ell)}\right)}{(1-q)(1-q^2)\cdots(1-q^{\ell-1})} \equiv \frac{(1-q^{\operatorname{lcm}(\ell-1)})^\ell}{(1-q)(1-q^2)\cdots(1-q^{\ell-1})} \pmod{\ell}.$$

We can now rewrite (6) using the notation from Section 2, where $K_\ell(q)$ is an anti-reciprocal polynomial of degree $d=\operatorname{lcm}(\ell)-\frac{\ell^2-\ell}{2}$:

$$\sum_{n=0}^{\infty}\left(p(n,\ell)-p(n-\operatorname{lcm}(\ell),\ell)\right)q^n \equiv \frac{q^\ell}{1-q^\ell}\cdot K_\ell(q) \pmod{\ell}. \tag{8}$$

We now apply the anti-reciprocal property to (8), so that we may consider two cases which arise depending on the parity of d.

Case 1: Suppose d is even. We use the anti-reciprocal property of $K_\ell(q)$ to organize all of its terms by first listing those q^α where $\alpha \equiv 0 \pmod{\ell}$. These are followed by the remaining $q^\beta k_\beta(q^\ell)$, a short-hand notation for all those terms of $K_\ell(q)$ such that $\beta \not\equiv 0 \pmod{\ell}$:

$$\begin{aligned} K_\ell(q) &= \left(1 + aq^\ell + bq^{2\ell} + \cdots + yq^{\frac{d}{2}-\ell} + 0q^{\frac{d}{2}} - yq^{\frac{d}{2}+\ell} - \cdots - aq^{d-\ell} - q^d\right) \\ &+ qk_1(q^\ell) + q^2k_2(q^\ell) + \cdots + q^{\ell-1}k_{\ell-1}(q^\ell) \\ &= (1-q^\ell)\left(\frac{1-q^d}{1-q^\ell} + aq^\ell\frac{1-q^{d-2\ell}}{1-q^\ell} + bq^{2\ell}\frac{1-q^{d-4\ell}}{1-q^\ell} + \cdots + yq^{\frac{d}{2}-\ell}\frac{1-q^{2\ell}}{1-q^\ell}\right) \\ &+ qk_1(q^\ell) + q^2k_2(q^\ell) + \cdots + q^{\ell-1}k_{\ell-1}(q^\ell). \end{aligned}$$

Let $L_e(q^\ell) = \left(\frac{1-q^d}{1-q^\ell} + aq^\ell\frac{1-q^{d-2\ell}}{1-q^\ell} + bq^{2\ell}\frac{1-q^{d-4\ell}}{1-q^\ell} + \cdots + yq^{\frac{d}{2}-\ell}\frac{1-q^{2\ell}}{1-q^\ell}\right)$ so that

$$K_\ell(q) = (1-q^\ell)L_e(q^\ell) + qk_1(q^\ell) + q^2k_2(q^\ell) + \cdots + q^{\ell-1}k_{\ell-1}(q^\ell).$$

Case 2: Suppose d is odd. Case 2 is done just as Case 1, the only difference being that there is no middle term of $L_o(q^\ell)$ (with notation $o =$ odd).

Writing $L(q^\ell)$ for $L_o(q^\ell)$ or $L_e(q^\ell)$, as appropriate, we have

$$K_\ell(q) = (1-q^\ell)L(q^\ell) + qk_1(q^\ell) + q^2k_2(q^\ell) + \cdots + q^{\ell-1}k_{\ell-1}(q^\ell). \tag{9}$$

We put together equations (8) and (9) to conclude the proof of Theorem 1.

$$\sum_{n=0}^{\infty}\left(p(n,\ell) - p(n-\mathrm{lcm}(\ell),\ell)\right)q^n \tag{10}$$

$$\begin{aligned} &\equiv \frac{q^\ell}{1-q^\ell}\cdot\left(L(q^\ell)(1-q^\ell) + qk_1(q^\ell) + q^2k_2(q^\ell) + \cdots + q^{\ell-1}k_{\ell-1}(q^\ell)\right) \pmod{\ell} \\ &\quad = q^\ell L(q^\ell) + q^\ell\left(\frac{qk_1(q^\ell) + q^2k_2(q^\ell) + \cdots + q^{\ell-1}k_{\ell-1}(q^\ell)}{1-q^\ell}\right). \end{aligned}$$

Notice that $\left(\frac{qk_1(q^\ell)+q^2k_2(q^\ell)+\cdots+q^{\ell-1}k_{\ell-1}(q^\ell)}{1-q^\ell}\right)$ contributes nothing to the coefficients of $q^{\ell j}$. Since $q^\ell L(q^\ell)$ is a polynomial of degree $d+\ell = \mathrm{lcm}(\ell) - \frac{\ell^2-3\ell}{2}$ it follows that for an odd prime ℓ and $n \geq \mathrm{lcm}(\ell-1) - \frac{\ell-3}{2}$, $p(n\ell,\ell) - p(n\ell - \mathrm{lcm}(\ell),\ell) \equiv 0 \pmod{\ell}$. ∎

4. Proof of the corollary to the main theorem

We now prove Corollary 1.

Corollary 1. *For ℓ an odd prime, $n \geq 0$, and $0 \leq t \leq \frac{\ell-3}{2}$,*

$$p(n\cdot\mathrm{lcm}(\ell) - t\ell,\ell) \equiv 0 \pmod{\ell}.$$

Proof. In Theorem 1, set $n = k\cdot\mathrm{lcm}(\ell-1) - t$ for k an integer so that

$$\begin{aligned} &p([k\cdot\mathrm{lcm}(\ell-1)-t]\cdot\ell,\ell) - p([k\cdot\mathrm{lcm}(\ell-1)-t]\cdot\ell - \mathrm{lcm}(\ell),\ell) \\ &= p(k\cdot\mathrm{lcm}(\ell) - t\ell,\ell) - p([k-1]\cdot\mathrm{lcm}(\ell) - t\ell,\ell) \equiv 0 \pmod{\ell}. \end{aligned}$$

Proceed by induction on k. Let $k=1$ so that

$$p(\mathrm{lcm}(\ell)-t\ell,\ell) = p(\mathrm{lcm}(\ell)-t\ell,\ell) - 0 = p(\mathrm{lcm}(\ell)-t\ell,\ell) - p(-t\ell,\ell) \equiv 0 \pmod{\ell}.$$

Now suppose $p(n \cdot \mathrm{lcm}(\ell) - t\ell, \ell) \equiv 0 \pmod{\ell}$ is true for all $n < k$. Hence,

$$p(k \cdot \mathrm{lcm}(\ell) - t\ell, \ell) - p([k-1] \cdot \mathrm{lcm}(\ell) - t\ell, \ell) \equiv 0 \pmod{\ell},$$

which implies that $p(k \cdot \mathrm{lcm}(\ell) - t\ell, \ell) \equiv 0 \pmod{\ell}$ by the induction hypothesis. Thus the corollary is proved. □

In conclusion, Corollary 1 gives us $\frac{\ell-1}{2}$ Ramanujan-like congruences for every odd prime ℓ. There are numerous related results arising from extensions of the methods of this paper. These will be explored in a subsequent publication.

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