Quantum Chromodynamics: History and Prospects, Oberwölz, Austria

Summation, Integration and Special Function Algorithms for Feynman Integrals

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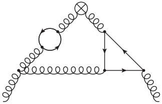
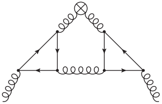
joint work with J. Blümlein, A. Hasselhuhn, F. Wissbrock (DESY),
J. Ablinger (RISC/DESY), M. Round (RISC)

September 3, 2012

Example: 3-loop topologies of gluonic massive operator matrix elements with two fermion lines (unpolarized case)

$D_\varepsilon(n) =$  [by axodraw (J. Vermaseren)] $+$  $+ \sim$**80 further diagrams**

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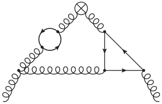
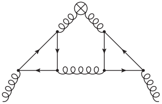
$D_\varepsilon(n) =$  [by axodraw (J. Vermaseren)] $+$  $+ \sim$**80 further diagrams**

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$$D_\varepsilon(n) = \sum_{j_1=0}^{n-5} \sum_{j_2=0}^{n-j_1-6} \pi 2^{\varepsilon+3} e^{-\frac{3\gamma\varepsilon}{2}} \Gamma(2-\varepsilon)\Gamma\left(\frac{\varepsilon}{2}+2\right)\Gamma\left(-\frac{3\varepsilon}{2}\right) \qquad (\sim\text{2GB})$$

$$\times \frac{(-1)^{j_1}(j_2+1)\Gamma\left(-\frac{\varepsilon}{2}+j_1+4\right)\Gamma(-j_1+n-2)\Gamma(\varepsilon-j_1-j_2+n-5)}{(\varepsilon-10)(\varepsilon-8)(\varepsilon-2)\varepsilon\Gamma\left(\frac{5}{2}-\varepsilon\right)\Gamma\left(\frac{\varepsilon+5}{2}\right)\Gamma\left(\frac{\varepsilon}{2}+n+1\right)\Gamma(-j_1-j_2+n-4)}$$

$+ \sim$**2400 further multi-sums**

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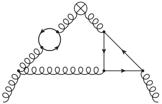
$D_\varepsilon(n) =$ 
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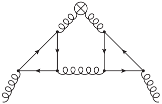
$+$  $+ \sim$**80 further diagrams**

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↓ (see talk)

$$D_\varepsilon(n) = \varepsilon^{-3}F_{-3}(n) + \varepsilon^{-2}F_{-2}(n) + \varepsilon^{-1}F_{-1}(n) + \varepsilon^{0}F_{0}(n) + \dots$$

General tactic

Feynman parameter integrals with one mass M and local operator insertions in $4+\varepsilon$-dimensional Minkowski space:

$$D_\varepsilon(n) = \int \frac{d^D p_1}{(2\pi)^D} \dots \int \frac{d^D p_k}{(2\pi)^D} \frac{N(p_1,\dots,p_k;p;M^2;\Delta,n)}{(-p_1^2+m_1^2)^{l_1}\dots(-p_k^2+m_k^2)^{l_k}} \prod_V \delta_V$$

$\downarrow$ (Blümlein/Stan/Schneider 2012)

Definite hypergeometric multi-sums:

$$D_\varepsilon(n) = \sum_{k_1=l_1}^{L_1(n)} \dots \sum_{k_v=l_v}^{L_v(n,k_1,\dots,k_{v-1})} \sum_{i=1}^{l} f(\varepsilon,n,k_1,\dots,k_v) + \dots$$

f: proper hypergeometric series in terms of Γ-functions
$L_v(n,k_1,\dots,k_{v-1})$: integer linear relation in the parameters (or ∞)

$\downarrow$ (symbolic summation)

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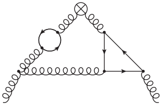
$D_\varepsilon(n) =$ 

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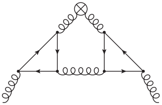
$+$  $+ \sim$**80 further diagrams**

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Step 1: Expansion of the summand

$$\sum_{j_1=0}^{n-5}\sum_{j_2=0}^{n-j_1-6}\pi 2^{\varepsilon+3}e^{-\frac{3\gamma\varepsilon}{2}}\Gamma(2-\varepsilon)\Gamma\left(\frac{\varepsilon}{2}+2\right)\Gamma\left(-\frac{3\varepsilon}{2}\right)$$

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$$=\varepsilon^{-3}\boxed{0}+\varepsilon^{-2}\boxed{\sum_{j_1=0}^{n-5}\sum_{j_2=0}^{n-j_1-6}f_{-2}(n,j_2,j_1)}+\varepsilon^{-1}\boxed{\sum_{j_1=0}^{n-5}\sum_{j_2=0}^{n-j_1-6}f_{-1}(n,j_2,j_1)}\dots$$

Step 2: Simplify sums from inside to outside

$$\sum_{j_1=0}^{n-5}\sum_{j_2=0}^{n-j_1-6}\frac{-8(j_1+2)(j_1+3)(n+1)(j_1-n+3)(j_1-n+4)(-1)^{j_1}(j_1+1)!(-j_1+n-5)!(j_2+1)}{135(n+1)!(j_1+j_2-n+5)}$$

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$$\times\boxed{\sum_{j_2=0}^{n-j_1-6}\frac{j_2+1}{j_1+j_2-n+5}}$$

$$(n+2)(-j_1+n-5)\mathbf{A}(j_1)+(j_1-n+4)\mathbf{A}(j_1+1)=j_1-n+5$$

recurrence finder

$$A(j_1) = \sum_{j_2=0}^{n-j_1-6} \frac{j_2+1}{j_1+j_2-n+5}$$

$$(n+2)(-j_1+n-5)\mathbf{A}(j_1)+(j_1-n+4)\mathbf{A}(j_1+1)=j_1-n+5$$

recurrence solver

$$A(j_1)=\sum_{j_2=0}^{n-j_1-6}\frac{j_2+1}{j_1+j_2-n+5} \in \Big\{(n-j_1-4)\sum_{i=1}^{j_1}\frac{1}{-3+n-i} + c\times(n-j_1-4) \,\Big|\, c\in\mathbb{R}\Big\}$$

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Difference field algorithms/theory

(see, e.g., Karr 1981, Bronstein 2000, Schneider 2001/2004/2005a–c/2007/2008/2010a–c)

$$\boxed{A(j_1)=\sum_{j_2=0}^{n-j_1-6}\frac{j_2+1}{j_1+j_2-n+5}}=\boxed{(n-j_1-4)\Big(\sum_{i=1}^{j_1}\frac{1}{-3+n-i}+\frac{\big(n^4-2n^3-7n^2+16n-6\big)}{(n-3)(n-2)(n-1)n}-S_1(n)\Big)}$$

where

$$S_1(n)=\sum_{i=1}^{n}\frac{1}{i}$$

More generally: Sigma's summation spiral

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$A(n) = \sum_{k=0}^{n} f(n,k);$$

$f(n,k)$: indefinite nested product-sum in k;
n: extra parameter

FIND a recurrence for $A(n)$

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2. Recurrence solving

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$a_0(n), \dots, a_d(n), h(n)$:
indefinite nested product-sum expressions.

$$a_0(n)A(n) + \dots + a_d(n)A(n+d) = h(n);$$

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3. Find a "closed form"

A(n)=combined solutions.

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$$||$$

$$-\frac{16\big(n^8+6n^7-6n^6-80n^5-81n^4+178n^3+274n^2-4n-96\big)}{45(n-2)(n-1)^2n^2(n+1)(n+2)^2(n+3)^2}$$

$$+\frac{16(-1)^n\big(3n^2+12n+11\big)}{135(n+1)(n+2)^2(n+3)^2}+\frac{16\big(n^2-n-8\big)}{45(n-1)n(n+2)(n+3)}S_1(n)$$

Mathematica Session:

In[1]:= << **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= << **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[3]:= << **EvaluateMultiSums.m**

EvaluateMultiSums by Carsten Schneider © RISC-Linz

In[4]:= **EvaluateMultiSum[**

$$\frac{-8(j_1+2)(j_1+3)(n+1)(j_1-n+3)(j_1-n+4)(-1)^{j_1}(j_1+1)!(-j_1+n-5)!(j_2+1)}{135(n+1)!(j_1+j_2-n+5)},$$

$$\{\{j_2,0,n-j_1-6\},\{j_1,0,N-5\}\},\{n\},\{5\},\mathrm{ExpandIn}\to\{\varepsilon,-3,-1\}]$$

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Out[4]=

$$\Big\{0,\quad \frac{16(-1)^n\left(3n^2+12n+11\right)}{135(n+1)(n+2)^2(n+3)^2}-\frac{16\left(n^8+6n^7-6n^6-80n^5-81n^4+178n^3+274n^2-4n-96\right)}{45(n-2)(n-1)^2n^2(n+1)(n+2)^2(n+3)^2}$$

$$\frac{16\left(n^2-n-8\right)}{45(n-1)n(n+2)(n+3)}S_1(n),\qquad -\frac{8\left(n^2-n-8\right)}{45(n-1)n(n+2)(n+3)}S_2(n)+\frac{2(-1)^n(187n+127)\left(3n^2+12n+11\right)}{2025(n+1)^2(n+2)^2(n+3)^2}$$

$$+\left(\frac{2\left(17n^6-231n^5+121n^4+2063n^3-1458n^2-2432n+960\right)}{675(n-2)(n-1)^2n^2(n+1)(n+2)(n+3)}-\frac{16(-1)^n\left(3n^2+12n+11\right)}{135(n+1)(n+2)^2(n+3)^2}\right)S_1(n)+$$

$$\frac{2\left(43n^{12}+112n^{11}+263n^{10}-216n^9-11309n^8-16476n^7+55837n^6+78164n^5-95178n^4-116688n^3+51784n^2+30624n-23040\right)}{675(n-2)^2(n-1)^3n^3(n+1)^2(n+2)^2(n+3)^2}\Big\}$$

Computer algebra and special functions:

Harmonic sums (J. Vermaseren, J. Blümlein; M. Hoffman, D. Broadhurst,...)

$$S_{2,1}(n) = \boxed{\sum_{i=1}^{n} \frac{1}{i^2} \sum_{j=1}^{i} \frac{1}{j}}$$

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Integral representation:

$$= \int_0^1 \frac{x^n - 1}{1 - x} \left(\int_0^x \frac{\int_0^y \frac{1}{1-z} dz}{y} dy - \zeta(2) \right) dx, \qquad \zeta(z) := \sum_{i=1}^{\infty} 1/i^z$$

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Asymptotic expansion:

$$= \left(\frac{1}{30n^5} - \frac{1}{6n^3} + \frac{1}{2n^2} - \frac{1}{n} \right) \ln(n)$$

$$- \tfrac{1}{100n^5} - \tfrac{1}{6n^4} + \tfrac{13}{36n^3} - \tfrac{1}{4n^2} - \tfrac{1}{n} + 2\zeta(3) + O(\tfrac{\ln(n)}{n^6}).$$

limit computations **numerical evaluation**

Computer algebra and special functions:

Generalization to cyclotomic harmonic sums

$$\boxed{\sum_{k=1}^{n}\frac{(-1)^k}{2k+1}} =$$

Integral representation:

$$=-(-1)^n\int_0^1\frac{x^{2n}}{x^2+1}dx+\frac{(-1)^n}{2n+1}-1+\frac{\pi}{4},$$

Asymptotic expansion:

$$=(-1)^n\left(-\tfrac{3}{64n^5}-\tfrac{1}{16n^4}+\tfrac{3}{16n^3}-\tfrac{1}{4n^2}+\tfrac{1}{4n}\right)+\tfrac{\pi}{4}-1+O(\tfrac{1}{n^6})$$

limit computations | **numerical evaluation**

(J. Ablinger, J. Blümlein, CS; J. Math. Phys. 2011 [arXiv:1105.6063 [math-ph]])

$$\sum_{i=1}^{n} \frac{\sum_{j=1}^{i} \frac{1}{j^2}}{(2i+1)^2} = \text{asymptotic expansion?}$$

In[5]:= << **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[6]:= **SExpansion[S[{{2, 1, 2}, {1, 0, 2}}, n], n, 10]**

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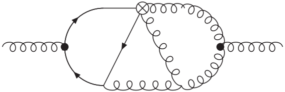
In[6]:= **SExpansion[S[{{2, 1, 2}, {1, 0, 2}}, n], n, 10]**

Out[6]=
$$\left(-\frac{16\ln 2^2}{3}+\frac{3}{128n^{10}}-\frac{367}{5760n^9}+\frac{7}{96n^8}-\frac{221}{2016n^7}+\frac{5}{24n^6}-\frac{127}{360n^5}+\frac{1}{2n^4}-\frac{11}{18n^3}+\frac{2}{3n^2}-\frac{2}{3n}-\frac{1936}{15}\right)\frac{1}{4}(\pi-4)^2+\left(-\frac{32\ln 2^2}{3}+\frac{3}{64n^{10}}-\frac{367}{2880n^9}+\frac{7}{48n^8}-\frac{221}{1008n^7}+\frac{5}{12n^6}-\frac{127}{180n^5}+\frac{1}{n^4}-\frac{11}{9n^3}+\frac{4}{3n^2}-\frac{4}{3n}-\frac{3872}{45}\right)\frac{1}{4}(\pi-4)-\frac{968}{45}\frac{1}{4}(\pi-4)^4-\frac{3872}{45}\frac{1}{4}(\pi-4)^3+8\text{li4half}+\frac{\ln 2^4}{3}-\frac{16\ln 2^2}{3}+7\ln 2\ z3+\frac{125891}{1075200n^{10}}-\frac{10259}{80640n^9}+\frac{92257}{645120n^8}-\frac{5507}{20160n^7}+\frac{2837}{5760n^6}-\frac{509}{720n^5}+\frac{161}{192n^4}-\frac{31}{36n^3}+\frac{19}{24n^2}-\frac{2}{3n}-\frac{968}{45}$$

So far, the most complicated massive 3–loop ladder graph

for Quarkonic Local Operator Matrix Elements [Nucl. Physics B.; 2012]



$$= F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + F_{-1}(n)\varepsilon^{-1} + \boxed{F_0(n)}$$

So far, the most complicated massive 3–loop ladder graph

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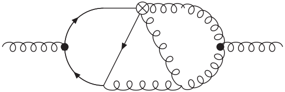

$$= F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + F_{-1}(n)\varepsilon^{-1} + \boxed{F_0(n)}$$

$$||$$

$$\sum_{j=0}^{n-3}\sum_{k=0}^{j}\sum_{l=0}^{k}\sum_{q=0}^{-j+n-3}\sum_{s=1}^{-l+n-q-3}\sum_{r=0}^{-l+n-q-s-3}(-1)^{-j+k-l+n-q-3}\times$$

$$\times \frac{\binom{j+1}{k+1}\binom{k}{l}\binom{n-1}{j+2}\binom{-j+n-3}{q}\binom{-l+n-q-3}{s}\binom{-l+n-q-s-3}{r}r!(-l+n-q-r-s-3)!(s-1)!}{(-l+n-q-2)!(-j+n-1)(n-q-r-s-2)(q+s+1)}$$

$$\Big[4S_1(-j+n-1) - 4S_1(-j+n-2) - 2S_1(k)$$

$$-(S_1(-l+n-q-2) + S_1(-l+n-q-r-s-3) - 2S_1(r+s))$$

$$+2S_1(s-1) - 2S_1(r+s)\Big] + \textbf{3 further 6–fold sums}$$

$$
\boxed{F_0(n)} =
$$

$$
\begin{aligned}
&\frac{7}{12}S_1(n)^4+\frac{(17n+5)S_1(n)^3}{3n(n+1)}+\Big(\frac{35n^2-2n-5}{2n^2(n+1)^2}+\frac{13S_2(n)}{2}+\frac{5(-1)^n}{2n^2}\Big)S_1(n)^2\\
&+\Big(-\frac{4(13n+5)}{n^2(n+1)^2}+\Big(\frac{4(-1)^n(2n+1)}{n(n+1)}-\frac{13}{n}\Big)S_2(n)+\Big(\frac{29}{3}-(-1)^n\Big)S_3(n)\\
&+\big(2+2(-1)^n\big)S_{2,1}(n)-28S_{-2,1}(n)+\frac{20(-1)^n}{n^2(n+1)}\Big)S_1(n)+\Big(\frac{3}{4}+(-1)^n\Big)S_2(n)^2\\
&-2(-1)^nS_{-2}(n)^2+S_{-3}(n)\Big(\frac{2(3n-5)}{n(n+1)}+\big(26+4(-1)^n\big)S_1(n)+\frac{4(-1)^n}{n+1}\Big)\\
&+\Big(\frac{(-1)^n(5-3n)}{2n^2(n+1)}-\frac{5}{2n^2}\Big)S_2(n)+S_{-2}(n)\Big(10S_1(n)^2+\Big(\frac{8(-1)^n(2n+1)}{n(n+1)}\\
&+\frac{4(3n-1)}{n(n+1)}\Big)S_1(n)+\frac{8(-1)^n(3n+1)}{n(n+1)^2}+\big(-22+6(-1)^n\big)S_2(n)-\frac{16}{n(n+1)}\Big)\\
&+\Big(\frac{(-1)^n(9n+5)}{n(n+1)}-\frac{29}{3n}\Big)S_3(n)+\Big(\frac{19}{2}-2(-1)^n\Big)S_4(n)+\big(-6+5(-1)^n\big)S_{-4}(n)\\
&+\Big(-\frac{2(-1)^n(9n+5)}{n(n+1)}-\frac{2}{n}\Big)S_{2,1}(n)+\big(20+2(-1)^n\big)S_{2,-2}(n)+\big(-17+13(-1)^n\big)S_{3,1}(n)\\
&-\frac{8(-1)^n(2n+1)+4(9n+1)}{n(n+1)}S_{-2,1}(n)-\big(24+4(-1)^n\big)S_{-3,1}(n)+\big(3-5(-1)^n\big)S_{2,1,1}(n)\\
&+32S_{-2,1,1}(n)+\left(\frac{3}{2}S_1(n)^2-\frac{3S_1(n)}{n}+\frac{3}{2}(-1)^nS_{-2}(n)\right)\zeta(2)
\end{aligned}
$$

Strategies 2,3,4: Find a recurrence for the integral/sum

$$D_\varepsilon(n) = \int_0^1 \dots \int_0^1 \Phi(\varepsilon, n, x_1, x_2, \dots, x_7) dx_1 dx_2 \dots dx_7$$
$$\stackrel{?}{=} F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + F_{-1}(n)\varepsilon^{-1} + \dots$$

multivariate
Almquist/Zeilberger
(Jakob Ablinger)

$$a_0(\varepsilon, n)D_\varepsilon(n) + \dots + a_d(\varepsilon, n)D_\varepsilon(n+d) = h(\varepsilon, n)$$

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multivariate
Almquist/Zeilberger
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$$\sum_{i_1} \cdots \sum_{i_7} f(\varepsilon, n, i_1, i_2, \dots, i_7)$$

MultiSum Package
(Flavia Stan)

$$a_0(\varepsilon, n) D_\varepsilon(n) + \dots + a_d(\varepsilon, n) D_\varepsilon(n+d) = h(\varepsilon, n)$$

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MultiSum Package
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Holonomic/difference field Approach
(Mark Round)

$$a_0(\varepsilon, n) D_\varepsilon(n) + \dots + a_d(\varepsilon, n) D_\varepsilon(n + d) = h(\varepsilon, n)$$

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ε-recurrence solver

multivariate
Almquist/Zeilberger
(Jakob Ablinger)

$$\sum_{i_1} \dots \sum_{i_7} f(\varepsilon, n, i_1, i_2, \dots, i_7)$$

MultiSum Package
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Holonomic/difference field Approach
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$$a_0(\varepsilon, n) D_\varepsilon(n) + \dots + a_d(\varepsilon, n) D_\varepsilon(n + d) = h(\varepsilon, n)$$

Ansatz (for power series)

$$
\begin{aligned}
&a_0(\varepsilon,n)\Big[F_0(n)+F_1(n)\varepsilon+F_2(n)\varepsilon^2+\dots\Big]\\
+&a_1(\varepsilon,n)\Big[F_0(n+1)+F_1(n+1)\varepsilon+F_2(n+1)\varepsilon^2+\dots\Big]\\
+&\\
&\vdots\\
+&a_d(\varepsilon,n)\Big[F_0(n+d)+F_1(n+d)\varepsilon+F_2(n+d)\varepsilon^2+\dots\Big]\\
&\qquad\qquad\qquad\qquad = h_0(n)+h_1(n)\varepsilon+h_2(n)\varepsilon^2+\dots
\end{aligned}
$$

given

Ansatz (for power series)

$$
\begin{aligned}
&a_0(\varepsilon,n)\Big[F_0(n)+F_1(n)\varepsilon+F_2(n)\varepsilon^2+\dots\Big]\\
+&a_1(\varepsilon,n)\Big[F_0(n+1)+F_1(n+1)\varepsilon+F_2(n+1)\varepsilon^2+\dots\Big]\\
+&\\
&\vdots\\
+&a_d(\varepsilon,n)\Big[F_0(n+d)+F_1(n+d)\varepsilon+F_2(n+d)\varepsilon^2+\dots\Big]\\
&\qquad\qquad\qquad\qquad = h_0(n)+h_1(n)\varepsilon+h_2(n)\varepsilon^2+\dots
\end{aligned}
$$

$\Downarrow$ constant terms must agree

$$
\boxed{a_0(0,n)F_0(n)+a_1(0,n)F_0(n+1)+\dots+a_d(0,n)F_0(n+d)=h_0(n)}
$$

Ansatz (for power series)

$$
\begin{aligned}
&a_0(\varepsilon,n)\Big[F_0(n)+F_1(n)\varepsilon+F_2(n)\varepsilon^2+\dots\Big]\\
&+a_1(\varepsilon,n)\Big[F_0(n+1)+F_1(n+1)\varepsilon+F_2(n+1)\varepsilon^2+\dots\Big]\\
&+\\
&\ \vdots\\
&+a_d(\varepsilon,n)\Big[F_0(n+d)+F_1(n+d)\varepsilon+F_2(n+d)\varepsilon^2+\dots\Big]\\
&\qquad\qquad=h_0(n)+h_1(n)\varepsilon+h_2(n)\varepsilon^2+\dots
\end{aligned}
$$

$\Downarrow$ constant terms must agree

$$
\boxed{a_0(0,n)F_0(n)+a_1(0,n)F_0(n+1)+\dots+a_d(0,n)F_0(n+d)=h_0(n)}
$$

If $F_0(n)$ (with required initial values) is not expressible in terms of indefinite nested sums and products: **game over**

Ansatz (for power series)

$$
\begin{aligned}
&a_0(\varepsilon,n)\Big[F_0(n)+F_1(n)\varepsilon+F_2(n)\varepsilon^2+\dots\Big]\\
+&a_1(\varepsilon,n)\Big[F_0(n+1)+F_1(n+1)\varepsilon+F_2(n+1)\varepsilon^2+\dots\Big]\\
+&\\
&\vdots\\
+&a_d(\varepsilon,n)\Big[F_0(n+d)+F_1(n+d)\varepsilon+F_2(n+d)\varepsilon^2+\dots\Big]\\
&\qquad\qquad = h_0(n)+h_1(n)\varepsilon+h_2(n)\varepsilon^2+\dots
\end{aligned}
$$

$\Downarrow$ constant terms must agree

$$
\boxed{a_0(0,n)F_0(n)+a_1(0,n)F_0(n+1)+\dots+a_d(0,n)F_0(n+d)=h_0(n)}
$$

Ansatz (for power series)

$$
\begin{aligned}
&a_0(\varepsilon,n)\Big[F_0(n)+F_1(n)\varepsilon+F_2(n)\varepsilon^2+\dots\Big]\\
&+a_1(\varepsilon,n)\Big[F_0(n+1)+F_1(n+1)\varepsilon+F_2(n+1)\varepsilon^2+\dots\Big]\\
&+\\
&\vdots\\
&+a_d(\varepsilon,n)\Big[F_0(n+d)+F_1(n+d)\varepsilon+F_2(n+d)\varepsilon^2+\dots\Big]\\
&\qquad\qquad= h_0(n)+h_1(n)\varepsilon+h_2(n)\varepsilon^2+\dots
\end{aligned}
$$

$\Downarrow$ constant terms must agree

$$
\boxed{a_0(0,n)F_0(n)+a_1(0,n)F_0(n+1)+\dots+a_d(0,n)F_0(n+d)=h_0(n)}
$$

$$
\begin{aligned}
&\;a_0(\varepsilon,n)\Big[F_1(n)\varepsilon+F_2(n)\varepsilon^2+\dots\Big]\\
&+a_1(\varepsilon,n)\Big[F_1(n+1)\varepsilon+F_2(n+1)\varepsilon^2+\dots\Big]\\
&+\\
&\;\vdots\\
&+a_d(\varepsilon,n)\Big[F_1(n+d)\varepsilon+F_2(n+d)\varepsilon^2+\dots\Big]\\
&\qquad\qquad\qquad\qquad=h_0'(n)+h_1'(n)\varepsilon+h_2'(n)\varepsilon^2+\dots
\end{aligned}
$$

$$
\begin{aligned}
&\quad a_0(\varepsilon,n)\Big[F_1(n)\varepsilon+F_2(n)\varepsilon^2+\dots\Big]\\
&+a_1(\varepsilon,n)\Big[F_1(n+1)\varepsilon+F_2(n+1)\varepsilon^2+\dots\Big]\\
&+\\
&\ \vdots\\
&+a_d(\varepsilon,n)\Big[F_1(n+d)\varepsilon+F_2(n+d)\varepsilon^2+\dots\Big]\\
&\qquad\qquad\qquad\qquad = \underbrace{h'_0(n)}_{\mathbf{=0}}+h'_1(n)\varepsilon+h'_2(n)\varepsilon^2+\dots
\end{aligned}
$$

$$
\begin{aligned}
&a_0(\varepsilon,n)\Big[F_1(n)\varepsilon+F_2(n)\varepsilon^2+\dots\Big]\\
&+a_1(\varepsilon,n)\Big[F_1(n+1)\varepsilon+F_2(n+1)\varepsilon^2+\dots\Big]\\
&+\\
&\vdots\\
&+a_d(\varepsilon,n)\Big[F_1(n+d)\varepsilon+F_2(n+d)\varepsilon^2+\dots\Big]\\
&\qquad\qquad\qquad=h'_1(n)\varepsilon+h'_2(n)\varepsilon^2+\dots
\end{aligned}
$$

Divide by ε

$$
\begin{aligned}
&\;a_0(\varepsilon,n)\Big[F_1(n)+F_2(n)\varepsilon+\dots\Big]\\
&+a_1(\varepsilon,n)\Big[F_1(n+1)+F_2(n+1)\varepsilon+\dots\Big]\\
&+\\
&\;\vdots\\
&+a_d(\varepsilon,n)\Big[F_1(n+d)+F_2(n+d)\varepsilon+\dots\Big]=h_1'(n)+h_2'(n)\varepsilon+\dots
\end{aligned}
$$

$$
\begin{aligned}
&a_0(\varepsilon,n)\Big[F_1(n)+F_2(n)\varepsilon+\dots\Big]\\
+&a_1(\varepsilon,n)\Big[F_1(n+1)+F_2(n+1)\varepsilon+\dots\Big]\\
+&\\
&\vdots\\
+&a_d(\varepsilon,n)\Big[F_1(n+d)+F_2(n+d)\varepsilon+\dots\Big]=h'_1(n)+h'_2(n)\varepsilon+\dots
\end{aligned}
$$

Now repeat for $F_1(n), F_2(n), \dots$

Remark: Works the same for Laurent series.

For math/CA details: J. Symbolic Comput. 2012 [arXiv:1011.2656v1]

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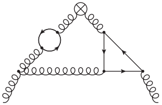
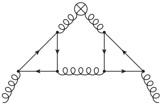
Example: 3-loop topologies of gluonic massive operator matrix elements with two fermion lines (unpolarized case)

$D_\varepsilon(n) =$  [by axodraw (J. Vermaseren)] $+$  $+ \sim$**80 further diagrams**

$\downarrow$ J. Blümlein, A. Hasselhuhn

$$D_\varepsilon(n) = \sum_{j_1=0}^{n-5}\sum_{j_2=0}^{n-j_1-6} \pi 2^{\varepsilon+3} e^{-\frac{3\gamma\varepsilon}{2}} \Gamma(2-\varepsilon)\Gamma\left(\frac{\varepsilon}{2}+2\right)\Gamma\left(-\frac{3\varepsilon}{2}\right) \quad (\sim 2\text{GB})$$

$$\times \frac{(-1)^{j_1}(j_2+1)\Gamma\left(-\frac{\varepsilon}{2}+j_1+4\right)\Gamma(-j_1+n-2)\Gamma(\varepsilon-j_1-j_2+n-5)}{(\varepsilon-10)(\varepsilon-8)(\varepsilon-2)\varepsilon\Gamma\left(\frac{5}{2}-\varepsilon\right)\Gamma\left(\frac{\varepsilon+5}{2}\right)\Gamma\left(\frac{\varepsilon}{2}+n+1\right)\Gamma(-j_1-j_2+n-4)}$$

$+ \sim$**2400 further multi-sums**

$\downarrow$ **time: $\sim$8 month**

$$D_\varepsilon(n) = \varepsilon^{-3}F_{-3}(n) + \varepsilon^{-2}F_{-2}(n) + \varepsilon^{-1}F_{-1}(n) + \varepsilon^{0}F_{0}(n) + \dots$$

Efficient calculation

1. Reduction to key sums

- Synchronize 2400 sums to 4 sums:

$$\sum_{i_2=5}^{n-5}\sum_{i_1=0}^{i_2} h_1(\varepsilon, n, i_2, i_1) \qquad \sum_{i_2=0}^{n-5}\sum_{i_1=0}^{n-i_2-5} h_2(\varepsilon, n, i_2, i_1)$$

$$\sum_{i_1=5}^{n-5} h_3(\varepsilon, n, i_1) \qquad \sum_{i_1=0}^{\infty} h_4(\varepsilon, n, i_1)$$

Note: 4 sums plus sum-free expression $>$ 2 GB

Efficient calculation

1. Reduction to key sums

- Synchronize 2400 sums to 4 sums:

$$\sum_{i_2=5}^{n-5}\sum_{i_1=0}^{i_2} h_1(\varepsilon,n,i_2,i_1) \qquad \sum_{i_2=0}^{n-5}\sum_{i_1=0}^{n-i_2-5} h_2(\varepsilon,n,i_2,i_1)$$

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- Eliminate algebraic relations among Γ/Pochhammer/binomial symbols

Efficient calculation

1. Reduction to key sums

- Synchronize 2400 sums to 4 sums:

$$\sum_{i_2=5}^{n-5}\sum_{i_1=0}^{i_2} h_1(\varepsilon,n,i_2,i_1) \qquad \sum_{i_2=0}^{n-5}\sum_{i_1=0}^{n-i_2-5} h_2(\varepsilon,n,i_2,i_1)$$

$$\sum_{i_1=5}^{n-5} h_3(\varepsilon,n,i_1) \qquad \sum_{i_1=0}^{\infty} h_4(\varepsilon,n,i_1)$$

Note: 4 sums plus sum-free expression $>$ 2 GB

- Eliminate algebraic relations among Γ/Pochhammer/binomial symbols

- Write the sums in the form

$\sum$ (product of Γ/Pochhammer/binomial symbols)*(rational function)

Note: 29 sums, total size: 7.6 MB

Efficient calculation

1. Reduction to key sums

In[7]:= $<<$ **SumProduction.m**

SumProduction - A summation package by Carsten Schneider © RISC-Linz

In[8]:= **expr** $=<<$ **DESYInput.txt;**

In[9]:= **compactExpr =**

ReduceMultiSums[expr, {n}, {5}];

7hours

2GB $\to$ **7.6**MB

Efficient calculation

1. Reduction to key sums

In[7]:= $<<$ **SumProduction.m**

SumProduction - A summation package by Carsten Schneider © RISC-Linz

In[8]:= $\mathbf{expr} =<< \mathbf{DESYInput.txt;}$

In[9]:= $\mathbf{compactExpr} =$

$\mathbf{ReduceMultiSums[expr, \{n\}, \{5\}];}$

7hours
2GB → **7.6**MB

2. (Parallel) calculation of the ε–expansion for each sum:

In[10]:= $\mathbf{ProcessEachSum[compactExpr, \{n\}, \{6\},}$
$\mathbf{ExpandIn \rightarrow \{ep, -3, 0\}]}$

90minutes

Efficient calculation

1. Reduction to key sums

In[7]:= $<< \mathbf{SumProduction.m}$

SumProduction - A summation package by Carsten Schneider © RISC-Linz

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7hours
2GB → **7.6**MB

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In[10]:= $\mathbf{ProcessEachSum[compactExpr, \{n\}, \{6\},}$

$$\mathbf{ExpandIn \to \{ep, -3, 0\}]}$$

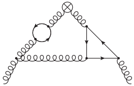
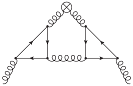
90minutes

3. Combine expansions (+eliminate algebraic relations):

In[11]:= $\mathbf{CombineExpression[compactExpr, \{n\}, \{6\}];}$

20seconds
7.6MB → **0.1**MB

Example: 3-loop topologies of gluonic massive operator matrix elements with two fermion lines (unpolarized case)

$D_\varepsilon(n) =$  [by axodraw (J. Vermaseren)] $+$  $+$ ∼80 further diagrams

$$\varepsilon^{-3}F_{-3}(n) + \varepsilon^{-2}F_{-2}(n) + \varepsilon^{-1}F_{-1}(n) + \varepsilon^{0}F_{0}(n) + \dots$$

where

$$F_{-3} = C_A\Big(\frac{512}{27}S_1(n) - \frac{64\big(3n^4+6n^3+19n^2+28n+28\big)}{27(n-1)n(n+1)(n+2)}\Big) - \frac{512C_F\big(n^2+n+2\big)^2}{9(n-1)n^2(n+1)^2(n+2)}$$

$$F_{-2} = C_A\Big(\frac{64\big(20n^2-20n+9\big)S_1(n)}{81(n-1)n} - \frac{16\big(3n^6+9n^5+367n^4+839n^3+1046n^2+568n+96\big)}{81(n-1)n^2(n+1)^2(n+2)}\Big)$$
$$+ C_F\Big(\frac{128\big(n^2+n+2\big)^2S_1(n)}{9(n-1)n^2(n+1)^2(n+2)} - \frac{128\big(14n^6+33n^5+59n^4+39n^3+55n^2+20n-12\big)}{27(n-1)n^3(n+1)^3(n+2)}\Big)$$

$$F_{-1} = C_A\Big(\zeta_2\Big(\frac{64}{9}S_1(n) - \frac{8\big(3n^4+6n^3+19n^2+28n+28\big)}{9(n-1)n(n+1)(n+2)}\Big) + \frac{64\big(20n^6+57n^5+12n^4-56n^3-61n^2-30n-16\big)S_1(n)}{27(n-1)n^2(n+1)^2(n+2)} -$$
$$- \frac{4\big(57n^8+228n^7+4044n^6+12486n^5+17787n^4+12342n^3+1952n^2-2368n-960\big)}{81(n-1)n^3(n+1)^3(n+2)}\Big)$$
$$+ C_F\Big(-\frac{160\big(n^2+n+2\big)^2S_1(n)^2}{9(n-1)n^2(n+1)^2(n+2)} + \frac{32\big(n^2+n+2\big)^2S_2(n)}{3(n-1)n^2(n+1)^2(n+2)} + \frac{64\big(16n^6+57n^5+268n^4+465n^3+410n^2+208n+4}{27(n-1)n^3(n+1)^3(n+2)}$$
$$- \frac{64\big(n^2+n+2\big)^2\zeta_2}{3(n-1)n^2(n+1)^2(n+2)} - \frac{64\big(185n^8+704n^7+1712n^6+2699n^5+3694n^4+3801n^3+2249n^2+744n+180\big)}{81(n-1)n^4(n+1)^4(n+2)}\Big)$$

$$F_0 = C_A\Big(\frac{8S_1(n)^3}{27(n-1)n} + \frac{4(16n^5+31n^4-38n^3-3n^2+50n+32)S_1(n)^2}{27(n-1)n^2(n+1)^2(n+2)} + \Big(\frac{8(6944n^8+26480n^7+24941n^6-7003n^5-247\ldots}{729(n-1)n^3(n+\ldots}$$

$$+\frac{8S_2(n)}{9(n-1)n}\Big)S_1(n) + \frac{4809n^{10}+24045n^9-224384n^8-1104398n^7-2105327n^6-2139551n^5-1133210n^4-209408n^3\ldots}{729(n-1)n^4(n+1)^4(n+2)}$$

$$+\zeta_3\Big(\frac{56(3n^4+6n^3+19n^2+28n+28)}{27(n-1)n(n+1)(n+2)} - \frac{448}{27}S_1(n)\Big) + \zeta_2\Big(\frac{8(20n^2-20n+9)S_1(n)}{27(n-1)n}$$

$$-\frac{2(3n^6+9n^5+367n^4+839n^3+1046n^2+568n+96)}{27(n-1)n^2(n+1)^2(n+2)}\Big) - \frac{4(40n^6+112n^5+7n^4-126n^3-251n^2-190n-96)S_2(n)}{27(n-1)n^2(n+1)^2(n+2)} + \ldots$$

$$+C_F\Big(\frac{112(n^2+n+2)^2S_1(n)^3}{27(n-1)n^2(n+1)^2(n+2)} - \frac{16(44n^6+123n^5+386n^4+543n^3+520n^2+248n+24)S_1(n)^2}{27(n-1)n^3(n+1)^3(n+2)}$$

$$+\Big(\frac{16S_2(n)(n^2+n+2)^2}{3(n-1)n^2(n+1)^2(n+2)} + \frac{32(205n^8+856n^7+3169n^6+6484n^5+7310n^4+4722n^3+1534n^2+48n-72)}{81(n-1)n^4(n+1)^4(n+2)}\Big)S_1(n)$$

$$-\frac{32(1976n^{10}+9385n^9+24088n^8+38989n^7+50214n^6+53872n^5+35219n^4+6890n^3-4233n^2-2844n-756)}{243(n-1)n^5(n+1)^5(n+2)} + \frac{44\ldots}{9(n-\ldots}$$

$$-\frac{16(14n^6+33n^5+59n^4+39n^3+55n^2+20n-12)}{9(n-1)n^3(n+1)^3(n+2)}\Big) + \frac{16(4n^6+3n^5-50n^4-129n^3-100n^2-56n-24)S_2(n)}{9(n-1)n^3(n+1)^3(n+2)} - \frac{160\ldots}{27(n\ldots}$$

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- Required time: 9 1/2 hours.
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- For another problem about 200000 quadruple sums have been reduced to about 80 key sums and have been calculated afterwards.