

RISC, Hagenberg, Austria

LHCPhenoNet School: Integration, Summation and Special Functions in QFT

Difference field algorithms for Feynman integrals (II)

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Summation Paradigms:

Telescoping, creative telescoping,
recurrence finding

Recall: Simplify

$$\sum_{k=1}^n S_1(k)$$

where $S_1(k) = \sum_{i=1}^k \frac{1}{i}$

Telescoping

GIVEN $f(k) = S_1(k)$.

FIND $g(k)$:

$$f(k) = g(k+1) - g(k)$$

for all $1 \leq k \leq n$ and $n \geq 0$.

Telescoping

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FIND $g(k)$:

$$f(k) = g(k+1) - g(k)$$

for all $1 \leq k \leq n$ and $n \geq 0$.

We compute

$$g(k) = (S_1(k) - 1)k.$$

Telescoping

GIVEN $f(k) = S_1(k)$.

FIND $g(k)$:

$$f(k) = g(k+1) - g(k)$$

for all $1 \leq k \leq n$ and $n \geq 0$.

Summing this equation over k from 1 to n gives

$$\begin{aligned} \sum_{k=1}^n S_1(k) &= g(n+1) - g(1) \\ &= (S_1(n+1) - 1)(n+1). \end{aligned}$$

Telescoping

FIND a closed form for

$$\sum_{k=1}^n S_1(k).$$

A $\Pi\Sigma^*$ -field for the **summand**

$$\text{const}_\sigma \mathbb{F} = \mathbb{Q}$$

Consider the rational function field

$$\mathbb{F} := \mathbb{Q}$$

with the automorphism $\sigma : \mathbb{F} \rightarrow \mathbb{F}$ defined by

$$\sigma(c) = c \quad \forall c \in \mathbb{Q},$$

Telescoping

FIND a closed form for

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A $\Pi\Sigma^*$ -field for the summand

$$\text{const}_\sigma \mathbb{F} = \mathbb{Q}$$

Consider the rational function field

$$\mathbb{F} := \mathbb{Q}(k)$$

with the automorphism $\sigma : \mathbb{F} \rightarrow \mathbb{F}$ defined by

$$\sigma(c) = c \quad \forall c \in \mathbb{Q},$$

$$\sigma(k) = k + 1,$$

$$S k = k + 1,$$

Telescoping

FIND a closed form for

$$\sum_{k=1}^n S_1(k).$$

A $\Pi\Sigma^*$ -field for the summand

$$\text{const}_\sigma \mathbb{F} = \mathbb{Q}$$

Consider the rational function field

$$\mathbb{F} := \mathbb{Q}(k)(h)$$

with the automorphism $\sigma : \mathbb{F} \rightarrow \mathbb{F}$ defined by

$$\sigma(c) = c \quad \forall c \in \mathbb{Q},$$

$$\sigma(k) = k + 1,$$

$$\sigma(h) = h + \frac{1}{k+1},$$

$$\mathcal{S}k = k + 1,$$

$$\mathcal{S}S_1(k) = S_1(k) + \frac{1}{k+1}.$$

Telescoping in the given difference field

FIND $g \in \mathbb{F}$:

$$\sigma(g) - g = h.$$

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$$g(k+1) - g(k) = S_1(k)$$

with

$$g(k) = (S_1(k) - 1)k.$$

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with

$$g(k) = (S_1(k) - 1)k.$$

Hence,

$$(S_1(n+1) - 1)(n+1) = \sum_{k=1}^n S_1(k).$$

Example

Simplify

$$\sum_{k=1}^n \binom{n}{k} S_1(k)$$

where $S_1(k) = \sum_{i=1}^k \frac{1}{i}$

Simplify

$$A(n) := \sum_{k=1}^n \binom{n}{k} S_1(k).$$

A difference field for the **summand**

Consider the rational function field

$$\mathbb{F} := \mathbb{Q}(n)(k)(h)(b)$$

with the automorphism $\sigma : \mathbb{F} \rightarrow \mathbb{F}$ defined by

$$\sigma(c) = c \quad \forall c \in \mathbb{Q}(n),$$

$$\sigma(k) = k + 1,$$

$$\sigma(h) = h + \frac{1}{k+1},$$

$$\sigma(b) = \frac{n-k}{k+1} b,$$

$$\mathcal{S} k = k + 1,$$

$$\mathcal{S} S_1(k) = S_1(k) + \frac{1}{k+1},$$

$$\mathcal{S} \binom{n}{k} = \frac{n-k}{k+1} \binom{n}{k}.$$

Creative telescoping

REPRESENT $f(n, k)$ in \mathbb{F} :

$$f(n, k) = S_1(k) \binom{n}{k} \longleftrightarrow h b =: f_0$$

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REPRESENT $f(n+i, k)$ in \mathbb{F} :

$$f(n, k) = S_1(k) \binom{n}{k} \longleftrightarrow hb =: f_0$$

$$f(n+1, k) = \frac{(n+1) S_1(k) \binom{n}{k}}{n+1-k} \longleftrightarrow \frac{(n+1)hb}{n+1-k} =: f_1$$

FIND $g \in \mathbb{F}$ and $c_0, c_1 \in \mathbb{Q}(n)$:

$$\sigma(g) - g = c_0 f_0 + c_1 f_1$$



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$$f(n+2, k) = \frac{(n+1)(n+2) S_1(k) \binom{n}{k}}{(n+1-k)(n+2-k)} \longleftrightarrow \frac{(n+1)(n+2)hb}{(n+1-k)(n+2-k)} =: f_2.$$

FIND $g \in \mathbb{F}$ and $c_0, c_1, c_2 \in \mathbb{Q}(n)$:

$$\sigma(g) - g = c_0 f_0 + c_1 f_1 + c_2 f_2$$



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FIND $g \in \mathbb{F}$ and $c_0, c_1, c_2 \in \mathbb{Q}(n)$:

$$\sigma(g) - g = c_0 f_0 + c_1 f_1 + c_2 f_2$$

We compute

$$c_0 := 4(1+n), \quad c_1 := -2(3+2n), \quad c_2 := 2+n,$$

$$g := \frac{(1+n)(-2+k-n+(2k-2k^2+kn)h)b}{(1-k+n)(2-k+n)}.$$

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This gives

$$\boxed{g(n, k+1) - g(n, k)} = \boxed{c_0(n)f(n, k) + c_1(n)f(n+1, k) + c_2(n)f(n+2, k)}$$

with

$$c_0(n) := 4(1+n), \quad c_1(n) := -2(3+2n), \quad c_2(n) := 2+n,$$

$$g(n, k) := \frac{(1+n)(-2+k-n+(2k-2k^2+kn)S_1(k)) \binom{n}{k}}{(1-k+n)(2-k+n)}.$$

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$$\boxed{g(n, k+1) - g(n, k)} = \boxed{c_0(n)f(n, k) + c_1(n)f(n+1, k) + c_2(n)f(n+2, k)}$$

Summing over k from 0 to n gives

$$\boxed{g(n, n+1) - g(n, 0)} = \boxed{\begin{aligned} &c_0(n)A(n) + \\ &c_1(n) [A(n+1) - f(n+1, n+1)] \\ &c_2(n) [A(n+2) - f(n+2, n+1) - f(n+2, n+2)]. \end{aligned}}$$

for $A(n) = \sum_{k=0}^n \binom{n}{k} S_1(k)$

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Summing over k from 0 to n gives

$$\boxed{1 = 4(1+n)S(n) - 2(3+2n)S(n+1) + (2+n)S(n+2)}$$

for $A(n) = \sum_{k=0}^n \binom{n}{k} S_1(k)$

Example

Summation paradigms

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a **definite** sum

$$A(n) = \sum_{k=0}^n f(n, k);$$

$f(n, k)$: indefinite nested product-sum in k ;
 n : extra parameter

FIND a **recurrence** for $A(n)$

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FIND a **recurrence** for $A(n)$

2. Recurrence solving

GIVEN a recurrence

$a_0(n), \dots, a_d(n), h(n)$:
 indefinite nested product-sum expressions.

$$a_d(n)A(n+d) + \dots + a_0(n)A(n) = h(n);$$

FIND **all solutions** expressible by indefinite nested products and sums
 (Abramov/Bronstein/Petkovšek/CS, in preparation)

Recurrence solving

Special case: homogeneous recurrences with $a_i(n) \in \mathbb{K}[n]$

$$a_d(n)A(n+d) + a_{d-1}(n)A(n+d-1) + \cdots + a_0(n)A(n) = 0$$

Recurrence solving

Special case: homogeneous recurrences with $a_i(n) \in \mathbb{K}[n]$

$$a_d(n)A(n+d) + a_{d-1}(n)A(n+d-1) + \cdots + a_0(n)A(n) = 0$$
$$\parallel$$
$$\left[a_d(n)S^d + a_{d-1}(n)S^{d-1} + \cdots + a_0(n)I \right] A(n)$$

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Hyper

$$\prod_{j=\lambda}^n b_1(j-1)$$


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$$a_d(n)A(n+d) + a_{d-1}(n)A(n+d-1) + \cdots + a_0(n)A(n) = 0$$

||

$$\left[\left(\tilde{a}_{d-1}(n)S^d + \tilde{a}_{d-2}(n)S^{d-2} + \cdots + \tilde{a}_0(n)I \right) \left(S - b_1(n) \right) \right] A(n)$$

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$$a_d(n)A(n+d) + a_{d-1}(n)A(n+d-1) + \cdots + a_0(n)A(n) = 0$$
$$\parallel$$
$$c(n) \left(S - b_d(n) \right) \cdots \left(S - b_2(n) \right) \left(S - b_1(n) \right) A(n)$$

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$$L_1(n) = \prod_{j=\lambda}^n b_1(j-1)$$

d linearly independent solutions

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⋮

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Example

⋮

$$L_d(n) = \prod_{j=\lambda}^n b_1(j-1) \sum_{i_1=\lambda}^{n-1} \frac{\prod_{j=\lambda}^{i_1} b_2(j-1)}{\prod_{j=\lambda}^{i_1+1} b_1(j-1)} \cdots \sum_{i_{d-1}=\lambda}^{i_{d-2}-1} \frac{\prod_{j=\lambda}^{i_{d-1}} b_d(j-1)}{\prod_{j=\lambda}^{i_{d-1}+1} b_{d-1}(j-1)}$$

Summation paradigms

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

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$$A(n) = \sum_{k=0}^n f(n, k);$$

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NOTE: By construction, the solutions are highly nested.

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3. Indefinite summation for simplification

Summation paradigms

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4. Find a "closed form"

$A(n)$ =combined solutions.

Warming up example

A warm up example

$$\begin{aligned}
 \text{GIVEN } F(n) &= \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{e^{-e\gamma}}{\Gamma(\varepsilon + 1)} \times \\
 &\times \left(\frac{\Gamma(k+1)}{\Gamma(k+2+n)} \frac{\Gamma(\frac{\varepsilon}{2})\Gamma(1-\frac{\varepsilon}{2})\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+1+\frac{\varepsilon}{2})\Gamma(k+j+1+n)}{\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+2+n)\Gamma(k+j+2)} \right. \\
 &\left. + \frac{\Gamma(k+1)}{\Gamma(k+2+n)} \frac{\Gamma(-\frac{\varepsilon}{2})\Gamma(1+\frac{\varepsilon}{2})\Gamma(j+1+\varepsilon)\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(k+j+1+\frac{\varepsilon}{2}+n)}{\Gamma(j+1)\Gamma(j+2+\frac{\varepsilon}{2}+n)\Gamma(k+j+2+\frac{\varepsilon}{2})} \right) \\
 &\quad \underbrace{\hspace{15em}}_{f(n, k, j)}.
 \end{aligned}$$

Arose in the context of

I. Bierenbaum, J. Blümlein, and S. Klein, *Evaluating two-loop massive operator matrix elements with Mellin-Barnes integrals*. 2006

A warm up example

$$\text{GIVEN } F(n) = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{e^{-\epsilon\gamma}}{\Gamma(\epsilon + 1)} \times$$

$$\times \left(\frac{\Gamma(k+1)}{\Gamma(k+2+n)} \frac{\Gamma(\frac{\epsilon}{2})\Gamma(1-\frac{\epsilon}{2})\Gamma(j+1-\frac{\epsilon}{2})\Gamma(j+1+\frac{\epsilon}{2})\Gamma(k+j+1+n)}{\Gamma(j+1-\frac{\epsilon}{2})\Gamma(j+2+n)\Gamma(k+j+2)} \right.$$

$$\left. + \frac{\Gamma(k+1)}{\Gamma(k+2+n)} \frac{\Gamma(-\frac{\epsilon}{2})\Gamma(1+\frac{\epsilon}{2})\Gamma(j+1+\epsilon)\Gamma(j+1-\frac{\epsilon}{2})\Gamma(k+j+1+\frac{\epsilon}{2}+n)}{\Gamma(j+1)\Gamma(j+2+\frac{\epsilon}{2}+n)\Gamma(k+j+2+\frac{\epsilon}{2})} \right).$$

$$f(n, k, j)$$

FIND the first coefficients of the ϵ -expansion

$$F(N) = F_0(n) + \epsilon F_1(n) + \dots$$

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$$\text{GIVEN } F(n) = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{e^{-\epsilon\gamma}}{\Gamma(\epsilon + 1)} \times$$

$$\times \left(\frac{\Gamma(k+1)}{\Gamma(k+2+n)} \frac{\Gamma(\frac{\epsilon}{2})\Gamma(1-\frac{\epsilon}{2})\Gamma(j+1-\frac{\epsilon}{2})\Gamma(j+1+\frac{\epsilon}{2})\Gamma(k+j+1+n)}{\Gamma(j+1-\frac{\epsilon}{2})\Gamma(j+2+n)\Gamma(k+j+2)} \right.$$

$$\left. + \frac{\Gamma(k+1)}{\Gamma(k+2+n)} \frac{\Gamma(-\frac{\epsilon}{2})\Gamma(1+\frac{\epsilon}{2})\Gamma(j+1+\epsilon)\Gamma(j+1-\frac{\epsilon}{2})\Gamma(k+j+1+\frac{\epsilon}{2}+n)}{\Gamma(j+1)\Gamma(j+2+\frac{\epsilon}{2}+n)\Gamma(k+j+2+\frac{\epsilon}{2})} \right).$$

$$f(n, k, j)$$

Step 1: Compute the first coefficients of the ϵ -expansion

$$f(n, k, j) = f_0(n, k, j) + \epsilon f_1(n, k, j) + \dots$$

Arose in the context of

I. Bierenbaum, J. Blümlein, and S. Klein, *Evaluating two-loop massive operator matrix elements with Mellin-Barnes integrals*. 2006

A warm up example

$$\begin{aligned} \text{GIVEN } F(n) &= \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{e^{-e\gamma}}{\Gamma(\varepsilon + 1)} \times \\ &\times \left(\frac{\Gamma(k+1)}{\Gamma(k+2+n)} \frac{\Gamma(\frac{\varepsilon}{2})\Gamma(1-\frac{\varepsilon}{2})\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+1+\frac{\varepsilon}{2})\Gamma(k+j+1+n)}{\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+2+n)\Gamma(k+j+2)} \right. \\ &\left. + \frac{\Gamma(k+1)}{\Gamma(k+2+n)} \frac{\Gamma(-\frac{\varepsilon}{2})\Gamma(1+\frac{\varepsilon}{2})\Gamma(j+1+\varepsilon)\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(k+j+1+\frac{\varepsilon}{2}+n)}{\Gamma(j+1)\Gamma(j+2+\frac{\varepsilon}{2}+n)\Gamma(k+j+2+\frac{\varepsilon}{2})} \right). \end{aligned}$$

$f(n, k, j)$

Step 2: **Simplify** the sums in

$$\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} f(n, k, j) = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} f_0(n, k, j) + \varepsilon \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} f_1(n, k, j) + \dots$$

Arose in the context of

I. Bierenbaum, J. Blümlein, and S. Klein, *Evaluating two-loop massive operator matrix elements with Mellin-Barnes integrals*. 2006

Simplify

$$\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \left(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} \right. \\ \left. + \frac{j!k!(j+k+n)!(-S_1(j) + S_1(j+k) + S_1(j+n) - S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!} \right) \\ \underbrace{\hspace{15em}}_{f(j)}$$

where

$$S_1(n) = \sum_{i=1}^n \frac{1}{i} (= H_n)$$

Simplify

$$\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \left(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} \right. \\ \left. + \frac{j!k!(j+k+n)!(-S_1(j) + S_1(j+k) + S_1(j+n) - S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!} \right) \\ \underbrace{\hspace{15em}}_{f(j)}$$

FIND $g(j)$:

$$f(j) = g(j+1) - g(j)$$

Simplify

$$\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \left(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} \right. \\ \left. + \frac{j!k!(j+k+n)!(-S_1(j) + S_1(j+k) + S_1(j+n) - S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!} \right) \\ \underbrace{\hspace{15em}}_{f(j)}$$

FIND $g(j)$:

$$\boxed{f(j) = g(j+1) - g(j)}$$

↑

$$g(j) = \frac{(j+k+1)(j+n+1)j!k!(j+k+n)!(S_1(j) - S_1(j+k) - S_1(j+n) + S_1(j+k+n))}{kn(j+k+1)!(j+n+1)!(k+n+1)!}$$

Simplify

$$\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \left(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} \right. \\ \left. + \frac{j!k!(j+k+n)!(-S_1(j) + S_1(j+k) + S_1(j+n) - S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!} \right) \\ \underbrace{\hspace{15em}}_{f(j)}$$

FIND $g(j)$:

$$f(j) = g(j+1) - g(j)$$

Summing the telescoping equation over j from 0 to a gives

$$\sum_{j=0}^a f(n, k, j) = g(a+1) - g(0) \\ = \frac{(a+1)!(k-1)!(a+k+n+1)!(S_1(a) - S_1(a+k) - S_1(a+n) + S_1(a+k+n))}{n(a+k+1)!(a+n+1)!(k+n+1)!} \\ + \frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)n!} + \frac{(2a+k+n+2)a!k!(a+k+n)!}{(a+k+1)(a+n+1)(a+k+1)!(a+n+1)!(k+n+1)!}$$

 $a \rightarrow \infty$

Simplify

$$\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \left(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} \right. \\ \left. + \frac{j!k!(j+k+n)!(-S_1(j) + S_1(j+k) + S_1(j+n) - S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!} \right) \\ \underbrace{\hspace{15em}}_{f(j)}$$

$$\sum_{j=0}^{\infty} f(n, k, j) = \frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)n!}$$

Simplify

$$\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \left(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} \right. \\ \left. + \frac{j!k!(j+k+n)!(-S_1(j) + S_1(j+k) + S_1(j+n) - S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!} \right) \\ \underbrace{\hspace{15em}}_{f(j)}$$

$$\sum_{k=1}^{\infty} \sum_{j=0}^{\infty} f(n, k, j) = \sum_{k=1}^{\infty} \frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)n!}$$

Telescoping

GIVEN

$$\text{SUM}(n) := \sum_{k=1}^a \frac{S_1(k) + S_1(n) - S_1(k+n)}{\underbrace{kn(k+n+1)n!}_{=: f(n,k)}}.$$

FIND $g(n, k)$:

$$\boxed{g(n, k+1) - g(n, k)} = \boxed{f(n, k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.no solution 

Zeilberger's creative telescoping paradigm

GIVEN

$$\text{SUM}(n) := \sum_{k=1}^a \frac{S_1(k) + S_1(n) - S_1(k+n)}{\underbrace{kn(k+n+1)n!}_{=: f(n,k)}}.$$

FIND $g(n, k)$ and $c_0(n), c_1(n)$:

$$\boxed{g(n, k+1) - g(n, k)} = \boxed{c_0(n)f(n, k) + c_1(n)f(n+1, k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

Zeilberger's creative telescoping paradigm

GIVEN

$$\text{SUM}(n) := \sum_{k=1}^a \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)n!}}_{=: f(n, k)}.$$

FIND $g(n, k)$ and $c_0(n), c_1(n)$:

$$\boxed{g(n, k+1) - g(n, k)} = \boxed{c_0(n)f(n, k) + c_1(n)f(n+1, k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

Sigma computes: $c_0(n) = -n, c_1(n) = (n+1)(n+2)$ and

$$g(n, k) = \frac{kS_1(k) + (-n-1)S_1(n) - kS_1(k+n) - 2}{(k+n+1)n!(n+1)^2}$$

Zeilberger's creative telescoping paradigm

GIVEN

$$\text{SUM}(n) := \sum_{k=1}^a \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)n!}}_{=: f(n, k)}.$$

FIND $g(n, k)$ and $c_0(n), c_1(n)$:

$$\boxed{g(n, k+1) - g(n, k)} = \boxed{c_0(n)f(n, k) + c_1(n)f(n+1, k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

Summing this equation over k from 0 to a gives:

$$\boxed{g(n, a+1) - g(n, 0)} = \boxed{c_0(n) \text{SUM}(n) + c_1(n) \text{SUM}(n+1)}$$

Zeilberger's creative telescoping paradigm

GIVEN

$$\text{SUM}(n) := \sum_{k=1}^a \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)n!}}_{=: f(n, k)}.$$

FIND $g(n, k)$ and $c_0(n), c_1(n)$:

$$\boxed{g(n, k+1) - g(n, k)} = \boxed{c_0(n)f(n, k) + c_1(n)f(n+1, k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.Summing this equation over k from 0 to a gives:

$$\begin{aligned} \boxed{g(n, a+1) - g(n, 0)} &= \boxed{c_0(n) \text{SUM}(n) + c_1(n) \text{SUM}(n+1)} \\ \parallel & \qquad \qquad \qquad \parallel \\ \frac{(a+1)(S_1(a)+S_1(n)-S_1(a+n))}{(n+1)^2(a+n+2)n!} & \qquad - n\text{SUM}(n) + (1+n)(2+n)\text{SUM}(n+1) \\ + \frac{a(a+1)}{(n+1)^3(a+n+1)(a+n+2)n!} & \end{aligned}$$

$$(n+2)\mathbf{A}(n+1) - n\mathbf{A}(n) = \frac{(n+1)S_1(n) + 1}{(n+1)^3}$$

recurrence finder

$$A(n) = \sum_{k=1}^{\infty} \frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}$$

$$(n+2)\mathbf{A}(n+1) - n\mathbf{A}(n) = \frac{(n+1)S_1(n) + 1}{(n+1)^3}$$

recurrence solver

$$A(n) = \sum_{k=1}^{\infty} \frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}$$

∈

$$\left\{ c \times \frac{1}{n(n+1)} + \frac{S_1(n)^2 + S_2(n)}{2n(n+1)} \mid c \in \mathbb{R} \right\}$$

where

$$S_1(n) = \sum_{i=1}^n \frac{1}{i}$$

$$S_2(n) = \sum_{i=1}^n \frac{1}{i^2}$$

$$(n+2)\mathbf{A}(n+1) - n\mathbf{A}(n) = \frac{(n+1)S_1(n) + 1}{(n+1)^3}$$

$$A(n) = \sum_{k=1}^{\infty} \frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)} = \frac{1}{2} \times \frac{1}{n(n+1)} + \frac{S_1(n)^2 + S_2(n)}{2n(n+1)}$$

where

$$S_1(n) = \sum_{i=1}^n \frac{1}{i} \quad S_2(n) = \sum_{i=1}^n \frac{1}{i^2}$$

Simplify

$$\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \left(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} \right. \\ \left. + \frac{j!k!(j+k+n)!(-S_1(j) + S_1(j+k) + S_1(j+n) - S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!} \right) \\ \underbrace{\hspace{15em}}_{f(j)}$$

$$\sum_{k=1}^{\infty} \sum_{j=0}^{\infty} f(n, k, j) = \sum_{k=1}^{\infty} \frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)n!} \\ = \frac{S_1(n)^2 + S_2(n)}{2n(n+1)!}$$

where

$$S_1(n) = \sum_{i=1}^n \frac{1}{i} \quad S_2(n) = \sum_{i=1}^n \frac{1}{i^2}$$

GIVEN

$$\begin{aligned} & \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{e^{-\epsilon\gamma}}{\Gamma(\epsilon+1)} \left(\frac{\Gamma(k+1)}{\Gamma(k+2+n)} \frac{\Gamma(\frac{\epsilon}{2})\Gamma(1-\frac{\epsilon}{2})\Gamma(j+1-\frac{\epsilon}{2})\Gamma(j+1+\frac{\epsilon}{2})\Gamma(k+j+1+n)}{\Gamma(j+1-\frac{\epsilon}{2})\Gamma(j+2+n)\Gamma(k+j+2)} \right. \\ & \left. + \frac{\Gamma(k+1)}{\Gamma(k+2+n)} \frac{\Gamma(-\frac{\epsilon}{2})\Gamma(1+\frac{\epsilon}{2})\Gamma(j+1+\epsilon)\Gamma(j+1-\frac{\epsilon}{2})\Gamma(k+j+1+\frac{\epsilon}{2}+n)}{\Gamma(j+1)\Gamma(j+2+\frac{\epsilon}{2}+n)\Gamma(k+j+2+\frac{\epsilon}{2})} \right). \\ & = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} f_0(n, k, j) + \end{aligned}$$

Sigma computes

$$\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} f_0(n, k, j) = \frac{S_1(n)^2 + 3S_1(n)}{2n(n+1)!}.$$

GIVEN

$$\begin{aligned} & \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{e^{-\varepsilon\gamma}}{\Gamma(\varepsilon+1)} \left(\frac{\Gamma(k+1)}{\Gamma(k+2+n)} \frac{\Gamma(\frac{\varepsilon}{2})\Gamma(1-\frac{\varepsilon}{2})\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+1+\frac{\varepsilon}{2})\Gamma(k+j+1+n)}{\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+2+n)\Gamma(k+j+2)} \right. \\ & \left. + \frac{\Gamma(k+1)}{\Gamma(k+2+n)} \frac{\Gamma(-\frac{\varepsilon}{2})\Gamma(1+\frac{\varepsilon}{2})\Gamma(j+1+\varepsilon)\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(k+j+1+\frac{\varepsilon}{2}+n)}{\Gamma(j+1)\Gamma(j+2+\frac{\varepsilon}{2}+n)\Gamma(k+j+2+\frac{\varepsilon}{2})} \right). \\ & = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} f_0(n, k, j) + \varepsilon \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} f_1(n, k, j) + \end{aligned}$$

Sigma computes

$$\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} f_0(n, k, j) = \frac{S_1(n)^2 + 3S_1(n)}{2n(n+1)!}.$$

$$\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} f_1(n, k, j) = \frac{-S_1(n)^3 - 3S_2(n)S_1(n) - 8S_3(n)}{6n(n+1)!}.$$

GIVEN

$$\begin{aligned} & \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{e^{-\varepsilon\gamma}}{\Gamma(\varepsilon+1)} \left(\frac{\Gamma(k+1)}{\Gamma(k+2+n)} \frac{\Gamma(\frac{\varepsilon}{2})\Gamma(1-\frac{\varepsilon}{2})\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+1+\frac{\varepsilon}{2})\Gamma(k+j+1+n)}{\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+2+n)\Gamma(k+j+2)} \right. \\ & \left. + \frac{\Gamma(k+1)}{\Gamma(k+2+n)} \frac{\Gamma(-\frac{\varepsilon}{2})\Gamma(1+\frac{\varepsilon}{2})\Gamma(j+1+\varepsilon)\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(k+j+1+\frac{\varepsilon}{2}+n)}{\Gamma(j+1)\Gamma(j+2+\frac{\varepsilon}{2}+n)\Gamma(k+j+2+\frac{\varepsilon}{2})} \right). \\ & = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} f_0(n, k, j) + \varepsilon \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} f_1(n, k, j) + \varepsilon^2 \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} f_2(n, k, j) + \end{aligned}$$

Sigma computes

$$\begin{aligned} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} f_2(n, k, j) &= \frac{1}{96n(n+1)} \left(S_1(n)^4 + (12\zeta_2 + 54S_2(n))S_1(n)^2 \right. \\ & \quad \left. + 104S_3(n)S_1(n) - 48S_{2,1}(n)S_1(n) + 51S_2(n)^2 + 36\zeta_2S_2(n) \right. \\ & \quad \left. + 126S_4(n) - 48S_{3,1}(n) - 96S_{1,1,2}(n) \right) \end{aligned}$$

GIVEN

$$\begin{aligned} & \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{e^{-\varepsilon\gamma}}{\Gamma(\varepsilon+1)} \left(\frac{\Gamma(k+1)}{\Gamma(k+2+n)} \frac{\Gamma(\frac{\varepsilon}{2})\Gamma(1-\frac{\varepsilon}{2})\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+1+\frac{\varepsilon}{2})\Gamma(k+j+1+n)}{\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+2+n)\Gamma(k+j+2)} \right. \\ & \left. + \frac{\Gamma(k+1)}{\Gamma(k+2+n)} \frac{\Gamma(-\frac{\varepsilon}{2})\Gamma(1+\frac{\varepsilon}{2})\Gamma(j+1+\varepsilon)\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(k+j+1+\frac{\varepsilon}{2}+n)}{\Gamma(j+1)\Gamma(j+2+\frac{\varepsilon}{2}+n)\Gamma(k+j+2+\frac{\varepsilon}{2})} \right). \\ & = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} f_0(n, k, j) + \varepsilon \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} f_1(n, k, j) + \varepsilon^2 \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} f_1(n, k, j) + \varepsilon^3 \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} f_1(n, k, j) + \dots \end{aligned}$$

Sigma computes

$$\begin{aligned} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} f_3(n, k, j) &= \frac{1}{960n(n+1)} \left(S_1(n)^5 + (20\zeta_2 + 130S_2(n))S_1(n)^3 + \right. \\ & (40\zeta_3 + 380S_3(n))S_1(n)^2 + (135S_2(n)^2 + 60\zeta_2S_2(n) + 510S_4(n))S_1(n) \\ & - 240S_{3,1}(n)S_1(n) - 240S_{1,1,2}(n)S_1(n) + 160\zeta_2S_3(n) + S_2(n)(120\zeta_3 \\ & + 380S_3(n)) + 624S_5(n) + (-120S_1(n)^2 - 120S_2(n))S_{2,1}(n) \\ & \left. - 240S_{4,1}(n) - 240S_{1,1,3}(n) + 240S_{2,2,1}(n) \right) \end{aligned}$$

Guessing and Finding

(J. Blümlein, M. Kauers, S. Klein, CS; Comput. Phys. Comm. 180, pp. 2143-2165. 2009; arXiv 0902.4091)

In the non-singlet (3-loop) case ~ 360 diagrams contribute. The integrals are of the form:

$$F(n, \varepsilon) = \int_0^1 dx_1 \dots \int_0^1 dx_7 \sum_{i=1}^K \frac{p_i(x_1, x_2, \dots, x_7)^{n+\dots+r_i\varepsilon+\dots}}{q_i(x_1, x_2, \dots, x_7)^{\dots+s_i\varepsilon+\dots}}$$

where $K \in \mathbb{N}$, $r_i, s_i \in \mathbb{Q}$, and p_i, q_i are polynomials in x_1, \dots, x_7 .

In the non-singlet (3-loop) case ~ 360 diagrams contribute. The integrals are of the form:

$$\begin{aligned}
 F(n, \varepsilon) &= \int_0^1 dx_1 \dots \int_0^1 dx_7 \sum_{i=1}^K \frac{p_i(x_1, x_2, \dots, x_7)^{n+\dots+r_i\varepsilon+\dots}}{q_i(x_1, x_2, \dots, x_7)^{\dots+s_i\varepsilon+\dots}} \\
 &= F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + F_{-1}(n)\varepsilon^{-1} + \boxed{F_0(n)}\varepsilon^0 + \dots
 \end{aligned}$$

The **3-loop anomalous dimensions** can be derived from the single pole part of $F(n, \varepsilon)$. The other poles are needed for the **renormalization**.

Vermaseren, Moch: 3-5 CPU years (2004)

In the non-singlet (3-loop) case ~ 360 diagrams contribute. The integrals are of the form:

$$\begin{aligned} F(n, \varepsilon) &= \int_0^1 dx_1 \dots \int_0^1 dx_7 \sum_{i=1}^K \frac{p_i(x_1, x_2, \dots, x_7)^{n+\dots+r_i\varepsilon+\dots}}{q_i(x_1, x_2, \dots, x_7)^{\dots+s_i\varepsilon+\dots}} \\ &= F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + F_{-1}(n)\varepsilon^{-1} + \boxed{F_0(n)}\varepsilon^0 + \dots \\ &\quad \downarrow \\ &\text{Initial values } F_0(i), i = 1, \dots, 5114 \end{aligned}$$

In the non-singlet (3-loop) case ~ 360 diagrams contribute. The integrals are of the form:

$$\begin{aligned}
 F(n, \varepsilon) &= \int_0^1 dx_1 \dots \int_0^1 dx_7 \sum_{i=1}^K \frac{p_i(x_1, x_2, \dots, x_7)^{n+\dots+r_i\varepsilon+\dots}}{q_i(x_1, x_2, \dots, x_7)^{\dots+s_i\varepsilon+\dots}} \\
 &= F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + F_{-1}(n)\varepsilon^{-1} + \boxed{F_0(n)}\varepsilon^0 + \dots
 \end{aligned}$$

↓

Initial values $F_0(i)$, $i = 1, \dots, 5114$

↓ Recurrence finder (M. Kauers)

$$a_0(n)F_0(n) + a_1(n)F_0(n+1) + \dots + a_{35}(n)F_0(n+35) = 0$$

$$a_0(n)F_0(n) + a_1(n)F_0(n+1) + \cdots + \boxed{a_{35}(n)}F_0(n+35) = 0$$

$$a_0(n)F_0(n) + a_1(n)F_0(n+1) + \cdots + \boxed{a_{35}(n)}F_0(n+35) = 0$$

$$a_{35}(n) = \boxed{A_0} + A_1n + A_2n^2 + \cdots + A_{938}n^{983} \in \mathbb{Z}[n]$$

$$a_0(n)F_0(n) + a_1(n)F_0(n+1) + \dots + \boxed{a_{35}(n)}F_0(n+35) = 0$$

$$a_{35}(n) = \boxed{A_0} + A_1n + A_2n^2 + \dots + A_{938}n^{983} \in \mathbb{Z}[n]$$

$$A_0 = 4640944309211313672503980223716264124200407085993854002412460315194$$

95765021269344971048446299722216293405285738333200767150194016391501666
 27950213807356109710952045603966273388757782697588602201277983560532017
 37487592671445911325765145271945214255462153147308420597210761595329365
 51563452998613135384718911305253299053198893606401464021608911620974192
 09001668029951620780182947258262939450801154511774527832503874341661898
 89167522107378468797979810265385510643937043867557563467523740406094658
 99100467933353731959645624977524424672990654427732309881685346483771128
 69020837147452024401528169079406933665344476181260243344172097691636706
 62803059675535809027169693064474147719610219849628486896079642312975136
 20776876867741883488363846944854496482629372436829699055391369178850397
 00381638011612302679580897488076647721311930634735316787779620757659951
 5202809978299053753901432067359626151

(885 decimal digits)

In the non-singlet (3-loop) case ~ 360 diagrams contribute. The integrals are of the form:

$$F(n, \varepsilon) = \int_0^1 dx_1 \dots \int_0^1 dx_7 \sum_{i=1}^K \frac{p_i(x_1, x_2, \dots, x_7)^{n+\dots+r_i\varepsilon+\dots}}{q_i(x_1, x_2, \dots, x_7)^{\dots+s_i\varepsilon+\dots}}$$

$$= F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + F_{-1}(n)\varepsilon^{-1} + \boxed{F_0(n)}\varepsilon^0 + \dots$$

↓

Initial values $F_0(i)$, $i = 1, \dots, 5114$

↓ Recurrence finder (M. Kauers)

$$a_0(n)F_0(n) + a_1(n)F_0(n+1) + \dots + a_{35}(n)F_0(n+35) = 0$$

↓

Sigma

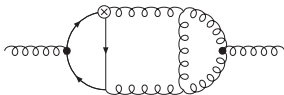
CLOSED FORM

Automatization

Example: All n -Results for 3-Loop Ladder Graphs

Joint work with J. Ablinger (RISC), J. Blümlein (DESY),
A. Hasselhuhn (DESY), S. Klein (RWTH)
(Nuclear Physics B, 2012; arXiv:1206.2252v1)

In total around 50 diagrams (for this class) have been calculated, like e.g.



(containing three massive fermion propagators)



Around 1000 sums have to be calculated for this diagram

$$\sum_{j=0}^{n-2} \sum_{r=0}^{j+1} \sum_{s=0}^{n-j+r-2} \frac{(-1)^{r+s} \binom{j+1}{r} \binom{-j+n+r-2}{s} (-j+n-2)! r!}{(n-s)(s+1)(-j+n+r)!}$$

Simple sum

$$\sum_{j=0}^{n-2} \sum_{r=0}^{j+1} \sum_{s=0}^{n-j+r-2} \frac{(-1)^{r+s} \binom{j+1}{r} \binom{-j+n+r-2}{s} (-j+n-2)! r!}{(n-s)(s+1)(-j+n+r)!}$$

||

$$\sum_{j=0}^{n-2} \sum_{r=0}^{j+1} \boxed{\sum_{s=0}^{n-j+r-2} \frac{(-1)^{r+s} \binom{j+1}{r} \binom{-j+n+r-2}{s} (-j+n-2)! r!}{(n-s)(s+1)(-j+n+r)!}}$$

$$\sum_{j=0}^{n-2} \sum_{r=0}^{j+1} \sum_{s=0}^{n-j+r-2} \frac{(-1)^{r+s} \binom{j+1}{r} \binom{-j+n+r-2}{s} (-j+n-2)! r!}{(n-s)(s+1)(-j+n+r)!}$$

||

$$\sum_{j=0}^{n-2} \sum_{r=0}^{j+1} \sum_{s=0}^{n-j+r-2} \frac{(-1)^{r+s} \binom{j+1}{r} \binom{-j+n+r-2}{s} (-j+n-2)! r!}{(n-s)(s+1)(-j+n+r)!}$$

||

$$\left(\binom{j+1}{r} \left(\frac{(-1)^r (-j+n-2)! r!}{(n+1)(-j+n+r-1)(-j+n+r)!} + \frac{(-1)^{n+r} (j+1)! (-j+n-2)! (-j+n-1)_r r!}{(n-1)n(n+1)(-j+n+r)! (-j-1)_r (2-n)_j} \right) \right)$$

$$\sum_{j=0}^{n-2} \sum_{r=0}^{j+1} \sum_{s=0}^{n-j+r-2} \frac{(-1)^{r+s} \binom{j+1}{r} \binom{-j+n+r-2}{s} (-j+n-2)! r!}{(n-s)(s+1)(-j+n+r)!}$$

$$\parallel$$

$$\sum_{j=0}^{n-2} \left(\sum_{r=0}^{j+1} \binom{j+1}{r} \left(\frac{(-1)^r (-j+n-2)! r!}{(n+1)(-j+n+r-1)(-j+n+r)!} + \frac{(-1)^{n+r} (j+1)! (-j+n-2)! (-j+n-1)_r r!}{(n-1)n(n+1)(-j+n+r)! (-j-1)_r (2-n)_j} \right) \right)$$

$$\sum_{j=0}^{n-2} \sum_{r=0}^{j+1} \sum_{s=0}^{n-j+r-2} \frac{(-1)^{r+s} \binom{j+1}{r} \binom{-j+n+r-2}{s} (-j+n-2)! r!}{(n-s)(s+1)(-j+n+r)!}$$

||

$$\sum_{j=0}^{n-2} \left(\sum_{r=0}^{j+1} \binom{j+1}{r} \left(\frac{(-1)^r (-j+n-2)! r!}{(n+1)(-j+n+r-1)(-j+n+r)!} + \frac{(-1)^{n+r} (j+1)! (-j+n-2)! (-j+n-1)_r r!}{(n-1)n(n+1)(-j+n+r)! (-j-1)_r (2-n)_j} \right) \right)$$

||

$$\left(\frac{n^2 - n + 1}{(n-1)^2 n^2 (n+1)(2-n)_j} + \frac{\sum_{i=1}^j \frac{(2-n)_i}{(-i+n-1)^2 (i+1)!}}{(n+1)(2-n)_j} + \frac{(-1)^{j+n} (-j-2)(-j+n-2)!}{(j-n+1)(n+1)^2 n!} \right) (j+1)! - \frac{1}{(n+1)^2 (-j+n-1)}$$

$$\sum_{j=0}^{n-2} \sum_{r=0}^{j+1} \sum_{s=0}^{n-j+r-2} \frac{(-1)^{r+s} \binom{j+1}{r} \binom{-j+n+r-2}{s} (-j+n-2)! r!}{(n-s)(s+1)(-j+n+r)!}$$

||

$$\sum_{j=0}^{n-2} \left(\left(\frac{n^2 - n + 1}{(n-1)^2 n^2 (n+1)(2-n)_j} + \frac{\sum_{i=1}^j \frac{(2-n)_i}{(-i+n-1)^2 (i+1)!}}{(n+1)(2-n)_j} + \frac{(-1)^{j+n} (-j-2)(-j+n-2)!}{(j-n+1)(n+1)^2 n!} \right) (j+1)! - \frac{1}{(n+1)^2 (-j+n-1)} \right)$$

$$\sum_{j=0}^{n-2} \sum_{r=0}^{j+1} \sum_{s=0}^{n-j+r-2} \frac{(-1)^{r+s} \binom{j+1}{r} \binom{-j+n+r-2}{s} (-j+n-2)! r!}{(n-s)(s+1)(-j+n+r)!}$$

||

$$\sum_{j=0}^{n-2} \left(\left(\frac{n^2 - n + 1}{(n-1)^2 n^2 (n+1)(2-n)_j} + \frac{\sum_{i=1}^j \frac{(2-n)_i}{(-i+n-1)^2 (i+1)!}}{(n+1)(2-n)_j} + \frac{(-1)^{j+n} (-j-2)(-j+n-2)!}{(j-n+1)(n+1)^2 n!} \right) (j+1)! - \frac{1}{(n+1)^2 (-j+n-1)} \right)$$

||

$$\frac{-n^2 - n - 1}{n^2 (n+1)^3} + \frac{(-1)^n (n^2 + n + 1)}{n^2 (n+1)^3} - \frac{2S_{-2}(n)}{n+1} + \frac{S_1(n)}{(n+1)^2} + \frac{S_2(n)}{-n-1}$$

Note: $S_a(n) = \sum_{i=1}^N \frac{\text{sign}(a)^i}{i^{|a|}}$, $a \in \mathbb{Z} \setminus \{0\}$

Example .

Mathematica Session:

In[1]:= << **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= << **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[3]:= << **EvaluateMultiSums.m**

EvaluateMultiSums by Carsten Schneider © RISC-Linz

In[4]:= **EvaluateMultiSum** $\left[\frac{(-1)^{r+s} \binom{j+1}{r} \binom{-j+n+r-2}{s} (-j+n-2)! r!}{(n-s)(s+1)(-j+n+r)!}, \{ \{s, 0, n-j+r-2\}, \{r, 0, j+1\}, \{j, 0, n-2\} \}\right]$

Out[4]= $\frac{-n^2 - n - 1}{n^2(n+1)^3} + \frac{(-1)^n (n^2 + n + 1)}{n^2(n+1)^3} - \frac{2S_{-2}(n)}{n+1} + \frac{S_1(n)}{(n+1)^2} + \frac{S_2(n)}{-n-1}$

A typical sum

$$\sum_{j=0}^{n-2} \sum_{s=1}^{j+1} \sum_{r=0}^{n+s-j-2} \sum_{\sigma=0}^{\infty} \frac{-2(-1)^{s+r} \binom{j+1}{s} \binom{-j+n+s-2}{r} (n-j)!(s-1)!\sigma! S_1(r+2)}{(n-r)(r+1)(r+2)(-j+n+\sigma+1)(-j+n+\sigma+2)(-j+n+s+\sigma)!}$$

A typical sum

$$\sum_{j=0}^{n-2} \sum_{s=1}^{j+1} \sum_{r=0}^{n+s-j-2} \sum_{\sigma=0}^{\infty} \frac{-2(-1)^{s+r} \binom{j+1}{s} \binom{-j+n+s-2}{r} (n-j)!(s-1)!\sigma! S_1(r+2)}{(n-r)(r+1)(r+2)(-j+n+\sigma+1)(-j+n+\sigma+2)(-j+n+s+\sigma)!}$$

$$= \frac{(2n^2 + 6n + 5) S_{-2}(n)^2}{2(n+1)(n+2)} + S_{-2,-1,2}(n) + S_{-2,1,-2}(n)$$

$$+ \dots$$

where, e.g.,

$$S_{-2,1,-2}(n) = \sum_{i=1}^n \frac{(-1)^i \sum_{j=1}^i \frac{\sum_{k=1}^j (-1)^k}{k^2}}{j^2}$$

Vermaseren 98/Blümlein/Kurth 99

A typical sum

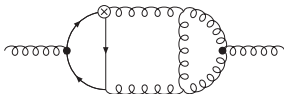
$$\begin{aligned}
& \sum_{j=0}^{n-2} \sum_{s=1}^{j+1} \sum_{r=0}^{n+s-j-2} \sum_{\sigma=0}^{\infty} \frac{-2(-1)^{s+r} \binom{j+1}{s} \binom{-j+n+s-2}{r} (n-j)!(s-1)!\sigma! S_1(r+2)}{(n-r)(r+1)(r+2)(-j+n+\sigma+1)(-j+n+\sigma+2)(-j+n+s+\sigma)!} \\
&= \frac{(2n^2 + 6n + 5) S_{-2}(n)^2}{2(n+1)(n+2)} + S_{-2,-1,2}(n) + S_{-2,1,-2}(n) \\
&+ \dots - S_{2,1,1,1}(-1, 2, \frac{1}{2}, -1; n) + S_{2,1,1,1}(1, \frac{1}{2}, 1, 2; n) \\
&+ \dots
\end{aligned}$$

where, e.g.,

145 S -sums occur

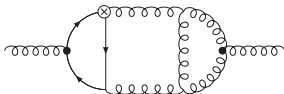
$$S_{2,1,1,1}(1, \frac{1}{2}, 1, 2; n) = \sum_{i=1}^n \frac{\sum_{j=1}^i \frac{\left(\frac{1}{2}\right)^j \sum_{k=1}^j \frac{\sum_{l=1}^k \frac{2^l}{l}}{k}}{j}}{i^2}$$

S. Moch, P. Uwer, S. Weinzierl 02



Sigma.m

Around 1000 sums are calculated containing in total 533 S -sums



Sigma.m

Around 1000 sums are calculated containing in total 533 S -sums

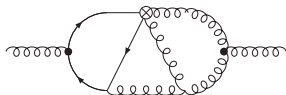


J. Ablinger's HarmonicSum.m

After elimination the following sums remain:

$$S_{-4}(n), S_{-3}(n), S_{-2}(n), S_1(n), S_2(n), S_3(n), S_4(n), S_{-3,1}(n), \\ S_{-2,1}(n), S_{2,-2}(n), S_{2,1}(n), S_{3,1}(n), S_{-2,1,1}(n), S_{2,1,1}(n)$$

So far, the most complicated 3-loop ladder graph:



$$= F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + F_{-1}(n)\varepsilon^{-1} + \boxed{F_0(n)}$$

So far, the most complicated 3-loop ladder graph:



$$= F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + F_{-1}(n)\varepsilon^{-1} + \boxed{F_0(n)}$$

||

$$\sum_{j=0}^{n-3} \sum_{k=0}^j \sum_{l=0}^k \sum_{q=0}^{-j+n-3-l+n-q-3} \sum_{s=1}^{-l+n-q-3-l+n-q-s-3} \sum_{r=0}^{-l+n-q-s-3} (-1)^{-j+k-l+n-q-3} \times$$

$$\times \frac{\binom{j+1}{k+1} \binom{k}{l} \binom{n-1}{j+2} \binom{-j+n-3}{q} \binom{-l+n-q-3}{s} \binom{-l+n-q-s-3}{r} r! (-l+n-q-r-s-3)! (s-1)!}{(-l+n-q-2)! (-j+n-1) (n-q-r-s-2) (q+s+1)}$$

$$\left[4S_1(-j+n-1) - 4S_1(-j+n-2) - 2S_1(k) \right.$$

$$\left. - (S_1(-l+n-q-2) + S_1(-l+n-q-r-s-3) - 2S_1(r+s)) \right.$$

$$\left. + 2S_1(s-1) - 2S_1(r+s) \right] + \mathbf{3 \text{ further 6-fold sums}}$$

$$\boxed{F_0(n)} =
\frac{7}{12}S_1(n)^4 + \frac{(17n+5)S_1(n)^3}{3n(n+1)} + \left(\frac{35n^2-2n-5}{2n^2(n+1)^2} + \frac{13S_2(n)}{2} + \frac{5(-1)^n}{2n^2}\right)S_1(n)^2
+ \left(-\frac{4(13n+5)}{n^2(n+1)^2} + \left(\frac{4(-1)^n(2n+1)}{n(n+1)} - \frac{13}{n}\right)S_2(n) + \left(\frac{29}{3} - (-1)^n\right)S_3(n)\right)
+ (2+2(-1)^n)S_{2,1}(n) - 28S_{-2,1}(n) + \frac{20(-1)^n}{n^2(n+1)}S_1(n) + \left(\frac{3}{4} + (-1)^n\right)S_2(n)^2
- 2(-1)^nS_{-2}(n)^2 + S_{-3}(n)\left(\frac{2(3n-5)}{n(n+1)} + (26+4(-1)^n)S_1(n) + \frac{4(-1)^n}{n+1}\right)
+ \left(\frac{(-1)^n(5-3n)}{2n^2(n+1)} - \frac{5}{2n^2}\right)S_2(n) + S_{-2}(n)\left(10S_1(n)^2 + \frac{8(-1)^n(2n+1)}{n(n+1)}\right)
+ \frac{4(3n-1)}{n(n+1)}S_1(n) + \frac{8(-1)^n(3n+1)}{n(n+1)^2} + (-22+6(-1)^n)S_2(n) - \frac{16}{n(n+1)}
+ \left(\frac{(-1)^n(9n+5)}{n(n+1)} - \frac{29}{3n}\right)S_3(n) + \left(\frac{19}{2} - 2(-1)^n\right)S_4(n) + (-6+5(-1)^n)S_{-4}(n)
+ \left(-\frac{2(-1)^n(9n+5)}{n(n+1)} - \frac{2}{n}\right)S_{2,1}(n) + (20+2(-1)^n)S_{2,-2}(n) + (-17+13(-1)^n)S_{3,1}(n)
- \frac{8(-1)^n(2n+1)+4(9n+1)}{n(n+1)}S_{-2,1}(n) - (24+4(-1)^n)S_{-3,1}(n) + (3-5(-1)^n)S_{2,1,1}(n)
+ 32S_{-2,1,1}(n) + \left(\frac{3}{2}S_1(n)^2 - \frac{3S_1(n)}{n} + \frac{3}{2}(-1)^nS_{-2}(n)\right)\zeta(2)$$

New Strategies

Find a recurrence for the integral/sum

$$D_\varepsilon(n) = \int_0^1 \dots \int_0^1 \Phi(\varepsilon, n, x_1, x_2, \dots, x_7) dx_1 dx_2 \dots dx_7$$
$$\stackrel{?}{=} F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + F_{-1}(n)\varepsilon^{-1} + \dots$$

multivariate
Almquist/Zeilberger
(Jakob Ablinger)

$$a_0(\varepsilon, n)D_\varepsilon(n) + \dots + a_d(\varepsilon, n)D_\varepsilon(n+d) = h(\varepsilon, n)$$

Find a recurrence for the integral/sum

$$D_\varepsilon(n) = \int_0^1 \dots \int_0^1 \Phi(\varepsilon, n, x_1, x_2, \dots, x_7) dx_1 dx_2 \dots dx_7$$

$$\stackrel{?}{=} F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + F_{-1}(n)\varepsilon^{-1} + \dots$$

multivariate
Almquist/Zeilberger
(Jakob Ablinger)

$$\sum_{i_1} \dots \sum_{i_7} f(\varepsilon, n, i_1, i_2, \dots, i_7)$$

MultiSum Package
(Flavia Stan)

$$a_0(\varepsilon, n)D_\varepsilon(n) + \dots + a_d(\varepsilon, n)D_\varepsilon(n+d) = h(\varepsilon, n)$$

Find a recurrence for the integral/sum

$$D_\varepsilon(n) = \int_0^1 \dots \int_0^1 \Phi(\varepsilon, n, x_1, x_2, \dots, x_7) dx_1 dx_2 \dots dx_7$$

$$\stackrel{?}{=} F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + F_{-1}(n)\varepsilon^{-1} + \dots$$

multivariate
Almquist/Zeilberger
(Jakob Ablinger)

$$\sum_{i_1} \dots \sum_{i_7} f(\varepsilon, n, i_1, i_2, \dots, i_7)$$

MultiSum Package
(Flavia Stan)

Holonomic/difference field Approach
(Mark Round)

$$a_0(\varepsilon, n)D_\varepsilon(n) + \dots + a_d(\varepsilon, n)D_\varepsilon(n+d) = h(\varepsilon, n)$$

Find a recurrence for the integral/sum

$$D_\varepsilon(n) = \int_0^1 \dots \int_0^1 \Phi(\varepsilon, n, x_1, x_2, \dots, x_7) dx_1 dx_2 \dots dx_7$$

$$\stackrel{?}{=} F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + F_{-1}(n)\varepsilon^{-1} + \dots$$

 ε -recurrence solver

multivariate
Almquist/Zeilberger
(Jakob Ablinger)

$$\sum_{i_1} \dots \sum_{i_7} f(\varepsilon, n, i_1, i_2, \dots, i_7)$$

MultiSum Package
(Flavia Stan)

Holonomic/difference field Approach
(Mark Round)

$$a_0(\varepsilon, n)D_\varepsilon(n) + \dots + a_d(\varepsilon, n)D_\varepsilon(n+d) = h(\varepsilon, n)$$

Ansatz (for power series)

$$\begin{aligned} & a_0(\varepsilon, n) \left[F_0(n) + F_1(n)\varepsilon + F_2(n)\varepsilon^2 + \dots \right] \\ & + a_1(\varepsilon, n) \left[F_0(n+1) + F_1(n+1)\varepsilon + F_2(n+1)\varepsilon^2 + \dots \right] \\ & + \\ & \vdots \\ & + a_d(\varepsilon, n) \left[F_0(n+d) + F_1(n+d)\varepsilon + F_2(n+d)\varepsilon^2 + \dots \right] \\ & \qquad \qquad \qquad = h_0(n) + h_1(n)\varepsilon + h_1(n)\varepsilon^2 + \dots \end{aligned}$$

given

Ansatz (for power series)

$$\begin{aligned} & a_0(\varepsilon, n) \left[F_0(n) + F_1(n)\varepsilon + F_2(n)\varepsilon^2 + \dots \right] \\ & + a_1(\varepsilon, n) \left[F_0(n+1) + F_1(n+1)\varepsilon + F_2(n+1)\varepsilon^2 + \dots \right] \\ & + \\ & \vdots \\ & + a_d(\varepsilon, n) \left[F_0(n+d) + F_1(n+d)\varepsilon + F_2(n+d)\varepsilon^2 + \dots \right] \\ & \qquad \qquad \qquad = h_0(n) + h_1(n)\varepsilon + h_1(n)\varepsilon^2 + \dots \end{aligned}$$

↓ constant terms must agree

$$a_0(0, n)F_0(n) + a_1(0, n)F_0(n+1) + \dots + a_d(0, n)F_0(n+d) = h_0(n)$$

Ansatz (for power series)

$$\begin{aligned}
 & a_0(\varepsilon, n) \left[F_0(n) + F_1(n)\varepsilon + F_2(n)\varepsilon^2 + \dots \right] \\
 & + a_1(\varepsilon, n) \left[F_0(n+1) + F_1(n+1)\varepsilon + F_2(n+1)\varepsilon^2 + \dots \right] \\
 & + \\
 & \vdots \\
 & + a_d(\varepsilon, n) \left[F_0(n+d) + F_1(n+d)\varepsilon + F_2(n+d)\varepsilon^2 + \dots \right] \\
 & \qquad \qquad \qquad = h_0(n) + h_1(n)\varepsilon + h_1(n)\varepsilon^2 + \dots
 \end{aligned}$$

↓ constant terms must agree

$$a_0(0, n)F_0(n) + a_1(0, n)F_0(n+1) + \dots + a_d(0, n)F_0(n+d) = h_0(n)$$

If $F_0(n)$ (with required initial values) is not expressible in terms of indefinite nested sums and products:

game over

Ansatz (for power series)

$$\begin{aligned}
 & a_0(\varepsilon, n) \left[F_0(n) + F_1(n)\varepsilon + F_2(n)\varepsilon^2 + \dots \right] \\
 & + a_1(\varepsilon, n) \left[F_0(n+1) + F_1(n+1)\varepsilon + F_2(n+1)\varepsilon^2 + \dots \right] \\
 & + \\
 & \vdots \\
 & + a_d(\varepsilon, n) \left[F_0(n+d) + F_1(n+d)\varepsilon + F_2(n+d)\varepsilon^2 + \dots \right] \\
 & \qquad \qquad \qquad = h_0(n) + h_1(n)\varepsilon + h_1(n)\varepsilon^2 + \dots
 \end{aligned}$$

⇓ constant terms must agree

$$a_0(0, n)F_0(n) + a_1(0, n)F_0(n+1) + \dots + a_d(0, n)F_0(n+d) = h_0(n)$$

Ansatz (for power series)

$$\begin{aligned}
 & a_0(\varepsilon, n) \left[F_0(n) + F_1(n)\varepsilon + F_2(n)\varepsilon^2 + \dots \right] \\
 & + a_1(\varepsilon, n) \left[F_0(n+1) + F_1(n+1)\varepsilon + F_2(n+1)\varepsilon^2 + \dots \right] \\
 & + \\
 & \vdots \\
 & + a_d(\varepsilon, n) \left[F_0(n+d) + F_1(n+d)\varepsilon + F_2(n+d)\varepsilon^2 + \dots \right] \\
 & \qquad \qquad \qquad = h_0(n) + h_1(n)\varepsilon + h_1(n)\varepsilon^2 + \dots
 \end{aligned}$$

↓ constant terms must agree

$$a_0(0, n)F_0(n) + a_1(0, n)F_0(n+1) + \dots + a_d(0, n)F_0(n+d) = h_0(n)$$

$$\begin{aligned} & a_0(\varepsilon, n) \left[F_1(n)\varepsilon + F_2(n)\varepsilon^2 + \dots \right] \\ + & a_1(\varepsilon, n) \left[F_1(n+1)\varepsilon + F_2(n+1)\varepsilon^2 + \dots \right] \\ + & \\ & \vdots \\ + & a_d(\varepsilon, n) \left[F_1(n+d)\varepsilon + F_2(n+d)\varepsilon^2 + \dots \right] \\ & = h'_0(n) + h'_1(n)\varepsilon + h'_2(n)\varepsilon^2 + \dots \end{aligned}$$

$$\begin{aligned}
 & a_0(\varepsilon, n) \left[F_1(n)\varepsilon + F_2(n)\varepsilon^2 + \dots \right] \\
 & + a_1(\varepsilon, n) \left[F_1(n+1)\varepsilon + F_2(n+1)\varepsilon^2 + \dots \right] \\
 & + \\
 & \vdots \\
 & + a_d(\varepsilon, n) \left[F_1(n+d)\varepsilon + F_2(n+d)\varepsilon^2 + \dots \right] \\
 & \qquad \qquad \qquad = \underbrace{h'_0(n) + h'_1(n)}_{=0} \varepsilon + h'_2(n)\varepsilon^2 + \dots
 \end{aligned}$$

Divide by ε

$$\begin{aligned} & a_0(\varepsilon, n) \left[F_1(n) + F_2(n)\varepsilon + \dots \right] \\ & + a_1(\varepsilon, n) \left[F_1(n+1) + F_2(n+1)\varepsilon + \dots \right] \\ & + \\ & \vdots \\ & + a_d(\varepsilon, n) \left[F_1(n+d) + F_2(n+d)\varepsilon + \dots \right] = h'_1(n) + h'_2(n)\varepsilon + \dots \end{aligned}$$

Now repeat for $F_1(n), F_2(n), \dots$

Example

Remark: Works the same for Laurent series.

(see J. Blümlein, S. Klein, CS, F. Stan. J. Symbolic Comput. 47, 2012; arXiv:1011.2656v2)